CERTAIN PROBLEMS IN MAGNETOHYDRODYNAMICS

CONSIDERING THE FINITE CONDUCTIVITY OF THE MEDIUM

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FOREWORD

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CERTAIN PROBLEMS IN MAGNETOHYDRODYNAMICS
CONSIDERING THE FINITE CONDUCTIVITY OF THE MEDIUM

Following is a translation of an article by I. I. Nochevinka entitled "Nekotoryye Zadachi Magnitnoy Gidrodinamiki s Uchetom konechnoy Provodimosti Sredy" (English version above) in Vestnik Moskovskogo Universiteta, Seriya III, Fizika, Astronomiya (Herald of Moscow University, Series 3, Physics, Astronomy), Vol 1961, No 1, Moscow, 1961.

An approximating method of finding the parameters of planar motion of a conducting gas is offered, taking into account "magnetic" viscosity in the presence of a magnetic field, perpendicular to the plane of flow.

An approximating technique is advanced for the solution of the equations which describe the planar, isentropic flow of an ultrarelativistic gas in an arbitrary magnetic field.

Examinations of the motion of conducting fluids in the environment of magnetic fields, with the consideration of magnetic viscosity, have always encountered substantial mathematical difficulties. Up till now only the special cases of motion of a fluid with finite conductivity in a magnetic field have been successfully investigated.

We will examine planar motion of an ideally compressible fluid with finite conductivity in a transverse magnetic
field $H(x, y, z)$, which can be described by the system of equations

$$\text{rot}[\vec{u} H] = -\gamma_i \nabla \vec{H}; \text{div} \vec{H} = 0;$$

$$(\vec{u} \nabla) \vec{u} = -\frac{1}{\gamma_i} \nabla P^* + \frac{f^*}{\gamma_i}; \text{div}(\rho \vec{v}) = 0; \quad p^* = f(u^*, s^*),$$

where $\gamma_i = \frac{c_i^2}{4\pi n_i}$ - coefficient of magnetic viscosity, $f^* = -\frac{1}{8\pi} \gamma_i H_{\text{vol}}^3$ - volume density of the electromagnetic force.

We introduce the dimensionless parameters

$$H = \frac{H^*}{a_0 \gamma_0}; \quad \vec{v} = \frac{\vec{v}^*}{a_0}; \quad \rho = \frac{\rho^*}{\rho_0}; \quad p = \frac{p^*}{\rho_0 a_0^2}; \quad x = \frac{a_0}{\gamma_0} x^*;$$

$$y = \frac{a_0}{\gamma_0} y^*; \quad \gamma_i = \frac{\gamma_i}{\gamma_0},$$

where by the zero index is signified the corresponding characteristic magnitudes for the given flow. System (1) in dimensionless variables takes the form

$$\text{rot}[\vec{u} H] = -\gamma_i \nabla \vec{H}; \text{div} \vec{H} = 0;$$

$$(\vec{v} \nabla) \vec{v} = -\frac{1}{\gamma_i} \nabla P; \text{div}(\rho \vec{v}) = 0; \quad P = f_1(u, s),$$

where $P = p + \frac{H^3}{8\pi}$ - the total pressure of the gas and of the magnetic field.

In the case of constant finite conductivity ($\eta = \text{const}$), considering that $\text{div} \vec{H} = 0$, the equation of induction to within the gradient of an arbitrary function
can be put in the form

$$[\bar{v} \overline{H}] = \gamma \text{rot}\overline{H}. \quad (4)$$

which is equivalent to the equations

$$
\begin{align*}
\nu_y H_z &= \frac{\partial}{\partial y} (\gamma H_z); & \nu_x H_y &= \frac{\partial}{\partial x} (\gamma H_y), \\
\text{i.e. } \nu &= \text{grad} \tau, \text{ where } \tau = \ln H_z. \quad (5)
\end{align*}
$$

Thus, the investigation of planar motion of an ideally compressible conducting fluid with a constant coefficient of conductivity in a transverse magnetic field reduces to the investigation, with the aid of well-cultivated methods, of purely hydrodynamical potential flows by the means of the transformed equation of state $\gamma 2$.

In a series of cases it is necessary to consider the variability of the coefficient of conductivity due to the presence of large conductivity gradients. If under these circumstances the conductivity gradient coincides in direction with the conductivity current, as for example in the cooling of the stream in a plasmatron, then $[\text{grad} \gamma \text{rot} \overline{H}] = 0$ and the equation of induction within the margin of an arbitrary function can be written as

$$[\bar{v} \overline{H}] = \gamma(x, y) \text{rot}\overline{H},$$

from which

$$\bar{v}(x, y) = \gamma(x, y) \text{grad} \tau. \quad (6)$$

where $\tau = \ln H_z$ and $\ln H_z = \text{const}$ form a family of surfaces, normal to the lines of current. This is equivalent to some vortical flow with a coefficient of proportionality.
In such cases, when from a series of factors giving rise to a variability in the magnetic viscosity, we can single out a basic one, i.e., we can consider the coefficient of magnetic viscosity as a function of one independent variable, e.g. \( \eta = \eta(T) \), the examining of the parameters can be conducted with the assistance of the method developed in work [1].

In the role of an application we will examine the problem of the outflow of an ideally compressible fluid in the presence of a conductivity gradient which coincides in direction with the conductivity current, from an infinite vessel with flat sides in the presence of a perpendicular magnetic field. We will accept the above mentioned suppositions and consider the equation of state in the form \( P(\rho, s) = A(s) \rho^2 - B \). Performing calculations analogous to [1], we will obtain equations for the determination of the functions of current \( \varphi \) and of the quasipotential \( \psi \) in the variables \( r(\eta), \theta (\Theta) \) - the angle of inclination of the velocity vector with the \( \chi \) - axis).

\[
\frac{\partial^2 \varphi}{\partial s^2} + \frac{d}{dr} \ln \left[ K \frac{r'(\chi) \left( 6 \rho^2 - 9 \rho \rho_p \cdot 4 \rho_p^2 \right)}{2 \rho^3 \left( \rho - \rho_p \right)^2} \right] \frac{\partial \varphi}{\partial r} + \frac{1}{K} \left[ \frac{2 \rho^3 \left( \rho - \rho_p \right)^2}{r'(\chi) \left( 6 \rho^2 - 9 \rho \rho_p \cdot 4 \rho_p^2 \right)} \right]^2 \frac{\partial^2 \varphi}{\partial \Theta^2} = 0. \tag{7}
\]

\[
\frac{\partial^2 \varphi}{\partial s^2} + \frac{d}{dr} \ln \left[ \frac{r'(\chi) \left( 6 \rho^2 - 9 \rho \rho_p \cdot 4 \rho_p^2 \right)}{2 \rho^3 \left( \rho - \rho_p \right)^2} \right] \frac{\partial \varphi}{\partial r} + \frac{1}{K} \left[ \frac{2 \rho^3 \left( \rho - \rho_p \right)^2}{r'(\chi) \left( 6 \rho^2 - 9 \rho \rho_p \cdot 4 \rho_p^2 \right)} \right]^2 \frac{\partial^2 \varphi}{\partial \rho^2} = 0. \tag{8}
\]

By the introduction of Chapygin's approximating function \( K(r) \), equations (7) and (8) can be put in the form [1].
\[ \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{D_1}{c_1^2} \right) \left( 1 - \frac{1}{r^2} \right) \frac{\partial \psi}{\partial \theta} = 0. \]  

(9)

where the coefficients $D_1$ and $c_1$ are chosen so that they will give the approximate equation of state's best approximation to what is asked for in the defined range of Mach numbers.

Let $AB$ and $A'B'$ be projections of the walls onto the plane $XY$ (Fig. 1). We will turn the $X$-axis perpendicular to the wall of the vessel through the middle of $BB'$. We will consider the volume outflow per second of the fluid $\Omega$, which by virtue of continuity must be one and the same in all sections of the stream. Let $\pi_0$ - a plane of fluid at infinity, $\pi_1$ - a plane outside the vessel, $\pi_0$ - a plane in an adiabatically restricted gas. We will select the line of current along the $X$-axis as zero, i.e., $\psi(\pi, 0) = 0$ for $0 < \pi < \pi_0$; $\pi_1$ - a plane in an adiabatically restricted gas. The boundary conditions $\psi(\pi, 0)$ for such a choice are obtained as

\[ \psi \left( \pi \pm \frac{\pi}{2} \right) = \frac{Q}{2} \text{ for } \pi \pm \frac{\pi}{2}; \]  

\[ \psi \left( \pi_0 - \pi_1, \theta \right) = \frac{Q}{2} \text{ for } -\frac{\pi}{2} < \theta < 0; \]  

\[ \psi \left( \pi_0 + \frac{\pi}{2} \right) = -\frac{Q}{2} \text{ for } \pi_0 + \frac{\pi}{2} > \pi_1; \]  

\[ \psi \left( \pi_0 - \pi_1, \theta \right) = -\frac{Q}{2} \text{ for } 0 < \theta < -\frac{\pi}{2}. \]  

(10)
Considering the region of variability of the plane of the fluid $z \sim r \sim r_1$, from the relationship connecting $r$ with $\sqrt{1/r}$, we will determine the range of variability of $r$. Then immediately with the help of (10) we obtain boundary conditions for the function $\psi(r, \theta)$.

$$\psi(r, -\frac{n}{2}) = \frac{Q}{2} \text{ for } r > r_1;$$

$$\psi(r_1, \theta) = \frac{Q}{2} \text{ for } \frac{n}{2} \leq \theta \leq 0;$$

$$\psi(r, \frac{n}{2}) = \frac{Q}{2} \text{ for } r_1 < r < r_1;$$

$$\psi (r, -r_1, 0) = 0 \text{ for } 0 < \theta < \frac{n}{2}. \quad (11)$$

Equation (9) can be solved by a Fourier method

$$\psi(r, \theta) = R(r) \Phi(\theta). \quad (12)$$

For the functions $R(r)$ and $\Phi(\theta)$ we obtain the equations

$$\frac{d^2 \Phi(\theta)}{d\theta^2} + \nu^2 \Phi(\theta) = 0. \quad (13)$$

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \nu^2 A^\times \left(\frac{1}{r^2} - 1\right) R(r) = 0. \quad (14)$$

where

$$A^\times = \frac{D_1}{C_1}; \quad 0.$$
From the condition of periodicity for the function \( \Phi(\theta) \):

\[
\Phi(\theta) = \Phi(\theta + 2\pi), \quad \Phi'(\theta) = \Phi'(\theta + 2\pi)
\]

it follows that \( \mu^n = n^2 \), where \( n \) is an integer. Introducing a new variable \( \tilde{r} = \sqrt{A^2 - nr} \) into Bessel's equation (14) we will obtain

\[
\frac{d^2 R(\tilde{r})}{d\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{dR(\tilde{r})}{d\tilde{r}} + \left(1 - \frac{1}{\beta^2}\right) R(\tilde{r}) = 0,
\]

whose solution is expressible by Bessel functions of imaginary argument.

\[
R(\tilde{r}) = C_1 I_1 + C_2 K_1.
\]

From the conditions of boundedness of the fluid plane at infinity \( \rho < N \), the boundedness of \( \tilde{r}(\eta) \) follows; from where \( C_1 = 0 \). The general solution of equation (14) we obtain in the form

\[
\psi(r, \theta) = \sum_{n} A_{1n} \cos(n \theta) + A_{2n} \sin(n \theta) J_1\left(\sqrt{\frac{A^2}{r}} \eta\right).
\]

The function \( \psi(r, \theta) \) can be analogously defined.

The transition to the physical plane can be accomplished with the aid of the formulas

\[
dx = \frac{1}{\nu} \left[ (\eta \cos \theta \frac{\partial r}{\partial \tilde{r}} - \frac{1}{\rho} \sin \theta \frac{\partial \tilde{r}}{\partial \tilde{r}}) \right] d\tilde{r} + \frac{1}{\nu} \left[ (\eta \sin \theta \frac{\partial \tilde{r}}{\partial \tilde{r}} - \frac{1}{\rho} \cos \theta \frac{\partial \tilde{r}}{\partial \tilde{r}}) \right] d\tilde{r}.
\]
Thus the methods developed in hydrodynamics can be effectively applied not only in the investigation of planar motion of a conducting fluid in a transverse magnetic field, when the conductivity is infinitely large, but also in a series of cases when it is necessary to concern oneself with the presence of magnetic viscosity, constant or variable with the gradient which coincides with the direction of the conductivity current.

The approximating method described can also be applied in the examination of planar problems in the ultrarelativistic case of motion of a conducting medium in the presence of arbitrary magnetic fields.

The motion of a relativistic gas in the presence of arbitrary magnetic fields can be represented in the form of an equality to zero of the divergence of the total mechanical and electromagnetic tensor of the energy-impulse. In the case of an infinite conductivity of the gas, the energy-impulse tensor can be written in the form

\[
T_{ik} = \frac{w^*}{V} U_i U_k + p^* \delta^{ik} (i, k = 1, 2, 3, 4),
\]  

(20)

where \( w^* = pV + pVc^2 + \omega \) - the total heat content of the gas and of the magnetic field, \( p^* = \rho + \rho' \) - the total pressure, \( V \) - the specific volume, \( U_i \) - 4-velocity, \( x_{1,2,3} = x, y, z; x_4 =ict \).

It can be proved that in the case of arbitrarily
chosen magnetic fields it is sufficient to limit oneself only to the investigation of the modified tensor of energy-impulse of macroscopic bodies, which include in themselves additional heat content \( w' \) and additional pressure \( P' \). In the general case \( w' \) and \( P' \) can easily be calculated by the formulas

\[
F_i\cdot F_j = \frac{u^2}{c^4} \frac{F^2_{im}}{4\varepsilon^2 n'_{ik}} U_i U_j; P' = - \frac{F^2_{im}}{16\varepsilon^4} \left( \theta = \sqrt{1 - \frac{u^2}{c^2}} \right),
\]

where \( F_{ik} \) — components of the electromagnetic field tensor in the system of representation \( K' \), relative to which the given element of gas travels with velocity \( u \).

The expediency of knowing the total energy-impulse tensor of the system, gas and electromagnetic field, compared to the tensor of the macroscopic bodies appears during the transition from one representation to the other, seeing that the Lorentz transformation for the components of the tensor of an electromagnetic field \( F_{ik} \) is much simpler than for the components of the electromagnetic tensor of energy-impulse \( T_{ik} \).

An investigation of the parameters of planar flow of an ultrarelativistic gas in a given magnetic field in the case of maintaining isentropic behaviour can be conducted with the aid of the method set forth in work \( \int_{l}^{m} \). We will conduct an investigation for an infinitesimally small element of gas in the laboratory system \( K' \). In the case of isentropic flows, there exists, as is known, a relativistic analogue of the potential

\[
\frac{\partial \phi}{\partial x_i} = U_i \phi^*.
\]

from which for planar flows we have for \( i = 1, 2 \)

\[
\omega^* U_1 = \frac{\partial \phi}{\partial x_1} \quad \omega^* U_2 = \frac{\partial \phi}{\partial x_2}.
\]
Introducing aiwo with the help of the relativistic equation of continuity the relativistic analogue of the function of current

$$\frac{U_1}{V} = \frac{\partial \phi}{\partial x_1}; \quad \frac{U_2}{V} = \frac{\partial \phi}{\partial x_2},$$

we obtain equations analogous to the equations for stationary, planar, vortical flow of a common gas \(1\)

$$\frac{1}{\omega^*} \frac{\partial \phi}{\partial x} = V \frac{\partial \phi}{\partial y}; \quad \frac{1}{\omega^*} \frac{\partial \phi}{\partial y} = -V \frac{\partial \phi}{\partial x} (x_1, y) = x, y,$$

(25)

where the role of the coefficient of proportionality is filled by the magnitude of the reciprocal of the total heat content. Thus, the problem reduces to the solution of the equations (25) with the use of the relativistic equation of Bernoulli

$$\frac{\omega^*}{\nu} = \omega^*_{0} - \text{const},$$

(26)

and of the equation of state

$$\rho^* = (\gamma - 1)\varepsilon^*,$$  \(1 \leq \gamma \leq 2\),

(27)

which in the case of an ultrarelativistic system is wholly determined by the fixing of one thermodynamic function (e.g. \(\omega^*\)). Introducing new independent variables \(\xi(\omega^*)\) and \(\eta\), where \(\eta\) - angle, formed by the 3-vector of velocity \(\frac{u}{u}\) with the X-axis, and noting that \(u = u(\omega^*)\), \(V = V(\omega^*)\), we can put the equations (25) in the form

$$\frac{1}{\omega^*} \frac{\partial \varphi}{\partial \xi} = \left(\frac{\partial V}{\partial \omega^*} - \frac{V}{u} \frac{\partial u}{\partial \omega^*}\right) \frac{du^*}{dx} \frac{\partial \xi}{\partial x} + \frac{1}{\omega^*} \left(\frac{1}{u} \frac{\partial u}{\partial \omega^*}\right) \frac{du^*}{dx} \frac{\partial \xi}{\partial x} = V \frac{\partial \xi}{\partial \xi},$$

(28)
With this we suppose that the functions \( q \) and \( q^* \) are continuous, finite, single-valued and that the Jacobian

\[
(x, y) = 0 \quad \text{in all regions of the flow. Placing in (28)}
\]

the quantities \( u(w^*) \) and \( \frac{\partial u}{\partial w^*} \), defined in (26), \( v(w^*) \) and

\[
\frac{\partial v}{\partial w^*} \quad \text{in (27), noting in the limiting case } \gamma = \frac{4}{3}, \text{ and}
\]

imposing on \( \zeta \) (arbitrary function of \( w^* \)) the simplified condition

\[
\frac{A_4(-3w^*+4)}{w^*(1-w^*)} \frac{dw^*}{\xi^*} = 1.
\]

(29)

we put equations (28) in the form

\[
\frac{\partial \zeta}{\partial s} = \frac{\partial \eta}{\partial s} \cdot \frac{\partial \eta}{\partial \zeta} = -K(w^*) \frac{\partial \zeta}{\partial \xi},
\]

(30)

where

\[
K(w^*) = \frac{A_4(-3w^*+4)}{w^*(1-w^*)}
\]

is the Chapygin function introduced by us.

The solution of equations (30) with the help of Chapygin's approximating method is put forth in work [1].

Isentropic flow of an ultrarelativistic gas in a perpendicular magnetic field is examined by the surmised method. By this it comes to light that a perpendicular magnetic field does not violate the isentropicity of an ultrarelativistic gas. In the case of a magnetic
field arbitrarily disposed in the plane of flow, isentropic behaviour is generally violated.

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