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L. CROCCO

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SUPERCRITICAL GASEOUS DISCHARGE WITH HIGH FREQUENCY OSCILLATIONS

LUIGI CROCCO (*)

RIASSUNTO. – *La conoscenza del comportamento di un effusore supercritico quando le condizioni a monte dell'effusore sono oscillatorie è importante in connessione con vari problemi relativi ai propulsori a reazione. Tsien ha recentemente analizzato la questione nel caso di piccola ampiezza di oscillazione e per una distribuzione lineare di velocità nella parte subsonica dell'effusore. Egli ha dato la soluzione nei due casi estremi di frequenze molto basse o molto alte facendo uso di alcune ipotesi restrittive. Qui lo studio viene esteso ad ipotesi più generali; inoltre nel caso di oscillazioni isentropiche viene data la soluzione per tutte le frequenze.*

Introduction

Many gas flow systems, with or without combustion, are terminated by a discharge nozzle. When the stability of these systems is studied, it is necessary to determine the behavior of the nozzle under oscillatory conditions, which may be quite different from the corresponding behavior in steady state. The problem has recently been treated by Tsien (ref. 1) for supercritical discharge, this case being the most interesting for its applications to the stability of combustion in rockets or in combustors for jet devises. Tsien has investigated the case in which the oscillations in the incoming flow are isothermal, and therefore non isentropic, and has computed the departures from the steady state behavior in the range of low frequencies, as well as the asymptotic response to very high frequencies. Unfortunately the frequencies appearing in the study of high frequency instability are likely to be in the intermediate range, where none of Tsien's solutions can be used. Also the isothermal condition is not representative of most of the actual cases. The purpose of the present paper is the extension of Tsien's treatment to the non-isothermal case, and especially the determination of the nozzle behavior in the intermediate range of frequencies.

The equations

Calling p, ρ, u the pressure, density and velocity in steady state, completely determined by the shape of the nozzle, and $p + p'$, $\rho + \rho'$, $u + u'$ the corresponding values in unsteady conditions; and assuming the perturbations p', ρ', u' to be small compared with the unperturbed quantities, Tsien has written the continuity and momentum equations in the following form, retaining only the first-order terms in the perturbations

$$\frac{\partial}{\partial t}\left(\frac{\rho'}{\rho}\right) + u\frac{\partial}{\partial x}\left(\frac{\rho'}{\rho} + \frac{u'}{u}\right) = 0 \qquad [1]$$

$$\frac{\partial}{\partial t}\left(\frac{u'}{u}\right) + \left(\frac{\rho'}{\rho} + 2\frac{u'}{u}\right)\frac{du}{dx} + u\frac{\partial}{\partial x}\left(\frac{u'}{u}\right) = \frac{p'}{p}\frac{du}{dx} - \frac{p}{\rho u}\frac{\partial}{\partial x}\left(\frac{p'}{p}\right) \qquad [2]$$

x being the distance along the nozzle and t the time. The third equation between the dependent variables p'/p, ρ'/ρ and u'/u is the energy equation or, more simply, the equation expressing the constance of entropy of any fluid mass when we follow its motion:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\frac{S'}{c_v} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{p'}{p} - \gamma\frac{\rho'}{\rho}\right) = 0 \qquad [3]$$

where S' is the entropy perturbation, c_v is the constant volume specific heat, and γ the adiabatic index.

In these equations u, du/dx and $p/\rho u$ are to be considered known functions of x, determined by the nozzle shape. Due to the linearity of equations [1], [2], [3] the harmonic form of oscillatory time dependence can be chosen, and, using the complex representation, the dependent variables can be written as

$$\begin{aligned} \frac{p'}{p} &= \varphi(x)\, e^{i\omega t} \\ \frac{\rho'}{\rho} &= \delta(x)\, e^{i\omega t} \\ \frac{u'}{u} &= \nu(x)\, e^{i\omega t} \end{aligned} \qquad [4]$$

where ω is the angular frequency and φ, δ, ν are complex functions of x alone. At the nozzle entrance, $x = x_e$, the three functions have certain values φ_e, δ_e, ν_e. The problem in which we are interested is finding the distributions of φ, δ, ν along the nozzle, but expecially determining the relations between φ_e, δ_e, ν_e. These relations will in fact constitute the boundary conditions to be applied to the rest of the flow system as a result of the nozzle presence (1).

Equation [3] is immediately integrated as

$$\frac{S'}{c_v} = \frac{p'}{p} - \gamma\frac{\rho'}{\rho} = f\left(t - \int_{x_e}^{x}\frac{dx}{u}\right)$$

the arbitrary function f being in general determined by the known time dependance of the entropy at $x = x_e$. For the exponential time dependance assumed

(*) Professore di Motori Aeronautici alla Scuola d'Ingegneria Aeronautica dell'Università di Roma. Attualmente Robert H. Goddard, Professor, Jet Propulsion Center, Princeton, N. Y.

(1) Instead of determining these boundary conditions, Tsien has preferred to introduce a « Transfer function » relating the fractional variation of mass flow to the fractional variation of pressure. The definition of Tsien's transfer function can be easily extended to the non isothermal case. However we prefer not to use this concept which has no direct utility in the applications. The present procedure is closely related to the conventional use of the « Impedence » in acoustic treatments.

in equations [4] we obtain

$$\nu(x) - \gamma\,\delta(x) = \sigma \exp\left(-i\,\omega \int_{x_e}^{x} \frac{dx}{u}\right) \qquad [5]$$

where the constant σ represents the amplitude of the entropy oscillation divided by c_p. With the assumption [4] and the relation [5] equations [1] and [2] are reduced to the following system of ordinary differential equations in ν and δ

$$u\frac{d\nu}{dx} + u\frac{d\,\delta}{d\,x} + i\,\omega\,\delta = 0$$

$$\begin{aligned} u\frac{d\nu}{dx} + \frac{c^2}{u}\frac{d\delta}{dx} + \left(2\frac{du}{dx} - i\,\omega\right)\nu - (\gamma-1)\frac{du}{dx}\,\delta = \\ = \sigma\left(\frac{du}{dx} + i\,\omega\,\frac{c^2}{\gamma u^2}\right)\exp\left(-i\,\omega\int_{x_e}^{x}\frac{dx}{u}\right) \end{aligned} \qquad [6]$$

in which c is the sound velocity. It is easily seen that these equations present a singularity at $u = c$, that is at the sonic throat; where therefore only one family of solutions remains regular.

Equations [6] could be discussed and solved numerically for general nozzle shape. However the numerical integration can be avoided if, following Tsien, we confine our attention to nozzles in which u increases linearly with x in the subsonic portion of the nozzle, this condition being not too restrictive since many actual nozzles have practically linear velocity distribution near the sonic throat; and since this is the region in which an analytical solution is particularly useful. We take therefore

$$\frac{du}{dx} = \frac{u}{x} = \frac{c_*}{x_*} = \frac{c_* - u_e}{l} \qquad [7]$$

with c_* representing the so-called critical sound speed reached at the throat, where $x = x_*$, and $l = x_* - x_e$ representing the length of the subsonic portion of the nozzle.

Moreover we take with Tsien a new independent variable

$$z = \left(\frac{x}{x_*}\right)^2 = \left(\frac{u}{c_*}\right)^2 \qquad [8]$$

in terms of which we have

$$c^2 = c_*^2\left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\,z\right)$$

$$\int_{x_e}^{x}\frac{dx}{u} = \frac{x_*}{2\,c_*}\log\frac{z}{z_e} \qquad [9]$$

where z_e represents the assigned value of [8] at the nozzle entrance; and we define a reduced angular frequency

$$\beta = \frac{x_*\,\omega}{c_*} = \frac{l\,\omega}{c_* - u_e}. \qquad [10]$$

Introducing [7], [8], [9], [10] in equations [6] we find, eliminating $d\nu/dx$:

$$\begin{aligned} (2 + i\,\beta)\,\nu = (\gamma - 1 + i\,\beta)\,\delta - (\gamma + 1)(1 - z)\frac{d\,\delta}{dz} + \\ + \sigma\left(\frac{z}{z_e}\right)^{-\frac{i\beta}{2}}\left[1 + \frac{i\,\beta(\gamma-1)}{2\gamma} - \frac{i\,\beta(\gamma-1)}{2\gamma}\,\frac{1}{z}\right] \end{aligned} \qquad [11]$$

and eliminating ν with the help of [11] from one of the equations [6] we obtain

$$\begin{aligned} z(1-z)\frac{d^2\delta}{dz^2} - \left(2 + \frac{2\,i\,\beta}{\gamma+1}\right)z\frac{d\,\delta}{dz} - \frac{i\,\beta\,(2 + i\,\beta)}{2(\gamma+1)}\,\delta = \\ = -\,i\,\beta\,\sigma\left(\frac{z}{z_e}\right)^{-\frac{i\beta}{2}}\left[\frac{1 - i\,\beta\,\frac{\gamma-1}{2\gamma}}{2(\gamma+1)} + \frac{2 + i\,\beta}{4\,\gamma}\,\frac{1}{z}\right]. \end{aligned} \qquad [12]$$

Equation [12] is a non homogeneous complex hypergeometric equation, with singularities at $z = 0$, $z = 1$, and $z = \infty$. Of these singularities only the one at $z = 1$ is important for our problem, since the other are out of the range of variability of z, which musts be contained between a non-vanishing minimum at the entrance of the nozzle and a finite maximum at its exit. The singular point $z = 1$ is a particular case of the more general singularity of equations [6].

The condition at the sonic throat

Tsien has observed that since the motion is supersonic in the diverging part of the nozzle no wave can be transmitted backwards to the throat, and therefore the propagation of the oscillations must always take place toward the downstream direction. He has therefore used the condition that the propagation velocity $U(x)$ must always be positive. If U is defined by the formula

$$\frac{\partial}{\partial t}\left(\frac{\rho'}{\rho}\right) + U\frac{\partial}{\partial x}\left(\frac{\rho'}{\rho}\right) = 0$$

substituting from [4] we have $U = -\,i\,\omega\,\delta/(d\,\delta/dx)$. The condition that U must remain of the same sign throughout the nozzle means that it must never vanish, and therefore $(d\,\delta/dx)/\delta$ must remain finite. This condition is not satisfied if the solution is singular, with the important consequence that only a solution which is regular is compatible with the condition of downstream propagation. This conclusion, which is true for the most general case considered in equations [6], allows a more concise and mathematically more definite expression of the condition at the throat. The same result is obtained on a more physical basis by considering that a wave of finite amplitude at the sonic throat cannot send but infinitesimal waves upstream, since the upstream propagation velocity is zero, and therefore only waves of infinite amplitude at the sonic throat can send finite waves upstream. The absence of upwards moving waves is therefore connected with the suppression of all singularities at the sonic throat (1).

(1) The Author is indebted to dr. L. Lees for a more rigorous discussion of the condition of non-singularity, based on conventional methods of non-steady one-dimensional gasdynamics.

Solution for low frequencies in the non-isothermal case

Tsien has treated the case of isothermal oscillations. His treatment is immediately extended to the non-isothermal case. Assuming a prescribed temperature oscillation at the entrance of the nozzle

$$\left(\frac{T'}{T}\right)_{z=z_e} = \theta\, e^{i\omega t} \qquad [13]$$

we have between φ_e and δ_e the relation

$$\varphi_e - \delta_e = 0. \qquad [14]$$

On the other hand equation [5] gives

$$\varphi_e - \gamma\,\delta_e = \sigma \qquad [15]$$

and therefore we obtain the relation

$$\delta_e = \frac{\theta - \sigma}{\gamma - 1} \qquad [16]$$

which determines the amplitude of the density oscillation at the entrance of the nozzle when θ and σ are assigned. Of course one could prescribe, instead of arbitrary values of θ and σ, arbitrary values of φ and δ, and determine the corresponding values of θ and σ.

In Tsien's case $\theta = 0$.

At low frequencies we can expand all quantities in powers of $i\beta$, since equations [11] and [12] contain only this combination of i and β.

We take therefore

$$\delta(z,\beta) = \delta^{(0)}(z) + i\beta\,\delta^{(1)}(z) + \dots \qquad [17]$$

$$v(z,\beta) = v^{(0)}(z) + i\beta\,v^{(1)}(z) + \dots \qquad [18]$$

Replacing [17] in equation [12] and equating the coefficients of the same powers of $i\beta$, the equation breaks up into

$$z(1-z)\frac{d^2\delta^{(0)}}{dz^2} - 2z\frac{d\,\delta^{(0)}}{dz} = 0 \qquad [19]$$

$$z(1-z)\frac{d^2\delta^{(1)}}{dz^2} - 2z\frac{d\,\delta^{(1)}}{dz} = \frac{2}{\gamma+1}\,z\,\frac{d\,\delta^{(0)}}{dz} + \frac{1}{\gamma+1}\,\delta^{(0)} - \frac{\sigma}{2}\left(\frac{1}{\gamma+1} + \frac{1}{\gamma z}\right). \qquad [20]$$

The solution of [19] which is non singular at $z = 1$ is $\delta^{(0)} = \text{const.}$ and therefore, since the solution [17] must hold at $\beta = 0$, from [16] we obtain

$$\delta^{(0)} = \frac{\theta - \sigma}{\gamma - 1}. \qquad [21]$$

Introducing this value in equation [20], and integrating this first order equation in $d\,\delta^{(1)}/dz$ we obtain the expression

$$(1-z)^2\frac{d\,\delta^{(1)}}{dz} = \frac{\theta}{\gamma^2-1}(\log z - z) - \frac{\sigma}{2}\left[\frac{1}{\gamma(\gamma-1)}\log z - \frac{1}{\gamma-1}\,z - \frac{1}{\gamma}\,\frac{1}{z}\right] + C \qquad [22]$$

where the integration constant C has to be determined in such a way that the right hand side of equation [22] vanishes at $z = 1$, so that $d\,\delta^{(1)}/dz$ may remain finite at this point. We obtain

$$\frac{d\,\delta^{(1)}}{dz} = \frac{\theta}{\gamma^2-1}\,\frac{\log z + 1 - z}{(1-z)^2} - \frac{\sigma}{2\gamma(\gamma-1)}\left[\frac{\log z + 1 - z}{(1-z)^2} - \frac{\gamma-1}{z}\right]. \qquad [23]$$

It is immediately checked that this expression in regular at $z = 1$. An additional integration, with the condition following from [16] and [21]

$$\delta^{(1)}{}_e = 0 \qquad [24]$$

would give the (non-singular) expression for $\delta^{(1)}(z)$.

Replacing now [17], [18] in equation [11] and again equating the coefficients of the same powers of $i\beta$ one obtains, with the use of [21], [23], [24]

$$v^{(0)} = v_e^{(0)} = \frac{\theta}{2};$$

$$v_e^{(1)} = -\frac{\theta}{4}\left(\frac{2}{\gamma-1}\,\frac{\log z_e}{1-z_e} + 1\right) + \frac{\sigma}{4\gamma}\left(\frac{\gamma+1}{\gamma-1}\,\frac{\log z_e}{1-z_e} + 1\right) \qquad [25]$$

and therefore, recalling [17], [18]:

$$\frac{v_e}{\delta_e} = \frac{v_e^{(0)} + i\beta\,v_e^{(1)} + \dots}{\delta_e^{(0)} + i\beta\,\delta_e^{(1)} + \dots} = \frac{\gamma-1}{2}\,\frac{\theta}{\theta-\sigma} + \frac{\gamma-1}{\theta-\sigma}\,v_e^{(1)}\cdot i\beta + \dots \qquad (\beta \ll 1) \qquad [26]$$

This quantity, representing the complex ratio between the fractional variations of velocity and density at the entrance of the nozzle, and analogous to the reciprocal impedence of acoustics, can be used as boundary condition for the rest of the flow system in oscillatory state. We see that [26] depends only on β, z and the ratio θ/σ. The first term of the series [26] applies to the steady state and contains as particular cases the isothermal case $\theta = 0$ where the velocity cannot change ($v_e = 0$); the isentropic case $\sigma = 0$, $v_e/\delta_e = (\gamma-1)/2$, and the isopicnic case $\theta = \sigma$, $\delta_e = 0$. Equation [26] shows that even for moderate β the boundary condition imposed by the nozzle can change considerably since both the phase and the amplitude ratio between velocity and density fluctuations are affected considerably.

The same procedure can be used, without substantial difficulty, to compute higher order terms in $i\beta$.

Solution for high frequencies in the non-isothermal case

Following a procedure similar to the one used by Tsien we first determine the particular solution δ^* of equation [12] taking

$$\delta^*(z) = Z(z)\left(\frac{z}{z_e}\right)^{-\frac{i\beta}{2}}$$

and replacing it in equation [12]. An equation for $Z(z)$ is obtained which can be solved by series taking

$$Z(z) = Z^{(0)}(z) + \frac{1}{i\beta} Z^{(1)}(z) + \ldots$$

and equating the coefficients of the same powers of $i\beta$. The result is

$$\delta^* = -\frac{\sigma}{\gamma}\left(\frac{z}{z_e}\right)^{-\frac{i\beta}{2}}\left(1 + \frac{1}{i\beta}\frac{1}{\frac{\gamma+1}{2}\frac{1}{z} - \frac{\gamma-1}{2}} + \ldots\right) \quad [27]$$

Replacing [27] in equation [11], the corresponding value of ν is found to be zero (by pushing the expansions to higher powers it is actually found that the first non vanishing term in the series for ν corresponding to the particular solution [27] is the term in $(i\beta)^{-2}$). The δ^* fulfils the condition of regularity at $z = 1$, but not the condition [16] at $z = z_e$.

Therefore a solution of the homogeneous equation corresponding to [12] must be determined. If we take $\delta = \exp\,(i\beta\lambda(z))$ and replace in this homogeneous equation, we find that the derivative $d\lambda/dz = y(z)$ satisfies the Riccati's equation

$$z(1-z)\frac{dy}{dz} = \frac{2+i\beta}{2(\gamma+1)} + 2\left(1 + \frac{i\beta}{\gamma+1}\right)z\,y - i\beta z(1-z)y^2.$$

Introducing in this equation the series

$$y(z) = y^{(0)}(z) + \frac{1}{i\beta}y^{(1)}(z) + \ldots$$

and equating the terms with equal powers of $i\beta$ we obtain

$$y^{(0)} = -\frac{1}{(\gamma+1)(1-z)}\left(\sqrt{1 + \frac{\gamma+1}{2}\frac{1-z}{z}} - 1\right) \quad [28]$$

$$y^{(1)} = \frac{\gamma+1}{2\sqrt{1 + \frac{\gamma+1}{2}\frac{1-z}{z}}}\left[(1-z)\frac{dy^{(0)}}{dz} - 2y^{(0)} - \frac{1}{\gamma+1}\frac{1}{z}\right] \quad [29]$$

where only the solutions remaining regular at $z = 1$ have been considered. The solution of equation [12] can be put now under the form

$$\delta = C\exp\left(i\beta\int_{z_e}^{z} y^{(0)}dz + \int_{z_e}^{z} y^{(1)}dz + \ldots\right) + \delta^* \quad [30]$$

where the constant C is determined by the condition [16] at $z = z_e$ and is expressed by

$$C = \frac{1}{\gamma-1}\left(\theta - \frac{\sigma}{\gamma}\right) + \frac{1}{i\beta}\frac{\sigma}{\gamma}\frac{1}{\frac{\gamma+1}{2}\frac{1}{z_e} - \frac{\gamma-1}{2}} + \ldots \quad [31]$$

Coming now to equation [11] and recalling that δ^* does not bring any contribution to ν up to terms of order $(i\beta)^{-2}$, up to this order we can express ν as

$$\nu = \left(\eta^{(0)} + \frac{1}{i\beta}\eta^{(1)} + \ldots\right)$$

$$C\exp\left(i\beta\int_{z_e}^{z} y^{(0)}dz + \int_{z_e}^{z} y^{(1)}dz + \ldots\right) \quad [32]$$

and, after substitution in equation [11] and comparison of terms with same powers of $i\beta$, we find

$$\eta^{(0)} = \sqrt{1 + \frac{\gamma+1}{2}\frac{1-z}{z}}; \qquad [33]$$
$$\eta^{(1)} = \gamma - 1 - 2\eta^{(0)} - (\gamma+1)(1-z)y^{(1)}$$

$y^{(1)}$ being given by [28], [29].

Finally computing ν at $z = z_e$ from [32] and recalling the condition [16] we find

$$\frac{\nu_e}{\delta_e} = \frac{\theta - \frac{\sigma}{\gamma}}{\theta - \sigma}\eta_e^{(0)} + \frac{1}{i\beta}\left(\frac{\gamma-1}{\gamma}\frac{\sigma}{\theta-\sigma}\frac{\eta_e^{(0)}}{\frac{\gamma+1}{2}\frac{1}{z_e} - \frac{\gamma-1}{2}} + \frac{\theta - \frac{\sigma}{\gamma}}{\theta - \sigma}\eta_e^{(1)}\right) + \ldots \quad (\beta \gg 1) \quad [34]$$

with $\eta_e^{(0)}$, $\eta_e^{(1)}$ given by equations [33], [28], [29] at $z = z_e$.

Higher order terms in expansions [27], [30], [32] and [34] can easily be computed. Again we see that ν_e/δ_e depends only on β, γ and θ σ. For $\theta = 0$ and $\beta \to \infty$ [34] gives Tsien's result.

Practical range of β

It is interesting to investigate if equations [26] or [34] can be used for stability computations of practical combustion systems. It seems to day established that high frequencies appearing in unstable combustion are close to the natural modes of oscillation of the combustion system, that is — for instance — to the organ-pipe frequencies if the combustion chamber is of elongated form. The fundamental organ-pipe mode has an angular frequency $\omega = \pi c_c/L$, c_c representing the sound velocity in the chamber and L its lenght. Hence from [10] we have $\beta = \pi c_c l/(c_e - u_e) L$. Now $c_c/(c_e - u_e)$ is close to one, so that, if the length of the subsonic portion of the nozzle is around 1/3 of the length of the chamber, β is around unity for the fundamental mode, around two for the second mode and so on·

This approximate computation shows therefore that unless the geometry of the system is quite different from the assumed one the results of the previous sections do not give much help if the frequencies are close to the natural modes of oscillation of the chamber, as — for instance — in the study of « screaming » conditions in rocket motors.

Solutions for all frequencies in the isentropic case

Fortunately it is possible to take advantage of the analytical properties of equation [12] to extend the computations to the whole range of frequencies. In order to avoid the complication connected with the determination of the particular solution of equation [12], we have here considered only the isentropic case, which probably represents quite closely the actual phenomenon. With $\sigma = 0$ equation [12] is reduced to an homogeneous hypergeometric equation, of which the solution remaining non-singular at $z = 1$ is given by the known hypergeometric series in powers of $1 - z$:

$$F(a, b; c; 1-z) = 1 + \frac{a\,b}{c}(1-z) + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{(1-z)^2}{2!} + \dots \qquad [35]$$

with a, b, c given by

$$c = a + b + 1 = 2\left(1 + \frac{i\beta}{\gamma+1}\right); \qquad ab = \frac{i\beta}{\gamma+1}\left(1 + \frac{i\beta}{2}\right). \qquad [36]$$

In principle, therefore, our problem is solved taking

$$\delta = C\,F(a, b; c; 1-z) \qquad [37]$$

and computing ν from equation [11] with $\sigma = 0$ and with dF/dz obtained from the differentiation of [35]. The ratio ν_1/δ_1 is therefore determined and is independent of C.

However this procedure cannot be followed in practical cases since z_1 is generally quite small (1) and $1 - z_1$ is close to one, in a region where the convergence of [35] is too poor. The difficulty can be overcome with the help of the properties of the solutions of the hypergeometric equations. Simpler developments are obtained using instead of series [35], the proportional series

$$f(a, b; c; 1-z) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; 1-z) = \sum_{s=0}^{\infty} (1-z)^s \frac{\Gamma(s+a)\Gamma(s+b)}{\Gamma(s+1)\Gamma(s+c)} \qquad [38]$$

where $\Gamma(x)$ represents the gamma function of argument x. This particular solution can be expressed as a linear combination of the series expressing two particular solutions of the hypergeometric equation in powers of z.

One such relationship, particularly useful in the present case, is (ref. 2)

$$f(a, b; c; 1-z) = z^{1-a-b}\,\frac{g(c-a, c-b; 1+c-a-b; z)}{\Gamma(c-a)\Gamma(c-b)\cos(c-a-b)\pi} \qquad [39]$$

where

$$g(a', b'; c'; z) = -\pi\cot(c'\pi)\,[f(a', b'; c'; z) - z^{1-c'} f(a'+1-c', b'+1-c'; 2-c'; z)] \qquad [40]$$

the two functions f being given by the corresponding series [38]. With the particular value [36] of c, the value of c' to use in [40] is given, following [39], by $c' = 2$. Now for integer $c' = n$ the quantity in the brackets of equation [40] vanishes (as it is easily checked) and the factor preceding the brackets becomes infinity. The corresponding value of g can be found through a limiting process to be (ref. 2), for $n > 1$,

$$\begin{aligned} g(a', b'; n; z) = {} & -f(a', b'; n; z)\log z + \sum_{s=-1}^{s=-(n-1)} (-z)^s \frac{\Gamma(-s)\Gamma(s+a')\Gamma(s+b')}{\Gamma(s+n)} - \\ & - \sum_{s=0}^{\infty} z^s \frac{\Gamma(s+a')\Gamma(s+b')}{\Gamma(s+1)\Gamma(s+n)}[\psi(s+a') + \\ & + \psi(s+b') - \psi(s+1) - \psi(s+n)] \end{aligned} \qquad [41]$$

where $\psi(x)$ represents, as usual, the logarithmic derivative of the gamma function with respect to the argument x.

Finally from equations [38], [39], [41] with a, b, c given by [36] and therefore $n = 2$ we find

$$f(a, b; c; 1-z) = \log z \sum_{s=1}^{\infty} A_s z^s + \sum_{s=1}^{\infty} A_s D_s z^s + \frac{2(\gamma+1)}{i\beta(2+i\beta)} \qquad [42]$$

with the coefficients A_s given by the recurrence formula

$$\frac{A_{s+1}}{A_s} = 1 - \frac{\beta^2}{2(\gamma+1)s(s+1)} + i\beta\frac{2s+1}{(\gamma+1)s(s+1)}; \quad A_1 = 1$$

and with

$$D_s = \psi(s+a) + \psi(s+b) - \psi(s) - \psi(s+1). \qquad [43]$$

The computation of D_s can be performed with the help of the series expression of $\psi(x)$:

$$\psi(x) = \log x - \frac{1}{2x} + \frac{1}{2}\sum_{r=1}^{\infty}\frac{(-1)^r B_r}{r\,x^{2r}}$$

where $B_1 = 1/6$, $B_2 = 1/30$..... are the Bernoulli's numbers.

(1) If the Mach number at the nozzle entrance is .2, $1-z_1$ is around .96. Even for Mach number .5, $1-z_1$ is close to .75.

We obtain

$$D_s = \log\frac{A_{s+1}}{A_s} + \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s+1} - \frac{2s+1+\dfrac{2i\beta}{\gamma+1}}{s(s+1)}\frac{A_s}{A_{s+1}}\right] +$$
$$+ \sum_{r=1}^{\infty}(-1)^r \frac{B_r}{2r}\left[(s+a)^{-2r} + (s+b)^{-2r} -\right.$$
$$\left. - s^{-2r} - (s+1)^{-2r}\right]. \qquad [44]$$

The convergence of the last series is very fast for high values of s. D_s has been therefore computed from [44] only for the highest value of s needed in the evaluation of [42]; and for the other values of D_s from the recurrence formula

$$D_{s+1} - D_s = \frac{1}{s+a} + \frac{1}{s+b} - \frac{1}{s} - \frac{1}{s+1}.$$

The solution [37] can be expressed as

$$\delta = C' f(a, b; c; 1-z)$$

with the constants C' and C connected through [38]; and [42] provides the series expansion suited for computations at $z = z_s$. Actually to solve the problem of the boundary condition equivalent to the presence of the nozzle we need only the calculation of v/δ and therefore, from equation [11], of the quantity $(1/f)\ (df/dz) = (1/F)\ (dF/dz)$, independent of the values of the integration constants C or C', and easy to calculate from the inverse ratio of series [42] or [35] and the series obtained through their differentiation. The real and the imaginary part of this quantity, the first divided by β^2 and the second by β for convenience of scale, are given for $\gamma = 1.2$ in figures 1 and 2. The number of terms used in the computation are sufficient to give very accurate values up to $z = .2$ for series [42] and reasonable accurate data up to $z = .3$; the corresponding limits for series [35] are $z = .8$ and $z = .7$. Between .3 and .7 the dotted curve is only interpolated. The lines for $\beta = 0$ are entirely computed from the equations

$$\lim_{\beta\to 0}\left[\frac{1}{\beta^2} R.P.\left(\frac{1}{\delta}\frac{d\delta}{dz}\right)\right] = -\frac{1}{(\gamma+1)^2(1-z)^2}$$
$$\left[2\int_1^z \frac{\log z}{1-z}dz + \frac{\gamma-1}{2}\cdot(1-z) +\right.$$
$$\left. + \left(z + \frac{\gamma+1}{2}\right)\log z - \frac{z\log z(\log z - 1 - z)}{1-z}\right]$$

$$\lim_{\beta\to 0}\left[\frac{1}{\beta} I.P.\left(\frac{1}{\delta}\frac{d\delta}{dz}\right)\right] = \frac{\log z + 1 - z}{(\gamma+1)(1-z)^2}$$

which can be derived with the procedure used previously for small β.

For $\beta \to \infty$ the first of the two quantities goes to zero, but the second one takes the expression

$$\lim_{\beta\to\infty}\left[\frac{1}{\beta} I.P.\left(\frac{1}{\delta}\frac{d\delta}{dz}\right)\right] = y^{(0)}$$

$y^{(0)}$ being given by equation [28]. Both quantities tend logarithmically to infinity at $z = 0$, a value

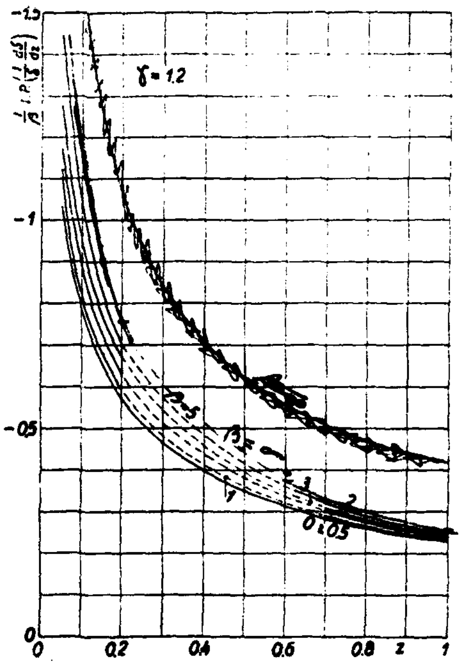

Fig. 1

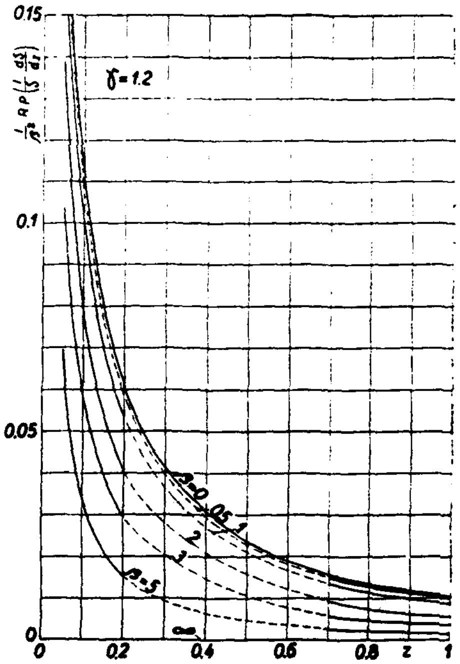

Fig. 2

which can never occur in practice. The behavior of $(1/\delta_e)(d\delta/dz)_e$ is better seen in figure 3 where this quantity divided by $-i\beta$ is represented in the complex plane for various values of $z_e = (u_e/c_*)^2$.

From $(1/\delta_e)(d\delta/dz)_e$ and equation [11] the quantity v_e/δ_e has been computed. Figures 4 and 5 show the real and the imaginary part of v_e/δ_e as a function of β for different values of z_e. Figure 6 gives a representation of v_e/δ_e in the complex plane, from which the phase and amplitude relations between the velocity and density fractional fluctuations are immediately deduced. For a given Mach number at the nozzle entrence the ratio of the amplitudes increases steadily with increasing frequencies, while

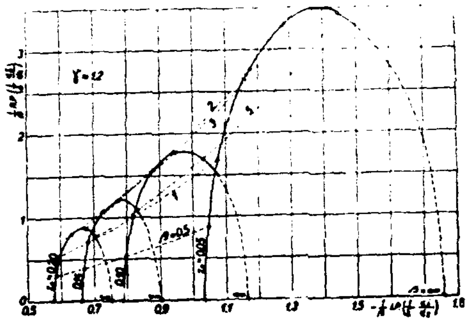

Fig. 3

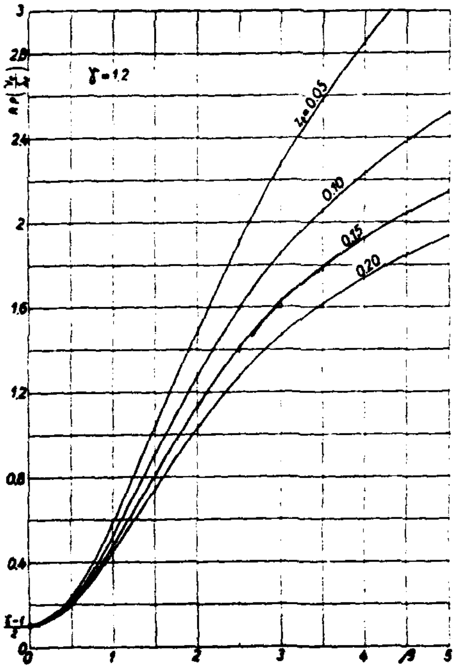

Fig. 4

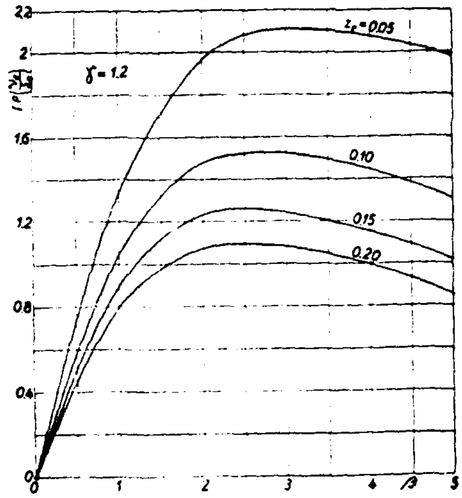

Fig. 5

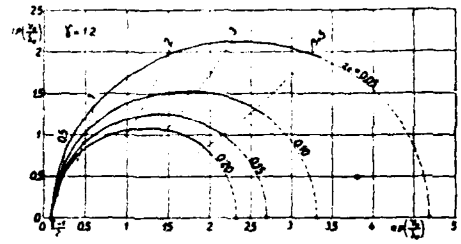

Fig. 6

the phase goes from zero to a maximum which is always an important fraction of $\pi/2$ and then decreases back to zero.

Final observations

The computed values of v_e/δ_e are sufficient to solve problems in the complete range of practical frequencies and Mach numbers in the isentropic problem and for linear velocity distribution. Additional computation are required in other cases.

The isentropic case with other velocity distributions can be computed starting from the solution for the linear distribution, which is still available near $z = 1$ provided $du/dx \neq 0$, and continuing with the numerical integration of equations [6] with $\sigma = 0$. In particular at the sonic point $z = 1$ we have the relation

$$\left(\frac{v}{\delta}\right)_{u=c_*} = \frac{\gamma - 1 + i\beta}{2 + i\beta}$$

that is the same equation given by [11] for finite $(1/\delta)(d\delta/dz)$ at $z = 1$, with $\beta = \omega/(du/dx)_{u=c_*}$. In terms of β the relation between v and δ at the sonic point is independent of the nozzle shape.

It is interesting to observe that the fractional oscillation of Mach number, given in the isentropic case by $\nu - (\gamma - 1)\,\delta/2$, is at the throat equal to

$$\delta_{x=x_0}\left[\left(\frac{\nu}{\delta}\right)_{x=x_0} - \frac{\gamma-1}{2}\right] = \delta_{x=x_0}\,\frac{(3-\gamma)\,i\,\beta}{2\,(2+i\,\beta)}$$

and is therefore different from zero for finite β. The sonic line is therefore oscillating around the throat, and the amplitude of the oscillations can be related to the amplitude of oscillation of pressure, density or temperature at the nozzle entrance.

Finally we observe with Tsien that, especially for short approaches to the throat the one-dimensional assumption can be very much in error, and that additional errors are to be expected from the fact that the fluid has been considered as a frictionless, insulating, non reacting gas.

REFERENCES

(1) H. S. Tsien: *The Transfer Functions of Rocket Nozzles*, « Journal of the American Rocket Society », vol. 22, n. 3, p. 139, may-june 1952.

(2) C. Snow: *The Hypergeometric and Legendre Functions with Applications to Integral Equations of Potential Theory*, Publication of « National Bureau of Standards » U. S. A. 1942.

SUPERCRITICAL GASEOUS DISCHARGE WITH HIGH FREQUENCY OSCILLATIONS

by L. CROCCO

For several problems connected with jet propulsion it is required to know the behavior of a supercritical discharge nozzle when the conditions upstream of the nozzle are oscillating. Tsien has recently analyzed the question in the case in which the oscillation amplitudes are small and the velocity axial distribution is linear in the subsonic portion of the nozzle. Under certain restrictive assumptions he has given the solution in the two extreme cases of very low and of very high frequency. Here the study is extended to more general assumptions. Moreover, in the case in which the upstream oscillations are *isentropic*, the solution is given in the complete range of frequencies.