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PREFACE

This research was conducted under NUWC Job Order Number A10020, R&D Project Number RR00N00, Performance Evaluation of Nonlinear Signal Processors with Mismatch, Principal Investigator Dr. Albert H. Nuttall (Code 302). This technical report was prepared with funds provided by the NUWC In-House Independent Research Program, sponsored by the Office of Naval Research. Also, the research presented in this report was conducted under NUWC Job Order Number D15010, Project Number S0219, AN/BQQ-5 Sonar Program, Principal Investigator Kevin C. Collins (Code 2191), sponsored by NAVSEA, CAPT G. Kent (PMS 425).

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The joint characteristic function of the real and imaginary parts of the complex cross spectrum estimate is derived in closed form for sinusoidal signals and arbitrary noise spectra. The corresponding joint probability density function for noise-only is also obtained in closed form, while that for signal present requires a numerical procedure involving a two-dimensional fast Fourier transform. These results are used to obtain the mean and deflection of the magnitude cross spectrum estimate. Comparisons are made with the deflection of an auto spectrum estimate available from the two channels by weighted summation. Potential limitations of restricting attention to the deflection criterion as a detectability measure are pointed out.
14. SUBJECT TERMS (continued)

fast Fourier transform
mean magnitude
deflection
auto spectrum estimate
Gaussian approximation
false alarm probability
detection probability
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2. Deflections $d_g$ and $d_a$ for $R_y = R_x/2$ 35

LIST OF SYMBOLS

- **bold** random variable
- $t$ time, (1)
- $s(t)$ input signal, (1)
- $A$ amplitude of input sinewave, (1)
- $f_0$ frequency of input sinewave, (1)
- $\phi$ phase of input sinewave, (1)
- $x(t)$ input noise process in one channel
- $y(t)$ input noise process in other channel
- $K$ number of pieces used for spectrum estimation, (2)
- $w(t)$ fundamental time weighting function, (2)
- $w_k(t)$ $k$-th time weighting function, (2)
- $T_k$ time delay of $k$-th time weighting function, (2)
- $\nu$ frequency variable, (3)
- $W(\nu)$ fundamental spectral window, (3)
- $W_k(\nu)$ $k$-th spectral window, (3)
- $f$ analysis frequency of interest, (4)
- $S_k$ $k$-th signal voltage density estimate, (4)
- $\alpha_k, \beta_k$ real and imaginary parts of $S_k$, (4)
- $X_k$ $k$-th noise voltage density estimate for $x(t)$, (5)
- $a_k, b_k$ real and imaginary parts of $X_k$, (5)
$Y_k$ k-th noise voltage density estimate for $y(t)$, (6)
$c_k^r, d_k^i$ real and imaginary parts of $Y_k$, (6)
$G$ complex cross spectrum estimate, (7)
$R, Q$ real and imaginary parts of $G$, (7)
$R_k, Q_k$ real and imaginary parts of k-th term, (8)
$\gamma$ nonrandom signal quantity, (9)
$R_x(\tau)$ covariance of process $x(t)$, (10)
$G_x(\nu)$ spectrum of process $x(t)$, (10)
$\sigma_x^2$ variance measure in $x$ channel, (12)
$G_y(\nu)$ spectrum of process $y(t)$, (13)
$\sigma_y^2$ variance measure in $y$ channel, (13)
$f_k(z, y)$ joint characteristic function of $R_k$ and $Q_k$, (14)
$\xi, \eta$ real arguments of joint characteristic function, (15)
$z, y$ auxiliary complex arguments, (15)
$f_{RQ}(z, y)$ joint characteristic function of $R$ and $Q$, (19)
$X_{mn}$ m,n-th joint cumulant of $R$ and $Q$, (20)
$R_x, R_y$ signal to noise ratio measures, (22)
$\sigma^2$ variance of $R$ and $Q$, (24)
$P_g$ Gaussian joint probability density function, (25)
$\mu_1$ mean magnitude of $G$, (26)
$\mu_1 g$ Gaussian approximation to $\mu_1$, (27), (28)
$I_0$ modified Bessel function, (28)
$1F_1$ confluent hypergeometric function, (28)
$V$ auxiliary variable, (29)
$\mu_1 g(A=0)$ value of $\mu_1 g$ for zero signal input, $A = 0$, (31)
$\mu_2$ mean square magnitude of $G$, (33)
dc deflection of magnitude $|G|$, (35)

dg Gaussian-approximation deflection, (36)

$\mu_1(K=1)$ value of $\mu_1$ for $K = 1$, (41)

$\mu_{1g}(K=1)$ value of $\mu_{1g}$ for $K = 1$, (42)

dc(K=1) value of $d_c$ for $K = 1$, (43)

dg(K=1) value of $d_g$ for $K = 1$, (44)

$P_{Rq}$ value of $f_{Rq}$ for $A = 0$, (45)

B $\sigma_x \sigma_y / \gamma$, (45)

$P_{Rq}^0$ value of $p_{Rq}$ for $A = 0$, (46)

$K_v(x)$ modified Bessel function of second kind, (46)

$R_{pq}$ aliased probability density function, (55)

$R_{pq}$ collapsed version of $\varepsilon_k f_{Rq}$, (56)

$k$ increment for $P_{Rq}$, in both $u$ and $v$, (57)

$D_2$ double integral for moment, (58)

S nonzero region of $P_{Rq}$, (59)

$K_m$ number of samples of $f_{Rq}$ in each dimension
$z(t)$  sum process, (60)
$\lambda$  scale factor in sum, (60)
$Z_k$  $k$-th voltage density estimate of $z(t)$, (61)
$G_z$  auto spectrum estimate of $z(t)$, (62)
$e_k, f_k$  auxiliary random variables, (63)
$\sigma^2_e$  variance of $e_k$ and $f_k$, (64)
$F_k(\xi)$  characteristic function of $k$-th term of sum, (66)
$F(\xi)$  characteristic function of $G_z$, (67)
$X_j$  $j$-th cumulant of $G_z$, (68)
$P_d$  detection probability for $G_z$, (69)
$\text{Pr}(E)$  probability of event $E$, (69)
v  threshold, (69)
$Q_K$  Q function, (69)
Pf  false alarm probability for $G_z$, (70)
da  deflection of auto spectrum estimate $G_z$, (74)
dx  deflection of random variable $x$, (78)
$\mathcal{H}_1$  hypothesis that signal is present
$\mathcal{H}_0$  hypothesis that signal is absent
$m_1$  mean under hypothesis $\mathcal{H}_1$, (79)
$\Phi$  error function, (79)
m_0  mean under hypothesis $\mathcal{H}_0$, (80)
$\tilde{\Phi}$  inverse error function, (81)
C  dimensionless constant, (82)
y  distorted random variable, (82)
dy  deflection of random variable $y$, (84)
STATISTICS OF COMPLEX CROSS SPECTRUM ESTIMATE FOR SINUSOIDAL SIGNALS AND ARBITRARY NOISE SPECTRA

INTRODUCTION

The analysis of the stability of the estimate of the complex cross spectrum usually proceeds on the basis that the two input processes have a slowly varying cross spectrum relative to the spectral window employed. See, for example, [1; (4) and sequel]. Here, we will eliminate that restriction and allow real input signals with arbitrarily narrow width, namely sinusoids, and allow real additive input noises with arbitrary spectra.

On the other hand, we will restrict consideration to the special case where the two input noise processes are zero mean Gaussian and are statistically independent of each other. Furthermore, the temporal weightings applied will be presumed nonoverlapping in time, thereby leading to (approximately) independent spectral estimates for each time segment.

We will derive the exact joint characteristic function of the real and imaginary parts R and Q, of the cross spectrum estimate $G = R + iQ$, with both signal and noise present. This result enables determination of the high-order joint cumulants of random variables R and Q, for arbitrary signal-to-noise ratios.

For the noise-only case, the corresponding joint probability density function of random variables R and Q will be derived in closed form. It can then be used to derive various moments of the magnitude of estimate G, such as the average magnitude $|G|$. 
When signal is present, the joint probability density function of $R$ and $Q$ cannot be found in closed form. Instead, a two-dimensional fast Fourier transform is utilized, followed by numerical integration to find the moments of interest. Comparison of these accurate results with a Gaussian approximation affords quantitative verification of the Gaussian approximation when the number of pieces, $K$, used in the finite average for cross spectrum estimate $G$ exceeds 10 approximately.

A deflection criterion of cross spectrum magnitude estimate $|G|$ is defined and evaluated, both numerically and by use of the same Gaussian approximation for the joint probability density function of $R$ and $Q$. Finally, the same statistics are evaluated for an auto spectrum estimate obtained from the two input processes optimally scaled prior to addition.
PROBLEM DEFINITION

This study is a follow-on to an earlier report [1], where the overlapped fast Fourier transform processing method of weighted data segments for the purpose of estimation of the cross spectrum was well documented. Familiarity with that material and results is presumed in the following development.

A common sinusoidal signal \( s(t) \) at frequency \( f_o \) is present in both input channels; that is,

\[
s(t) = A \cos(2\pi f_o t + \phi),
\]

where \( \phi \) is a random variable uniformly distributed over \( 2\pi \). The input noise processes in the two channels are \( x(t) \) and \( y(t) \), respectively. The \( k \)-th time weighting function and its spectral window (Fourier transform) are given by

\[
w_k(t) = w(t - T_k) \quad \text{for} \quad 1 \leq k \leq K,
\]

\[
W_k(v) = \int dt \exp(-i2\pi vt) w_k(t) = \exp(-i2\pi vT_k) W(v),
\]

where \( K \) is the total number of pieces used in the cross spectrum estimate \( G \), and \( W(v) \) is the window (Fourier transform) corresponding to fundamental time weighting \( w(t) \). Time delays \( \{T_k\} \) are taken widely enough spaced that individual temporal weightings \( \{w_k(t)\} \) do not overlap on the time axis. (Integrals without limits are over the range of nonzero integrand.)

The \( k \)-th voltage density estimate, at analysis frequency \( f \), of the signal component is, for both channels,
The k-th voltage density estimate, at analysis frequency f, of the noise component x(t) is

\[ X_k = \int dt \exp(-i2\pi ft) w_k(t) x(t) = a_k + ib_k , \]  

for 1 \leq k \leq K. Random variables a_k and b_k are zero mean Gaussian, since the two input processes x(t) and y(t) are zero mean Gaussian processes. The corresponding quantities for the other channel are

\[ Y_k = \int dt \exp(-i2\pi ft) w_k(t) y(t) = c_k + id_k . \]

The complex cross spectrum estimate at analysis frequency f is therefore given by

\[ G = \frac{1}{K} \sum_{k=1}^{K} (S_k + X_k)(S_k + Y_k)^* = \]

\[ = \frac{1}{K} \sum_{k=1}^{K} (\alpha_k + i\beta_k + a_k + ib_k)(\alpha_k - i\beta_k + c_k - id_k) = \]

\[ = \frac{1}{K} \sum_{k=1}^{K} (R_k + iQ_k) = R + iQ , \]

where real and imaginary parts

\[ TR 10709 \]

\[ S_k = \int dt \exp(-i2\pi ft) w_k(t) A \cos(2\pi f_0 t + \phi) = \]

\[ \approx \frac{A}{2} e^{i\phi} \int dt w_k(t) \exp(-i2\pi(f-f_0)t) = \frac{A}{2} e^{i\phi} W_k(f-f_0) = \alpha_k + i\beta_k , \]
\[ R_k = (a_k + a_k)(a_k + c_k) + (\beta_k + b_k)(\beta_k + d_k) , \]
\[ Q_k = (\beta_k + b_k)(\alpha_k + c_k) - (a_k + a_k)(\beta_k + d_k) . \]  
\[ (8) \]

It is important to notice from (4) and (3) that
\[ |S_k|^2 = |\alpha_k + i\beta_k|^2 = a_k^2 + \beta_k^2 = \frac{1}{4} \lambda^2 |W(f-f_o)|^2 = \gamma , \]  
\[ (9) \]

where the latter quantity \( \gamma \) is a constant, not a random variable, and that, furthermore, \( \gamma \) is independent of \( k \), the segment number.

STATISTICS OF \( a_k \) AND \( b_k \)

From (5), we observe that ensemble average
\[ |X_k|^2 = |a_k + ib_k|^2 = a_k^2 + b_k^2 = \]
\[ = \int \int dt du \exp(-i2\pi f(t-u)) w_k(t) w_k^*(u) R_x(t-u) = \]
\[ = \int dv G_x(v) |W_k(f-v)|^2 = \int dv G_x(v) |W(f-v)|^2 , \]  
\[ (10) \]

where \( R_x(\tau) \) and \( G_x(\nu) \) are, respectively, the covariance and spectrum of random process \( x(t) \), and we used (3). Again, notice that this average is independent of \( k \). Also, the result in (10) holds regardless of the relative widths and variations of window \( |W(\nu)|^2 \) and spectrum \( G_x(\nu) \).

At the same time, from (5) and (3),
\[
\overline{x_k^2} = (a_k + ib_k)^2 = \frac{\overline{a_k}^2 - \overline{b_k}^2}{2} + i2 \frac{a_k b_k}{2} = \\
\int dv \ G_x(v) \ W_k(f-v) \ W_k(f+v) = \\
\exp(-i4\pi ft_k) \int dv \ G_x(v) \ |W(f-v)|^2 = c_x^2 , \quad \overline{a_k b_k} = 0 , 
\]

if analysis frequency f is not near zero frequency. Combining (10) and (11), we find properties

\[
\overline{a_k^2} = \overline{b_k^2} = \frac{1}{2} \int dv \ G_x(v) \ |W(f-v)|^2 = c_x^2 , \quad \overline{a_k b_k} = 0 , 
\]

which are independent of k. Thus, Gaussian random variables \(a_k\) and \(b_k\) are statistically independent of each other.

**STATISTICS OF \(c_k\) AND \(d_k\)**

In an entirely similar fashion, but working instead from (6),

\[
\overline{c_k} = \overline{d_k} = 0 , \quad \overline{c_k d_k} = 0 , \\
\overline{c_k^2} = \overline{d_k^2} = \frac{1}{2} \int dv \ G_y(v) \ |W(f-v)|^2 = c_y^2 , 
\]

all quantities being independent of k. Furthermore, the time delays \(\{T_k\}\) in (2) are widely enough separated that all the random variables for weighting \(k\) are independent of all those for weighting \(m\), when \(k \neq m\).
JOINT CHARACTERISTIC FUNCTION OF R AND Q

The joint characteristic function of the k-th pair of random variables $R_k$ and $Q_k$ in (7) and (8) is given by ensemble average

$$f_k(z,y) = \exp(zR_k + yQ_k) = \exp\left[z(a_k+a_k)(a_k+c_k) + z(\beta_k+b_k)(\beta_k+d_k) + y(\beta_k+b_k)(\alpha_k+c_k) - y(\alpha_k+a_k)(\beta_k+d_k)\right],$$

(14)

where we have let variables $z = i\xi$ and $y = i\eta$, $\xi$ and $\eta$ real, (15)

for shorthand purposes. At the same time, from (12) and (13), the joint probability density function of $a_k$, $b_k$, $c_k$, $d_k$ is, for all $k$, given by

$$p(a,b,c,d) = \left(2\pi\sigma_x^2\right)^{-1} \left(2\pi\sigma_y^2\right)^{-1} \exp\left[-\frac{a^2 + b^2}{2\sigma_x^2} - \frac{c^2 + d^2}{2\sigma_y^2}\right].$$

(16)

When we employ this result for $p$ in the average required by (14), holding random variables $\alpha_k$ and $\beta_k$ fixed for now, the resulting four-fold integral can be evaluated by first evaluating the double integral on $a$ and $b$, followed by the double integral on $c$ and $d$, by means of the following result:

$$\int \int dx \ dy \ \exp\left(-\frac{1}{2}ax^2 - \frac{1}{2}by^2 + \gamma xy + \mu x + \nu y\right) =$$

$$= 2\pi \left(\alpha\beta - \gamma^2\right)^{-\frac{1}{2}} \exp\left(\frac{\beta\mu^2 + \alpha\nu^2 + 2\gamma\mu\nu}{2(\alpha\beta - \gamma^2)}\right)$$

(17)

for $\alpha > 0$, $\beta > 0$, $\alpha \beta > \gamma^2$. The end result, after much
manipulation, is the joint characteristic function of $R_k$ and $Q_k$, conditioned on given fixed values of random variables $a_k$ and $b_k$, namely

$$f_k(z,y) = \frac{1}{1 - \sigma_x^2 \sigma_y^2 (z^2 + y^2)} \exp\left(\gamma \frac{z + (z^2 + y^2)(\sigma_x^2 + \sigma_y^2)/2}{1 - \sigma_x^2 \sigma_y^2 (z^2 + y^2)}\right) \tag{18}$$

where we used relation (9).

But, since (18) contains no random variables, it is actually the unconditional joint characteristic function of $R_k$ and $Q_k$. Also, since the constant $\gamma$ is independent of $k$, we see that $f_k(z,y)$ is independent of $k$. This leads to the desired result, namely the joint characteristic function of summation variables $R$ and $Q$ in (7), as

$$f_{RQ}(z,y) = \exp(\gamma R + Q) = \exp\left(\gamma \frac{\sum_{k=1}^{K} R_k + \frac{\gamma}{K} \sum_{k=1}^{K} Q_k}{K}\right) =

= \prod_{k=1}^{K} f_k\left(\frac{z}{K},\frac{y}{K}\right) = f_k\left(\frac{z}{K},\frac{y}{K}\right)^K

= \left[1 - \sigma_x^2 \sigma_y^2 (z^2 + y^2)/K^2\right]^{-K} \exp\left(\gamma \frac{z + (z^2 + y^2)(\sigma_x^2 + \sigma_y^2)/(2K)}{1 - \sigma_x^2 \sigma_y^2 (z^2 + y^2)/K^2}\right) \tag{19}$$

This is an exact result for the joint characteristic function of $R$ and $Q$, under the conditions cited above, such as disjoint temporal weightings $\{w_k(t)\}$, common sinusoidal signal $s(t)$, and independent input noise processes $x(t)$ and $y(t)$. The complex cross spectrum estimate is given by (7) as $G = R + iQ$. 

8
JOINT CUMULANTS OF R AND Q

When we expand $\ln f_{RQ}(z,y)$ in (19) in a power series in $z$ and $y$ according to

$$\ln f_{RQ}(z,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \chi_{mn} \frac{z^m y^n}{m! n!}, \quad \chi_{00} = 0,$$

then, $\chi_{mn}$ is the $m,n$-th joint cumulant of $R$ and $Q$. There follows

$$\chi_{10} = y, \quad \chi_{01} = 0, \quad \chi_{mn} = 0 \text{ for } m + n = 3, 5, 7, \ldots$$

$$\chi_{20} = \chi_{02} = \frac{2 \sigma_x^2 \sigma_y^2}{K} (1 + R_x + R_y), \quad \chi_{11} = 0,$$

$$\chi_{22} = \frac{4 \sigma_x^4 \sigma_y^4}{K^3} (1 + 2R_x + 2R_y),$$

$$\chi_{40} = \chi_{04} = 3 \chi_{22}, \quad \chi_{31} = \chi_{13} = 0,$$

where we have defined, with the help of (9) and (12),

$$R_x = \frac{Y}{2\sigma_x^2} = \frac{1}{4} A^2 \left| W(f-f_o) \right|^2 \int dv \, G_x(v) \left| W(f-v) \right|^2,$$

$$R_y = \frac{Y}{2\sigma_y^2} = \frac{1}{4} A^2 \left| W(f-f_o) \right|^2 \int dv \, G_y(v) \left| W(f-v) \right|^2.$$

These latter quantities are measures of the signal-to-noise power ratios at the outputs of window $|W(v)|^2$ in (3).
The results in (21) indicate that as the number of pieces $K \to \infty$, the two random variables $R$ and $Q$ tend to joint Gaussian. For example, we find

$$\frac{X_{40}}{X_{20}^2} = \frac{3}{K} \frac{1 + 2R_x + 2R_y}{(1 + R_x + R_y)^2} \to 0 \text{ as } K \to \infty. \quad (23)$$

Since the joint third-order moments are all zero, this result indicates a rather rapid approach to the Gaussian approximation.
GAUSSIAN APPROXIMATION

The quantities $X_{10}$ and $X_{01}$ in (21) are the means of $R$ and $Q$, respectively. Also, $X_{20}$ and $X_{02}$ are the respective variances, which are equal and will be denoted by $\sigma^2$; that is,

$$\sigma^2 = \frac{2 \sigma_x^2 \sigma_y^2}{K} (1 + R_x + R_y).$$

(24)

Since covariance $X_{11}$ is zero, the Gaussian approximation to the two-dimensional probability density function of $R$ and $Q$ is given by

$$p_{g}(u,v) = \frac{1}{2\pi \sigma^2} \exp\left\{-\frac{(u - \gamma)^2 + v^2}{2\sigma^2}\right\} \text{ for all } u, v.$$  

(25)

MEAN MAGNITUDE OF $G$

With this approximation at hand, we can now evaluate some moments of $G$ that are not available directly from joint characteristic function (19) or joint cumulants (21). In particular, we are interested in the mean magnitude of complex cross spectrum estimate $G = R + iQ$; namely,

$$\mu_1 = |G| = \frac{1}{(R^2 + Q^2)^{1/2}} \int \int du \, dv \, (u^2 + v^2)^{1/2} \, p_{RQ}(u,v) = \mu_1 g.$$  

(26)

$$= \int \int du \, dv \, (u^2 + v^2)^{1/2} \, p_{g}(u,v) = \mu_1 g =$$

(27)
\[
\int \int du \ dv \ \frac{(u^2 + v^2)^{\frac{1}{2}}}{2\pi \sigma^2} \ \exp\left(-\frac{(u - \gamma)^2 + v^2}{2\sigma^2}\right) =
\]

\[
\int \int du \ dv \ \frac{(u^2 + v^2)^{\frac{1}{2}}}{2\pi \sigma^2} \ \exp\left(-\frac{r^2}{2\sigma^2} + \frac{\gamma^2}{2}\right) =
\]

\[
\int \int du \ dv \ \frac{(u^2 + v^2)^{\frac{1}{2}}}{2\pi \sigma^2} \ \exp\left(-\frac{r^2}{2\sigma^2} + \frac{\gamma^2}{2}\right) =
\]

\[
\pi \sigma \exp(-\gamma) \ {}_1F_1(1.5;1;\gamma) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sigma \ {}_1F_1(-.5;1;\gamma) , \quad (28)
\]

where we used \([2; 6.631 1], [3; 13.1.27]\), and defined

\[
\gamma = \frac{\sigma^2}{2\sigma^2} = k \ \frac{R_x}{1 + R_x + R_y} . \quad (29)
\]

The function \(_1F_1\) is the confluent hypergeometric function\([3; 13.1.2\) and 13.1.10].

It must be repeated that (28) is an approximation for the desired average \(\mu_1\), because the Gaussian density function \(p_g(u,v)\), employed in (27) and the sequel, is itself an approximation to the true (unknown) probability density function \(p_{RQ}(u,v)\) of \(R\) and \(Q\). The result of the Gaussian approximation in (27) and (28) has been denoted by \(\mu_{1g}\).

Upon use of (24), the mean value approximation, \(\mu_{1g}\) in (28), takes the form

\[
\mu_{1g} = \sigma_x \sigma_y \left(\frac{\pi}{k}\right)^{\frac{1}{2}} (1 + R_x + R_y)^{\frac{1}{2}} {}_1F_1(-.5;1;\gamma) , \quad (30)
\]
where the various parameters are defined in (12), (13), (22), and (29). If the signal is absent at the input, then \( A = 0, R_x = 0, R_y = 0, V = 0 \), giving

\[
\mu_{1g}(A=0) = \sigma_x \sigma_y \left( \frac{\pi}{K} \right)^{\frac{1}{2}},
\]

which decays to 0 as \( K^{-\frac{1}{2}} \) for large \( K \).

On the other hand, for \( A > 0 \), suppose that \( K \) is large enough that parameter \( V \) in (29) is large compared with 1. Then, we have the asymptotic result [4; A.1.16b]

\[
\mu_{1g} \sim V \left( 1 + \frac{1}{4V} \right) = \frac{1}{4} A^2 |W(f-f_0)|^2 \left( 1 + \frac{1}{4V} \right) \text{ as } V \to \infty,
\]

where we used (9). That is, the mean magnitude \( \mu_{1g} \) approaches the signal-only output \( \gamma \), with an additive term that decays as \( K^{-1} \), not \( K^{-\frac{1}{2}} \) as in (31). These results are expected to be most accurate for large \( K \), where the Gaussian approximation is best.

**MEAN SQUARE MAGNITUDE OF G**

The mean square magnitude of cross spectrum estimate \( G \) is

\[
\mu_2 = |G|^2 = R_x^2 + Q^2 = R_x^2 + \sigma_R^2 + \sigma_Q^2 = \gamma^2 + 2\sigma^2,
\]

where we used (21) and (24). Upon additional use of (22), this develops into

\[
\mu_2 = 4 \sigma_x^2 \sigma_y^2 \left( R_x R_y + \frac{1 + R_x + R_y}{K} \right).
\]
This result is exact, having been developed directly from the exact joint characteristic function (19) of R and Q; hence, there is no need to add subscript g to \( \mu_2 \). However, it can be noted that use of the Gaussian probability density approximation (25) yields exactly the same result (34).

**DEFLECTION OF MAGNITUDE |G|**

The deflection of the magnitude of the complex cross spectrum estimate |G| is defined here as

\[
d_c = \frac{\mu_1 - \mu_1(A=0)}{\left(\mu_2(A=0) - \mu_1^2(A=0)\right)^{1/2}}. \tag{35}
\]

However, since only the approximate result \( \mu_{1g} \) is available, we also define the Gaussian-approximation deflection as

\[
d_g = \frac{\mu_{1g} - \mu_{1g}(A=0)}{\left(\mu_2(A=0) - \mu_{1g}^2(A=0)\right)^{1/2}}. \tag{36}
\]

Substitution of (30), (31), and (34) (with \( A = 0 \)) yields

\[
d_g = \left(\frac{\pi}{4\pi}\right)^{1/2} \left[ (1 + R_x + R_y)^{1/2} I_F(-.5;1;-V) - 1 \right], \tag{37}
\]

where \( V \) is given by (29).

If the number of pieces \( K \) is so large that \( V > 1 \), use of the asymptotic behavior of \( I_F \) [4; A.1.16b] yields
Thus, the Gaussian deflection increases as $K^{1/2}$ for large $K$ and is proportional to the geometric mean of the individual signal-to-noise ratios.

A comparison of the Gaussian deflection $d_g$ in (37) with some exact results for the desired deflection $d_c$ in (35) will be made in the next section for selected values of $K$ and $A$. The asymptotic behavior (38) will not be employed for that comparison, since it is valid only for larger values of $K$. 

\[
d_g \sim \left( \frac{4}{4-\pi} K R_x R_y \right)^{1/2} - \left( \frac{\pi}{4-\pi} \right)^{1/2} = 2.16 \left( K R_x R_y \right)^{1/2} - 1.91 \quad \text{as} \quad K \to \infty .
\]
EXACT RESULTS

Some special cases for mean magnitude $\mu_1$ defined in (26) can be carried out in closed form. These results complement the earlier approximations, furnish a check on the Gaussian approximations, and establish their regions of accuracy.

MEAN MAGNITUDE FOR ONE PIECE, $K = 1$

When $K = 1$, the magnitude of complex estimate $G$ follows immediately from (7) as

$$|G| = \left( (\alpha_1 + a_1)^2 + (\beta_1 + b_1)^2 \right)^{\frac{1}{2}} \left( (\alpha_1 + c_1)^2 + (\beta_1 + d_1)^2 \right)^{\frac{1}{2}} .$$  \hfill (39)

The desired average over the six random variables involved is conducted by first holding random variables $\alpha_1$ and $\beta_1$ fixed. The conditional average over the remaining four random variables then factors into the product of two averages. The first conditional average can be expressed as

$$\int \int \frac{da db}{2\pi \sigma_x^2} \exp \left( -\frac{a^2 + b^2}{2\sigma_x^2} \right) \left( (\alpha_1 + a)^2 + (\beta_1 + b)^2 \right)^{\frac{1}{2}} =$$

$$\int \int \frac{dt du}{2\pi \sigma_x^2} \exp \left( -\frac{(t - \alpha_1)^2 + (u - \beta_1)^2}{2\sigma_x^2} \right) (t^2 + u^2)^{\frac{1}{2}} =$$
\[
\int_0^\infty dr \int_0^{\pi} d\theta \frac{r}{2\pi \sigma_x^2} \exp \left\{- \frac{r^2 - 2\alpha r \cos \theta - 2\beta r \sin \theta + \alpha^2 + \beta^2}{2\sigma_x^2} \right\} = \\
\int_0^\infty dr \frac{r^2}{\sigma_x^2} \exp \left\{- \frac{r^2 + \gamma}{2\sigma_x^2} \right\} I_0 \left( \frac{\sqrt{r^2}}{\sigma_x} \right) = \left[ \frac{\pi}{2} \right]^\frac{1}{2} \sigma_x \, {}_1F_1(-.5;1;-R_x), \quad (40)
\]

where we used (16), (9), [2; 6.631 1, 9.210 1, 9.212 1], and (22). But, it must now be observed that no random variables remain in the end result, meaning that no further averaging is required! A similar approach can be used for the second term in (39), yielding the desired exact result for the mean of \(|G|\) as

\[
\mu_1(K=1) = \frac{\pi}{2} \sigma_x \sigma_y \, {}_1F_1(-.5;1;-R_x) \, {}_1F_1(-.5;1;-R_y). \quad (41)
\]

This can be compared with the corresponding Gaussian approximation according to (30) and (29), namely

\[
\mu_{1g}(K=1) = \pi^\frac{1}{2} \sigma_x \sigma_y \, (1+R_x+R_y)^\frac{1}{2} \, {}_1F_1\left(-.5;1;\frac{R_x R_y}{1+R_x+R_y}\right). \quad (42)
\]

A short table follows; for \(K = 1\), the Gaussian approximation is anywhere from 6% to 13% in error, over this range of values.

**Tabulation of Mean Magnitude for \(K = 1\)**

<table>
<thead>
<tr>
<th>(R_x)</th>
<th>(R_y)</th>
<th>(\mu_1(K=1))</th>
<th>(\mu_{1g}(K=1))</th>
<th>(\mu_{1g}(K=1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.57</td>
<td>1.77</td>
<td>1.13</td>
</tr>
<tr>
<td>.5</td>
<td>.25</td>
<td>2.18</td>
<td>2.43</td>
<td>1.12</td>
</tr>
<tr>
<td>.5</td>
<td>.5</td>
<td>2.40</td>
<td>2.66</td>
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</tr>
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<td>1</td>
<td>.25</td>
<td>2.55</td>
<td>2.80</td>
<td>1.10</td>
</tr>
<tr>
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<td>.5</td>
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<td>3.08</td>
<td>1.10</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
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</tr>
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<td>2</td>
<td>.5</td>
<td>3.52</td>
<td>3.77</td>
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</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4.12</td>
<td>4.38</td>
<td>1.06</td>
</tr>
</tbody>
</table>
DEFLECTION FOR ONE PIECE, $K = 1$

The deflection of interest was defined in (35). When we use exact results (41) and (34), we find, that for $K = 1$,

$$d_c(K=1) = \frac{\pi}{(16 - \pi^2)^{\frac{1}{4}}} \left[ \frac{1}{1+F_1(-0.5;1;-R_x)} \frac{1}{1+F_1(-0.5;1;-R_y)} - 1 \right]. \quad (43)$$

On the other hand, the Gaussian approximation to the deflection is given by (37) and (29) in the form

$$d_g(K=1) = \left( \frac{\pi}{4 - \pi} \right)^{\frac{1}{4}} \left[ (1+R_x+R_y)^{\frac{1}{4}} \frac{1}{1+F_1(-0.5;1;\frac{R_x R_y}{1+R_x+R_y})} - 1 \right]. \quad (44)$$

A comparison of these two results is given in the following table; the Gaussian approximation overestimates the deflection by about 40% for $K = 1$. This is not too surprising when we recall that the Gaussian approximation cannot be expected to be valid for $K = 1$, but rather to be best for large $K$, where summation variables $R$ and $Q$ in (7) are tending toward Gaussian.

<table>
<thead>
<tr>
<th>$R_x$</th>
<th>$R_y$</th>
<th>$d_c(K=1)$</th>
<th>$d_g(K=1)$</th>
<th>$\frac{d_g(K=1)}{d_c(K=1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0+</td>
<td>0+</td>
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<td>1.41</td>
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</tr>
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<td>1.57</td>
<td>2.16</td>
<td>1.37</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.06</td>
<td>2.81</td>
<td>1.37</td>
</tr>
</tbody>
</table>
JOINT PROBABILITY DENSITY FUNCTION FOR SIGNAL ABSENT, ANY K

When amplitude $A$ in input signal (1) is zero, then parameter $\gamma$ in (9) is zero, and the exact joint characteristic function of $R$ and $Q$ in (19) reduces to

$$f_{RQ}^0(i\xi,i\eta) = \left[1 + B^2(\xi^2 + \eta^2)\right]^{-K}, \quad (45)$$

where we used (15) and defined $B = \sigma_x \sigma_y / K$. The corresponding exact joint probability density function of $R$ and $Q$ is then

$$p_{RQ}^0(u,v) = \frac{1}{4\pi^2} \iint d\xi \, d\eta \exp(-iu\xi - iv\eta) \left[1 + B^2(\xi^2 + \eta^2)\right]^{-K} =$$

$$= \frac{1}{4\pi^2} \int_0^\infty dr \int_{-\pi}^\pi d\theta \frac{\exp(-ir(ucos\theta + vsin\theta))}{\left[1 + B^2r^2\right]^K} =$$

$$= \frac{1}{2\pi} \int_0^\infty dr \frac{J_0(\rho r)}{\left[1 + B^2r^2\right]^K} = \frac{\rho^{K-1}}{\pi 2^K (K-1)! B^{K+1} K^{K-1} \rho(B)}, \quad (46)$$

where we have defined $\rho = (u^2 + v^2)^{1/2}$ and used [2; 6.565 4]. The function $K_v(x)$ is a modified Bessel function of the second kind and order $v$ [3; 9.6]. Relation (46), which applies only for signal amplitude $A = 0$, is valid for all argument values $u,v$ and any number of pieces $K$; this joint density is seen to be a function only of radius $(u^2 + v^2)^{1/2}$. 

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MOMENTS OF MAGNITUDE ESTIMATE $|G|$ FOR SIGNAL ABSENT, ANY K

The exact mean magnitude $\mu_1 = |G|$ is given by (26). When signal amplitude $A = 0$, this becomes

$$\mu_1(A=0) = \iint du \, dv \, (u^2 + v^2)^{\frac{1}{2}} \, f_{RQ}(u,v) =$$

$$= \int_0^\pi d\phi \int_0^\infty d\rho \, \frac{\rho^{K-1}}{2^K (K-1)!} \, B^{K+1} \, K_{K-1}(\rho \, B) =$$

$$= \pi \frac{B \, \Gamma(K+\frac{1}{2})}{\Gamma(K)} = \sigma_x \sigma_y \frac{(1/2)_K}{(1)_K} \text{ for all } K, \quad (47)$$

where we used (46), [2; 6.561 16], and $B = \sigma_x \sigma_y / K$. More generally, the $2v$-th moment of $|G|$ for signal absent is available according to

$$\mu_{2v}(A=0) = \iint du \, dv \, (u^2+v^2)^v \, f_{RQ}(u,v) = \frac{\Gamma(K+v)}{\Gamma(K)} \left[ \frac{2\sigma_x \sigma_y}{K} \right]^{2v} \quad (48)$$

which is exact for all $K$ and $v$. As checks on (48), we have:

1 for $v = 0$; result (47) for $v = \frac{1}{2}$; and $4 \sigma_x^2 \sigma_y^2 / K$ for $v = 1$. The last result is the mean square value $\overline{R^2 + Q^2}$ and agrees with exact result (34) when signal amplitude $A = 0$ there. The asymptotic behavior of moment (48) is given by

$$\mu_{2v}(A=0) \sim \Gamma(v+1) \left[ \frac{4 \sigma_x^2 \sigma_y^2}{K} \right]^{v} \text{ as } K \to \infty, \quad (49)$$

where we used [3; 6.1.47].
The Gaussian approximation, $\mu_1(A=0)$, to $\mu_1(A=0)$ is given by (31). A comparison of the two is given below; it reveals that the Gaussian approximation is excellent for $K > 10$.

### Tabulation of Mean Magnitude for $A = 0$

<table>
<thead>
<tr>
<th>K</th>
<th>$\mu_1(A=0)/(\sigma_x\sigma_y)$</th>
<th>$\mu_{1g}(A=0)/(\sigma_x\sigma_y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5708</td>
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</tr>
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</tr>
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</tr>
<tr>
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<td>.1868</td>
</tr>
<tr>
<td>100</td>
<td>.1770</td>
<td>.1772</td>
</tr>
</tbody>
</table>
JOINT PROBABILITY DENSITY FUNCTION FOR SIGNAL PRESENT, ANY K

We have analytically determined the exact joint characteristic function of R and Q in (19), for arbitrary signal amplitude A and number of pieces K. However, the corresponding joint probability density function \( p_{RQ}(u,v) \) is not generally available in closed form. When (19) is substituted into the double Fourier transform for \( p_{RQ}(u,v) \), and a change to polar coordinates is made, the following single integral results:

\[
p_{RQ}(u,v) = \frac{1}{2\pi} \int_{0}^{\infty} dr \frac{r}{B(r)^{K}} \exp \left( -\frac{\gamma \left( \frac{\sigma_x^2 + \sigma_y^2}{2K} \right) r^2}{B(r)} \right) J_0(D(r; u, v)) , \tag{50}\]

where

\[
B(r) = 1 + \frac{\sigma_x^2 \sigma_y^2 r^2}{K^2} , \quad D(r; u, v) = r \left( \left[ u - \frac{\gamma}{B(r)} \right]^2 + v^2 \right)^{\frac{1}{2}} . \tag{51}\]

Although numerical values could be obtained from (50), a more efficient approach is to use a two-dimensional fast Fourier transform directly on characteristic function (19).

We begin by defining, for numerical convenience, the normalized random variables

\[
r = \frac{R}{\sigma_x \sigma_y} , \quad q = \frac{Q}{\sigma_x \sigma_y} . \tag{52}\]

The joint characteristic function of \( r \) and \( q \) is then
\[ f_{\text{rq}}(i\xi, i\eta) = \exp(i\xi R + i\eta q) = \exp\left(\frac{i\xi R + i\eta Q}{\sigma_x \sigma_y} \right) = f_{\text{RQ}}\left(\frac{i\xi}{\sigma_x \sigma_y}, \frac{i\eta}{\sigma_x \sigma_y}\right) = \]
\[
[1 + (\xi^2 + \eta^2)/K^2]^{-K} \exp\left(\frac{i\xi^2 R^2 R_y^2 - (\xi^2 + \eta^2)(R_x + R_y)/K}{1 + (\xi^2 + \eta^2)/K^2}\right), \tag{53}
\]

where we used (19) and (22). There follows
\[
\begin{align*}
\bar{r} &= 2R_x^2 R_y^2, \quad \bar{q} = 0, \quad \bar{r} q = 0, \quad \sigma_r^2 = \sigma_q^2 = 2(1 + R_x + R_y)/K. \tag{54}
\end{align*}
\]

The joint probability density function of \( r \) and \( q \) is
\[
\begin{align*}
\hat{P}_{\text{rq}}(u, v) &= \frac{1}{4\pi^2} \int \int d\xi \, d\eta \exp(-iu\xi - iv\eta) f_{\text{rq}}(i\xi, i\eta) = \\
&= \frac{1}{2\pi^2} \Re \int_0^\infty d\xi \int_{-\infty}^\infty d\eta \exp(-iu\xi - iv\eta) f_{\text{rq}}(i\xi, i\eta) = \\
&= \frac{\Delta^2}{2\pi^2} \Re \sum_{k=0}^\infty \varepsilon_k \sum_{n=-\infty}^\infty \exp(-i\Delta f(uk + v\lambda)) f_{\text{rq}}(ik\Delta f, i\lambda\Delta f), \tag{55}
\end{align*}
\]

where we used the conjugate symmetry of \( f_{\text{rq}} \), and took a common sampling increment \( \Delta f \) for \( f_{\text{rq}} \), in both \( \xi \) and \( \eta \). Coefficient \( \varepsilon_k \) is associated with the trapezoidal rule and is 1 for all \( k \) except for \( \varepsilon_0 = 1/2 \). It should be observed that the resulting approximation in (55), which will be denoted by \( \hat{P}_{\text{rq}}(u, v) \), is periodic in both \( u \) and \( v \), with period \( 2\pi/\Delta f \). This aliased probability density function, \( \hat{P}_{\text{rq}}(u, v) \), is the quantity that will be evaluated.
We now take samples of the function $P_{r q}(u,v)$ over full periods in $u$ and $v$, that is, $0 \leq m,n \leq N-1$, according to

$$P_{r q}\left(\frac{2\pi m}{NA_f},\frac{2\pi n}{NA_f}\right) = \frac{\Delta_f^2}{2\pi^2} \Re \sum_{k=0}^{\infty} \varepsilon_k \sum_{\lambda=-\infty}^{\infty} \exp\left(-\frac{2\pi}{N}(mk+n\lambda)\right) f_{r q}(i k \Delta f, i \lambda \Delta f)$$

$$= \frac{\Delta_f^2}{2\pi^2} \Re \sum_{k=0}^{N-1} \sum_{\lambda=0}^{N-1} \exp\left(-\frac{2\pi}{N}(mk+n\lambda)\right) f_a(i k \Delta f, i \lambda \Delta f) , \quad (56)$$

where $\{f_a(i k \Delta f, i \lambda \Delta f)\}$ is the collapsed (or prealiased) version of $\{\varepsilon_k f_{r q}(i k \Delta f, i \lambda \Delta f)\}$. No approximation is involved in the last step in (56), in reducing the infinite sums to finite sums. The double sum in (56) will be recognized as a two-dimensional fast Fourier transform, when $N$ is taken as a power of 2.

The common increment in the two arguments of the aliased probability density function $P_{r q}(u,v)$ in (56) is

$$\Delta_p = \frac{2\pi}{NA_f} . \quad (57)$$

Sampling increment $\Delta_f$ in $\xi$ and $\eta$ must be small enough that the resulting aliasing in periodic function $P_{r q}(u,v)$ is insignificant. Also, sampling increment $\Delta_p$ in $u$ and $v$ must be small enough to track important variations in $P_{r q}(u,v)$. This will generally require large values of $N$.  

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The desired moment is mean magnitude

\[ \mu_1 = (R^2 + Q^2)^{1/2} = \sigma_x \sigma_y \left( r^2 + q^2 \right)^{1/2} = \]

\[ = \sigma_x \sigma_y \iint du \, dv \left( u^2 + v^2 \right)^{1/2} p_{rq}(u,v) = \sigma_x \sigma_y D_2. \quad (58) \]

In order to evaluate double integral \( D_2 \), three approximations must be accepted. First, the doubly infinite range must be replaced by a square of size \( 2\pi/\Delta \xi \) covering the region where \( p_{rq}(u,v) \) is essentially nonzero; this region, to be denoted by \( S \), is roughly centered at \( u,v = \bar{r},\bar{q} \). Then, \( p_{rq}(u,v) \) must be replaced by \( p_{rq}(u,v) \), since the former function cannot be evaluated. Finally, the double integral must be replaced by a double sum, using the sample points furnished by (56). The accuracy of these three replacements depends critically on the ability to accomplish the goals listed under (57), and therefore on the ability to utilize large values of \( N \) in (56). The resulting approximation to \( D_2 \) is

\[ D_2 \approx \Delta^3 \sum_{m,n} \sum_{r \in S} (m^2 + n^2)^{1/2} p_{rq}(m\Delta, n\Delta). \quad (59) \]

One final nuance is that since region \( S \) can encompass negative values for \( m \) and/or \( n \), whereas (56) is typically evaluated only for \( 0 \leq m,n \leq N-1 \), the lookup for the appropriate value of \( p_{rq}(m\Delta, n\Delta) \) to use with \( (m^2 + n^2)^{1/2} \) in (59) is in bins.
m modulo N and n modulo N, respectively. A program that achieves all of these features is listed in the appendix; it includes some diagnostic plots that keep track of the aliasing and attempt to control the error inherent in (59).

An example for $K = 1$, $R_x = R_y = 10$ yielded $\mu_1/(\sigma_x \sigma_y) = D_2 = 21.026487893$, when done exactly by means of (41). As an illustration of the accuracy of (59), it yielded $D_2 \approx 21.026487800$ for the same parameter values, using increment $\Delta_f = .06$ and $N = 128$. Also, $K_m = 200$ samples of characteristic function $f_{rq}(i\xi, i\eta)$ in each dimension were used, thereby minimizing termination error.

Another check on the above procedure and program was accomplished by deliberately taking, as a test case, a Gaussian two-dimensional characteristic function, and subjecting it to the above numerical techniques. The exact answer for the mean magnitude is furnished by (30) for this Gaussian example. In particular, for $K = 10$, $R_x = R_y = 1$, (30) yielded $\mu_1g/(\sigma_x \sigma_y) = 2.1577687$. On the other hand, for $\Delta_f = .6$, $N = 128$, $K_m = 50$, numerical procedure (59) yielded 2.1577675, an error of 1.2E-6.

**DEFLECTION OF MAGNITUDE $|G|$, ANY $K$**

We now have the ability to exactly evaluate the deflection $d_c$ of magnitude estimate $|G|$ defined in (35), and to compare it with the Gaussian approximation defined in (36) and evaluated in (37). A numerical comparison is presented in the table below.
<table>
<thead>
<tr>
<th>K</th>
<th>R_x</th>
<th>R_y</th>
<th>(\mu_1)</th>
<th>(\mu_{1g})</th>
<th>(d_c)</th>
<th>(d_g)</th>
<th>(\Delta_f)</th>
<th>(K_m)</th>
</tr>
</thead>
<tbody>
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<td>.5</td>
<td>1.2131</td>
<td>1.2241</td>
<td>2.1558</td>
<td>2.2650</td>
<td>.5</td>
<td>50</td>
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<tr>
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<td>.5</td>
<td>1.5978</td>
<td>1.6068</td>
<td>3.4135</td>
<td>3.5712</td>
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<td>50</td>
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<tr>
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<td>1</td>
<td>2.1515</td>
<td>2.1578</td>
<td>5.2231</td>
<td>5.4517</td>
<td>.5</td>
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<tr>
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<td>.5</td>
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<td>2.1859</td>
<td>5.3148</td>
<td>5.5477</td>
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<td>2.9742</td>
<td>7.8990</td>
<td>8.2384</td>
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<tr>
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<td>2</td>
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<td>4.1272</td>
<td>11.672</td>
<td>12.174</td>
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<td>1</td>
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<td>4.1533</td>
<td>11.757</td>
<td>12.263</td>
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<td>4</td>
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<td>8.1133</td>
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<td>25.779</td>
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<td>1</td>
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<td>2.0151</td>
<td>19.748</td>
<td>19.836</td>
<td>1.5</td>
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</table>

For \(K = 10\), the agreement of mean magnitude \(\mu_1\) and Gaussian approximation \(\mu_{1g}\) is very good over the entire range of parameter values considered, with the Gaussian approximation being a slight overestimate by less than 1%. On the other hand, the agreement between deflections \(d_c\) and \(d_g\) is not quite as good, with the Gaussian case overestimating by about 4.5%. High accuracy in the deflection for \(K = 10\) can only be achieved through the detailed numerical procedure presented above; the Gaussian approximation has some limitations at this low value of \(K\), the number of independent pieces.

On the other hand, for \(K = 100\), the means are virtually identical, while the deflections differ by 0.5%. This is an illustration of the approach of summation variables \(R\) and \(Q\) in (7) to Gaussian for large numbers of pieces, \(K\).
In this section, we consider adding the two received processes together and estimating the resulting auto spectrum. We then evaluate the deflection of this auto spectrum estimate and compare it with the deflection for the magnitude of the complex cross spectrum estimate, $|G|$.

Since the two input noises $x(t)$ and $y(t)$ utilized in (6) and (7) can have different levels, we scale them and sum according to

$$z(t) = [s(t) + x(t)] + \lambda [s(t) + y(t)] =$$
$$= (1 + \lambda) s(t) + x(t) + \lambda y(t). \quad (60)$$

Scale factor $\lambda$ will be chosen to maximize the deflection of the auto spectrum estimate of process $z(t)$. (More generally, we should filter the two processes and add.)

**CHARACTERISTIC FUNCTION OF AUTO SPECTRUM ESTIMATE**

Analogous to (4), (5), and (6), the $k$-th voltage density estimate, at analysis frequency $f$, of process $z(t)$ is

$$Z_k \equiv \int dt \exp(-i2\pi ft) w_k(t) z(t) =$$
$$= (1 + \lambda) \frac{\bar{A}}{2} e^{i\phi} W_k(f-f_0) + X_k + \lambda Y_k =$$
$$= (1 + \lambda)(a_k + i\beta_k) + a_k + ib_k + \lambda(c_k + id_k). \quad (61)$$
The auto spectrum estimate at analysis frequency $f$ is given by

$$G_z = \frac{1}{K} \sum_{k=1}^{K} |z_k|^2 = \frac{1}{K} \sum_{k=1}^{K} |(1 + \lambda)(a_k + i\beta_k) + e_k + if_k|^2,$$  

(62)

where independent Gaussian random variables

$$e_k = a_k + \lambda c_k, \quad f_k = b_k + \lambda d_k,$$  

(63)

with properties

$$\bar{e}_k = 0, \quad \bar{f}_k = 0, \quad \bar{e}_k^2 = \bar{f}_k^2 = \sigma_x^2 + \lambda^2 \sigma_y^2 = \sigma_e^2.$$  

(64)

Here, we used (12) and (13). An alternative form for (62) is

$$G_z = \frac{1}{K} \sum_{k=1}^{K} \left[ \left( (1 + \lambda) a_k + e_k \right)^2 + \left( (1 + \lambda) \beta_k + f_k \right)^2 \right].$$  

(65)

We now hold the set of random variables $\{a_k\}$ and $\{\beta_k\}$ fixed and compute the conditional characteristic function of the $k$-th term of (65). Using (64), the Gaussian property of the random variables $\{e_k\}$ and $\{f_k\}$, and (17), the desired quantity is

$$F_k(i\xi) = \exp \left[ i\xi \left( (1+\lambda) a_k + e_k \right)^2 + i\xi \left( (1+\lambda) \beta_k + f_k \right)^2 \right] =$$

$$= \int de df \left[ \frac{1}{2\pi \sigma_e^2} \exp \left[ -\frac{e^2 + f^2}{2\sigma_e^2} + i\xi \left( (1+\lambda) a_k + e \right)^2 + i\xi \left( (1+\lambda) \beta_k + f \right)^2 \right] =$$

$$= \frac{1}{1 - i\xi 2\sigma_e^2} \exp \left( \frac{i\xi (1 + \lambda)^2 \left( a_k^2 + \beta_k^2 \right)}{1 - i\xi 2\sigma_e^2} \right).$$  

(66)
But, by use of (9), the end result in (66) is not a random variable at all, and furthermore, does not depend on \( k \).
Therefore, the characteristic function of auto spectrum estimate \( G_z \) in (65) is given by

\[
F(i\xi) = F_1\left(\frac{i\xi}{K}\right)^K = \left[1 - i\xi2\sigma_e^2/K\right]^{-K} \exp\left\{i\xi(1 + \lambda)^2\gamma\right\}.
\] (67)

This result in (67) is exact. By expanding \( \ln F(i\xi) \) in a power series in \( i\xi \), the \( j \)-th cumulant of estimate \( G_z \) is found to be

\[
\lambda_j = \frac{(i-1)!}{K^j-1} \left(2\sigma_e^2\right)^j \left[1 + j\frac{(1 + \lambda)^2 R_x R_y}{\lambda^2 R_x + R_y}\right] \quad \text{for } j \geq 1,
\] (68)

where we used (64) and (22).

**DISTRIBUTION OF AUTO SPECTRUM ESTIMATE**

The exceedance distribution function corresponding to characteristic function (67) is the detection probability for random variable \( G_z \) and is given by [5]

\[
P_d = \Pr(G_z > \nu) = Q_K\left(\frac{2K (1 + \lambda)^2 R_x R_y}{\lambda^2 R_x + R_y}\right)^{1/2}, \left(\frac{K \nu}{\sigma_x^2 + \lambda^2 \sigma_y^2}\right)^{1/2},
\] (69)

where we used (64) and (22). The false alarm probability is obtained by setting \( R_x = R_y = 0 \), thereby yielding
We now want to choose scale factor $\lambda$ in (60) so as to maximize the detection probability while holding the false alarm probability fixed. This latter requirement means holding the argument of the exponential in (70) fixed, which makes threshold $v$ a function of $\lambda$. It also makes the second argument of the $Q_K$ function in (69) constant. Therefore, maximization of $P_d$ is achieved by maximizing the first argument in (69), or equivalently by maximizing the quantity

$$
\frac{(1 + \lambda)^2}{\lambda^2 R_x + R_y} \tag{71}
$$

by choice of scale factor $\lambda$. The best choice is

$$
\lambda = \frac{R_y}{R_x} \tag{72}
$$

leading to maximum value $1/R_x + 1/R_y$ for (71). Substitution of these results in (69) yield the maximum detection probability as

$$
P_d = Q_K \left( \left( 2K (R_x + R_y) \right) \frac{1}{2} \left( \frac{Kv}{\sigma_x^2 + \sigma_y^2} \right) \frac{1}{2} \frac{\sigma_y}{\sigma_x} \right). \tag{73}
$$

(The nonsymmetry in the second argument can be eliminated by using the symmetric combination $\eta[s(t) + x(t)]/\sigma_x + \eta^{-1}[s(t) + y(t)]/\sigma_y$ instead of (60), and choosing $\eta$ optimally.) The corresponding false alarm probability is obtained by replacing the first argument in (73) by zero, that is, $R_x = R_y = 0$. 

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DEFLECTION OF AUTO SPECTRUM ESTIMATE

The deflection of auto spectrum estimate $G_z$ is defined analogously to that for magnitude cross spectrum estimate $|G|$ in (35), namely

$$d_a = \frac{X_1 - X_1(A=0)}{X_2(A=0)^{\frac{1}{2}}} = \left(1 + \lambda\right)^2 \frac{R_x R_y}{\lambda^2 R_x + R_y},$$

(74)

where we used the relevant cumulants in (68). But, this quantity has exactly the same dependence on scale factor $\lambda$ as does (71). Therefore the best choice of $\lambda$ is again (72), leading to the maximum deflection, which is exact for all $K$, of

$$d_a = K^{\frac{1}{2}} (R_x + R_y).$$

(75)

Thus, the choices of $\lambda$ that maximize the deflection and the detection probability coincide for the auto spectrum estimate.

The maximum deflection $d_a$ in (75) is not always larger than the magnitude cross spectrum deflection $d_c$ considered earlier, even though (75) has utilized the best scale factor $\lambda$ in summation (60). For example, for large $K$, we have from (38) the very good approximation,

$$d_g \sim K^{\frac{1}{2}} \left(\frac{4}{4-\pi} R_x R_y\right)^{\frac{1}{2}} \text{ as } K \to \infty.$$

(76)

The ratio of deflections is therefore given by

$$\frac{d_a}{d_g} \sim \left(\frac{4-\pi}{4}\right)^{\frac{1}{2}} \frac{R_x + R_y}{R_x^{\frac{1}{2}} R_y^{\frac{1}{2}}} = .463 \left[\left(\frac{R_x}{R_y}\right)^{\frac{1}{2}} + \left(\frac{R_y}{R_x}\right)^{\frac{1}{2}}\right] \text{ as } K \to \infty.$$
The minimum value of this ratio is .926, reached when \( R_y = R_x \).

However, if \( R_y/R_x > 2.207 \) or if \( R_y/R_x < .453 \), then ratio (77) is always larger than 1. Thus, which deflection is larger (for large K) depends on the ratio \( R_y/R_x = \sigma_x^2/\sigma_y^2 \). For small K, direct numerical evaluation reveals that \( d_a \) is usually larger than \( d_c \).

**GRAPHICAL COMPARISON OF DEFLECTIONS**

It was demonstrated in the previous section that the Gaussian approximation is rather accurate for evaluating the deflection of the magnitude cross spectrum estimate \(|G|\), when K is larger than 10. The resulting Gaussian deflection \( d_g \) was given by (37) along with (29). On the other hand, the deflection \( d_a \) for the auto spectrum estimate \( G_z \) is given exactly by (75), and is valid for all K.

Plots of deflections \( d_g \) and \( d_a \) are presented in figures 1 and 2 for \( R_y/R_x = 1 \) and 1/2, respectively. They confirm the general behavior predicted earlier. For example, figure 1 for \( R_y = R_x \) shows \( d_g \) to be larger than \( d_a \) for large K, but the curves cross for smaller values of K. On the other hand, figure 2 for \( R_y = R_x/2 \) has \( d_a \) generally larger than \( d_g \), except when K gets very large. The ratio \( R_y/R_x = 1/2 \) is not smaller than the breakpoint .453 (above) that would guarantee \( d_a \) greater than \( d_g \) for large K.

The fact that cross spectrum deflection \( d_g \) (or \( d_c \)) is greater than auto spectrum deflection \( d_a \) for some ranges of the parameter
Figure 1. Deflections $d_g$ and $d_a$ for $R_y = R_x$

Figure 2. Deflections $d_g$ and $d_a$ for $R_y = R_x/2$
values does not necessarily reflect in the relative detection capability of the two processing techniques. After all, the deflection criterion only involves moments up through second order, whereas the full detection and false alarm probabilities involve all orders of moments. An example where the deflection of a random variable can be artificially accentuated is illustrated in the next section.
ACCENTUATION OF DEFLECTION

The deflection of a random variable is based upon its two lowest order moments, and can therefore be a misleading statistic regarding detectability. That is, the detection and false alarm probabilities depend on the entire probability density functions for signal present and absent, respectively, not just their first two moments.

To illustrate these points, consider detection of a Gaussian random variable $x$ with mean $m_0$ under signal-absent hypothesis $H_0$, and mean $m_1 (> m_0)$ under signal-present hypothesis $H_1$. Also, let the standard deviations have a common value $\sigma$ under both hypotheses. Then, the deflection of random variable $x$ is

$$d_x = \frac{m_1 - m_0}{\sigma}. \quad (78)$$

The detection probability, for threshold $v$, is given by

$$P_d = \Pr(x > v | H_1) = \int_v^\infty \frac{du}{(2\pi)^{\frac{1}{2}} \sigma} \exp\left( -\frac{(u - m_1)^2}{2\sigma^2} \right) = \int_{(v-m_1)/\sigma}^\infty dt \frac{1}{(2\pi)^{\frac{1}{2}}} \exp(-t^2/2) \equiv \Phi\left(\frac{m_1 - v}{\sigma}\right). \quad (79)$$

Similarly, the false alarm probability is given by

$$P_f = \Pr(x > v | H_0) = \Phi\left(\frac{m_0 - v}{\sigma}\right). \quad (80)$$

For a given false alarm probability $P_f$, (80) can be solved for
threshold \( v \), and then substituted into (79). The result is

\[
P_d = \Phi \left( d_x + \Phi^{-1}(P_f) \right), \tag{81}
\]

where \( \Phi \) is the inverse \( \Phi \) function, and we used (78). Thus, given a specified performance level \( P_f \), \( P_d \) the single parameter \( d_x \) completely quantifies performance. Observe that we are still using the entire probability density functions of \( x \) under \( H_0 \) and \( H_1 \), as we must in order to evaluate the exceedance distribution functions in (79) and (80); however, the receiver operating characteristic depends on only the single parameter \( d_x \), through rule (81).

Now, let us consider a monotonic nonlinear distortion of random variable \( x \), yielding new random variable \( y \) according to

\[
y = \exp \left( \frac{C}{\sigma} x \right), \tag{82}
\]

where scaling \( C (> 0) \) is an unspecified constant at the moment. Obviously, the receiver operating characteristic for random variable \( y \) will be identical with that determined for \( x \) above in (79), (80), and (81); only the thresholds will change.

However, let us now consider the deflection of random variable \( y \). Since \( x \) is Gaussian, we have under hypothesis \( H_k \), the \( n \)-th moment of \( y \) in the form

\[
\overline{y^n} = \exp \left( n \frac{C}{\sigma} \right) = \int \frac{du}{(2\pi)^{\frac{1}{2}} \sigma} \exp \left( -\frac{(u - m_k)^2}{2 \sigma^2} + n\frac{C}{\sigma} u \right) =
\]

\[
= \exp \left( n \frac{C}{\sigma} \frac{m_k}{\sigma} + \frac{1}{2} n^2 C^2 \right). \tag{83}
\]
The deflection of random variable $y$ follows immediately as

$$d_y = \frac{\exp(Cd_x) - 1}{\left(\exp(C^2) - 1\right)^{1/2}}. \quad (84)$$

Deflection $d_y$ depends on $d_x$ and the dimensionless scaling $C$. If scaling $C$ is very small, then we have $d_y = d_x$; this agrees with the observation that distortion (82) is virtually linear then.

However, if parameter $C$ is substantial, deflection $d_y$ can be much greater than $d_x$. In fact, given a value of $d_x$, there is a value of $C$, namely $C = d_x$, at which $d_y$ peaks, with value

$$\max_C d_y = \left(\exp(d_x^2) - 1\right)^{1/2}. \quad (85)$$

As an example, the value of $d_y$ is greater than 1000 if $d_x > 3.72$. Thus, the deflection of random variable $y$ can be greatly accentuated relative to the deflection of $x$, merely by performing a monotonic nonlinear distortion. The ability to achieve this artificial improvement in deflection strongly cautions against relying on the deflection as a reliable measure of detectability.
SUMMARY

The joint characteristic function of the real and imaginary parts of the complex cross spectrum estimate has been derived in closed form, for arbitrary signal strength and noise spectra. For noise-only, the corresponding joint probability density function has also been derived in closed form and used to obtain exact results for fractional moments of the magnitude of the cross spectrum estimate. For signal present, an efficient two-dimensional fast Fourier transform numerical procedure has been utilized to get accurate probability density functions and moments.

When the number of pieces, K, used in the estimate of the cross spectrum is large, a Gaussian approximation has been employed for the joint probability density function of the real and imaginary parts. Numerical computations reveal that this Gaussian approximation is adequate if $K > 10$, and is very accurate for $K > 100$. This Gaussian approximation has then been used to determine the deflection of the magnitude of the cross spectrum estimate.

Comparisons of the deflections for the magnitude of the cross spectrum estimate and for the auto spectrum estimate reveal that they are rather close to each other. However, even though one deflection may be larger than the other for some ranges of parameter values, that does not necessarily make the corresponding processor a better detector. An example is presented to show how the deflection may be artificially enhanced
merely by nonlinear transformation of the decision variable, but without any change in the fundamental detectability of the signal.

A program is furnished in BASIC which enables calculation of the joint probability density function of the real and imaginary parts of the cross spectrum estimate, for arbitrary signal strength. In addition, it calculates the mean magnitude of the cross spectrum estimate and compares it with the Gaussian approximation.
APPENDIX - PROGRAM FOR CALCULATION OF $\mu_1$

This appendix contains a listing of a BASIC program for the evaluation of the aliased joint probability density function $p_{rq}$ given by (56), in addition to the normalized moment $D_2$ defined by (59). Inputs required of the user are $D_e$ ($A_f$) in line 10, $P (K)$ in line 20, $R_x$ ($R_x$) in line 30, $R_y$ ($R_y$) in line 40, $N (N)$ in line 50, and $K_m$ ($K_m$) in line 60. An explanation of each of these symbols is given in the program listing.

The first plot produced is a slice of the magnitude of the aliased characteristic function $f_a(i\xi, i\eta)$ in (56) along the $\xi$ axis; this affords a determination of whether adequate decay has been realized. The next plot displays the real and imaginary parts of $f_a$; this indicates whether the sampling rate is adequate to track the variations in these two functions.

Then, the sum of the sampled probability density function is computed and subtracted from 1; this error furnished a measure of the accuracy with which the density has been calculated. Next, a slice of density $p(u,v)$ along the $u$ axis is plotted; this indicates whether sufficient decay has been achieved before the aliasing shows up. (Strictly, this observation replaces the one above on the real and imaginary parts of $f_a$.) The user must also note the maximum location of the density and enter this number into the program at this point. Finally, a slice of $p(u,v)$ in $v$, for $u$ equal to the maximum location, is plotted; this guarantees that adequate decay in the other dimension of the density has been achieved.
The values of $\mu_1$, $\mu_{1g}$, $\mu_1(A=0)$, $d_c$, and $d_g$ are then printed out. This complete procedure furnishes a measure of accuracy of the Gaussian approximation results, provided that sufficient decays have been realized in all the plots indicated above.

```
10  Delf=.5       ! INCREMENT FOR CHARACTERISTIC FUNCTION
20  P=10.        ! NUMBER OF PIECES, K
30  Rx=1.        ! MEASURE OF SIGNAL-TO-NOISE RATIO IN x
40  Ry=1.        ! MEASURE OF SIGNAL-TO-NOISE RATIO IN y
50  N=128        ! SIZE OF FAST FOURIER TRANSFORM
60  Km=50        ! NUMBER OF SAMPLES IN EACH DIMENSION
70  DOUBLE N,Km,N1,K,L,Kt,Lt,K1    ! INTEGERS
80  DIM Fr(127,127),Fi(127,127),X(127),Y(127),Cos(32)
90  N1=N-1
100 REDIM Fr(0:N1,0:N1),Fi(0:N1,0:N1),X(0:N1),Y(0:N1),Cos(0:N/4)
110  T1=Delf*Delf/(P*P)
120  T2=(Rx+Ry)*Delf*De1f/P
130  T3=2.*SQR(Rx*Ry)*Delf
140  A=2.*PI/N
150  FOR K=0 TO N/4
160  Cos(K)=COS(A*K)    ! QUARTER-COSINE TABLE IN Cos(*)
170  NEXT K
180  FOR K=0 TO Km
190  Kt=K MODULO N
200  K2=K*K
210  T4=T3*K
220  FOR L=-Km TO Km
230  Lt=L MODULO N
240  Sq=K2+L*L
250  T=1.+T1*Sq
260  A=-P*LOG(T)-T2*Sq/T
270  IF A<-500. THEN 320
280  E=EXP(A)
290  A=T4/T
300  Fr(Kt,Lt)=Fr(Kt,Lt)+E*COS(A)    ! COLLAPSING
310  Fi(Kt,Lt)=Fi(Kt,Lt)+E*SIN(A)
320  NEXT L
330  NEXT K
340  GINIT
350  GRAPHICS ON
360  WINDOW 0,N,-10,0
370  GRID N/8,1
380  FOR K=0 TO N1
390  Fr=Fr(K,0)
400  Fi=Fi(K,0)
410  Fs=Fr*Fr+Fi*Fi
420  IF Fs>0. THEN 450
430  PENUP
440  GOTO 460
450  PLOT K,LGT(Fs)*.5
460  NEXT K
470  PENUP
480  PRINT " |f(xi,0)|"
490  PAUSE
500  PRINT " Re f(xi,0) and Im f(xi,0)"
```
510  GCLEAR
520  WINDOW 0,N,-1,1
530  GRID N/8,.2
540  FOR K=0 TO N1
550  PLOT K,Fr(K,0)  ! Re f(xi,0)
560  NEXT K
570  PENUP
580  LINE TYPE 3
590  FOR K=0 TO N1
600  PLOT K,Fi(K,0)  ! Im f(xi,0)
610  NEXT K
620  PENUP
630  LINE TYPE 1
640  FOR K=0 TO N1
650  FOR L=0 TO N1
660  X(L)=Fr(K,L)
670  Y(L)=Fi(K,L)
680  NEXT L
690  IF K>0 THEN 720
700  MAT X=X*.5
710  MAT Y=Y*.5
720  CALL Fft14(N,Cos(*),X(*),Y(*))
730  FOR L=0 TO N1
740  Fr(K,L)=X(L)
750  Fi(K,L)=Y(L)
760  NEXT L
770  NEXT K
780  FOR L=0 TO N1
790  FOR K=0 TO N1
800  X(K)=Fr(K,L)
810  Y(K)=Fi(K,L)
820  NEXT K
830  CALL Fft14(N,Cos(*),X(*),Y(*))
840  FOR K=0 TO N1
850  Fr(K,L)=X(K)
860  ! Fi(K,L)=Y(K)  ! Fi(*) unnecessary
870  NEXT K
880  NEXT L
890  MAT Fr=Fr*(Delf*Delf/(2.*PI*PI))
900  Delp=2.*PI/(N*Delf)
910  S=SUM(Fr)*Delp*Delp
920  PRINT "P =";P;" Rx =";Rx;" Ry =";Ry
930  PRINT "ERROR =";S-1.
940  PRINT " p(u,v)"
950  GCLEAR
960  WINDOW 0,N,-12,0
970  GRID N/8,1
980  FOR K=0 TO N1
990  Fr=Fr(K,0)
1000  IF Fr<>0. THEN 1030
1010  PENUP
1020  GOTO 1040
1030  PLOT K,LGT(ABS(Fr))
1040  NEXT K
1050  PLOT N,LGT(Fr(0,0))
1060  PENUP
1070  INPUT "MAXIMUM LOCATION OF Fr(K,O):",K1
1080  PRINT " p(u1,v1)"
1090  GCLEAR
1100  WINDOW 0,N,-12,0
GRID N/8,1
FOR L=0 TO N1
Fr=Fr(K1,L)
IF Fr>0 THEN 1170
PENUP
GOTO 1180
PLOT L,LGTCABS(Fr))
NEXT L
PLOT N,LGTCABS(Fr(K1,0))
PENUP
PAUSE
GCLEAR
M1=0.
FOR K=K1-N/2 TO K1+N/2
Kt=K MODULO N
K2=K*K
FOR L=-N/2 TO N/2
Lt=L MODULO N
M1=M1+SQR(K2+L*L)*Fr(Kt,Lt)
NEXT L
NEXT K
M1=M1*Delp*Delp*Delp
PRINT "Mu1 =";M1
V=P*Rx*Ry/(1.+Rx+Ry)
CALL F11(-.5,1.,-V,F11,I)
Mlg=SQR(P1*(1.+Rx+Ry)/P)*F11
PRINT "Mu1(GAUSS) =";Mlg
M10=PI
FOR K=1 TO P
M10=M10*(K-.5)/K
NEXT K
PRINT "Mu1(S) =";M10
M2=4./P
Dc=(M1-M10)/SQR(M2-M10*M10)
PRINT "Dc =";Dc
Dg=SQR((P1*(1.+Rx+Ry)*F11-1.))
PRINT "Dc(GAUSS) =";Dg
PRINT
PAUSE
END
SUB Ff't14(DOUBLE N,REAL Cos(*),X(*),Y(*))
N=2^14=16384; 0 SUBS
DOUBLE Log2n,N1,N2,N3,N4,J,K
INTEGER < 2^31 = 2,147,483,648
DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
IF N=1 THEN SUBEXIT
IF N>2 THEN 1640
A=X(0)+X(1)
X(1)=A-X(0)
A=X(0)
A=Y(0)+Y(1)
Y(1)=A-Y(0)
A=Y(0)
SUBEXIT
A=LOG(N)/LOG(2.)
Log2n=A
1660 IF ABS(A-Log2n)<1.E-8 THEN 1690
1670 PRINT "N =";N;"IS NOT A POWER OF 2; DISALLOWED."
1680 PAUSE
1690 N1=N/4
1700 N2=N1+1
1710 N3=N2+1
1720 N4=N3+N1
1730 FOR I1=1 TO Log2n
1740 I2=2^(Log2n-I1)
1750 I3=2*I2
1760 I4=N/I3
1770 FOR I5=1 TO I2
1780 I6=(I5-1)*I4+1
1790 IF I6<=N2 THEN 1830
1800 A1=-Cos(N4-I6-1)
1810 A2=-Cos(I6-N1-1)
1820 GOTO 1850
1830 R1=Cos(I6-1)
1840 R2=Cos(N3-I6-1)
1850 FOR I7=0 TO N-I3 STEP I3
1860 I8=I7+I5-1
1870 I9=I8+I2
1880 T1=X(I8)
1890 T2=X(I9)
1900 T3=Y(I8)
1910 T4=Y(I9)
1920 A3=T1-T2
1930 A4=T3-T4
1940 X(I8)=T1+T2
1950 Y(I8)=T3+T4
1960 X(I9)=A1*A3-A2*A4
1980 NEXT I7
1990 NEXT I5
2000 NEXT I1
2010 I1=Log2n+1
2020 FOR I2=1 TO 14
2030 L(I2-1)=1
2040 IF I2>Log2n THEN 2060
2050 L(I2-1)=2^(I1-I2)
2060 NEXT I2
2070 K=0
2080 FOR I1=1 TO L(13)
2090 FOR I2=11 TO L(12) STEP L(13)
2100 FOR I3=12 TO L(11) STEP L(12)
2110 FOR I4=13 TO L(10) STEP L(11)
2120 FOR I5=14 TO L(9) STEP L(10)
2130 FOR I6=15 TO L(8) STEP L(9)
2140 FOR I7=16 TO L(7) STEP L(8)
2150 FOR I8=17 TO L(6) STEP L(7)
2160 FOR I9=18 TO L(5) STEP L(6)
2170 FOR I10=19 TO L(4) STEP L(5)
2180 FOR I11=10 TO L(3) STEP L(4)
2190 FOR I12=11 TO L(2) STEP L(3)
2200 FOR I13=12 TO L(1) STEP L(2)
2210 FOR I1=I13 TO L(O) STEP L(1)
2220 J=I14-1
2230 IF K>J THEN 2300
2240 A=X(K)
2250 X(K)=X(J)
2260 X(J)=A
2270 A=Y(K)
2280 Y(K)=Y(J)
2290 Y(J)=A
2300 K=K+1
2310 NEXT I14
2320 NEXT I13
2330 NEXT I12
2340 NEXT I11
2350 NEXT I10
2360 NEXT I9
2370 NEXT I8
2380 NEXT I7
2390 NEXT I6
2400 NEXT I5
2410 NEXT I4
2420 .NEXT I3
2430 NEXT I2
2440 NEXT I1
2450 SUBEND
2460 !
2470 SUB F11(A,B,X,F11,D) ! POWER SERIES
2480 Error=1.E-16 ! RELATIVE ERROR TOLERANCE
2490 Number=1000 ! MAXIMUM NUMBER OF TERMS IN SERIES
2500 DOUBLE Number,N ! INTEGERS
2510 Bi=B-1.
2520 IF X<0. THEN 2640
2530 A1=A-1.
2540 F11=T=Big=1.
2550 FOR N=1 TO Number
2560 Fn=FLT(N)
2570 T=T*X*(Fn+A1)/(Fn*(Fn+Bi))
2580 F11=F11+T
2590 Af=ABS(F11)
2600 Big=MAX(Big,Af)
2610 IF Abs(T)<=Error*Af THEN 2750
2620 NEXT N
2630 GOTO 2740
2640 Bi=B-A-1.
2650 F11=T=Big=EXP(X)
2660 FOR N=1 TO Number
2670 Fn=FLT(N)
2680 T=-T*X*(Fn+Bi)/(Fn*(Fn+Bi))
2690 F11=F11+T
2700 Af=ABS(F11)
2710 Big=MAX(Big,Af)
2720 IF Abs(T)<=Error*Af THEN 2750
2730 NEXT N
2740 PRINT Number;"TERMS IN SUB F11 AT ";A;B;X
2750 D=15.-LGT(Big/Af) ! NUMBER OF SIGNIFICANT DIGITS
2760 SUBEND

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