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DEFINITION AND REALIZATION OF A GLOBAL VERTICAL DATUM

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April 1994

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The requirements for defining and realizing a global vertical datum (GVD) with a precision on the dm - or even cm - level has become essential now, due to the improved measurement accuracy provided by the space geodetic techniques. This study identifies the different approaches in which such a global vertical datum could be evolved.

The first approach, which can be considered as an ideal approach, uses the best available geocentric station positions from present and future space geodetic networks, highly accurate geopotential model and surface gravity data around the space stations for defining and realizing a global vertical datum. The other simplified approach, which can be considered more of an operational development with a short time framework, is mainly dependent on the widely available GPS and DORIS tracking networks and accurate geoid height models for the GVD development.

After a review of the various heights and height systems that are in use today, detailed mathematical procedures for setting up the observation equations to define a global vertical datum in a least-squares sense, under different data scenarios, are discussed. A test computation to realize the first iteration global vertical datum with available data at 17 space geodetic stations in six regional vertical datums is also included. The gravimetric height anomaly/undulation computations at the space geodetic stations required in the test computation have been performed using both Modified Stokes' technique and least-squares collocation technique combining surface gravity data in a small cap around the stations with potential coefficients from OSU91A model. The test results show that in an idealistic approach the global vertical datum can be realized to an accuracy of ±5 cm and its connection to the regional vertical datums to an accuracy of ±5 cm to ±23 cm. The estimated values of separation between different regional vertical datums agree mostly well with the results reported by various geodesists and oceanographers based on their regional studies.

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CHAPTER 1

INTRODUCTION

Fundamental to positioning is the concept of height or elevation above some vertical reference datum. Since the process of determining heights (or height differences) using spirit leveling demands that the vertical reference surface be an equipotential surface, historically the geoid has been used for that purpose. By its very definition, the geoid refers to an equipotential surface of the gravity field of the earth that most closely approximates the mean sea level in the spatial least squares sense, at a given time.

There are more than three hundred regional vertical datums in the world today, most of them are based on the long term mean value of the local mean sea level observed at one or more primary tide gauge stations, with an implicit assumption that the mean sea level at these tide gauge sites coincide with the geoid. With the present day accuracy requirement of ±10 cm for the vertical datum definition (Rapp, 1983), the mean value of the local mean sea level observed at the tide gauge sites cannot be considered to coincide with the geoid. The departure of the mean sea level (with respect to a specific time period) from the geoid is termed as Sea Surface Topography (SST), which has a magnitude of up to ±2m globally even over long time scales. SST can be considered as a function of the geographic location (ϕ, λ) and the time (t). Therefore it varies temporally and spatially. Most of the temporal variations within a specific period can be averaged out using appropriate filters, but those which are not cyclic in nature must be considered for the computation of SST at the epoch for which the mean sea level is computed. This leads to the concept of an epoch-dependent datum (Cross et al., 1987). This datum is established by taking tide gauge readings at one or more tide gauge stations over a particular time span and reducing these measurements to a specific epoch within the time interval. All heights are then referred to this adopted vertical reference datum. Mean sea level at a later epoch will not coincide with this reference surface.

Since the vertical datums adopted by different countries are based on mean sea levels computed for different epochs, using different procedures, and tide gauge measurements, they differ from each other as well as with the geoid to an extent of ±2m. Estimated vertical datum inconsistencies δh (Laskowski, 1983) and its propagated effects on terrestrial gravity anomalies over continental plates are given in Heck (1990, Table 1). The magnitude of the effects of vertical datum inconsistencies on gravity anomalies are on the order of ±0.5 mgal (ibid.). We see this effect reflected as systematic regional biases in gravity anomaly data banks.

A few years back, geodetic applications were not affected by such discrepancies in vertical datums as they have played a small role in the problems of geodesy. As greater accuracy is required and more accurate measurements are made now, the problems caused by vertical datum inconsistencies have started surfacing more dramatically than before (Rapp, 1987). The requirements for defining and realizing a global vertical datum with a precision on the dm- or even cm-level has become essential now, more than before, due to the improved measurement accuracy provided by the space geodetic techniques. Several arguments emphasizing the need for a global vertical datum have been discussed by Heck and Rummel (1990) and Carter et al. (1989). A few of the strong arguments are as follows:

- The establishment of a global vertical datum will provide a means to accurately connect national and continental vertical datums.
- Removal of the systematic regional biases in gravity anomaly data banks, noted earlier due to referencing heights to different level surfaces in different vertical datum zones, requires the definition of a global vertical datum, so that the gravity anomalies can now refer to one unique geopotential surface.
- Comparisons between the results of geodetic leveling and oceanographic procedures for determining sea surface slopes over long distances require a consistent vertical datum.

In addition to the above arguments, the rapid growth in positioning through the Global Positioning System (GPS) has created a need for better understanding of orthometric heights globally and how they can be determined using GPS.

1.1 Problem and the Background

Attempts to come up with a global vertical datum or a world vertical network, either by connecting the existing regional vertical networks or by defining a global datum irrespective of any specific regional vertical datum, is not new to the geodetic community. A special study group 1.75 on the World Vertical Reference System was established at the 1983 Hamburg IUGG/IAG meeting, with the primary purpose to study methods for the determination of a unique vertical reference surface that could be used on a global basis, under the chairmanship of Prof. Richard H. Rapp of The Ohio State University. The study group recommended that the need to study the vertical datum problem is more relevant in this space geodetic age and we should take advantage of the new data and programs that are now on the horizon (Rapp, 1987). "The First International Conference on Geodetic Aspects of the Law of the Sea" organized by the IAG Subcommission GALOS, at Denpasar, Bali-Indonesia 8-13 June 1992, by their resolution no. 3 noted the importance of consistency in defining sea coast positions for boundary purposes, urged the International Association of Geodesy to pursue with other interested organizations the possibility of advocating a global vertical datum and also to investigate a suitable definition of such a datum.

In addition to several International commission reports and seminar proceedings, numerous papers have also been published in the past twenty years to document the problem of the global vertical datum definition and proposed solutions. The relative merits and demerits of the oceanographic approach suggested by Cartwright (1985), Altimetric-Gravimetric approach by Mather et al. (1978), and Integrated Geodesy approach of Hein and Eissfeller (1985) are discussed in detail by Pavlis (1991) and partially by Heck and Rummel (1990).

With respect to the global vertical datum definition, the combination of geometrical and geodetic data is proposed in many ways. Colombo (1980), Rapp (1980), Hajela (1985) and Pavlis (1991) have suggested techniques for vertical datum connection by estimating the geopotential differences over intercontinental locations using satellite and terrestrial measurements. They use three-dimensional geocentric coordinates of at least one fundamental point in each vertical datum system. The geoid height at these locations can be calculated from some high-degree geopotential model, and the influence of local differences between the actual gravity field and the global model is determined from gravity anomalies given in some spherical cap around the fundamental point. The estimated accuracy of vertical datum connection between different continents is about ±15 to ±17 cm (Pavlis, 1991).

Another approach described by Rummel and Teunissen (1988), Heck and Rummel (1990), and Xu and Rummel (1991), uses globally distributed gravity anomalies. These values are given referring to different vertical datum realizations for different parts of the earth. Fixing one datum as reference, all the others may be determined if the orthometric heights of the stations in local vertical datum together with their geocentric coordinates are known for at least one fundamental benchmark in each vertical datum zone.

The ideas described in the above two paragraphs clearly show that the studies so far made lead to either a connection between selected vertical datums or suggest a procedure to determine the potential difference between selected benchmarks on two different regional datums which can lead to a vertical datum connection. The concept of defining and realizing a global vertical datum, independent of any particular regional datum, has been discussed by Rapp and Balasubramania (1992). This dissertation work enhances the ideas formulated in the report (ibid.) to come up with a practically realizable global vertical datum.

1.2 Organization and Scope of the Present Study

From the above sections we see that most of the countries in attempting to define a vertical reference surface have relied on local mean sea level as the surface that is accessible through tide gauge measurements. Since we are now moving into an era where more precise and more consistent height datum determinations are essential, there is a necessity to come up with a vertical datum that can be used on a global basis. There are basically two issues involved in this global vertical datum development, that is, connection of existing vertical datums and definition of a global datum independent of any particular regional datum. As mentioned earlier, these issues have been discussed in the literature in the past few years, but no specific action has been taken so far.

This dissertation will attempt to address these issues. In addition to detailed mathematical modeling to define and realize a global vertical datum to an accuracy of ±10cm after considering all the error sources in the observed data, a test calculation has also been included to come up with the first iteration global vertical datum to an accuracy permitted by the available data today.

Two different approaches are developed for this purpose. The first approach uses the best available station positions from the present and future space geodetic networks, highly accurate geopotential models and surface gravity data around the space stations for defining and realizing a global vertical datum. This approach can be considered as an ideal solution of the problem, but needs to be treated as a long term proposal since the data required are not available all over the world and/or to required precision.

The other approach can be considered more of an operational development for immediate implementation but less accurate than the first approach. This approach is mainly dependent on the widely available space geodetic data from GPS (Global Positioning System) and DORIS (Doppler Orbitography and Radiopositioning Integrated by Satellite) tracking networks and highly accurate regional geoid height models.

The dissertation structure is as follows:

Chapter 2 explains the height datum fundamentals, bringing out the different types of heights that are in use today with their definition, computation and intercomparisons. Procedures for establishing a more accurate and meaningful regional vertical datums, which in turn can help in realizing an accurate global vertical datum, are also discussed in this chapter.

The primary step in the development of global vertical datum involves generating mathematical models and defining required observations. Since the data availability in the various parts of the world is not uniform, the development of mathematical models under different data scenarios is discussed in Chapter 3. Consideration of different error sources in the available terrestrial gravity anomalies and their impact on computed gravimetric undulations are included in this chapter. Also included are the observation equation model development and the adjustment procedure to estimate the parameters defining the global vertical datum.

In Chapter 4, a test calculation for implementing the Global Vertical Datum based on the available data on six of the regional vertical datums is included. The six regional vertical datums considered in the test calculations are viz., North American Vertical Datum (NAVD 88), Ordnance Datum Newlyn (ODN, UK), IGN 69 (France), NN (Germany), AHD:71 (Australia) and Scandinavian datums. Though there is no single regional vertical datum as Scandinavian datum, the individual datums RH70 (Sweden), N60 (Finland) and NN1954, NNN1957 (Norway) were combined to serve as a single regional datum for computational convenience. More details on the Scandinavian height datum are given in Chapter 4. Two different computational procedures, namely, Least Squares Collocation technique and modified Stokes' integration technique, have been used to compute the gravimetric undulation. These results have been compared with undulation values from regional geoid height models (for U.S. and Scandinavian space geodetic stations only) which are generally developed using Fast-Fourier Transform (FFT) techniques.

Chapter 5 deals mainly with the simplified approach in realizing a global vertical datum. The theory behind the model development along with test results and plots in the area of GPS traverses in the U.S.A., Canada and Western Europe are included in this chapter. Height bias modeling has been done in four test areas in Canada and the U.S.A.

While Chapter 6 brings out the summary and conclusions drawn from this dissertation work, Chapter 7 lists the recommendations for the acquisition and use of data for defining and realizing a global vertical datum to a precision of a few centimeters.

CHAPTER 2

PRELIMINARIES AND DEFINITIONS

There are fundamentally two different height systems available today which can be realized with approximately the same order of accuracy (Torge, 1987). The first one is the gravity field related height system which is based on leveling and gravity data along the leveling lines. These observations lead to geopotential heights, if referred to a zero height level surface. The other one is the ellipsoidal height system realized through satellite techniques such as SLR, GPS-positioning etc. Original results are the cartesian coordinates of the stations in a near geocentric coordinate system which can be transferred to heights called ellipsoidal heights over the reference level ellipsoid. Knowing the gravimetrically derived geoid information from geoid height models, the orthometric height of the station can be computed in very good approximation using the relation:

$$H = h - N \tag{2.1}$$

In the above equation, the ellipsoidal height, h, is the distance from a defined ellipsoid to a point, measured along the straight line that is perpendicular to the ellipsoid and passes through the point. The orthometric height, H, is the separation between the geoid and the point measured along the curved vertical defined by the equipotential surfaces of the earth's gravity field and N, the geoid undulation, is the separation between the defined ellipsoid and geoid (Rapp, 1992).

The relative orthometric height accuracy that can be achieved using the above two height systems are given by Niemeier (1987, Table 3) and shown in Table 2.1.

Table 2.1 Height System and Accuracy For Commonly Used Observation Techniques

Height system	Observation technique	Additional information	Quantities available for ortho. ht. computation	Relative accuracy of computed orthometric hts.
1. Gravity field related	geometric leveling	- Surface gravity	height increments potential differences	±1-2.5 cm/10 km
	trigonometric leveling	- Surface gravity	height increments potential differences	±2-2.5 cm/10 km
2. Satellite based	GPS	- geoid	ellipsoidal height differences geoidal undulation differences	±2 cm/10 km

Though several papers have been published on the topic of obtaining the orthometric heights using GPS and other space geodetic techniques in the recent years, no country in the world has so far adopted the orthometric heights obtained from space geodetic techniques for defining their vertical network. The main reason for this is the insufficient knowledge of the regional geoid heights to the accuracy of leveled heights. With constant efforts to improve the regional geoid height models and also the ellipsoidal height accuracy for GPS observations, it may be possible in the future to replace conventional leveling by satellite based techniques. But as of today, since the conventional leveling is the only technique accepted by the geodetic community for defining precise heights, the following sections discuss the various heights and height systems that are used in different parts of the world today which are based mainly on the 'gravity related height system'.

Detailed discussions on height systems, their measuring procedures and the treatment of height networks may be found in the books of Heiskanen and Moritz (1967, Chapter 4), Torge (1991, Chapter 6), Vanicek and Krakiwsky (1986, Chapter 19), Bomford (1980, Chapter 3) and others. Additional details on various kinds of heights and their computation may be found in Krakiwsky (1965) and exhaustive treatise only on orthometric heights in Rapp (1961).

2.1 Height Definitions and Height Systems

For decades the well known geometric or spirit leveling was the only technique used for determination of precise heights. New leveling techniques such as motorized geometric leveling, motorized trigonometric leveling and motorized 3D-traversing, etc., have been developed now, to achieve the same accuracy as conventional leveling but to reduce the time and expenses involved to perform conventional leveling for extended networks (Niemeier, 1986). All these measurements (leveling, theodolite measurements, etc.) are almost exclusively referred to the system of level surfaces and plumb lines, the geoid playing an essential part.

2.1.1 Level Surfaces and Geopotential Numbers

Level instruments and theodolites, by virtue of their dependence on the local horizon with respect to their orientation by the actual gravity vector $\vec{g}$, are influenced by the actual earth's gravity field. The surfaces

$$W(x, y, z) = W_0 = \text{const.}, \qquad (2.2)$$

on which the potential W is constant, are called equipotential surfaces or level surfaces. Due to the figure of the earth, the earth rotation and inhomogeneous mass distribution the level surfaces near the earth's surface are not parallel and the lines that intersect all equipotential surfaces orthogonally are not exactly straight but slightly curved (refer Figure 2.1).

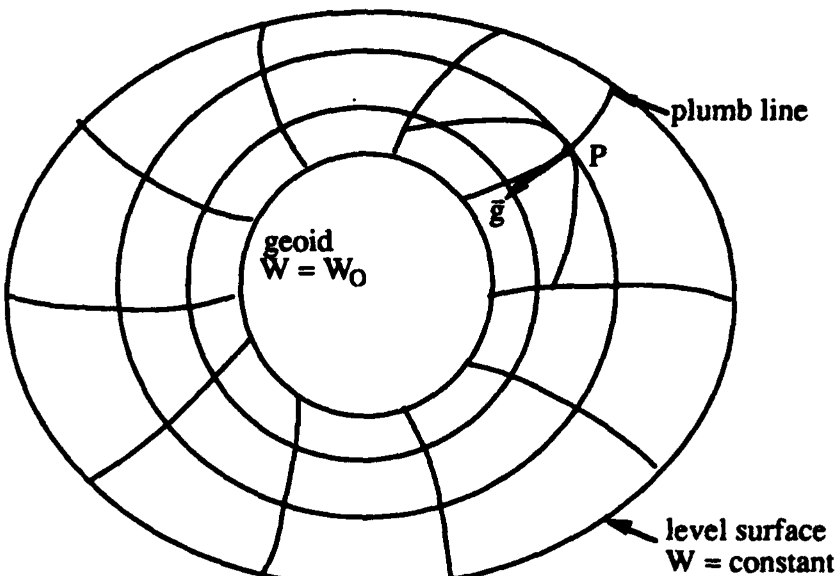


Figure 2.1 Level Surfaces and Plumb Lines

For a displacement along the level surface, the potential difference dW = 0, for displacement along the outer surface normal n the potential difference is

$$dW = -gdn \tag{2.3}$$

where dn is the separation between the level surfaces and g the magnitude of gravity at the plumbline (see Figure 2.3 also). Equation (2.3) provides the link between the potential difference dW and the difference in height of neighboring level surfaces.

As reference for most height systems the geoid plays an important role. Connected with the geoid is the zero point or datum problem. One has to define which point P_0 is located on geoid having a potential W_0. Since it was tacitly believed that the mean sea level should theoretically coincide with the geoid, or that the departure of the two surfaces was negligible (or unknown), locating the geoid or fixing the point P_0 with respect to a reference benchmark on the shore was reduced to the task of determining the position of the mean sea level. In most of the regional vertical datums in the world today, the point P_0 has been identified to lie on the long term mean value of the local mean sea level (generally with 18.6 years of sea level data) with assigned potential $W = W_0$. Establishment of a reference benchmark with respect to P_0 is shown in Figure 2.2 (given as Figure 19.1, Vanicek and Krakiwsky, 1986).

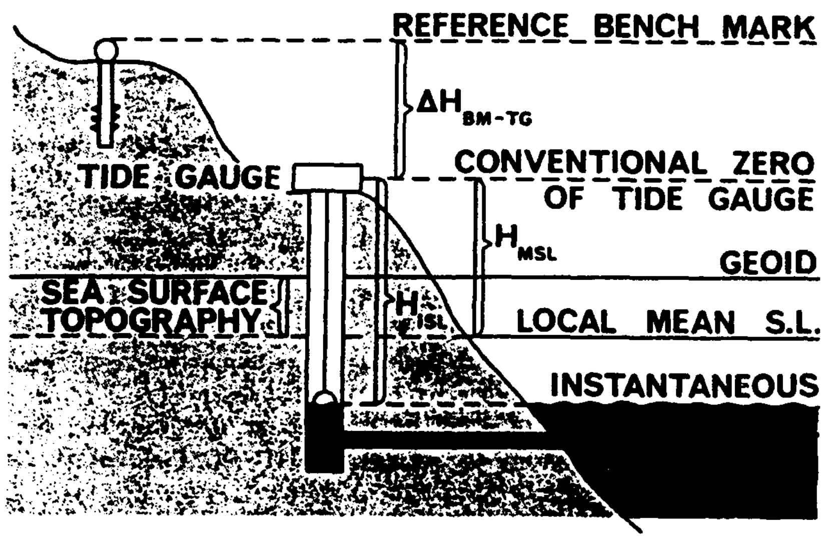


Figure 2.2 Establishment of Height of Reference Benchmark
(from Vanicek and Krakiwsky, 1986)

Procedures for establishing the fundamental benchmark to serve as the origin for the regional vertical datum and the problems inherent in the determination of local mean sea level for that purpose are discussed in Vanicek and Krakiwsky (1986, Chapter 19, p. 424-425).

For a surface point P, the potential difference to the geoid (local vertical reference datum through the point P_0 with potential $W = W_0$) is,

$$C_P = W_0 - W_P = -\int_{P_0}^{P} dW = \int_{P_0}^{P} g\,dn \tag{2.4}$$

where C_P is defined as the geopotential number, which can be determined by leveling and surface gravity values, starting from the zero point P_0. For the definition of other variables used in equation (2.4) refer to Figure 2.3 below:

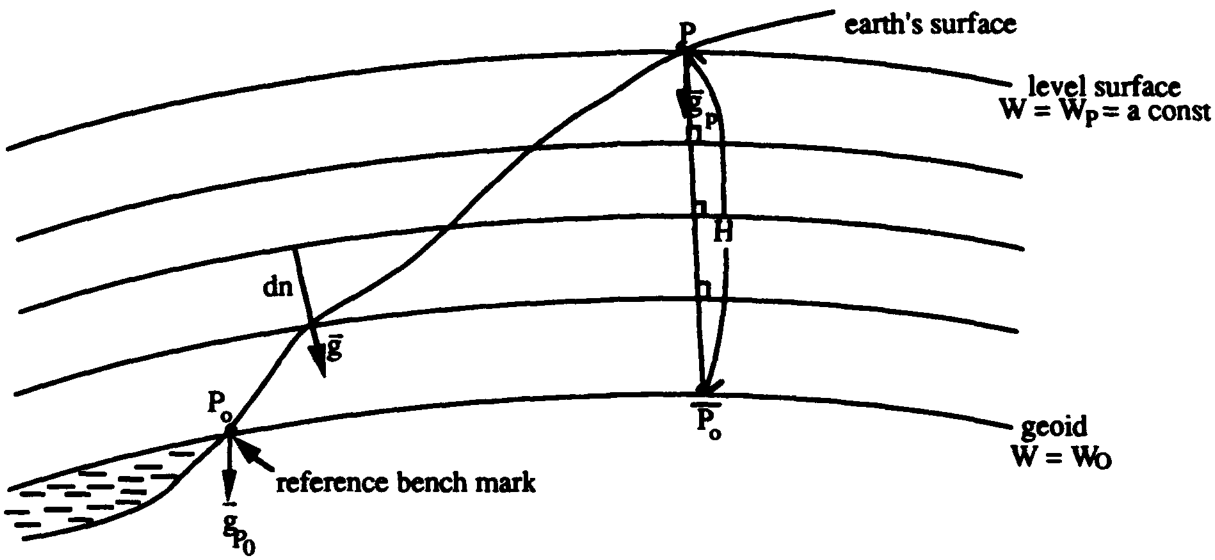


Figure 2.3 Geoid, Level Surfaces, Height Increments dn and Gravity g

To compute C_P it is not necessary to know the mean gravity $\bar{g}$ between P and $\bar{P}_0$ along the plumb line. Geopotential numbers are, therefore, the only quantities that can be determined without any density hypothesis. The geopotential number is independent of the particular leveling line used for relating the point P to the reference benchmark. The geopotential number is the same for all points of a level surface and can thus be considered as a natural measure of height, even if it does not have the dimension of a length (Heiskanen and Moritz, 1967). Some of the height datums in the world today, for example NAVD88, UELN73, etc., consider geopotential number as the basic quantity for the definition and computation of their height systems. The geopotential number C is measured in geopotential units (g.p.u.) where

1 g.p.u. = 1 kgal meter = 1000 gal meter

and 1 g.p.u. corresponds to a change in height of about 1 meter.

2.1.2 Orthometric Heights

The orthometric height H_P of a point P is defined as (Heiskanen and Moritz, 1967, (4.21))

$$H_p = \frac{C_P}{\bar{g}} \tag{2.5}$$

where C_P is the geopotential number at the point P and $\bar{g}$ the mean gravity along the plumbline (refer to Figure 2.4) between the surface P and its corresponding point P_0 on the geoid. Since we need the density distribution between the surface of the earth and the geoid for computing $\bar{g}$, the practical determination of this quantity is impossible without additional assumptions.

Several hypotheses have been made about the variation of gravity between the terrain surface and the geoid to approximate this mathematically rigorous quantity $\bar{g}$. Depending upon the method/rigorousness of estimating $\bar{g}$, numerous types of orthometric heights are defined and used in various parts of the world today. When working in a country or area where a specific type of orthometric height is used, we need to understand its definition and accuracy limitations. Some of the different kinds of orthometric heights used in practice are discussed in the following sections.

2.1.2.1 Rigorously Computed Orthometric Heights

Rigorous orthometric heights are those where there has been actual attempts to determine the actual gravity g, along the plumb line from the surface point to the corresponding point on the geoid, as closely as possible.

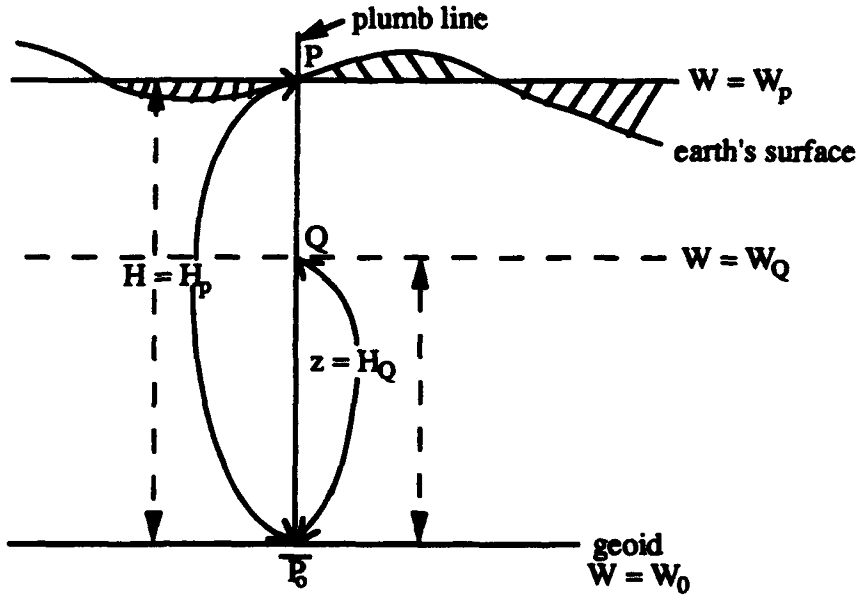


Figure 2.4 Reference Surfaces Used in the Reduction of Poincaré and Prey

From equation (2.5) we know that the orthometric height at the point P is

$$H_P = \frac{C_P}{\bar{g}}$$

where C_P is the geopotential number at the point P, $C_P = W_0 - W_P$, and $\bar{g}$ is the mean gravity along the plumb line from P to $\overline{P}_0$. The mean gravity $\bar{g}$ can be written as

$$\bar{g} = \frac{1}{H}\int_0^H g(z)dz \tag{2.6}$$

where g(z) is the actual gravity at the arbitrary point Q which has the height z (refer to Figure 2.4).

The simplest method to compute an approximation for the actual gravity g_Q at Q is by using the Poincaré and Prey reduction procedure discussed in Section 4.3, Heiskanen and Moritz (1967). This procedure is based on the assumption that the area around the gravity station P is completely flat and horizontal (Figure 2.4) with the masses between the geoid and the earth surface having constant density (2760 kg/m^3). This flat plate with thickness equal to the height of the station is also called a Bouguer plate. Based on the Prey reduction procedure we get a value for the actual gravity at Q as:

$$g_Q = g_P + 0.0848\,(H_P - H_Q) \tag{2.7}$$

with g in gals and H in kilometers.

If we calculate $\bar{g}$ as the mean of gravity g_P, measured at the surface point P, and of gravity g_Q computed at the corresponding geoid point Q ($= \overline{P}_0$, $H_Q = 0$) by Prey reduction then,

$$\bar{g} = \frac{1}{2}(g_P + g_Q) = \frac{1}{2}(g_P + g_P + 0.0848H_P) = g_P + 0.0424H_P \tag{2.8}$$

From equations (2.5) and (2.8), we get the orthometric height at the point P approximately as

$$H_P = \frac{C_P}{g_P + 0.0424H} \tag{2.9}$$

The orthometric heights, H_P, computed using equation (2.9) are called 'Helmert heights', with H_P in meters, the geopotential number C_P given in g.p.u., g_P in gals and the H is H_P in kilometers.

Helmert heights computed using equation (2.9) can be considered as the sufficient approximation to true orthometric heights for most of the applications. Sometimes in mountainous areas and for highest precision, it will be necessary to consider the variation in topography in addition to the infinite Bouguer plate. The heights computed taking topography into account, assuming normal gradient for free-air reduction and a constant density of 2670 kg/m^3 are called 'Niethammer heights'. Detailed discussion on Niethammer heights are given in Krakiwsky (1965).

Helmert heights and Niethammer heights are rigorously computed approximations for orthometric heights used in some parts of the world, where dense gravity data and terrain data are available.

Another (not so rigorously computed, but making use of geopotential number) orthometric height was proposed by J. Vignal (1954); this so-called 'Vignal height' is used in some countries. In this case, the mean gravity along the plumb line is computed in two steps (Krakiwsky, 1965, pp. 96-97).

- The normal gravity, γ_{ϕ_P}, is computed on the spheroid for the latitude, ϕ_P, of point P.
- This value is then reduced upward to the mid-point of the plumb line by a negative free air correction, $\frac{\delta^F_{gP}}{2}$.

Thus, we get the mean gravity to be approximately,

$$\bar{g} = \gamma_{\phi_P} - \frac{\delta^F_{gP}}{2} = \gamma_{\phi_P} - 0.3086 \cdot \frac{H_P}{2}$$

Then the Vignal height at point P will be

$$H_v = \frac{C_P}{\bar{g}} = \frac{C_P}{\gamma_{\phi_P} - 0.1543 \cdot H_P} \tag{2.10}$$

where γ_{ϕ_P} is in gals, H_P is in km and H_v in meters.

2.1.2.2 Normal Orthometric Heights

Normal orthometric heights can be considered as a coarse approximation to true orthometric heights. In determining this kind of orthometric height geopotential numbers are not used. Most of the countries which lack dense gravity data coverage adopt this kind of simple orthometric height obtained by the following relation

$$H_i = H_{i-1} + \Delta h + dH \tag{2.11}$$

where H_i is the normal orthometric height of the i^{th} station,

H_{i-1} is the normal orthometric height of the previous station in the level network,

Δh is the leveled difference corrected for all systematic errors, and

dH is called the normal orthometric correction, which accounts for the effect of convergence of equipotential surfaces of the normal field and is given by Rapp (1961) as

$$dH = -f^* \sin 2\phi H d\phi \tag{2.12}$$

In equation (2.12), f^* is the gravity flattening defined as equation (2.98) in Heiskanen and Moritz (1967). The negative sign in the expression indicates that the equipotential surfaces converge towards the poles. Also the ϕ, H in eqn. (2.12) refers to the mean latitude and elevation between the two stations i and (i-1), respectively, and $d\phi$ is the difference in latitude between the two stations.

2.1.3 Normal Heights

Normal heights are defined in the normal gravity field, that is, with respect to a level ellipsoid with normal potential $U = U_0$, by the formula

$$H^* = \frac{C_P}{\bar{\gamma}}, \quad \bar{\gamma} = \frac{1}{H^*}\int_0^{H^*} \gamma dH^* \tag{2.13}$$

The mean normal gravity value $\bar{\gamma}$ can be computed without any hypothesis. The integration from '0' implies that the normal height is zero at the reference surface from which the potential difference, C_P, was determined by leveling.

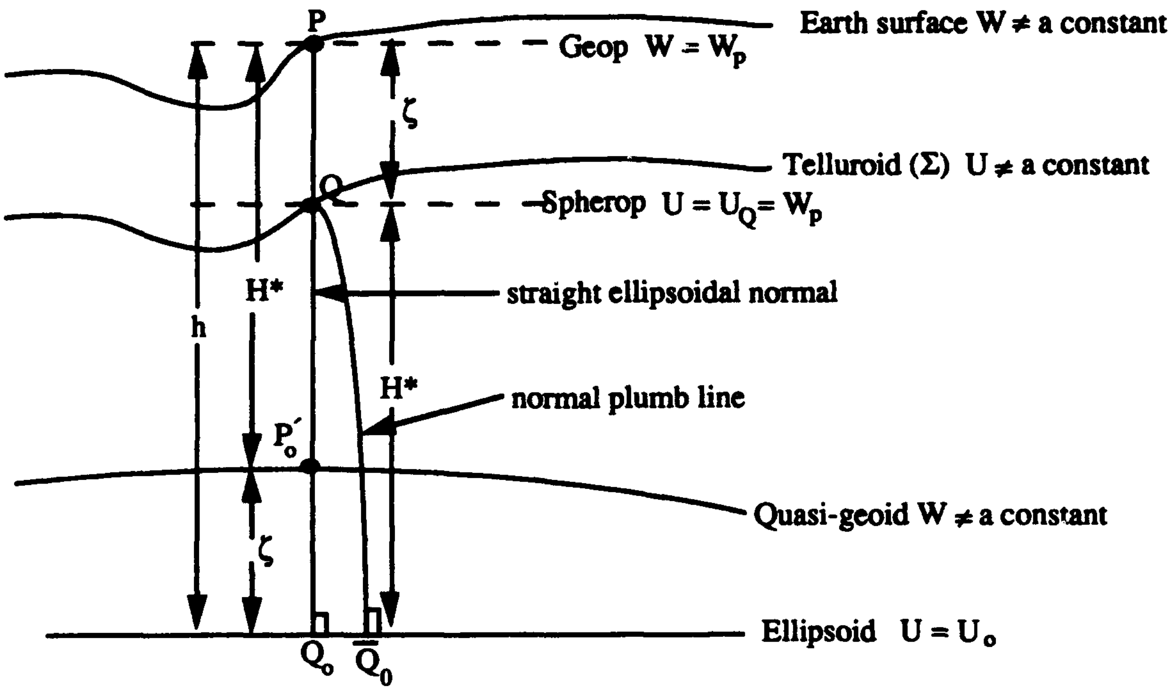


Figure 2.5 The Telluroid (Σ), Normal Height H* and Height Anomaly ζ

As depicted in Figure 2.5, the normal height H* of the point P is defined as the height of the corresponding point Q on the telluroid (Σ) above the reference ellipsoid measured along the normal plumb line from Q to $\bar{Q}_0$. However, since the curvature of the normal plumb line is very small and regular (Heiskanen and Moritz, 1967), the length measured along the normal plumb line can be considered to be equal to the straight ellipsoidal normal height to sufficient accuracy. The normal height H* is also interpreted as the height of the surface point P leading to the point P_0' on the so-called quasi-geoid which is not a level surface but is considered to be located close to the geoid since $\zeta \approx N$. The telluroid (Σ) is defined as the surface, sufficiently close to the earth's surface, whose normal potential U at every point Q is equal to the actual potential W at the corresponding point P, so that $U_Q = W_P$, corresponding points P and Q being situated on the same ellipsoidal normal (Heiskanen and Moritz, 1967). The height anomaly ζ is the distance of the telluroid point Q form the earth's surface point P, or the height of the quasi-geoid point P_0' from the reference ellipsoid.

The normal height H*, and hence the telluroid Σ, can approximately be determined by leveling combined with gravity measurements according to Heiskanen and Moritz (sec. 4-5, 1967). First the geopotential number of P, $C_P = W_0 - W_P$ is computed by

$$C_P = \int_{P_0}^{P} g\,dn \tag{2.14}$$

where g is the locally measured gravity and dn is the leveling increment. The normal height H* is then related to C_P by the analytical expression given as equation (4-44) (ibid.).

$$H^* = \frac{C_P}{\gamma_0}\left[1 + \left(1 + f + m - 2f\sin^2\phi\right)\frac{C_P}{a\gamma_0} + \left(\frac{C_P}{a\gamma_0}\right)^2\right] \tag{2.15}$$

where γ_0 is the normal gravity at the ellipsoid, for the same latitude ϕ. a, f are the semi-major axis and flattening, respectively, for the ellipsoid and m is the constant defined by eqn. (2-70) in Heiskanen and Moritz (1967). From equation (2.13) it is clear that H* can be computed independent of the density assumption unlike the orthometric height H (see equation (2.5)). This property of computed normal heights makes it preferable to adopt them as official heights for certain regional vertical networks. For example, some Eastern European countries, France, Germany, etc., use normal heights in their national height systems.

2.1.4 Inter Comparison Between Various Heights

As discussed in the previous two sections, the basic quantity for height determination is the geopotential number C_P. Depending upon the technique, purpose and type of gravity data used we arrive at different heights. Most of the countries in the world today adopt orthometric heights of one kind or the other for their national height systems. Due to sparse and not very accurate gravity coverage a few years ago, normal orthometric heights (ref. section 2.1.2.2) have been the preferred and adopted heights in many countries. NGVD29 (U.S.A.), AHD71 (Australia), ODN (U.K.) are few examples of national height systems adopting normal orthometric heights in their regional vertical networks.

Recently a new vertical datum called the North American Vertical Datum of 1988 (NAVD88) based on a minimum constraint adjustment of Canada-Mexico-US leveling observations, has been established for implementation in the North American countries (Zilkowski et al., 1992). This vertical datum adopts Helmert orthometric heights (refer to Section 2.1.2.1) computed using observed gravity data and precise level differences, which can be considered closer to the true orthometric heights. Orthometric heights determined using simple, empirical but sufficiently accurate relations between geopotential numbers and orthometric heights are also used in countries like Austria (Erker and Sünkel, 1989).

Rapp (1961) has carried out a numerical computation of orthometric heights at the benchmarks along a precise leveling line, about 125 km long, described as REUN, 211 München-Fasangarten-J-A-D-2-Mittenwald, in Report no. 59 of the German Geodetic Commission. The orthometric heights have been computed both with rigorous techniques like Niethammer method and Helmert method (refer to Sec. 2.1.2.1) and also using the quicker method of applying normal orthometric correction to leveled differences. The results given as Tables 14 and 15 of his report (ibid.) clearly show that the difference between the heights computed by Niethammer formula and Helmert formula is on the order of 1.5 cm. But the difference between the Helmert formula and one obtained by applying normal orthometric correction to leveled differences shows a considerable difference of about 8 cm which is greater than what could be allowed on the basis of leveling accuracy, over a leveling line length of 125 km. This shows that the normal orthometric correction does not satisfactorily take into account the effect of gravity on the leveling measurements, but it does reduce the discrepancy in elevations due to non-parallelism of equipotential surfaces.

In countries belonging to former Soviet Union, East European countries, France, Germany, etc., the normal height, H*, (discussed in Section 2.1.3) is used as official height in their national height systems. The relationship between the normal height H* and the orthometric height H is discussed in Sections (8-12) and (8-13) of Heiskanen and Moritz (1967) and also in Section 2.3 of Rapp and Balasubramania (1992).

2.2 Establishment of Regional Vertical Datum:

2.2.1 Existing Procedure

The current practice of establishing a regional vertical datum in a country or area, with some exceptions like NAVD88, UELN73, etc., is to connect the precise leveling nets to several tide gauges widely dispersed around the perimeter of the net and carry out an adjustment solution constraining the mean sea level at these tide gauges as zero, with the implicit assumption that the geoid and mean sea level coincide at these tide gauges.

As discussed in Chapter I and also with reference to Figure 2.2, we see that the mean sea levels computed at the tide gauges do not coincide with the geoid due to the existence of sea surface topography (SST) caused by ocean currents, temperature, and density variations, as well as air pressure and wind stress. In addition, close to the shore, the sea bed topography and river discharge may also play a significant role. Since the magnitude of the estimated SST varies from one tide gauge to another, the resulting adjustment of the leveling data to fit these different mean sea levels will lead to distortion in the level networks.

To understand the extent of variation in mean sea levels due to SST at different tide gauges, refer to Figure 2.6 below (refers to Figure 9, Zilkowski, et al., 1992).

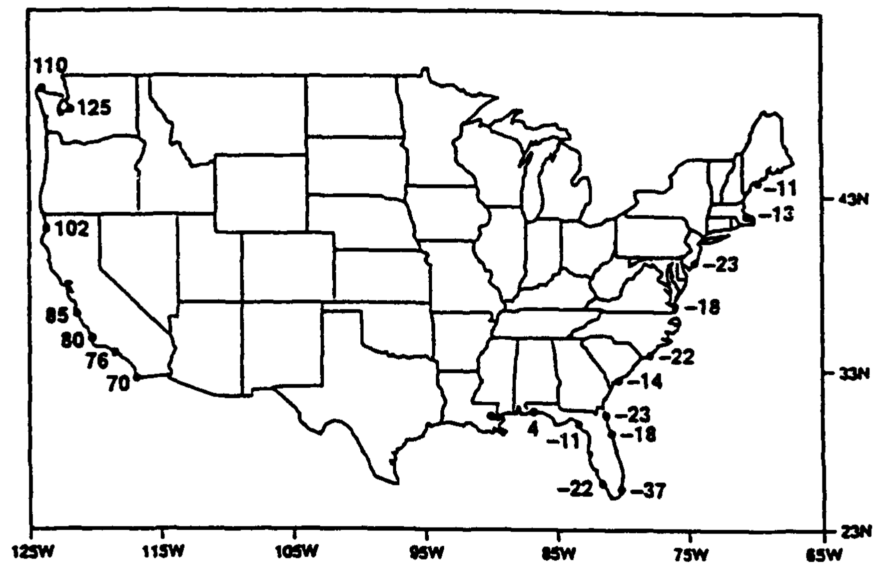


Figure 2.6 Height Differences Between NAVD88 and the Orthometric Height of Tidal Benchmarks Above LMSL (epoch 1960-78) (units = cm)

Figure 2.6 shows the height differences between the orthometric heights of tidal bench marks above local mean sea level (epoch 1960-78) at different tide gauge sites along the U.S. Coast, and the recently made available NAVD88 heights. NAVD88 vertical datum has been established by a minimum constraint adjustment of Canadian-Mexican-U.S. leveling observations holding fixed the height of primary benchmark at Father Point/Rimouski, Quebec, Canada as zero height. From Figure 2.6, we can understand that though there can be several reasons for the noted height differences at the tidal benchmarks, the main contribution to the differences is the SST, which varies to some extent from one tide gauge to another along the same coast and vary to a greater extent from one coast to another. Without removing the SST effects at tidal stations, it would be incorrect to constrain the heights of tidal benchmarks to zero height in the adjustment solution for establishing a regional vertical datum.

2.2.2 Modeling for Ideal Regional Vertical Datum Definition

2.2.2.1 Estimation of Sea Surface Topography

One obvious solution to the problem of relating the reference surface at the tide gauge sites to the mean sea level is to model the SST.

Several global and regional models have been developed so far mainly using altimetric data to estimate the SST. The Ohio State 1991 global SST models to degree 10 and 15 have been developed based on analysis of one year Geosat altimeter data from the orbits computed with GEM-T2 potential coefficient model and consistent station coordinates (Rapp et al., 1991). The commission error in the estimation of SST in the 'ocean areas' is about ±10 cm in the case of 10,10 model while the error for the 15,15 solution is higher because twice as many coefficients are included in the degree 15 model. The extrapolation of SST at the tide gauge points from these models is not recommended at this time since the inaccuracy in the estimate of SST in shallow coastal areas will be too high to contribute anything to adjusted level heights.

Efforts have been made by few researchers to model the local SST variation by considering monthly mean sea level and other data for several tide gauges. Merry and Vanicek (1983) proposed a technique which makes use of the response of sea level to external effects such as atmospheric pressure, temperature, wind, and so on. Their experiment with relevant data at various tide gauge sites in the Maritime Provinces of Canada concluded that their results do not represent the entire SST at the tide gauges but only reflect the local variations of SST. Similar studies have been carried out by Merry (1990) to determine the variation in mean sea levels at the tide gauge sites along the coast of South Africa. Heck and Rummel (1990) have developed formulas for extrapolating current and SST information along shallow coastal waters from locally measured data. The main difficulties with these methods are:

- They are better suited to the determination of SST differences between adjacent tide gauges rather than to the determination of absolute SST.
- Due to the inadequacy of the hypothesis and imperfect modeling of frictional forces in the continental shelf areas (eg. bottom friction and surface windstress) the SST required for relating vertical reference surface to the tide gauge sites cannot be estimated to intended precision (Heck and Rummel, 1990).
- The effort required to obtain the necessary data is significant and the methods may prove to be non-viable purely on economic grounds.

2.2.2.2 Combined Adjustment Solution for Defining Regional Vertical Datum

Several techniques for defining a regional vertical datum have been put forward by various authors in recent years. The general consensus amongst them is that, for establishing a regional vertical datum, we need to consider the potential differences derived from a combination of leveling and gravity data, precise altimeter data from missions like ERS-1, TOPEX/POSEIDON, etc., and the tide gauge records at the primary tide gauge stations and come up with a combined adjustment solution.

Groten and Müller (1990) have suggested such a combined adjustment model using gravity field as well as leveling data, altimetry and tide gauge data and three- dimensional coordinates directly determined by GPS. The ongoing Bass Strait vertical datum definition project in Australia (Rizos et al., 1991) is also another effort to integrate geometric information (from space geodetic techniques) and gravity field information (from surface gravity data and accurate regional geoid models) for regional vertical datum definition. Vanicek (1991) has proposed five different approaches for defining a regional vertical datum depending on availability of data and rigorousness required in defining the vertical datum.

Based on the assumption that realistic models to estimate the SST at reference tide gauges may be available in the near future, the following procedure can be adopted to come up with an adjustment solution for defining the vertical datum. Assuming that we have the following data:

- observed geopotential differences between the leveling bench marks expressed in geopotential units (gpu), $\Delta C_{ij} = C_j - C_i$ for all i, j (except i = j)
- geopotential numbers at the reference tide gauge sites obtained from realistic models to estimate the sea surface topography,

We can set up an observation equation model, introducing the estimated geopotential numbers at the tide gauge sites as stochastic constraints The formulation of the model will be

$$\begin{aligned} y &= AX + e, D\{y\} = \sigma_0^2 P^{-1} \\ z_0 &= KX + e_0, D\{z_0\} = \sigma_0^2 P_0^{-1} \end{aligned} \tag{2.17}$$

where
- y is the nx1 observation vector (of geopotential differences ΔC_{ij})
- A is the nxm coefficient matrix of observations; rank $A = q \leq m \leq n$
- m is the number of parameters
- X is the mx1 parameter vector (unknown)
- e is the nx1 random error vector
- z_0 is the ℓx1 vector of stochastic constraints having the same structure of observation vector with
- e_0 as corresponding ℓx1 random error vector
- P is the weight matrix of observations
- P_0 is the weight matrix of the stochastic constraints
- σ_0^2 is the common variance component (unknown)
- K is the ℓxm coefficient matrix of the constraints; rank $K = \ell < m$ and rank $\begin{bmatrix} A \\ K \end{bmatrix} = m$
- D is the dispersion operator.

Assuming no correlation between y and z_0, we can write the least squares solution for the adjusted parameters to be (B. Schaffrin, 1990)

$$\hat{X} = \left(A^T PA + K^T P_0 K\right)^{-1}\left(A^T Py + K^T P_0 z_0\right) \tag{2.18}$$

and the dispersion matrix of the estimated parameters to be

$$D\{\hat{X}\} = \sigma_0^2\left(A^T PA + K^T P_0 K\right)^{-1} \tag{2.19}$$

The problems with solution (2.18) at present conditions as pointed out by Zilkoski et al. (1992) are:

- Realistic models to compute the SST at the tide gauge sites to required accuracy are unavailable at this time, and
- An a priori estimate of standard errors for tidal stations where SST effects were not removed completely would be too large relative to precise leveling differences to allow any contribution to the final adjusted heights.

To overcome the inconsistencies in the estimated SST values, the 'robust sequential updating procedure' suggested by B. Schaffrin (1987, 1989, 1990) can provide an excellent adjustment technique in the presence of 'weak prior information'. This procedure provides solutions when the stochastic prior information, included as stochastic constraints in our observation equation model, is allowed to have a certain (guessed) bias β_0 or even certain suspected scaling error in the bias β_0. As discussed in B. Schaffrin (1987), the adjusted solution suppresses the inconsistencies in the estimated geopotential numbers at the tide gauges without destroying the homogeneity of the geopotential differences observed at the leveling benchmarks.

Vanicek (1991) has suggested using orthometric heights computed with equation (2.1) at space geodetic stations can be included with leveling data in a leveling network adjustment, in order to homogenize the error distribution within the network. But since the uncertainties of geoid height differences used to convert the ellipsoidal height differences to orthometric height differences are presently large compared to formal errors of leveling height differences (Zilkoski et al., 1992), such a suggestion is impracticable as of today.

2.2.2.3 Summary and Conclusions

1. The primary reference surface for heights is the geoid. Since it is approximated as the mean sea level (that is, as a temporal mean of all the instantaneous sea level heights at the epoch 't'), it is imperative that the tide gauge records be included in the regional vertical datum definition.

2. The mean sea level departs from the geoid due to the existence of the quantity called Sea Surface Topography (SST). Since this quantity depends on meteorological, geophysical, hydrological and oceanographic components at the tide gauge sites and its magnitude is a combination of all these components, it varies considerably from one region to another. Therefore, it is impracticable to include SST for a Global vertical datum modeling. But it is possible that the change in SST from one tide gauge location to another in a given datum can be determined to a reasonable accuracy, which could enable multiple relative tide gauge information to be incorporated into a regional vertical datum definition without creating the distortion problem noted earlier.

3. At present suitable techniques/realistic models to determine the SST exactly to intended precision are not available. But without removing SST effects at the tidal stations, it would also be incorrect to constrain the heights of tidal benchmarks to zero height in the establishment of regional vertical datums. In such a scenario, the suggested approach will be to estimate the SST as accurately as possible and then set the tidal benchmark height to be equal to (MSL-SST). Now the tide gauge heights (or geopotential number) can be used as stochastic constraints, keeping in mind the noise at these stations will be too high due to the undetermined bias in the SST values. To accommodate for the bias in the estimated SST values at the tide gauge sites and possible scaling error also in the bias, the robust sequential updating procedure suggested by B. Schaffrin (1987) can be adopted for adjustment of regional vertical datum.

4. The suggestion to include the orthometric heights, computed using space geodetic techniques and global/regional geopotential models, in order to homogenize the error distribution within the regional vertical datum adjustment seem to be impracticable as of today, as the uncertainties of geoid heights (or geoid height differences) are presently large compared to formal errors of leveling height differences (Zilkoski et al., 1992).

CHAPTER 3

GLOBAL VERTICAL DATUM DEFINITION - AN IDEAL APPROACH

Defining and realizing a Global Vertical Datum to an accuracy of ±10 cm is essential today for the various reasons cited in Chapter 1. With the existing measuring techniques and available improved models and procedures, the question is not whether we need to establish a precise vertical datum that can be used on a global basis, but to what accuracy and how soon.

Recognizing the fact that the data are neither uniformly available all over the world nor to the required precision, modelling procedures for a Global Vertical Datum that can be realized are discussed in this chapter. The available data as of today can be used in the models developed for estimating the parameters that will define the Global Vertical Datum, but only as a first iteration attempt. As understandable, it may require several iterations before achieving an Ideal Global Vertical Datum that could meet all the requirements stated in Chapter 1.

The modelling procedures have been developed for different data scenarios but with the assumption that the data are available everywhere and also to the required precision, that a Global Vertical Datum established to an accuracy of ±10 cm would warrant. It is also assumed that the regional vertical datums have been established based on the procedures discussed in Chapter 2.

3.1 Data Requirements for Modelling Purposes

Basically four kinds of data are essential for modelling the Global Vertical Datum. They are accurately determined free air gravity anomalies, precise heights of the stations above the regional vertical reference datums, an accurate global geopotential model, and accurate ellipsoidal heights of the stations above an adopted geocentric ellipsoid. In this section a full description and accuracy requirements of required data types are given.

3.1.1 Free-air Gravity Anomalies

Two different approaches are used in defining the free-air gravity anomalies. The first approach is termed as 'a classical approach' (Chapter 2-13, Heiskanen and Moritz, 1967) and the other as 'Molodensky approach' (Section 42, Moritz, 1980).

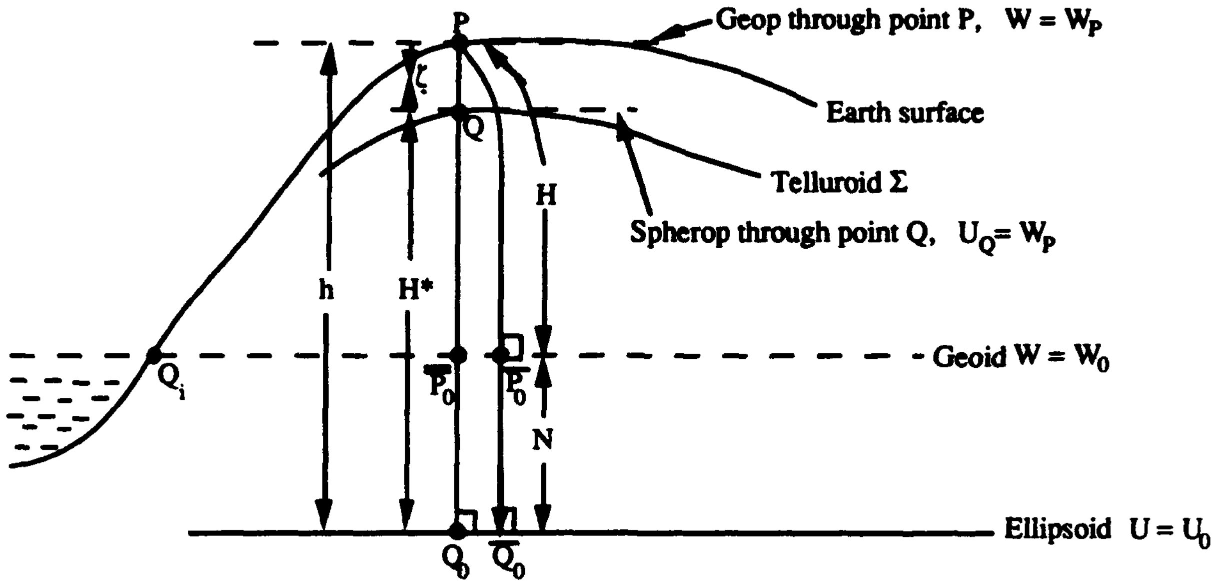


Figure 3.1 Various Equipotential Surfaces and Heights Used in Defining the Free-air Gravity Anomalies

In the classical approach the gravity anomaly Δg_C is defined as the difference in magnitude of the gravity vector at $\overline{P}_0$ and the normal gravity vector at $\overline{Q}_0$ (refer to Figure 3.1).

$$\Delta g_c = g_{\overline{P}_0} - \gamma_{\overline{Q}_0} \tag{3.1}$$

where $g_{\overline{P}_0}$ is the magnitude of the observed gravity at P reduced to geoid point $\overline{P}_0$, and $\gamma_{\overline{Q}_0}$ is the magnitude of normal gravity at the point $\overline{Q}_0$ computed on the reference ellipsoid.

The observed gravity at the point P is reduced to the point $\overline{P}_0$ on the geoid using free-air reduction procedures discussed in Heiskanen and Moritz (1967, Section 2.23). If g_P is the observed gravity at the surface of the earth then the value of $g_{\overline{P}_0}$ at the geoid may be obtained as a Taylor expansion:

$$g_{\overline{P}_0} = g_P - \left.\frac{\partial g}{\partial h}\right|_P H - \ldots \tag{3.2}$$

where $\frac{\partial g}{\partial h}$ is the vertical gradient of the gravity at P and H is the orthometric height of the station above the geoid. Considering only the linear term, the quantity $-\frac{\partial g}{\partial h}|_P H$ accounts for the free-air reduction of the observed gravity to the geoid. Here we have assumed that there are no masses above the geoid or that such masses have been removed beforehand so that this reduction is indeed carried out in "free-air". From equations (2-215) and (2.217) in Heiskanen and Moritz (1967), we note that the vertical gradient of gravity can be computed by:

$$\frac{\partial g}{\partial h} = \frac{\partial \gamma}{\partial h} + \frac{\partial \Delta g}{\partial h} \tag{3.3}$$

where $\frac{\partial g}{\partial h}$ is the normal gravity gradient given by equation (2-121) and $\frac{\partial \Delta g}{\partial h}$ is the anomalous part which can be computed using equation (2-217) in Heiskanen and Moritz (1967). Due to lack of dense gravity coverage required for computing the actual vertical gradient of gravity using equation (3.3), it is generally approximated by normal gradient of gravity, setting the anomalous part equal to zero. Based on this approximation, equation (3.2) can be written to second-order as:

$$g_{P_0} = g_P - \frac{\partial \gamma_{\bar{Q}_0}}{\partial h} H - \frac{1}{2!}\frac{\partial^2 \gamma_{\bar{Q}_0}}{\partial h^2} H^2 \tag{3.4}$$

Substituting the values for the first and second derivative of normal gravity from equations (2-121) and (2-122) from Heiskanen and Moritz (1967), in equation (3.4), we can write the free-air gravity anomaly in 'classical sense' as:

$$\Delta g_c = g_P - \gamma_{\bar{Q}_0}\left(1 - 2(1 + f + m - 2f\sin^2\phi)\frac{H}{a} + 3\left(\frac{H}{a}\right)^2\right) \tag{3.5}$$

In the Molodensky approach, the gravity anomaly is defined as (refer to Figure 3.1):

$$\Delta g_M = g_P - \gamma_Q \tag{3.6}$$

where g_P is the gravity observed at the surface point P and the γ_Q is the normal gravity at the corresponding point Q on the telluroid (Σ). The normal gravity at the point Q, γ_Q, is computed from the normal gravity at the ellipsoid γ_{Q_0}, by the normal free-air reduction, but now applied upwards.

$$\gamma_Q = \gamma_{Q_0} + \frac{\partial \gamma_{Q_0}}{\partial h} H^* + \frac{1}{2!}\frac{\partial^2 \gamma_{Q_0}}{\partial h^2} H^{*2} + \ldots \tag{3.7}$$

where H* is the normal height of the point P. A direct formula for computing γ_Q at Q is also given by equation (2-123) in Heiskanen and Moritz (1967) as:

$$\gamma_Q = \gamma_{Q_0}\left(1 - 2(1 + f + m - 2f\sin^2\phi)\frac{H^*}{a} + 3\left(\frac{H^*}{a}\right)^2\right) \tag{3.8}$$

From (3.6) and (3.8) we write:

$$\Delta g_M = g_P - \gamma_{Q_0}\left(1 - 2(1 + f + m - 2f\sin^2\phi)\frac{H^*}{a} + 3\left(\frac{H^*}{a}\right)^2\right) \tag{3.9}$$

A reasonable estimate for the numerical difference between the free-air anomalies computed using equations (3.5) and (3.9) can be determined as follows:

Considering only up to the first order normal gravity gradient, $\frac{\partial \gamma}{\partial h} \doteq 0.3086\,\text{mgal/meter}$, we can write from equations (3.7) and (3.4) as:

$$
\begin{aligned}
\Delta g_M &= g_P - \gamma_{Q_0} + 0.3086H^* \\
\Delta g_c &= g_P - \gamma_{\bar{Q}_0} + 0.3086H \\
\Delta g_M - \Delta g_c &= 0.3086(H^* - H) \qquad \left(\text{with}\, \gamma_{Q_0} \doteq \gamma_{\bar{Q}_0}\right) \\
&= 0.3086(h - \zeta - h + N) = 0.3086(N - \zeta)\,\text{mgals}
\end{aligned}
\tag{3.10}
$$

where N is the geoid undulation and ζ the height anomaly. From equation (8-104) in Heiskanen and Moritz (1967), we get:

$$
(\zeta - N)_{\text{meters}} = +0.1 H_{\text{km}}^{\text{av}} \cdot H_{\text{km}} \tag{3.11}
$$

where H is the elevation of the station above vertical reference datum and H_{av} the average height of the area considered.

For an elevation of 1000 meters, substituting (3.11) in (3.10) we get:

$$
\Delta g_M - \Delta g_c = -0.03086\ \text{mgals} \tag{3.12}
$$

Though the numerical difference between the two anomalies as implied by equation (3.12) is small, conceptually they are very different. While the free-air anomalies computed using equation (3.9) are referred to ground level, the classical free-air anomalies from (3.5) are referred to sea level (Heiskanen and Moritz, 1967, p. 293). This distinction should be carefully kept in mind when using the different types of free-air anomalies in the definition of global vertical datum.

The free-air anomalies defined and computed using either approach need to be corrected for the following systematic corrections:

<u>(a) Atmospheric Correction (δg_A):</u>

In the problem of physical geodesy, whether following the Molodensky approach or classical approach, it is assumed that the anomalous potential T outside the boundary surface is a harmonic function or, in other terms, that the space outside the surface is empty. Therefore the effect of atmosphere on Δg must be removed by computation. Detailed discussions on the effect of atmosphere on gravity anomalies and the amount of correction needed to be applied on observed gravity can be found in Moritz (1980, pp. 422-425), Pavlis (1988, 1991) and Rapp and Pavlis (1990).

We can make use of the following relation given by Pavlis (1991, p. 33) for computing the atmospheric correction that needs to be added to all gravity anomalies derived from terrestrial observations.

$$
\delta g_A = 0.8658 - 9.727 \times 10^{-5} H_P + 3.482 \times 10^{-9} H_P^2 (\text{mgal}) \tag{3.13}
$$

where H_P is the orthometric height of the gravity station in meters.

Since the indirect effect of atmospheric corrections on the computed height anomaly/geoid undulation is in the order of -0.7 cm (Moritz, 1980, p. 425), it can be safely neglected.

(b) Horizontal Datum Inconsistency Correction (δg_H):

In the computation of gravity anomalies, the normal gravity calculations require the values of geodetic latitude for the points of gravity observation. Since in the gravity formula the latitude values used are defined in the local horizontal datum (which is not generally consistent with global geocentric datum), a systematic error is introduced in the normal gravity computed and thus in the gravity anomaly. As the transformation parameters from regional horizontal datum to global geocentric datum can be determined to ±0.1 arc second accuracy in geodetic latitude and longitude (refer to Tables 9.1 and 9.2, DMA Tech. Report, Part I, Dec. 1987), the error in gravity anomalies due to horizontal datum inconsistancy can be easily computed and applied. A detailed discussion on the effect of horizontal datum inconsistency on observed gravity anomalies and hence on undulations computed using them are discussed in Chapter 4. The magnitude of the correction to gravity anomalies, δg_H, is about 0.3 mgals (Heck, 1990, Table 4) and the correction to the corresponding undulation computed using them is about 10 cm.

(c) Gravity Formula Correction (δg_G):

The computed free-air anomalies in different parts of the world should correspond to the same normal gravity formula used for computing the normal gravity values. If they are computed with different gravity formulas then corrections are needed to be applied to make them all compatible to a single reference formula. For example, if some of them referred to GRS67 formula and the rest to GRS80 formula, then we can use the following relation given by Moritz (1992):

$$(\gamma_{1980} - \gamma_{1967}) = (0.8316 + 0.782 \sin^2\phi - 0.0007 \sin^4\phi) \text{ mgal} \tag{3.14}$$

to compute the correction to be applied to the gravity anomalies computed with GRS67 formula. More discussions on the correction to be applied to currently available free-air anomaly data sets are included in Chapter 4.

(d) Ellipsoidal Corrections (ε_h, ε_γ, ε_P):

In most of the formulas in physical geodesy relating to gravity anomalies and geoidal heights such as Stokes' formula, Molodensky series etc., one usually neglects the flattening of the reference ellipsoid, arriving at expressions which are formally valid on the sphere only. Since this spherical approximation can cause an error of about ±20 cm in the computed undulation and in our problem of global vertical datum definition we need higher accuracies, ellipsoidal corrections to free-air gravity anomalies should be applied.

Rapp and Pavlis (1990) have given the necessary analytical expressions for the computation of ellipsoidal correction terms that are to be applied to the free-air gravity anomalies. The above terms are of the order of tens of microgals and of long wavelength in nature. Their effects have been studied in detail by Cruz (1986) and Pavlis (1988, 1991).

3.1.2 Precise Heights Above Regional Vertical Reference Datum

The different types of heights that are used in the various regional vertical networks in the world are discussed in Chapter 2, the most common heights being the normal heights, Helmert orthometric heights and normal orthometric heights. As discussed in Chapter 2, the basic quantity for height determination is the geopotential number C which is given by the relation:

$$C = \sum_{0}^{P} g_i \Delta n_i \tag{3.15}$$

where g_i is the measured gravity and Δn_i the leveling increments. In this modelling effort, it is assumed that the leveling increments meet the first order leveling procedure requirements and the gravity observations refer to the International Gravity Standardization Network 1971 (IGSN71) which defines the primary global gravity datum.

3.1.3 Ellipsoidal Heights

Precise ellipsoidal heights at the fundamental stations used in the modelling are obtained from observations with currently available space geodetic techniques like SLR, VLBI, GPS and DORIS networks. The original results from these observations will be the cartesian coordinates X, Y, Z of the stations with respect to a specific coordinate system. Considering a global geocentric reference system such as International Terrestrial Reference Frame 1990 (ITRF 1990) as a basic reference system, the individual station coordinates in their respective coordinate frame can be transformed to the global reference frame. Boucher and Altamimi (1992) give the seven parameters necessary to convert between some general reference systems and ITRF90. For more discussions refer to Rapp and Balasubramania (1992).

3.1.4 Global Geopotential Models

The gravitational potential field at any point outside the earth can be expressed as an infinite series of spherical harmonics as follows:

$$V_E(r,\theta,\lambda) = \frac{GM}{r}\left(1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n (\bar{c}_{nm}\cos m\lambda + \bar{s}_{nm}\sin m\lambda)\bar{P}_{nm}(\cos\theta)\right) \tag{3.16}$$

where GM = product of the gravitational constant and mass of the earth,
a = mean equatorial radius of the earth,
$\bar{c}_{nm}, \bar{s}_{nm}$ = fully normalized spherical harmonic coefficients,
$\bar{P}_{nm}(\cos\theta)$ = fully normalized Legendre polynomial,
n, m = degree and order of spherical harmonic expansion, and
r, θ, λ = spherical coordinates.

In practice since the gravitational potential has to be determined from the combination of satellite and terrestrial information, we can represent the potential only by a truncated series with a finite number of terms. Such a finite expansion of potential coefficients, complete to degree and order N_{max}, will be called as a global geopotential model and equation (3.16) becomes:

$$V(r,\theta,\lambda) = \frac{GM}{r}\left(1 + \sum_{n=2}^{N_{max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n \left(\hat{\bar{c}}_{nm}\cos m\lambda + \hat{\bar{s}}_{nm}\sin m\lambda\right)\bar{P}_{nm}(\cos\theta)\right) \tag{3.17}$$

The higher the maximum degree, N_{max}, in the series, the greater the resolution we have in representing the potential and quantities derived from it. For a series that is complete (i.e., all coefficients are known) to degree N, the resolution (R) of the geoid model can be estimated from (Rapp, 1991):

$$R(km) = \frac{111\,km \times 180}{N} = \frac{19980(km)}{N} \tag{3.18}$$

Pertinent information on available global geopotential models are given in Featherstone (1992, Table 3.1), and they are reproduced in Table 3.1 below.

Table 3.1: Information on Global Geopotential Models.

global geopotential model	M_{max}	GM_{ggm} ($m^3 s^{-2}$)	a_{ggm} (m)	*data used*
OSU91A (Rapp *et al*, 1991)	360	3986004.36×10^8	6378137	st,sa,tg
OSU89A (Rapp & Pavlis, 1990)	360	3986004.36×10^8	6378137	st,sa,tg
OSU89B (Rapp & Pavlis, *ibid*)	360	3986004.36×10^8	6378137	st,sa,tg,pg
OSU86E (Rapp & Cruz, 1986)	360	3986004.36×10^8	6378138	st,sa,tg
OSU86F (Rapp & Cruz, *ibid*	360	3986004.36×10^8	6378138	st,sa,tg,pg
OSU81 (Rapp, 1981c)	180	3986004.36×10^8	6378137	st,sa,tg
GEM-T1 (Marsh *et al*, 1987)	36	3986004.36×10^8	6378137	st
GEM-T2 (Marsh *et al*, 1990a/b)	36	3986004.36×10^8	6378137	st
GEM-T3 (Nerem *et al*, 1991)	50	3986004.36×10^8	6378137	st,sa,tg
GRIM3-L1 (Reigber *et al*, 1985)	36	3986005×10^8	6378140	st,sa,tg
GRIM4-S1 (Reigber & Balmino, 1991)	36	3986004.4×10^8	6378136	st,sa,tg
WGS 84-EGM (Kumar, 1984)	18	3986005×10^8	6378137	st,sa,tg
GPM2 † (Wenzel, 1985)	200	—	—	st,sa,tg
IFE88E2 †‡ (Bašić *et al*, 1989)	360	—	—	st,sa,tg
GRS 80 (Moritz, 1980a)	—	3986005×10^8	6378137	—

in table 3.1: st — satellite orbit perturbation information,

sa — satellite altimeter derived information,

tg — observed terrestrial gravity anomalies, and

pg — predicted terrestrial gravity anomalies.

† not currently held, ‡ tailored model.

3.2 Modelling for Vertical Datum Definition

3.2.1 Procedure when Free-air Anomalies in Molodensky Sense and Normal Heights of the Stations are Known

The following procedure can be adopted in setting up the observation equations for defining the Global Vertical Datum when normal heights of the stations referred to regional vertical datum and the free-air anomalies in the Molodensky sense are known:

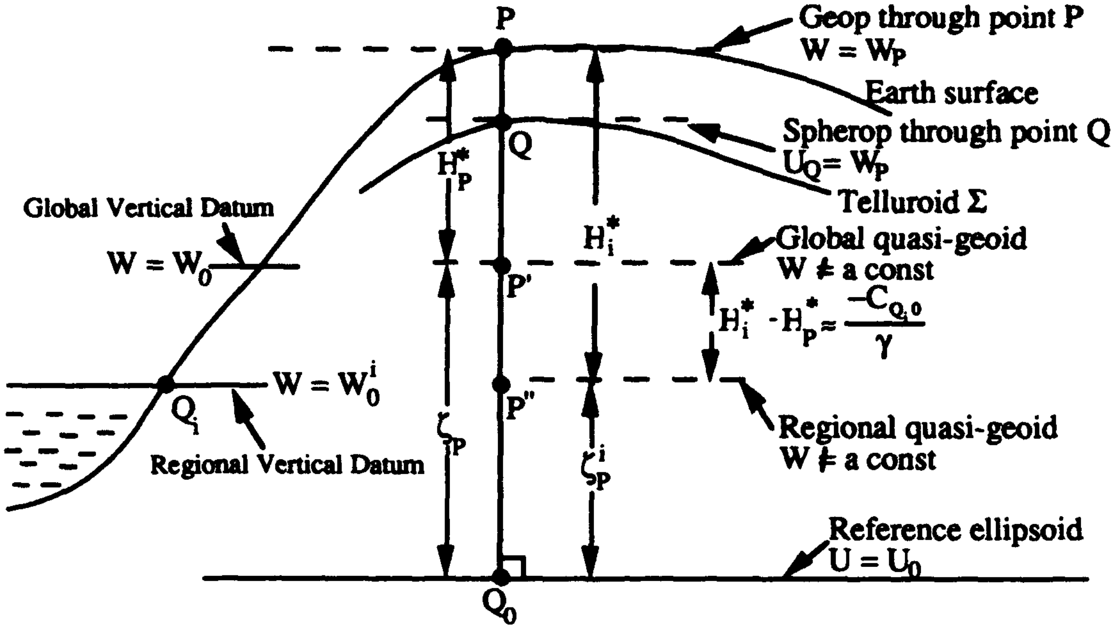


Figure 3.2 Various Reference Surfaces and Normal Height

In the above figure, Q_i is the fundamental benchmark serving as the origin for the regional vertical datum 'i'. The procedure for establishment of this benchmark is discussed in Chapter 2. Let H_P^* be the normal height of the point P computed with geopotential number $C_P = W_0 - W_P$, where W_0 is the geopotential of the unique (but unknown) fundamental surface referred to as the Global Vertical Datum, and H_i^* is the normal height of the same point P, but computed with geopotential number $C_P^i = W_0^i - W_P$, referring to ith regional vertical datum. From equation (2.15):

$$H_P^* = \frac{C_P}{\gamma_{Q_0}}\left[1+\left(1+f+m-2f\sin^2\phi\right)\frac{C_P}{a\gamma_{Q_0}}+\left(\frac{C_P}{a\gamma_{Q_0}}\right)^2\right]$$
$$H_i^* = \frac{C_P^i}{\gamma_{Q_0}}\left[1+\left(1+f+m-2f\sin^2\phi\right)\frac{C_P^i}{a\gamma_{Q_0}}+\left(\frac{C_P^i}{a\gamma_{Q_0}}\right)^2\right] \tag{3.19}$$

and $H_i^* - H_P^* \approx \left(\frac{C_P^i}{\gamma_{Q_0}} - \frac{C_P}{\gamma_{Q_0}}\right) = \frac{1}{\gamma_{Q_0}}\left(C_P^i - C_P\right)$

where γ_{Q_0} is the normal gravity at the ellipsoid, for the same latitude ϕ.

Denoting $C_P - C_P^i = W_0 - W_0^i = C_{Q_i0}$, then

$$H_i^* - H_P^* \approx -\frac{C_{Q_i0}}{\gamma} \tag{3.20}$$

From equation (8-8) in Heiskanen and Moritz (1967), the normal gravity γ_Q for the telluroid point Q is given by

$$\gamma_Q = \gamma_{Q_0} + \left[\frac{\partial \gamma}{\partial h}\right]_{Q_0} H_P^* + \frac{1}{2!}\left[\frac{\partial^2 \gamma}{\partial h^2}\right]_{Q_0} H_P^{*2} + \ldots \tag{3.21}$$

Neglecting higher order terms,

$$\begin{aligned} &= \gamma_{Q_0} + \left[\frac{\partial \gamma}{\partial h}\right]_{Q_0} H_i^* + \frac{1}{2!}\left[\frac{\partial^2 \gamma}{\partial h^2}\right] H_i^{*2} \\ &\quad + \left[\left(\frac{\partial \gamma}{\partial h}\right)_{Q_0} + \frac{1}{2}\left(\frac{\partial^2 \gamma}{\partial h^2}\right)_{Q_0} (H_P^* + H_i^*)\right](H_P^* - H_i^*) \\ &= \gamma_Q^{(i)} + \left[\left(\frac{\partial \gamma}{\partial h}\right)_{Q_0} + \left(\frac{\partial^2 \gamma}{\partial h^2}\right)_{Q_0} \left(\frac{H_P + H_i^*}{2}\right)\right](H_P^* - H_i^*) \end{aligned} \tag{3.22}$$

where $\gamma_Q^{(i)}$ is the normal gravity computed from elevation information referring to the regional vertical datum (i.e. H_i^*).

From the previous section we know the free-air anomaly in Molodensky sense, Δg_M, is defined as:

$\Delta g_M = g_P - \gamma_Q$ (from equation (3.6))

where g_P is the magnitude of the actual gravity acceleration at the surface point P, and

γ_Q is the magnitude of the normal gravity acceleration at the corresponding telluroid point Q.

Therefore, we can write from equations (3.20) and (3.22)

$$\Delta g_M = \Delta g_M^{(i)} - \frac{1}{\gamma_{Q_0}}\left[\left(\frac{\partial \gamma}{\partial h}\right)_{Q_0} + \left(\frac{\partial^2 \gamma}{\partial h^2}\right)\left(\frac{H_P^* + H_i^*}{2}\right)\right] C_{Q_i 0} \tag{3.23}$$

where $\Delta g_M, \Delta g_M^{(i)}$ are the gravity anomalies computed with normal gravity γ_Q determined using normal heights H_P^*, H_i^* of the point P referring to Global Vertical Datum and regional vertical datum 'i', respectively.

$C_{Q_i 0}$ is the potential difference between the geopotential surfaces referred to as global and regional vertical datums.

Now define $$q^j = \frac{1}{\gamma_{Q_0}}\left[\left(\frac{\partial \gamma}{\partial h}\right)_{Q_0} + \left(\frac{\partial^2 \gamma}{\partial h^2}\right)\left(\frac{H_P^* + H_j^*}{2}\right)\right] \tag{3.24}$$

From equations (2-121) and (2-122) in Heiskanen and Moritz (1967) with spherical approximation we have, $\frac{1}{\gamma_{Q_0}}\left(\frac{\delta\gamma}{\delta h}\right)=-\frac{2}{R}$ and,

$$\frac{1}{\gamma_{Q_0}}\left(\frac{\delta^2\gamma}{\delta h^2}\right)\cdot\left(\frac{H_P^* + H_i^*}{2}\right)=\frac{6}{R^2}\left(\frac{H_P^* + H_i^*}{2}\right)$$ (it is small and can be neglected).

Therefore, we can write equation (3.24) to be,

$$q^j \approx -\frac{2}{R} \tag{3.24'}$$

From equation (45.42) in Moritz (1980), the computation of the height anomaly ζ can be done using the following expression:

$$\zeta_P=\zeta_0+\frac{R}{4\pi\gamma}\iint_\sigma \Delta g_M S(\psi)d\sigma+\sum_{n=1}^{\infty}\frac{R}{4\pi\gamma}\iint_\sigma g_n S(\psi)d\sigma \tag{3.25}$$

where $\zeta_0=\frac{\delta(GM)}{R\gamma}-\frac{\Delta W_0}{\gamma}$ (refers to equation (2-181) in Heiskanen and Moritz (1967)),

$\delta(GM) = GM - (GM)_0$ is the difference between the true geocentric gravitational constant and that of adopted reference ellipsoid and $\Delta W_0 = W_0 - U_0$,

R is the mean radius of earth,

σ is the spherical surface,

γ is the mean gravity of the earth,

Δg_M are the free-air gravity anomalies defined in Molodensky sense,

ψ is the spherical distance between the computation point P and the variable points K,

S (ψ) is the Stokes' function,

g_n are the correction terms evaluated by the recursive formula.

$$g_n=-\sum_{r=1}^{n} z^r L_r(g_{n-r}) \tag{3.26}$$

In equation (3.26), z is the height difference between the station point P and varying points K, $z = h_K - h_P$. The operator L_n is evaluated recursively as (refers to equation (45-47) in Moritz (1980))

$$L_n(\Delta g)=\frac{1}{n}L_1[L_{n-1}(\Delta g)] \tag{3.27}$$

In planar approximation, L_1 is given by

$$L_1(\Delta g) = \frac{\partial \Delta g}{\partial z} = \frac{R^2}{2\pi} \iint_\sigma \frac{\Delta g - \Delta g_P}{\ell_0^3} d\sigma \tag{3.28}$$

with $\ell_0 = 2R \sin\frac{\psi}{2}$

Starting from $g_0 = \Delta g_M$, from equation (3.26) we can write:

$$\begin{aligned} g_1 &= -(h_K - h_P)\, L_1(g_0) \\ g_2 &= -(h_K - h_P)\, L_1(g_1) - (h_K - h_P)^2\, L_2(g_0) \end{aligned} \tag{3.29}$$

and so on. The terms g_1, g_2, etc. are called as Molodensky correction terms.

The evaluation of the equations (3.26) to (3.29) by formulating convolution integrals for the planar approximation and then solving them in frequency domain using Fast Fourier Transform techniques are discussed in Sideris and Schwarz (1986, 1988).

The free-air gravity anomalies derived in Molodensky sense and used in equation (3.25) are to be corrected for atmospheric and other corrections listed in Section 3.1.1. We can denote these corrected anomalies as Δg_M^C.

Based on the discussions in Sideris and Schwarz (1988), for a highly accurate computation of the height anomaly ζ, using equation (3.25), consideration of Molodensky's correction terms up to g_2 seems to be significant. But numerical tests (ibid.) also show that these correction terms are very sensitive to the high frequencies of the gravity field and thus should be computed with the densest grid of data available. Presently dense grids of gravity data required for computing the correction terms accurately are available only in the United States and in some countries in Western Europe. Therefore consideration of correction terms beyond the g_1 term in the computation of the height anomaly all over the globe seems impossible as of today (1994).

Non-consideration of the g_2 term in the height anomaly may lead to an error of about 2 cm in a moderately undulating terrain and more in rough mountainous areas. Sideris and Schwarz (1986) have pointed out that the use of free-air gravity anomalies corrected for terrain (also called as Faye anomalies) in the computation of the height anomaly makes the g_n terms smaller, smoother and easier to interpolate. Hence the use of Faye anomalies in the computation can minimize the error due to neglecting the g_2 term.

Now considering only up to Molodensky's correction term g_1, equation (3.25) simply becomes:

$$\zeta_P = \zeta_0 + \frac{R}{4\pi\gamma} \iint_\sigma \left(\Delta g_M^C + g_1\right) S(\psi)\, d\sigma \tag{3.30}$$

If terrain corrected free-air anomalies are available, then from equation (24) of Wang (1993), we can rewrite equation (3.30) as:

$$\zeta_P = \zeta_0 + \frac{R}{4\pi\gamma} \iint_\sigma \left(\Delta g_M^C + C\right) S(\psi)\, d\sigma - \frac{\Delta g_B}{\gamma} H_P^* - \frac{\pi G \rho}{\gamma} H_P^{*2} - \frac{\pi G \rho}{\gamma} \delta h^2 \tag{3.31}$$

where $\left(\Delta g_M^C + C\right)$ is the gravity anomaly corrected for terrain and other systematic effects listed in Section 3.1.1,

H_P^* is the normal height of the point P at which the height anomaly is computed,

$\delta h^2 = 0.453 - 0.018 \sin\phi + 0.087 \cos\phi \cos\lambda + 0.204 \cos\phi \sin\lambda$ (km^2) where ϕ, λ are the geographic latitude and longitude of the point P, respectively, and

$\Delta g_B = \left(\Delta g_P - 2\pi G\rho H_P^*\right)$ is the Bouguer anomaly, Δg_P is the surface free-air anomaly, G is Newtonian gravitational constant and ρ is the density of topographic mass.

Equation (3.31) is based on the assumption that the gravity anomaly Δg_M^C is linearly correlated with elevation, and it provides an accurate solution for computing the height anomaly when terrain-corrected free-air anomalies are available.

Combining equations (3.23) and (3.31), we can write:

$$\zeta_P = \zeta_0 + \frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_M^C + C\right)^j S(\psi)d\sigma_j - \frac{R}{4\pi\gamma}\sum_{j=1}^{n} q^j C_{Q_j0}\iint_{\sigma_j} S(\psi)d\sigma_j$$
$$-\frac{\Delta g_B}{\gamma}H_P^* - \frac{\pi G\rho}{\gamma}H_P^{*2} - \frac{\pi G\rho}{\gamma}\delta h^2 \qquad (3.32)$$

where $q^j = \frac{1}{\gamma_{Q_0}}\left[\left(\frac{\partial\gamma}{\partial h}\right)_{Q_0} + \left(\frac{\partial^2\gamma}{\partial h^2}\right)_{Q_0}\frac{H_P^* + H_j^*}{2}\right]$

In equation (3.32) n denotes the number of regional vertical datums considered in the Global Vertical Datum definition.

From Figure 3.2, we note that approximately,

$$H_P^* = H_i^* + \frac{C_{Q_i0}}{\gamma} \qquad (3.33)$$

where H_P^* is the normal height of the station referred to the global vertical datum,
H_i^* is the normal height of the station referred to the regional vertical datum, and
$\frac{C_{Q_i0}}{\gamma} = W_0 - W_0^i$

also

$$h_P = H_P^* + \zeta_P \qquad (3.34)$$

where h_P is the ellipsoidal height of the point P.

Substituting $\zeta_0 = \frac{\delta(GM)}{R\gamma} - \frac{\Delta W_0}{\gamma}$ in equation (3.32) where $\delta(GM) = GM - (GM)_0$ is the difference between the true geocentric gravitational constant and that of adopted reference ellipsoid, we get:

$$\zeta_P = \frac{\delta(GM)}{R\gamma} - \frac{\Delta W_0}{\gamma} + \frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_M^C + C\right)^j S(\psi)d\sigma_j - \frac{R}{4\pi\gamma}\sum_{j=1}^{n} q^j C_{Q_j0}\iint_{\sigma_j} S(\psi)d\sigma_j$$
$$- \frac{\Delta g_B}{\gamma}H_P^* - \frac{\pi G\rho}{\gamma}H_P^{*2} - \frac{\pi G\rho}{\gamma}\delta h^2 \qquad (3.35)$$

Since the GM value is known to an accuracy of approximately 0.003 ppm, GM shall be regarded as known in the discussion and one can assume $\delta(GM) = 0$ (Rapp and Balasubramania, 1992) in equation (3.35).

From equation (3.34) we can write:

$$\zeta_P = h_P - H_P^* \qquad (3.36)$$

Comparing equations (3.35) and (3.36):

$$h_P - H_P^* = \frac{-\Delta W_0}{\gamma} + \frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_M^C + C\right)^j S(\psi)d\sigma_j - \frac{R}{4\pi\gamma}\sum_{j=1}^{n} q^j C_{Q_j0}\iint_{\sigma_j} S(\psi)d\sigma_j$$
$$- \frac{\Delta g_B}{\gamma}H_P^* - \frac{\pi G\rho}{\gamma}H_P^{*2} - \frac{\pi G\rho}{\gamma}\delta h^2 \qquad (3.37)$$

From equation (3.33) $H_P^* = H_i^* + \frac{C_{Q_i0}}{\gamma}$.

Substituting (3.33) *in* (3.37) we get:

$$h_P - H_i^* = \frac{-\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_M^C + C\right)^j S(\psi)d\sigma_j - \frac{R}{4\pi\gamma}\sum_{j=1}^{n} q^j C_{Q_j0}\cdot\iint_{\sigma_j} S(\psi)d\sigma_j$$
$$- \frac{\Delta g_B}{\gamma}H_P^* - \frac{\pi G\rho}{\gamma}H_P^{*2} - \frac{\pi G\rho}{\gamma}\delta h^2 \qquad (3.38)$$

Substituting (3.24') in (3.38) and rewriting the known and unknown terms, we get:

$$h_P - H_i^* - \frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_M^C + C\right)^j S(\psi)d\sigma_j + \frac{\Delta g_B}{\gamma}H_P^* + \frac{\pi G\rho}{\gamma}H_P^{*2} + \frac{\pi G\rho}{\gamma}\delta h^2$$
$$= -\frac{\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \frac{2}{\gamma}\sum_{j=1}^{n} C_{Q_j0}\left\{\frac{1}{4\pi}\iint_{\sigma_j} S(\psi)d\sigma_j\right\} \qquad (3.39)$$

In equation (3.39) the left hand side can be computed using the observations. Let it be denoted by Y. Then,

$$Y = h_P - H_i^* - \frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_M^C + C\right)^j S(\psi)d\sigma_j + \frac{\Delta g_B}{\gamma}H_P^* + \frac{\pi G\rho}{\gamma}H_P^{*2} + \frac{\pi G\rho}{\gamma}\delta h^2 \qquad (3.40)$$

where h_P is the ellipsoidal height of the point P, and H_i^* is the normal height of the station referred to the i[th] vertical datum in which it lies.

Since in equation (3.40) the terms $\frac{\Delta g_B}{\gamma}, \frac{\pi G\rho}{\gamma}$, etc. are small, and the difference between H_i^* and H_P^* is also very small, we can replace H_P^*, H_P^{*2} in the right hand side of equation (3.40) by H_i^* and H_i^{*2}, respectively.

Denoting the gravimetric height anomaly (without zero height anomaly term) by ζ_i^i, we can write equation (3.39) as:

$$\boxed{Y = h_P - H_i^* - \zeta_i^i = \frac{-\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \frac{2}{\gamma}\sum_{j=1}^{n} C_{Q_j0}\left\{\frac{1}{4\pi}\iint_{\sigma_j} S(\psi)d\sigma_j\right\}} \tag{3.41}$$

where $$\zeta_i^i = \frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_M^C + C\right)^j S(\psi)d\sigma_j - \left(\frac{\Delta g_B}{\gamma}H_i^* + \frac{\pi G\rho}{\gamma}H_i^{*2} + \frac{\pi G\rho}{\gamma}\delta h^2\right) \tag{3.42}$$

Equation (3.41) is the required observation equation when free-air anomalies in Molodensky's sense and normal heights of the station above regional vertical datum are given.

3.2.2 Procedure When Free-air Anomalies in Classical Sense and Orthometric Heights of the Stations are Known

From equation (2.5) we know that the orthometric height of a surface point P with respect to an equipotential surface $W = W_0$ (in our case the global vertical datum) is given by:

$$H_P = \frac{C_P}{\bar{g}}$$

where C_P is the geopotential number at point P ($C_P = W_0 - W_P$) and $\bar{g}$ the mean gravity along the curved vertical between the surface point and its corresponding point P_0 on the global vertical datum (Refer to Figure 3.3). In the same way the orthometric height of the point P with respect to the regional vertical datum 'i' ($W = W_i$) can be written as:

$$H^i = \frac{C_{P_i}}{\bar{g}} \qquad \text{where } C_{P_i} = W_0^i - W_P \tag{3.43}$$

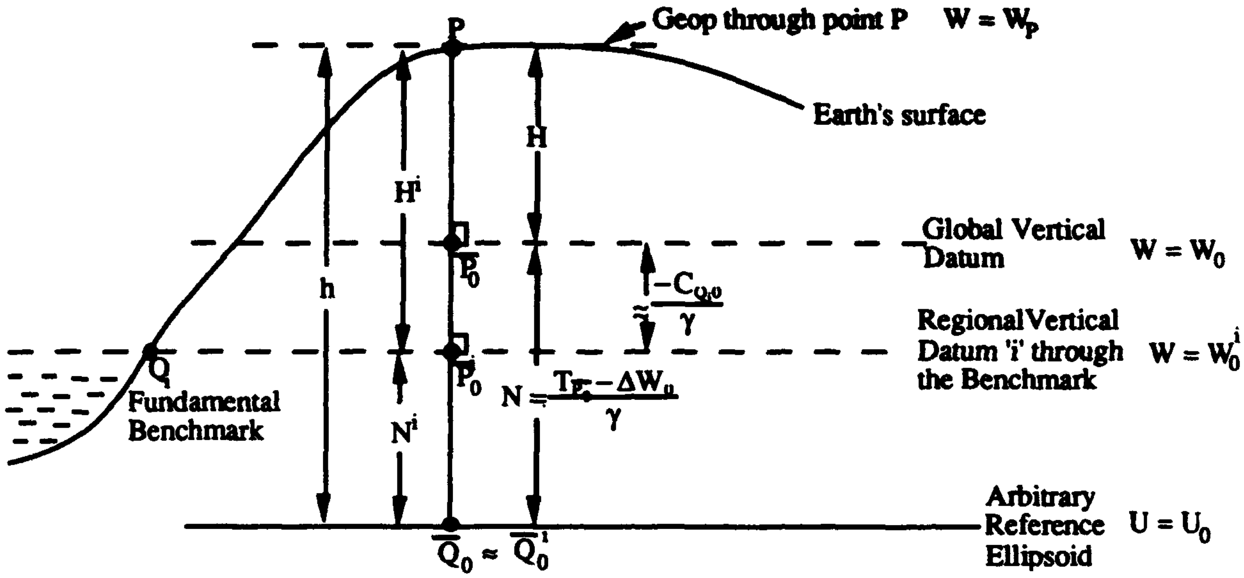


Figure 3.3 Various Equipotential Surfaces and Orthometric Height

In Figure 3.3, Q_i can be considered as the fundamental benchmark of the i-th regional vertical datum. The potential of the reference surface for the global vertical datum is W_0. We define as before $C_{Q_i0} = W_0 - W_0^i$ as the potential difference between the equipotential surfaces of the regional datum i and the global vertical datum. Introducing an arbitrary ellipsoid where U_0 is the potential on the surface, we define $\Delta W_0 = W_0 - U_0$. Also in the above figure the distance between the global vertical datum and the arbitrary ellipsoid is given. This depends on the disturbing potential at the point $\bar{P}_0, T_{\bar{P}_0}$, and $\Delta W_0 = W_0 - U_0$.

From equations (3.1) and (3.4) we know the free-air anomaly in a classical approach including terms only up to the second order is written as:

$$\Delta g_C = \left(g_P - \frac{\partial \gamma_{\bar{Q}_0}}{\partial h} H - \frac{1}{2!} \frac{\partial^2 \gamma_{\bar{Q}_0}}{\partial h^2} H^2 \right) - \gamma_{\bar{Q}_0} \tag{3.44}$$

where H is the orthometric height and $\gamma_{\bar{Q}_0}$ is the normal gravity on the ellipsoid. Since the orthometric heights of the stations are known with respect to the regional vertical datum i, H^i, the gravity anomalies computed using equation (3.44) will refer to the regional vertical datum only. Denoting the free-air anomalies referring to the regional vertical datum i as Δg_C^i, we can derive the relationship between Δg_C^i and Δg_C that refer to the global vertical datum as follows:

For an arbitrary reference ellipsoid, following a simple generalization of Brun's formula we can write (refer to Figure 3.3)

$$N = \frac{T_{\bar{P}_0} - \Delta W_0}{\gamma} \tag{3.45}$$

where $T_{\bar{P}_0}$ refers to the disturbing potential at the point $\bar{P}_0$ on the global vertical datum.

Similarly for the regional vertical datum'i' we can write

$$N^i = \frac{T_{\bar{P}_0^i} - \Delta W_0^i}{\gamma} \tag{3.46}$$

where $T_{\bar{P}_0^i}$ refers to the disturbing potential at the point $\bar{P}_0^i$ on the regional vertical datum, and $\Delta W_0^i = W_0^i - U_0$ is the potential difference.

From (3.45) and (3.46) we get

$$N - N^i = \frac{1}{\gamma}\left[\left(T_{\bar{P}_0} - T_{\bar{P}_i}\right) - \left(\Delta W_0 - \Delta W_0^i\right)\right]$$

since $T_{\bar{P}_0} \approx T_{\bar{P}_0^i}$, we can write

$$N - N^i \approx -\frac{C_{Q_i0}}{\gamma} \tag{3.47}$$

with eqn. (3.47), eqn. (3.45) can be modified as

$$N^i = N + \frac{C_{Q_i0}}{\gamma} = \frac{T_{\bar{P}_0} - \Delta W_0 + C_{Q_i0}}{\gamma} \tag{3.48}$$

From equation (2-148') in Heiskanen and Moritz (1967) we have the relation:

$$\Delta g_c = -\frac{\partial T_{\bar{P}_0}}{\partial n} + \frac{\partial \gamma}{\partial n} N \tag{3.49}$$

where Δg_c is the gravity anomalies defined in a classical sense referring to the global vertical datum,
N is the geop-spherop separation between the GVD and the arbitrary reference ellipsoid,
γ is the normal gravity, and
$\frac{\partial}{\partial n}$ is the derivative along the ellipsoidal normal.

Similarly for the case of the regional vertical datum 'i', we can write

$$\Delta g_c^i = -\frac{\partial T_{\bar{P}_0^i}}{\partial n} + \frac{\partial \gamma}{\partial n} N^i \tag{3.50}$$

Now substituting for N^i from equation (3.48), we get

$$\Delta g_c^i = \frac{\partial \gamma}{\partial n} \cdot \frac{T}{\gamma} - \frac{\partial \gamma}{\partial n} \cdot \frac{\Delta W_0}{\gamma} + \frac{\partial \gamma}{\partial n} \cdot \frac{C_{Q_i0}}{\gamma} - \frac{\partial T}{\partial n} \tag{3.51}$$

In spherical approximation, setting r = R,

$$\frac{\partial\gamma}{\partial n} = -\frac{2\gamma}{R} \text{ and } \frac{\partial T}{\partial n} = \frac{\partial T}{\partial R} \tag{3.52}$$

Substituting (3.52) in (3.51)

$$\Delta g_c^i = \frac{2}{R}\cdot\Delta W_0 - \frac{2}{R}\cdot C_{Q_i0} - \left(\frac{2}{R} + \frac{\partial}{\partial R}\right)T \tag{3.53}$$

From eqn. (2-179') in Heiskanen and Moritz

$$\Delta g_c = -\frac{\partial T}{\partial R} - \frac{2}{R}T + \frac{2}{R}\Delta W_0 \tag{3.54}$$

From equations (3.53) and (3.54) we get

$$\boxed{\Delta g_c = \Delta g_c^i + \frac{2}{R}C_{Q_i0}} \tag{3.55}$$

From equation (2-181) in Heiskanen and Moritz (1967), we can write:

$$N(P) = \frac{\delta(GM)}{R\gamma} - \frac{\Delta W_0}{\gamma} + \frac{R}{4\pi\gamma}\iint_\sigma \Delta g_C S(\psi)d\sigma \tag{3.56}$$

Substituting (3.47) and (3.55) in (3.56) and assuming that $\delta(GM)$ is equal to zero because of the high accuracy in the determination of GM today (1994) as noted earlier, we get:

$$N^i(P) = \frac{-\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \sum_{j=1}^{n}\frac{R}{4\pi\gamma}\iint_{\sigma_j} S(\psi)\left[\Delta g_c^j + \frac{2}{R}C_{Q_j0}\right]d\sigma_j \tag{3.57}$$

where 'i' refers to the i^{th} regional vertical datum and
'j' = 1, 2...n, n being the number of regional vertical datums considered.

The free-air anomaly used in equation (3.57) should be corrected for the systematic errors listed in Section 3.1.1 and also for the terrain effect. If Faye anomalies are used in the Stokes' formula in equation (3.57) then by Helmert's second method of condensation, an indirect effect correction is needed to be applied to the computed undulation. The required equation for computing the indirect effect as given by Grushinsky's formula (Wichiencharoen, 1982) is:

$$\delta N^i = \frac{-\pi G\rho}{\gamma}H^{i^2} \qquad \text{(independent of } j = 1,\ldots n) \tag{3.58}$$

where the quantities G, ρ, γ are defined earlier and H^i is the height of the station in the regional vertical datum.

Assuming the free-air anomalies Δg_C are corrected for systematic corrections and also for terrain, equation (3.57) gets modified as:

$$N^i(P)=\frac{-\Delta W_0}{\gamma}+\frac{C_{Q_i0}}{\gamma}+\frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}S(\psi)\left[\left(\Delta g_C^c+C\right)^j+\frac{2}{R}C_{Q_j0}\right]d\sigma_j-\frac{\pi G\rho}{\gamma}H^{i^2} \tag{3.59}$$

where $\left(\Delta g_C^c+C\right)^j$ refers to the free-air anomalies defined in a classical sense, corrected for terrain and other systematic corrections referring to regional vertical datum 'j', for j = 1, 2, ...n where 'n' is the number of regional vertical datum considered in the global vertical datum definition.

Assuming that we have a set of space geodetic stations on regional vertical datum 'i', for which we know the ellipsoidal heights 'h' above an arbitrary geocentric reference ellipsoid and the orthometric heights H^i above the local vertical datum, then we can write from equation (2.1) and Fig. 3.3:

$$N^i = h - H^i \tag{3.60}$$

comparing equations (3.59) and (3.60)

$$h-H^i=\frac{-\Delta W_0}{\gamma}+\frac{C_{Q_i0}}{\gamma}+\frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}S(\psi)\left[\left(\Delta g_C^c+C\right)^j+\frac{2}{R}C_{Q_j0}\right]d\sigma_j-\frac{\pi G\rho}{\gamma}H^{i^2}$$

$$=\frac{-\Delta W_0}{\gamma}+\frac{C_{Q_i0}}{\gamma}+\frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}S(\psi)\left(\Delta g_C^c+C\right)^j d\sigma_j+\frac{2}{\gamma}\sum_{j=1}^{n}C_{Q_i0}\left\{\frac{1}{4\pi}\iint_{\sigma_j}S(\psi)d\sigma_j\right\}-\frac{\pi G\rho}{\gamma}H_i^2$$

Re-arranging the known/observable and unknown terms, we get:

$$\begin{aligned} h-H^i-\frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_C^c+C\right)^j S(\psi)d\sigma_j+\frac{\pi G\rho}{\gamma}H_i^2 \\ =\frac{-\Delta W_0}{\gamma}+\frac{C_{Q_i0}}{\gamma}+\frac{2}{\gamma}\sum_{j=1}^{n}C_{Q_j0}\left\{\frac{1}{4\pi}\iint_{\sigma_j}S(\psi)d\sigma_j\right\} \end{aligned} \tag{3.61}$$

Denoting the left hand side of equation (3.61) as Y, we write:

$$Y=h-H^i-\left[\frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_C^c+C\right)^j S(\psi)d\sigma_j-\frac{\pi G\rho}{\gamma}H^{i^2}\right] \tag{3.62}$$

The third term on the right hand side of the equation refers to the geoid undulation computed using terrain corrected gravity anomalies without the zero undulation (N_0) term. Denoting the third term by N_1^i and from equation (3.61) and (3.62) we can write:

$$\boxed{Y=h-H^i-N_1^i=\frac{-\Delta W_0}{\gamma}+\frac{C_{Q_i0}}{\gamma}+\frac{2}{\gamma}\sum_{j=1}^{n}C_{Q_i0}\left\{\frac{1}{4\pi}\iint_{\sigma_j}S(\psi)d\sigma_j\right\}} \tag{3.63}$$

where $N_1^i=\frac{R}{4\pi\gamma}\sum_{j=1}^{n}\iint_{\sigma_j}\left(\Delta g_C^c+C\right)^j S(\psi)d\sigma_j-\frac{\pi G\rho}{\gamma}H_i^2$ (3.64)

Equation (3.63) is the required observation equation when free-air anomalies in the classical sense and orthometric heights of the station above the regional vertical datums are given.

3.2.3 Procedure When Free-air Anomalies in Classical Sense and Normal Orthometric Heights of Stations are Known

As explained in section 2.1.2.2, normal orthometric heights are used in countries which lack dense gravity data coverage. However, they are considered for all practical purposes as a coarser approximation to true orthometric heights. Therefore, the procedure described in Section 3.2.2 can be used for setting up the required observation equations in this case also. Since normal orthometric heights are not true orthometric heights, such heights can produce a systematic error in the definition of global vertical datum.

3.3 Computation of Gravimetric Height Anomaly/Undulation

To achieve the highest accuracy in the computation of gravimetric height anomaly, ζ_1^i, in equation (3.42) or the gravimetric undulation, N_1^i, in equation (3.64), the procedure that can be followed is to combine the surface gravity anomaly data defined in a regional vertical datum limited to a small cap (100 to 200 km) about the computation point and coefficients from an accurate geopotential model. Some of the currently used techniques for computing gravimetric height anomaly/geoid undulation are discussed in the following paragraphs.

3.3.1 Modified Stokes' Technique

This technique makes use of gravity anomaly data in a cap around the point of interest and a set of potential coefficients from an accurate global geopotential model to compute the height anomaly/geoid undulation. This technique has been discussed by several authors; to mention a few, Despotakis (1987), Pavlis (1991), and Sjöberg (1991). From equation (2.40) in Despotakis (1987), we can write the modified Stokes' equation (here incorporating the indirect effect term):

$$N = \frac{R}{4\pi\gamma}\iint_{\sigma_C} \Delta g S(\psi) d\sigma + \frac{R}{2\gamma}\sum_{n=0}^{\infty} Q_n(\psi)\Delta g_n(\theta,\lambda) - \frac{\pi G\rho}{\gamma}H^2 \tag{3.65}$$

where Δg are the free-air gravity anomalies corrected for systematic errors listed in Section 3.1.1 and considered in a cap radius ψ_C,
Q_n are the Molodensky truncation coefficients defined by equation (7-34) in Heiskanen and Moritz (1967), and
$\Delta g_n(\theta, \lambda)$ the n^{th} degree harmonic representation of $\Delta g(\theta, \lambda)$.

The first term on the right hand side of equation (3.65) corresponds to the contribution of gravity anomalies, in a cap ($\psi = \psi_C$), to the undulation computation and the second term to the remote zone contribution to the undulation computation. When terrain corrected gravity anomalies are used, equation (3.65) gets modified as:

$$\begin{aligned} N &= \frac{R}{4\pi\gamma}\iint (\Delta g + C) S(\psi) d\sigma - \frac{\pi G\rho}{\gamma}H^2 = \\ &= \frac{R}{4\pi\gamma}\iint_{\sigma_C} (\Delta g + C) S(\psi) d\sigma + \frac{R}{2\gamma}\sum_{n=0}^{\infty} Q_n(\psi)\Delta g_n^T(\theta,\lambda) - \frac{\pi G\rho}{\gamma}H^2 \end{aligned} \tag{3.66}$$

where $\Delta g_n^T(\theta,\lambda)$ is the n[th] degree harmonic term of $(\Delta g + C)$.

From discussions in Rapp (1993a), $\Delta g_n^T(\theta,\lambda)$ in the above equation is related to $\Delta g_n(\theta, \lambda)$ in equation (3.65) as follows:

$$\Delta g_n^T(\theta,\lambda) = \Delta g_n(\theta,\lambda) + C_n(\theta,\lambda) \qquad (3.67)$$

where C is the terrain correction, the coefficients of C can be found by replacing $\Delta g(\theta, \lambda)$ in equation (4) (Rapp, 1993a) with C, the terrain correction.

Since potential coefficient models available for use have yielded estimates of the potential, we can calculate $\Delta g_n^T(\theta,\lambda)$ from potential coefficients as:

$$\Delta g_n^T(\theta,\lambda) = \frac{GM}{r^2}(n-1)\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\overline{P}_{nm}(\cos\theta) \qquad (3.68)$$

Using (3.66) and assuming gravity anomalies used in the cap refer to only one datum 'i' in which the station lies, from equation (3.42) we get:

$$\begin{aligned}\zeta_i^i = {} & \frac{R}{4\pi\gamma}\iint_{\sigma_C}\left(\Delta g_M^C + C\right)^i S(\psi)d\sigma_c + \frac{R}{2\gamma}\sum_{n=0}^{\infty}Q_n(\psi)\Delta g_n^T(\theta,\lambda) - \\ & -\left(\frac{\Delta g_B}{\gamma}H_i^* + \frac{\pi G\rho}{\gamma}H_i^{*2} + \frac{\pi G\rho}{\gamma}\delta h^2\right)\end{aligned} \qquad (3.69)$$

Similarly, using (3.66) and assuming gravity anomalies used in the cap refer only to the datum 'i' in which the station lies, from equation (3.64) we get:

$$N_i^i = \frac{R}{4\pi\gamma}\iint_{\sigma_C}\left(\Delta g_C^c + C\right)^i S(\psi)d\sigma_c + \frac{R}{2\gamma}\sum_{n=0}^{\infty}Q_n(\psi)\Delta g_n^T(\theta,\lambda) - \frac{\pi G\rho}{\gamma}H_i^2 \qquad (3.70)$$

The gravimetric height anomaly ζ_i^i and the gravimetric undulation N_i^i computed by equations (3.69) and (3.70) with the Modified Stokes' technique are based on spherical approximation. But to get the solution with respect to an ellipsoid as reference surface we need to apply ellipsoidal corrections to the height anomaly/undulation computed. Rapp (1981) has developed procedures to compute the correction needed to fully refer the solution to an ellipsoid surface (equation (31), ibid.). Despotakis (1987, Chapter VI) has also derived the ellipsoid correction term to be applied to the height anomaly/undulation computed using gravity information in a cap surrounding the computation point combined with potential coefficient information, following the similar lines as in Rapp (1981). Denoting the ellipsoidal correction term as ΔN, equations (3.69) and (3.70) get modified as:

$$\begin{aligned}\zeta_i^i = {} & \frac{R}{4\pi\gamma}\iint_{\sigma_C}\left(\Delta g_M^C + C\right)^i S(\psi)d\sigma_c + \frac{R}{2\gamma}\sum_{n=0}^{\infty}Q_n(\psi)\Delta g_n^T(\theta,\lambda) - \\ & -\left(\frac{\Delta g_B}{\gamma}H_i^* + \frac{\pi G\rho}{\gamma}H_i^{*2} + \frac{\pi G\rho}{\gamma}\delta h^2\right) + \Delta N\end{aligned} \qquad (3.71)$$

$$N_i^i = \frac{R}{4\pi\gamma}\iint_{\sigma_C}\left(\Delta g_c^C + C\right)^i S(\psi)d\sigma_c + \frac{R}{2\gamma}\sum_{n=0}^{\infty}Q_n(\psi)\Delta g_n^T(\theta,\lambda) - \frac{\pi G\rho}{\gamma}H_i^2 + \Delta N \qquad (3.72)$$

and

$$\Delta N = \frac{e^2}{2\gamma}\sum_{n=0}^{m}(Q_n - X_n)\sum_{m=0}^{n}\left(G_{nm}\cos m\lambda + H_{nm}\sin m\lambda\right)P_{nm}(\sin\bar{\phi}) + e^2\left[\frac{1}{4}-\frac{3}{4}\sin^2\phi\right]N_0 \tag{3.73}$$

where Q_n are Molodensky's truncation coefficients (Heiskanen and Moritz (1967, Section 7.4)),

X_n are the Fourier coefficients of Stokes' function beyond the cap radius ψ_c, and

G_{nm}, H_{nm} are the coefficients defined in equation (39-81) in Moritz (1980).

The first term in equation (3.73) arises from the fact that the ellipsoidal effect $e^2\Delta g^1$, where Δg^1 is defined by equation (39-80) in Moritz (1980), has to be removed from the terrestrial gravity anomalies Δg_C^c used in the cap. As shown in Despotakis (1987) this correction term is dependent on the cap radius ψ_c in which the gravity anomaly data are used and its magnitude amounts to about 1-2 cm when $\psi_c = 2°$.

The second term in equation (3.73) corresponds to the ellipsoidal correction to the spherical undulation itself and is given as equation (39-21) in Moritz (1980) also. This term has been derived assuming average normal gravity being used at the station. If actual normal gravity computed at the stations are used in computing the undulation, this term becomes zero.

3.3.2 Least Squares Collocation Technique

As discussed in section 16 of Moritz (1980), Least Squares Collocation (LSC) technique provides the most accurate result obtainable on the basis of given data when prior information is included. Prediction of height anomaly ζ_1^i or gravimetric undulation N_1^i, based on the corrected free-air anomalies given in a regional datum can be done accurately using least squares collocation technique provided the required covariances are known.

The statistical approach to collocation is based on the theory of stochastic processes in which the gravity anomaly field is treated as a stationary stochastic process on a sphere with the assumption that both the observations and signals represent ergodic processes with zero expectation everywhere (Moritz, 1980, p. 76) and the average product of two signal quantities is the covariance between them. Therefore, in the case of gravity anomalies it is assumed that,

$$M\{\Delta g\} = 0 = E\{\Delta g\} \text{ and } M\{\Delta g(P)\cdot\Delta g(Q)\} = C(P,Q) = E\{\Delta g(P)\cdot\Delta g(Q)\} \tag{3.74}$$

The definition of the mean operator $M\{\cdot\}$ as a homogeneous and isotropic average over the whole sphere is an important assumption in this technique.

On a global scale, the anomalous gravity field exhibits a seemingly random behavior necessary for a statistical interpretation of the field. But when the area of interest is of limited extent, as in our case with limited surface gravity data, the global gravity field behavior represents a systematic component and short wavelength features become prominent from the statistical point of view (refer to Figure 3.4 given as Figures 1(a) and 1(b) in Collier and Leahy, 1992). Therefore, the conditions for equation (3.74) need to be modified with the requirement that the local average behavior must be zero and a local covariance function must be used.

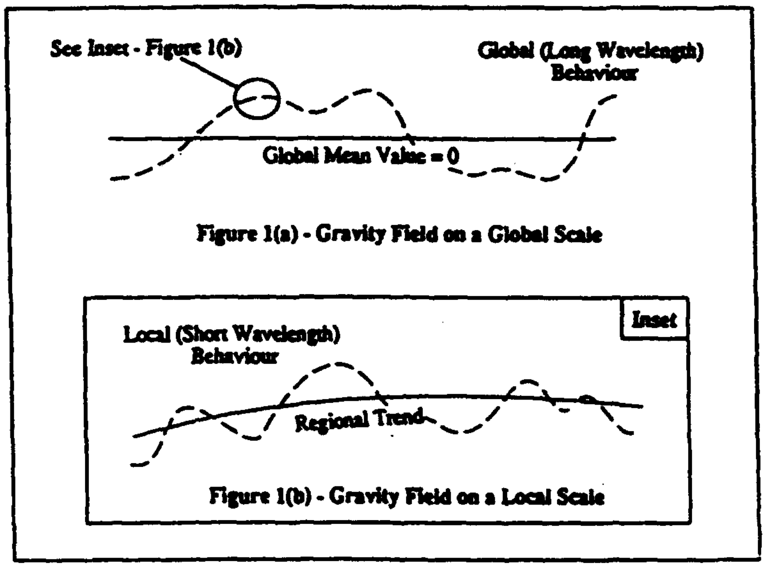


Figure 3.4 Gravity Field on a Global and Local Scale
(from Collier and Leahy, 1992)

To satisfy the requirement of zero expectation as well as that of isotropy and homogeneity for the covariance function, it is essential that the presence of all systematic trends in the local behavior of the gravity field be eliminated from the gravity data prior to generating an empirical covariance function. After the removal of the trend component, the remainder of the gravity field shall here be considered to behave in a stochastic fashion. Global geopotential models up to degree and order 360 can be used for removing the trend in the gravity field. Practical computation of reference gravity anomalies from a set of potential coefficients (up to degree 360), computation of a global covariance function and then scaling it to fit local data are discussed in detail in Bašić and Rapp (1992).

From equation (18-21) in Moritz (1980), the required expression for predicting the residual geoidal undulation/height anomaly at the space geodetic stations, using the LSC technique is given by

$$\hat{s} = C_{st}\overline{C}^{-1}Y \tag{3.75}$$

where $\hat{s}$ is the predicted residual geoid undulation/height anomaly,
Y are the residual gravity anomalies obtained after removing the reference anomalies computed using global geopotential model from the observed free-air anomalies corrected for terrain and other systematic effects,
C_{st} is the cross covariance matrix between unobserved and observed quantities, and
$\overline{C}$ is sum of the covariance matrices of the signal t and noise n $(C_{tt}+C_{nn})=D\{Y\}$.

In addition to providing an accurate prediction of height anomaly/geoid undulation, least squares collocation technique can also provide the accuracy of the predicted quantities which can be computed using the error covariance matrices given as equation (17-44), (17-47) and (17-48) in Moritz (1980).

$$E_{ss} = C_{ss} - C_{st}\bar{C}^{-1}C_{ts} \tag{3.76}$$

where E_{ss} is the error covariance matrix of predicted geoid undulation,

C_{ss} is the auto covariance matrix of unobserved geoid undulation,

C_{st}, C_{ts} are the cross covariance matrices as before C_{st}=C{s,Y}, and

$\bar{C}$ is the total covariance matrix of observations $(C_{tt} + C_{nn})$=D{Y}.

The height anomaly/geoid undulations predicted using LSC technique are to be corrected for ellipsoidal corrections following the procedure described in Section 39 of Moritz (1980). The discussions given in there indicate that the ellipsoidal correction computed and applied in Section 3.3.1 for the Modified Stokes' Technique can be applied in this case also. As pointed out earlier, the magnitude of ellipsoidal correction is in the order of 1-2 cm. The geoid undulation predicted using LSC technique should also be corrected for indirect effect discussed in Section 3.2.2 (eq. (3.58)).

The practical problem in using this procedure (mainly with available software) may be the inversion of the matrix $\bar{C}$ in equation (3.75) as well as the equation (3.76). The size of the matrix to be inverted will be (nxn) where n is the number of observations. Procedures to circumvent the problem of inverting the matrix have also been developed for predicting using the LSC technique. However, depending on the limitations of the computers used, in terms of CPU time and memory, the number of observations used for predicting will be curtailed.

3.3.3 Fast Fourier Transform (FFT) Technique

The conversion of gravity anomalies, defined in a regional vertical datum to geoid heights can be done using FFT method also, after removing the long wavelength part of the gravity field using an accurate geopotential model, complete to degree and order 360. This method takes advantage of the possibility of expressing the planar approximation of Stokes' integral in a two dimensional convolution form. According to the convolution theorem (Bracewell, 1978), the Fourier transform of a convolution is the product of the spectra of the two functions being convolved. Detailed discussions on expressing the Stokes' integral in a planar form and its transformation to spectral form are given in Sections 2.3 and 2.5 of Zhao (1989). The planar approximation underlying this approach is permissible due to the use of a high-order spherical harmonic reference field, yielding near-negligible indirect effects (Forsberg, 1990). However, it is also possible to apply FFT directly to the spherical Stokes' formula. This gives more accurate results and therefore the extension of the area of integration can be enlarged (Strang van Hees, 1990). A review of FFT methodology may be found in Schwarz et al. (1990) and the different sources of error in the use of FFT methods for conversion of gravity anomalies to geoid heights in Farelly (1991).

Though the high computation speed of FFT makes this technique a very efficient one in various scientific fields, computation of height anomaly/undulation at one station (not falling at any grid corner of given gravity anomalies) does not give any specific advantage for this method. Accuracy of prediction is almost the same as modified Stokes' technique or even less due to interpolation of the final geoid undulation to the station from grid corners, where N are predicted.

Most of the regional geoid height models such as GEOID93 (Milbert, 1993), GSD91 (Mainville and Véronneau, 1990), and NKG-89 (Forsberg, 1990) makes use of FFT technique for their development. The required values for ζ_i^i and N_i^i for equations (3.42) and (3.64) can also be interpolated from the regional geoid height models where it is available.

3.4 Adjustment Process

The discussions in this section follow the procedure suggested by Rapp and Balasubramania (1992).

From equations (3.41) and (3.63), we get the observation equations at each of the space geodetic stations 'k' where k = 1, 2...z and 'z' is the total number of stations used in the global vertical datum definition:

$$Y_k = \frac{-\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma} + \frac{2}{\gamma}\sum_{j=1}^{n} C_{Q_j0}\left\{\frac{1}{4\pi}\iint_{\sigma_j} S(\psi)d\sigma_j\right\} \tag{3.77}$$

we now define the integral:

$$J(\psi) = \frac{1}{4\pi}\iint_{\sigma_j} S(\psi)d\sigma_j \tag{3.78}$$

Considering terrestrial gravity anomalies only around a small cap ($\psi_C = 2°$) we write:

$$J(\psi_C) = \frac{1}{2}\int_0^{\psi_C} S(\psi)\sin\psi d\psi \tag{3.79}$$

Equation (3.79) is also equation (2-225) in Heiskanen and Moritz (1967). The integration is given by Lambert and Darling (1936, p. 103). We have:

$$J(\psi_C) = \frac{1}{2}\left\{1 + 4\sin\frac{\psi_c}{2} - \cos\psi_c - 6\sin^3\frac{\psi_c}{2} - \frac{7}{4}\sin^2\psi_c - \frac{3}{2}\sin^2\psi_c \log_e\left(\sin\frac{\psi_c}{2} + \sin^2\frac{\psi_c}{2}\right)\right\} \tag{3.80}$$

Now assuming that the anomalies in the cap refer to only one vertical datum as discussed in Section 3.3.1, in which the station lies, then for j = i equation (3.77) reduces to:

$$Y_k = \frac{-\Delta W_0}{\gamma} + \frac{C_{Q_i0}}{\gamma}(1 + 2J(\psi_c)) \tag{3.81}$$

If gravity anomalies in other vertical datums are involved, additional terms are needed in equation (3.81).

The general form of equation (3.81) can be written as:

$$Y = AX + e \tag{3.82}$$

Where Y is the misclosure vector from equations (3.41) or (3.63), A is the coefficient matrix, X are the parameters of the model (ΔW_0 and C_{Q_i0}), and e the observation error vector. The coefficient of ΔW_0 will be $-1/\gamma$ and the coefficient of C_{Q_i0} will be $(1+2J(\psi))/\gamma$, where we assume the same cap size for all stations.

In our global vertical datum problem, there will be z observation equations and I+1 parameters where I is the number of vertical datums considered. In this system there will be a rank defect of 1 because nothing has been set to define a fixed datum or surface. This defect can be removed by introducing a constraint. One such procedure as discussed by Rapp and Balasubramania (1992) is to carry out a free network adjustment as is done for some types of horizontal networks or to introduce a physically meaningful constraint on the system. We define the constraint equation such that:

$$\sum_{i=1}^{I}\left(W_0 - W_0^i\right)^2 = \sum_{i=1}^{I}\left(C_{Q_i0}\right)^2 = \text{a minimum} \tag{3.83}$$

To implement equation (3.83) we introduce a datum constraint such that

$$KX = 0 \qquad \text{with } K:=[0, 1, 1.....1] \tag{3.84}$$

where K is the 1xm coefficient matrix of the constraints with rank K = 1, and

$$\text{rank}\begin{bmatrix} A \\ K \end{bmatrix} = m = I + 1.$$

Defining the Lagrange target function as:

$$\phi = (Y - AX)^T \Sigma_y^{-1}(Y - AX) + 2\lambda(KX) \tag{3.85}$$

a least squares solution based on equation (3.85) leads to the following system of equations:

$$\begin{bmatrix} A^T\Sigma_y^{-1}A & K^T \\ K & 0 \end{bmatrix}\begin{bmatrix} \hat{X} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} A^T\Sigma_y^{-1}Y \\ 0 \end{bmatrix} \tag{3.86}$$

The solution of the above equation yields the following estimated parameter vector:

$$\hat{X} = \left(A^T\Sigma_y^{-1}A + K^TK\right)^{-1} A^T\Sigma_y^{-1}Y \tag{3.87}$$

The error covariance matrix of the estimated parameters would be:

$$\Sigma_{\hat{x}} = \left(A^T\Sigma_y^{-1}A + K^TK\right)^{-1} A^T\Sigma_y^{-1}A\left(A^T\Sigma_y^{-1}A + K^TK\right)^{-1} \tag{3.88}$$

The elements of K are in principle arbitrary. The simplest selection of the elements would be to set all but one element to 1 as shown in equation (3.84). This would imply that all vertical datums would contribute equally to the determination of the reference surface of the global vertical datum. An alternative solution would be one where some vertical datums would be given greater weight in defining the global vertical than others. Proper solution of the actual numerical values for the K elements depends on various factors such as the number and accuracy of space geodetic stations considered in each datum, number, type and accuracy of tide gauges considered in defining the regional vertical datum etc. Also we note that W_0 does not directly enter the constraint defined by equation (3.84). Therefore, the K value corresponding to ΔW_0 parameter must be chosen as zero.

Equations (3.87) and (3.88) are the key equations in estimating the adjusted parameters defining the global vertical datum and their accuracy estimates. Actual estimation of parameters and their accuracy are carried out in the next chapter based on available data in various regional vertical datums.

CHAPTER 4

REALIZATION OF THE FIRST ITERATION GLOBAL VERTICAL DATUM

Based on the modeling procedures discussed in Chapter 3, a first iteration test computation to estimate the parameters defining the Global Vertical Datum was carried out with available data from six regional vertical datums. The different regional vertical datums considered for realizing the Global Vertical Datum (GVD) are: North American Vertical Datum 88 (NAVD88), Ordnance Datum Newlyn (England), IGN 69 (France), NN (Germany), AHD71 (Australia), and Scandinavian Datum. Data used in the numerical investigations, corrections applied to them, different techniques used in evaluating certain gravimetric quantities and the results obtained are discussed in the following sections.

4.1 Data Used in Different Regional Vertical Datums

The ITRF91 coordinate system was selected as the consistent coordinate system for defining the station coordinates of the space geodetic stations such as VLBI, SLR, GPS, etc., used in the realization of GVD. Station coordinates in the ITRF91 (for epoch 1988.0) for all space stations, except for one, used in this study were obtained from Table T6 in IERS Technical Note 12 (Boucher et al., 1992(a), pp. 81-86). An ideal ellipsoid with the following parameters was adopted as the reference ellipsoid for this study:

$$\begin{aligned} a &= 6378136.3 \text{ m} \\ f &= 1/298.257222101 \\ GM &= 3986005 \times 10^{8} \text{m}^3\text{s}^{-2} \\ \omega &= 7292115 \times 10^{-11} \text{ rad s}^{-1} \end{aligned} \qquad (4.1)$$

The last three parameters are the same as the parameters defining the GRS80 ellipsoid. The list of stations used, their ellipsoidal coordinates, ellipsoidal heights and heights in local vertical datum are given in Table 4.1.

4.1.1 North American Vertical Datum 88 (NAVD88)

This vertical reference datum has been recently established (Zilkoski et al., 1992) based on a minimum-constraint adjustment of Canadian-Mexican-US leveling observations, holding fixed the height of the primary tidal bench mark, referenced to the local mean sea level height value at Father Point/Rimouski, Quebec, Canada. NAVD88 values are given in Helmert orthometric height units. This height system will eventually replace the National Geodetic Vertical Datum 29 (NGVD29) height system used in the United States during the next five to seven years. The NGVD29 height system was established by constraining the local mean sea level at 26 tide gauge stations (21 in the US and 5 in Canada) held fixed at 0.0 elevation. The list of distortions present in the NGVD29 height system are given in Table 1 of Zilkoski et al. (1985, p. 23).

A total of nine space geodetic stations (seven SLR stations and two VLBI stations) falling in this regional vertical datum are considered in this study. Location of these space geodetic stations are shown in Figure 4.1. The precise heights of these stations as listed in the NASA Space Geodesy Program Catalogue of Site Information (NASA Tech. memo 4482, 1993) refer to the NGVD29 height system. The VERTCON software (Version 1.00, 1992) received from Vertical Network Branch, NGS which computes the modeled difference in the orthometric height between the NAVD88 and the NGVD29 for a given location specified by latitude and longitude, was used to convert the NGVD29 normal orthometric heights to NAVD88 Helmert orthometric heights. The amount of correction to be applied at each of the geodetic stations is shown in Table 4.7. The 3'x3' gravity anomaly grid for the conterminous US, used as input data set for the development of GEOID90 Geoid Height Model by NGS was also used in this study. These gravity values are atmospherically corrected Helmert anomalies based on the International Gravity Standardization Net 1971 (IGSN71). Detailed documentation on this gravity data set can be found in Milbert (1991a). As quoted by Milbert (1991b) these 3'x3' mean anomalies have a standard deviation of ±1.5 mgal.

4.1.2 Ordnance Datum Newlyn, ODN (England)

ODN has been established based on the tidal observations taken between 1st May, 1915 to 30th April 1921 at the tide gauge on the South pier of Newlyn, Cornwall, England (Wilson, J.I., personal communication, March 1993). Heights are referred to the Tide Gauge Bench Mark which is defined to be 4.751 meters above the datum. Heights determined from this datum are based on either the 2nd Geodetic leveling of England and Wales (1920 to 1952) or the 3rd Geodetic leveling of England and Wales (1953 to the present).

The SLR station (CDP No. 7840) at Herstmonceux, England is the lone station falling in this regional datum to be used in this study. The station's orthometric height referring to the SLR axis has been accurately determined in 1991 by first order leveling.

The gravity data used around this station are taken from the same 6'x10' atmospherically corrected gravity data set used by Despotakis (1987) in his dissertation work, and they are not corrected for terrain.

4.1.3 Institut Géographique National 69, IGN69 (France)

The SLR station (CDP No. 7835) of Grasse, France is the only station in IGN69 used in this study. IGN69 is a normal height system using GRS67, with zero at the Marseille tide gauge. The height of the axis of the SLR station used resulted from an adjustment of the 1st order leveling net re-observed during 1961-1969 (C. Boucher, personal communication, April 1993).

The 6'x10' atmospherically corrected (but not corrected for terrain) data set used by Despotakis (1987) was also used in this study.

4.1.4 Normal-Null (NN), (Sea Level Datum), Germany

The heights in the German height system (NN) are referred to the level surface through the reference point of mean sea level. This reference point is situated 37.000m below the standard benchmark which was established via leveling lines from the tide gauge at Amsterdam ("Normaal Amsterdamsch Peil, NAP). Details regarding this height system are discussed in Torge (1991, Section 6.2.3, p. 228).

The SLR station (CDP No. 7834) at Wettzell, Germany is used in this study for realizing the GVD. The gravity data used are from the 6'x10' mean anomaly data set used by Despotakis (1987).

4.1.5 Australian Height Datum 71 (AHD71), Australia

The Australian Height Datum was established on May 5, 1971 by the division of National Mapping, Australia (National Mapping Council, 1979) carrying out a simultaneous adjustment of 97320 km two-way leveling, holding mean sea level for 1966-68 fixed at zero at thirty tide gauge stations around the coast of the Australian continent. The important characteristics of this datum as discussed by Morgan (1992) and Rizos et al. (1991) are listed as follows:

- ADH71 is primarily a homogeneous third order or mapping datum with an estimated internal precision of about ± 8 mm $\sqrt{k}$ where k is the length of level run in kilometers.

- Data making up the AHD71 are not accurate enough for geodetic, geodynamic and oceanographic studies. A mid seventies re-leveling program indicates that substantial errors exist in the original data.

- The difference between the constrained solution AHD71 and a free adjustment suggests a rise of mean sea level up the east coast of about 1.5m, though this result is in disagreement with physical oceanographic measurements (Sturges, 1974).

Two SLR stations, one at Yarragadee (CDP No. 7090) in Western Australia and another at Orroral Valley, Canberra (CDP No. 7943) in Eastern Australia are considered in this study for realizing the GVD. Figure 4.3, shows the locations of these stations. The gravity data used are from the 2'x2' mean anomaly data used by Despotakis (1987). Again these data also are corrected only for atmosphere and not for terrain.

4.1.6 Scandinavian Datum

Though there is no single regional vertical datum known as the Scandinavian datum, the individual datums RH70 (Sweden), N60 (Finland), and NN1954, NNN1957 (Norway) were combined to serve as a single regional datum in this study mainly for computational convenience. The following factors also prompted the decision:

- These three datums are in the same peninsular region and also have been included in the UELN-73 adjustment.

- In the three systems, the elevations can, in principle be reduced to any epoch by using published land uplift rates (Cross et al., 1987, p. 207).

- Most of the ongoing projects of scientific interest in the Nordic area consider these datums together. Fennoscandian Land uplift studies (Ekman and Mäkinen, 1990), the Baltic sea level project with GPS (Pan and Sjöberg, 1993) are some examples for the same.

- We have a precise regional geoid model NKG-89 (Forsberg, 1990) covering all these three datums.

The VLBI site (CDP No. 7601) of Metsahovi, Finland, the VLBI station (CDP No. 7602) at Tromsø, Norway and the GPS station identified as NDG (Sweden) in the European north-south GPS traverse proposed by IAG SSG 3.88 to control and improve the European geoid (Torge et al., 1989) are the three space geodetic stations considered in this study. The ellipsoidal coordinates referring to WGS84 and the normal height referring to UELN-73 for the GPS station in Sweden were obtained from the Appendix to the paper by Torge et al. (1989). The precise heights of the VLBI stations at Metsahovi and Tromsø referring to their local height systems were obtained from NASA Tech Memo 4482. Since the GPS station height is referred to UELN-

73, and in general Swedish heights are approximately equal to UELN heights (within 1-2 cm) (Forsberg, personal communication, May 1993) the other station heights were also converted to Swedish heights using the following conversion:

- Tromsø, Norway local height converted to Swedish height by subtracting 9 cm (Forsberg, personal communication, May 1993).

- Metsahovi, Finland local height to Swedish height by subtracting 8 cm (Pan and Sjöberg, 1993).

The Faye anomaly data in a 5°x5° area around each station in Norway, Sweden and Finland were received from René Forsberg. They were point anomaly data sets selected closest to the center of every 3'x6' cell in the area. Using the GEOGRID program, in collocation mode, and after removing the reference field using OSU91A geopotential model, the anomaly data were predicted at every 2'x2' grid corner. The reference anomaly at the grid corners was then added to the residual anomaly predicted and the final data set was used in the computations.

MAP OF GEODETIC SITES - UNITED STATES

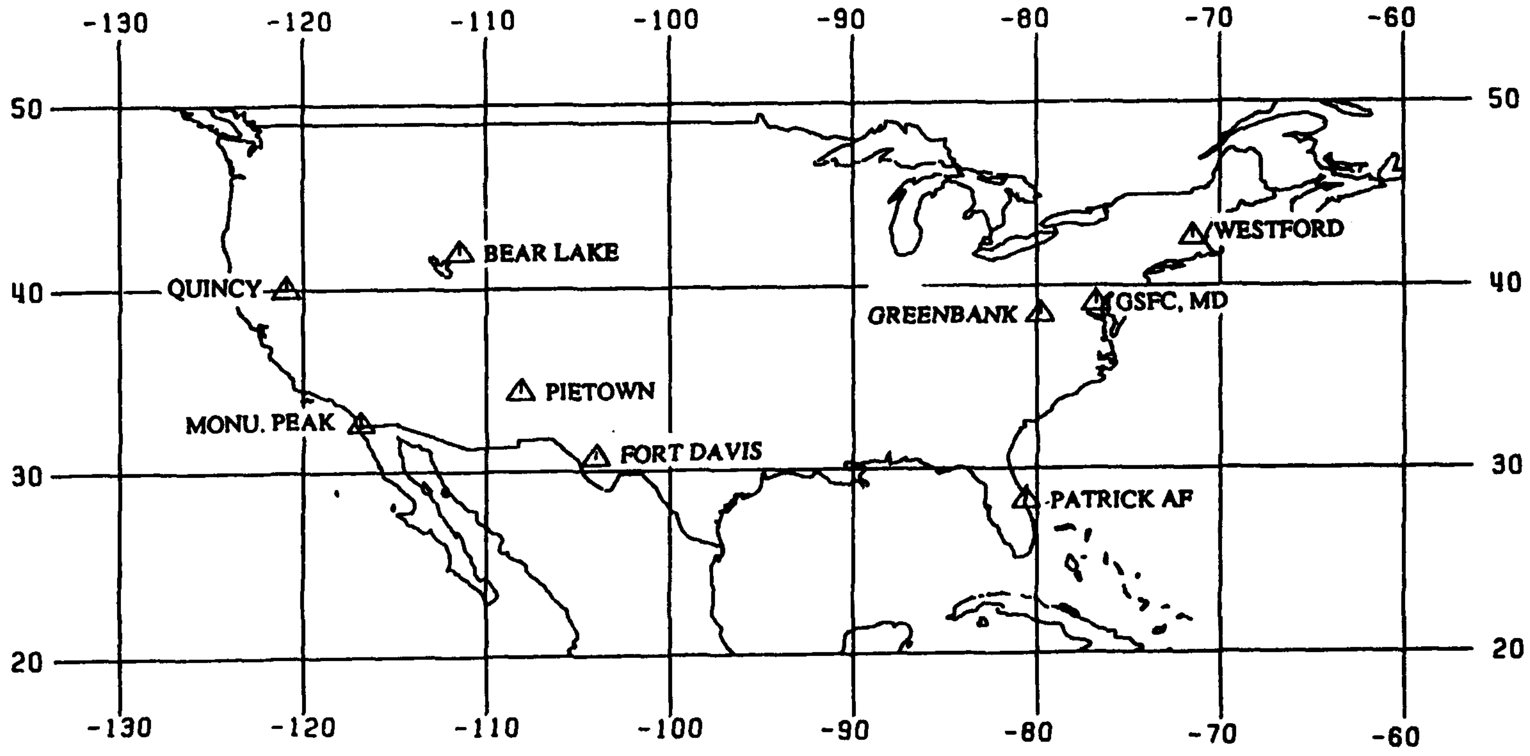


Figure 4.1 Location of Space Geodetic Stations in U.S.A. Used in This Study (NAVD88 Datum)

LOCATION OF SPACE GEODETIC STATIONS - EUROPE

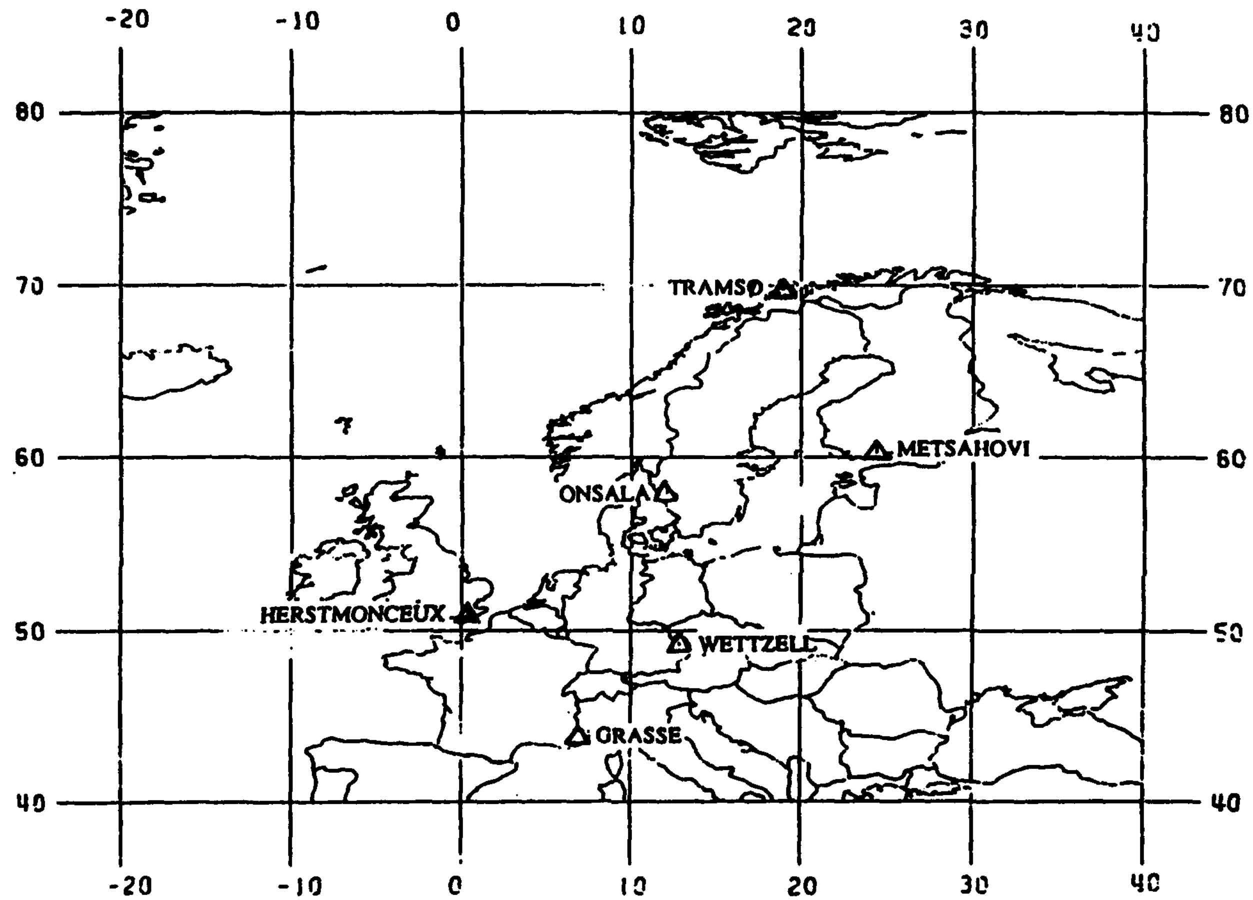


Figure 4.2 Location of Space Geodetic Stations in Europe (covering Scandinavian, ODN, IGN69, NN regional vertical datums) Used in This Study

LOCATION OF SPACE GEODETIC STATIONS
- AUSTRALIA

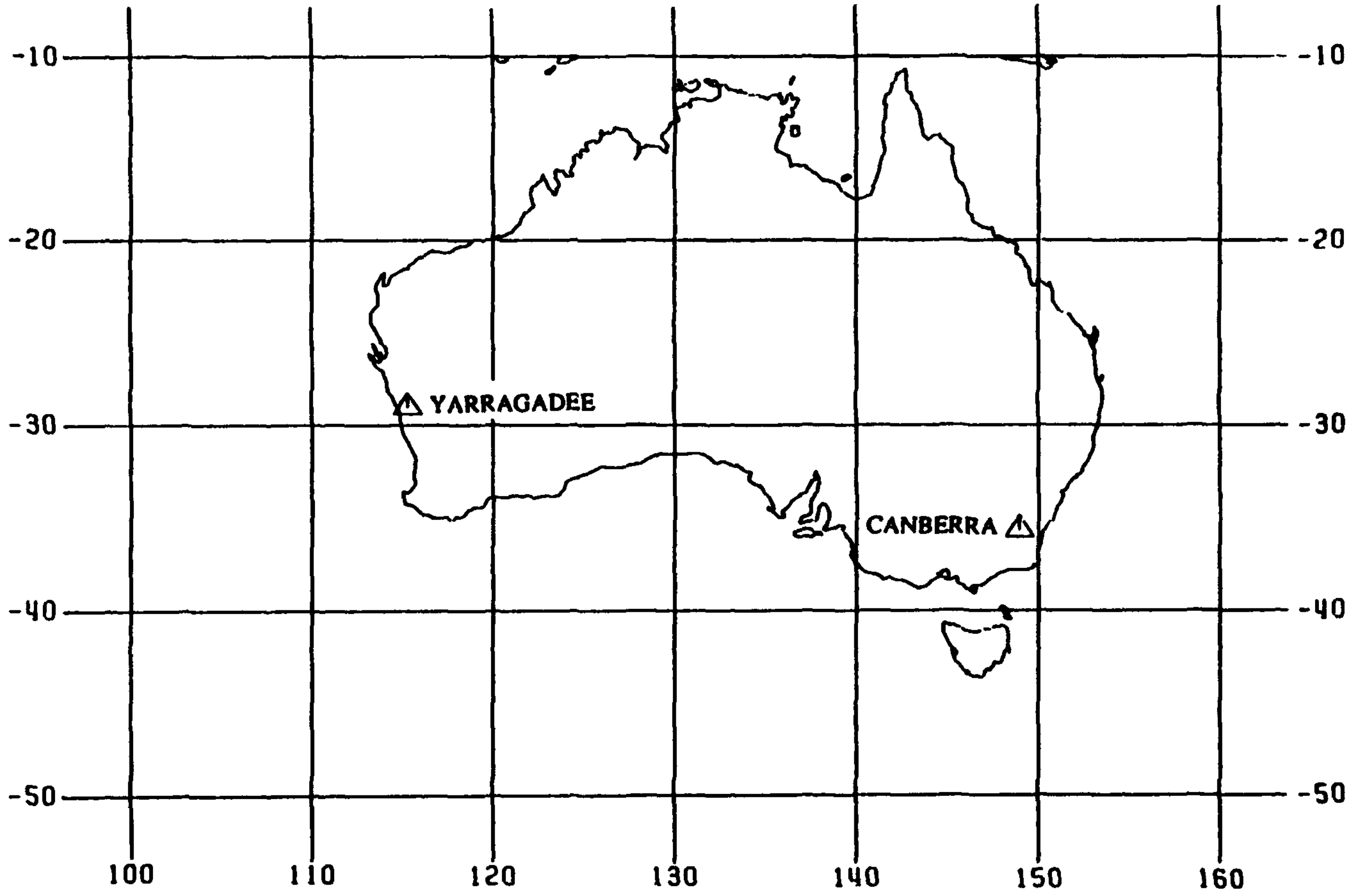


Figure 4.3 Location of Space Geodetic Stations in Australia Used in This Study (AHD71 Datum)

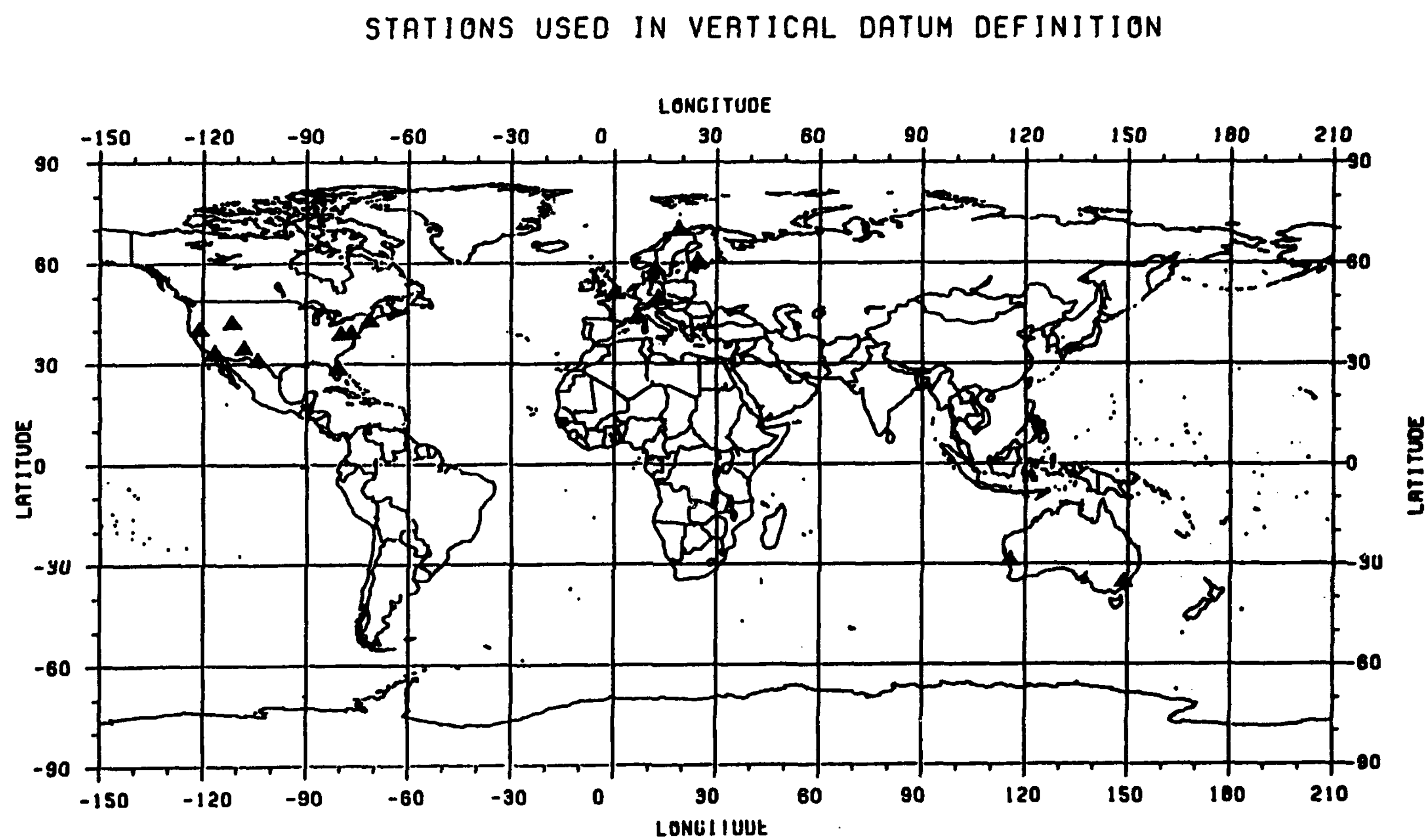


Figure 4.4 Location of Space Geodetic Stations Used in Global Vertical Datum Definition

Table 4.1 Stations Used in Global Vertical Datum Definition Study

Station		Latitude			Longitude			Ellip.ht (a=6378136.3) (m)	Elevation above Local Datum (m)
7051	QUINCY	39	58	24.577	239	3	37.432	1060.686	1083.480
7062	SANDDI	32	36	2.667	243	9	32.660	989.219	1022.000
7082	BEARLA	41	56	0.899	248	34	45.417	1963.700	1976.515
7091	WESTFD	42	37	21.701	288	30	44.233	92.794	120.502
7086	FTDAVS	30	40	37.311	255	59	2.369	1962.178	1983.161
7105	GSFCMD	39	1	14.176	283	10	20.050	19.890	53.050
7204	GREENB	38	26	16.164	280	9	50.944	813.222	844.715
7110	MOPEAK	32	53	30.254	243	34	38.140	1839.702	1870.778
7234	PITOWN	34	18	3.666	251	52	50.665	2365.390	2385.940
7069	PATRIK	28	13	40.669	279	23	39.191	-22.937	6.533
7602	TROMSO	69	39	46.702	18	56	21.964	133.599	101.400
7601	METSAH	60	14	31.069	24	23	2.999	60.269	40.598
1001	ONSGPS	57	53	50.713	12	2	46.382	39.959	2.736
7835	GRASSE	43	45	16.883	6	55	16.034	1323.560	1272.100
7834	WETZLL	49	8	41.775	12	52	41.142	661.836	614.440
7840	RGOUK1	50	52	2.565	0	20	10.040	76.098	30.693
7943	CANBRR	-35	-37	-29.751	148	57	17.296	949.635	929.500
7090	YARRAG	-29	-2	-47.410	115	20	48.272	242.051	266.568

4.2 Systematic Corrections to Gravity Anomalies

The systematic corrections needed to be applied to the terrestrial gravity anomalies were mainly the horizontal datum inconsistency correction (δg_H) and the gravity formula correction (δg_G), as they have been already corrected for atmospheric effects. Since we didn't have dense DTM data for applying the terrain correction to the gravity anomaly data sets that were not corrected for the terrain, they were used as such. Despotakis (1989), based on his undulation computations with data at 12 space stations in the western United States, has shown that the mean and RMS difference of computed undulations decreased by about 50 cm when the terrain corrections were taken into account. The standard deviation decreased about 20 cm when terrain-corrected gravity data was used. The extent of δg_H and δg_G corrections needed/applied to the various gravity data sets are discussed below:

4.2.1 Horizontal Datum Inconsistancy Correction (δg_H)

As mentioned in section 3.1.1(b), this correction is needed because of the systematic error introduced in the computation of normal gravity due to erroneous latitude values. Detailed theory behind such a correction is given in Heck (1990, Section 3.3, pp. 96-101). In Table 4.2, the details regarding the size and shape of the reference ellipsoids associated with the local geodetic datums are given. Also changes in the equatorial radius Δa and in the flattening Δf between the locally adopted reference ellipsoids and WGS84 reference ellipsoid are given. Table 4.3 gives the datum transformation parameters from the various local horizontal datums on which the stations used in this study lie and the ITRF91 reference frame. The transformation is done in two stages. In the first stage, the transformation is done from Local geodetic datum to WGS84 datum using the parameters given in Table 7.13 in DMA Technical Report Part 1 and in the second stage WGS84 to ITRF91 combining the parameters given in Boucher and Altamimi (1991, Table 4, p. 53) and IERS Tech. Note 12 (Boucher et al., 1992a, Table 2, p. 27).

Table 4.2 Local Geodetic System - To - WGS84; Comparison of Ellipsoidal Parameters

Regional Vertical Datum	Local Geodetic Datum and Associated Ref. Ellipsoid	Semi-major Axis a (m)	Inverse Flattening 1/f	Δa (m)	Δf (10^{-4})
North American	NAD83 (GRS80)	6378137.0	298.257	0.0	0.0
England	Ord. Survey of Great Britain (Airy)	6378563.396	299.32	-426.396	0.12
Scandinavian German French	European 1950 (International)	6378388.0	297.0	-251.0	0.141
Australian	Australian Geodetic 1966 (Australian National)	6378160.0	298.25	-23.0	0.0008

Table 4.3 Horizontal Datum Transformation Parameters from Local Geodetic Datum to ITRF91

Regional Vertical Datum	Δx (m)	Δy (m)	Δz (m)	ε "	ψ "	ω "	Δs (10^{-6})
Stage (i): From Local Datum to WGS84							
North American	0.0	0.0	0.0	0.0	0.0	0.0	0.0
England	446	-99	544	-0.945	-0.261	-0.435	20.8927
Scandinavian							
German							
French	-102	-102	-129	0.413	-0.184	0.385	2.4664
Australian	-127	-50	153	0.058	-0.018	-0.089	1.2065
Stage (ii): From WGS84 to ITRF91							
	-0.061	+0.513	+0.208	-0.0179	-0.0001	-0.0061	0.01102

The effect of horizontal datum inconsistencies on the computed gravimetric quantities are given in Table 4.4. Since the latitude values of the stations in the same regional vertical datums vary considerably, the extent of correction also changes from one station to another. The details given in Table 4.4 were computed only for one station in each of the different datums considered. The name of the station at which the computation was carried out is given in parenthesis.

Table 4.4 Errors Introduced due to Horizontal Datum Inconsistencies on Certain Gravimetric Quantities

Regional Datum	Error in Latitude (arc seconds)	Corrections to gravity anomaly (mgals) (δgH)	Corrections to the computed undulation (cm)
England (Herstmonceux)	1.5	0.036	1.2
Scandinavian, German (Wettzell), French	3.2	0.078	1.8
Australian (Yarragadee)	4.5	-0.097	-2.3

From the table above, it is clear that the correction required due to horizontal datum inconsistencies is quite substantial and must be taken into account when ±10 cm accuracy is sought in the definition of the GVD. The computed values in Table 4.4 compare well with the values given in Tables 2a and 4 of Heck (1990). The gravity data used in this study were all corrected for the Horizontal datum inconsistencies (δg_H).

4.2.2 Gravity Formula Correction (δg_G)

The gravity formula used in the computation of gravity anomalies in NAVD88 and in the Scandinavian datums refer to the formula implied by the GRS80 constants, in which the semi-major axis is taken as 6378137.0m. Similarly in other European and Australian stations, as discussed in Despotakis (1987), the gravity formula is computed using a = 6378136.0m. But in our study for the adopted ideal reference ellipsoid we have a = 6378136.3m. Therefore, the gravity anomalies are to be corrected for change in semi-major axis using the procedure explained in Despotakis (ibid., p. 51). The computed δg_G corrections for the gravity data sets in NAVD88 and the Scandinavian datums amounts to -0.216 mgal and for other gravity data sets in ODN, IGN69, NN and AHD71 datums, it is +0.093 mgal.

4.3 Computation of Gravimetric Height Anomaly/Geoid Undulation

Two different techniques, namely, Modified Stokes' technique and Least Squares Collocation (LSC) technique were used in the computation of gravimetric height anomaly/undulation at the space geodetic stations used in the GVD definition study.

4.3.1 Modified Stokes' Technique

As explained in section 3.3.1, this technique makes use of terrestrial gravity anomaly data in a cap around the station and a set of potential coefficients from an accurate geopotential model to compute the height anomaly/undulation. Equations (3.71) and (3.72) are the key equations used for this purpose.

4.3.1.1 Selection of Cap Size

Accuracy of computed height anomaly/undulation depends largely on the extent of the terrestrial gravity data used around the computation point. As shown in Table 8 of Pavlis (1991, p. 119), the grid spacing has only a minor effect on the quality of the result. Computation of undulation/height anomaly using Modified Stokes' technique with (3.71) or (3.72), involves four main error sources in its estimation. They are:

1. Errors associated with the gravity anomalies being given at discrete locations (or as mean values) instead of a continuous function (called sampling error or discretion error and denoted as δN_1).
2. Errors associated with gravity data in the cap surrounding the station (called propagated error and denoted as δN_2).
3. Errors associated with the potential coefficients of the reference field (called commission error and denoted as δN_3).
4. Errors associated with the neglected potential coefficients above degree M (called omission error and denoted as δN_4).

The total global root mean square error assuming uncorrelated error sources is:

$$\delta N = (\delta N_1^2 + \delta N_2^2 + \delta N_3^2 + \delta N_4^2)^{1/2} \tag{4.2}$$

Table 4.5 below shows the estimated accuracy of gravimetric undulation computed using equation (4.2) for different cap sizes of terrestrial gravity anomaly data. Detailed discussions on the different error sources listed above and the required expressions for computing the same are given in Section 3.1 of Pavlis (1991) and in Section 2 of Despotakis (1987). With potential coefficients from the OSU91A model and anomaly error degree variances referring to mean earth radius of 6371 kilometers, the values in Table 4.5 were computed using the reciprocal distance error covariance model, described in Pavlis (ibid., Section 3.4.2), with variance 4 mgal^2 and correlation length $0\overset{\circ}{.}1$. As can be seen from the results, due to the increase in the propagated error with an increase in cap size, beyond certain cap radius use of terrestrial gravity data does not improve but degrades the result. The estimated total error in the computed undulation for cap size $\psi = 2°$ and $\psi = 3°$ are almost the same indicating the use of Δg data in a 2° cap size for computing the undulations should be adequate.

Table 4.5 Estimated Accuracy of Computed Gravimetric Undulations Using Terrestrial Gravity Data in Different Cap Sizes Along with the OSU91A Global Geopotential Model (degree 360). Units are in cm.

Cap size radius (ψ_c)	Sampling error (δN_1)	Propagated error (δN_2)	Commission error (δN_3)	Omission error (δN_4)	Total error (δN)
0°	±0.0	±0.0	±56.0	±16.6	±58.4
1°	±4.0	±7.6	±18.8	±2.6	±20.8
2°	±4.0	±10.7	±14.6	±1.9	±18.6
3°	±4.0	±12.8	±12.2	±1.5	±18.3
4°	±4.0	±14.5	±10.5	±1.2	±18.4

As mentioned in the previous section, the gravity data around the space geodetic stations in the Scandinavian datum were available only in an area of 5°x5° in the latitude and longitudinal directions. Since these stations are located in extreme northern latitudes, the gravity data required in the longitudinal direction is almost double the latitude extent, which can be computed using the following relation:

$$\Delta\lambda = \frac{\Delta\phi}{\cos\phi} \tag{4.3}$$

where ϕ is the latitude of the station, $\Delta\phi$ and $\Delta\lambda$ are the latitude and longitudinal extents. As the data availability in longitudinal direction from the computation point is limited to 2° to 2°.5, terrestrial gravity data to a smaller latitude extent alone can be used. Table 4.6 shows the extent of the cap size radius of the terrestrial gravity data that can be used for estimating height anomaly/undulation with Modified Stokes' technique, with available data.

Table 4.6 Computation of Possible Cap Sizes of Terrestrial Gravity Data That Can Be Used in Scandinavian Datum

	Stations	Latitude ϕ	Gravity data available in longitude direction $\Delta\lambda$	Max. permissible in latitude extent $\Delta\phi = \Delta\lambda\cos\phi$	Selected cap size radius of Δg data
7601	Metsahovi, Finland	60°14'	2°.5	1°.241	1°
1001	Onsala (GPS), Sweden	57°23'	1°.9	1°.045	0°.5
7602	Tromsø, Norway	69°39'	2°.0	0°.869	0°.5

4.3.1.2 Numerical Results

Tables 4.7 and 4.9 show the results of the gravimetric height anomaly/undulation computations carried out at the space geodetic stations in the United States, Europe and Australia. Since ODN (England) values are given as orthometric heights, the gravimetric undulation computed at the Herstmonceux SLR station (CDP No. 7840) alone is shown separately in Table 4.9(b). For other European stations the results are given in Table 4.9(a).

The first column in all the tables refers to the number and name of the stations and the next five columns show the various components of the height anomaly/undulation computed using equations (3.71) and (3.72). The various components shown in column (2) to (6) are: the cap contribution in (2), remote zone contribution computed using potential coefficients in (3), indirect effect or corrections to height anomaly depending on computed quantity N_i^i or ζ_i^i in (4), ellipsoidal corrections in (5) and the total gravimetric undulation/height anomaly in (6).

In Table 4.7, the geometric undulation at the U.S. stations computed with NGVD29 heights, corrections to convert these undulations to NAVD88 height system computed using VERTCON (version 1.00) software and the resulting undulations in NAVD88 height systems are also given in columns (7), (8) and (9) respectively. Similarly, in Table 4.9(a), the geometric height anomaly computed with respect to the local height datum, corrections needed to convert to Swedish height systems in the case of station numbers 1001 and 7602 and the final height anomaly are given in columns (7) to (9). Since the other European stations and Australian stations are assumed to lie in their local height datum, no correction was needed to the computed geometric height anomaly/undulation at these stations.

Table 4.7 Computation of Gravimetric Undulation Using Modified Stokes' Technique and Geometric Undulation at the U.S. Space Geodetic Stations. Units are in meters.

Stations	Gravimetric Undulation (N_g)					Geometric Undulation (N_{GEOM}) (a = 6378136.3 m)		
(1)	N_1 (2)	N_2 (3)	N_3 (4)	N_4 (5)	N_g (6)	$h-H_{29}$ (7)	Corrections (VERTCON) (8)	N_{GEOM} (9)
7051 Quincy	3.423	-26.428	-0.067	-0.010	-23.082	-22.794	-0.94	-23.734
7082 Bearlake	4.581	-17.626	-0.223	+0.002	-13.268	-12.815	-1.28	-14.095
7091 Westford	0.873	-29.445	0.000	-0.017	-28.589	-28.546	0.25	-28.296
7086 Ft. Davis	2.411	-23.954	-0.225	0.010	-21.758	-20.972	-0.67	-21.642
7105 GSFC5	0.263	-33.328	0.000	-0.014	-33.079	-33.160	0.22	-32.940
7204 Green Bank	1.750	-32.955	-0.041	-0.009	-31.255	-31.492	0.05	-31.442
7110 Monu. Peak	0.878	-32.266	-0.200	-0.006	-31.594	-31.076	-0.74	-31.816
7234 Pie Town	3.440	-24.190	-0.325	0.006	-21.069	-20.549	-1.02	-21.569
7069 Patrick AF	0.910	-30.464	0.000	0.008	-29.546	-29.470	0.44	-29.030

Table 4.8 Computation of Gravimetric Undulation and Geometric Undulations at Australian Stations. Units are in meters.

Stations	Gravimetric Undulation (N_g)					Geometric Undulation (N_{GEOM})		
(1)	N_1 (2)	N_2 (3)	N_3 (4)	N_4 (5)	N_g (6)	$h - H_{AHD71}$ (7)	Corrections (8)	N_{GEOM} (9)
7090 Yarragadee	-0.990	-24.310	-0.004	0.000	-25.304	-24.517	---	-24.517
7943 Canberra	7.447	10.584	-0.049	0.014	17.996	20.105	---	20.105

Table 4.9(a) Computation of Gravimetric Height Anomaly and Geometric Height Anomaly at the European Space Geodetic Stations (except at Herstmonceux, England). Units are in meters.

Stations	Gravimetric Height Anomaly (ζ_g)					Geometric Height Anomaly (ζ_{GEOM})		
(1)	ζ_1 (2)	ζ_2 (3)	ζ_3 (4)	ζ_4 (5)	ζ_g (6)	$h - H^{*}_{Local}$ (7)	Corrections (8)	ζ_{GEOM} (9)
7601 Metsahovi ($\psi = 1°$)	-0.698	20.082	0.000	-0.010	19.374	19.671	0.08	19.751
1001 Onsala ($\psi = 0.°5$)	-0.214	36.094	-0.029	0.015	35.866	37.223	0.00	37.223
7602 Tromsø ($\psi = 0.°5$)	-1.241	32.050	-0.039	0.016	30.786	32.199	0.09	32.289
7834 Wettzell	4.533	42.253	-0.045	0.020	46.761	47.396	---	47.396
7835 Grasse	6.106	44.950	-0.127	0.014	50.943	51.460	---	51.460

Table 4.9(b) Computation of gravimetric undulation and geometric undulation at Herstmonceux SLR station (England). Units are in meters.

Station	Gravimetric Undulation (N_g)					Geometric Undulation (N_{GEOM})		
(1)	N_1 (2)	N_2 (3)	N_3 (4)	N_4 (5)	N_g (6)	$h - H_{ODN}$ (7)	Corrections (8)	N_{GEOM} (9)
7840 Herstmonceux	-2.433	47.781	0.000	0.011	45.359	45.405	---	45.405

4.3.2 Least-Squares Collocation Technique (LSC)

4.3.2.1 Covariance Functions

Since the LSC technique is based on the ergodic assumption discussed in Section 3.3.2, the basic covariance functions referred to in equations (3.75) and (3.76) are assumed to be homogeneous and isotropic, depending only on the spherical distance ψ between two points. In this study theoretical covariances were computed based on the noise model of the reference field and on a hypothetical model for the behavior of the anomaly degree variances (C_N) of the Earth's gravity field. The expressions for the fundamental covariance functions used in this study, as given in Bašić and Rapp (1992), are as follows:

$$C(\Delta g, \Delta g) = \sum_{n=2}^{360} \delta C_n s^{n+1} P_n(\cos\psi) + \sum_{361}^{\infty} C_n s^{n+2} P_n(\cos\psi) \tag{4.4}$$

$$C(\Delta g, N) = \frac{R_E}{\gamma}\left[\sum_{n=2}^{360} \frac{\delta C_n}{n-1} s^{n+1} P_n(\cos\psi) + \sum_{361}^{\infty} \frac{C_n}{n-1} s^{n+2} P_n(\cos\psi)\right] \tag{4.5}$$

$$C(N, N) = \frac{R_E^2}{\gamma^2}\left[\sum_{n=2}^{360} \frac{\delta C_n}{(n-1)^2} s^{n+1} P_n(\cos\psi) + \sum_{361}^{\infty} \frac{C_n}{(n-1)^2} s^{n+2} P_n(\cos\psi)\right] \tag{4.6}$$

where δC_n are the error degree variances related to the reference field defined through a set of geopotential coefficients computed as,

$$\delta C_n = \left(\frac{GM}{R_E^2}\right)^2 (n-1)^2 \left(\frac{R_E}{R_B}\right)^{2(n+2)} \sum_{m=0}^{n} \left(\bar{\varepsilon}_{cnm}^2 + \bar{\varepsilon}_{snm}^2\right) \tag{4.7}$$

$\bar{\varepsilon}_{cnm}, \bar{\varepsilon}_{snm}$ are the standard deviations of the potential coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$, R_E is the mean earth radius (6371 km), R_B is the radius of Bjerhamar's sphere and n, m are degree and order of the harmonics,

C_n are the anomaly degree variances computed by the well known Tscherning/Rapp (1974) (T/R) anomaly degree variance model,

$$C_n = \frac{A(n-1)}{(n-2)(n+B)} \tag{4.8}$$

with parameters A = 343.3408 mgal^2, B = 24 and $s = \left(R_B^2 / R_E^2\right) = 0.9988961$ recommended by Jekeli (1978).

The covariance functions computed with the error degree variances of the OSU91A reference field up to degree 360 and beyond in the T/R model with Jekeli parameters and used by Bašić and Rapp (1992) were also used in this study. The behavior of the covariance functions are shown in (ibid., Table 4.1 and Figure 4.4). For the prediction of residual geoid undulation/height anomaly using the local gravity data around the computation point, the global covariance function should be scaled to the local covariance function so as to reflect appropriately the accuracies of predicted quantities.

The scaling procedure described below follows closely the discussion in Hwang (1989, Section 3.3).

For the prediction area, the variance of residual anomalies is computed by,

$$\sigma_{\Delta g}^2 = \sum_{i=1}^{n} \frac{\left(\Delta g_i^{res} - \overline{\Delta g}\right)^2}{n} \tag{4.9}$$

where n is the total number of points used for prediction and $\overline{\Delta g}$ is the mean of the residual anomalies.

Let CGG(0) be the variance of anomalies based on (4.4). We have to apply a scaling factor α to this quantity so that the resultant will be identical to the variance of anomalies given by equation (4.9). Thus α is:

$$\alpha = \frac{\sigma_{\Delta g}^2}{CGG(0)} \tag{4.10}$$

The local covariance functions then can be constructed by multiplying equations (4.4), (4.5) and (4.6) by the scaling factor α.

To ensure a realistic scaling factor, the lowest value of α was taken to be 0.120 and the maximum value limited to 1.5. In this range, it was assumed that the accuracy of predicted undulation will be within acceptable limits.

Based on the above discussions, the equations (3.75) and (3.76) are modified for this study as follows:

$$\hat{s} = C_{st}\left(C_{tt} + \frac{1}{\alpha}C_{nn}\right)^{-1} Y \tag{4.11}$$

$$E_{ss} = \alpha\left[C_{ss} - C_{st}\left(C_{tt} + \frac{1}{\alpha}C_{nn}\right)^{-1} C_{ts}\right] \tag{4.12}$$

Equations (4.11) and (4.12) are similar to the expressions given by Hwang (1989) as equations (3.24) and (3.25).

4.3.2.2 Numerical Results

The residual undulations computed at the space geodetic stations in the United States are shown in Table 4.10. The computations were carried out with different grid sizes and number of points to understand their effects on computed undulations. Residual undulations computed with gravity data in 6'x6' grid size and covering an area of 4°x4° around the computation point were selected for this study. This was done not only for the reason that it covered the same extent of gravity data used by the Modified Stokes' technique to compute the height anomaly/undulation, but of the three cases considered they had the least standard deviations in their estimates. Table 4.11 shows the estimated height anomaly along with the accuracy estimate at the Scandinavian datum stations as well as at the other European space geodetic stations considered in the study. The number of points used in predicting the residual height anomaly/undulation are shown within the parenthesis under each computed value. The undulation computed at the Australian stations are shown in Table 4.12.

Table 4.10 Computation of Residual Undulation with Different Grid Sizes and Total Undulation at the U.S. Space Geodetic Stations Using LSC Technique. Units are in meters.

Station	3'x3' Δg in 2°x2° area	6'x6' Δg in 2°x2° area	6'x6' Δg in 4°x4° area	Ellipsoid correction + Indirect effect	Reference undulation OSU91A	Total undulation N_{LSC}
7051 Quincy	-0.443 ±0.274 (1600)	-0.501 ±0.280 (412)	-0.381 ±0.186 (1600)	-0.077	-22.500	-22.958
7082 Bear Lake	0.221 ±0.276 (1600)	0.221 ±0.273 (432)	0.677 ±0.187 (1600)	-0.221	-14.356	-13.900
7091 Westford	0.152 ±0.080 (1600)	0.187 ±0.077 (450)	-0.084 ±0.062 (1600)	-0.017	-28.634	-28.735
7086 Ft. Davis	0.189 ±0.141 (1512)	0.257 ±0.140 (378)	0.319 ±0.098 (1512)	-0.215	-21.777	-21.673
7105 GSFC 5	0.018 ±0.079 (1600)	0.000 ±0.078 (414)	-0.028 ±0.053 (1600)	-0.014	-32.983	-33.025
7204 Green Bank	0.176 ±0.095 (1600)	0.185 ±0.095 (414)	0.391 ±0.053 (1600)	-0.050	-32.173	-31.832
7110 Monu. Peak	0.529 ±0.261 (1526)	0.610 ±0.274 (378)	0.846 ±0.181 (1548)	-0.206	-32.629	-31.989
7234 Pie Town	0.087 ±0.167 (1600)	-0.099 ±0.158 (396)	0.137 ±0.182 (1548)	-0.319	-21.058	-21.240
7069 Patrick AF	-0.325 ±0.079 (1476)	-0.303 ±0.077 (378)	-0.031 ±0.052 (1476)	0.008	-29.815	-29.838

Table 4.11 Computation of Height Anomaly/Undulation at Scandinavian and Other European Stations Using LSC Technique. Units are in meters.

Station	Residual height anomaly/ undulation 6'x6' Δg in 4°x4° area	Ellipsoidal corrections + corrections to height anomaly	Reference height anomaly/ undulation	Total height anomaly/ undulation. ζ_{LSC}/N_{LSC}
7601 Metsahovi	0.091 ±0.08 (1031)	-0.010	19.497	19.578
1001 Onsala (GPS)	0.171 ±0.079 (1600)	-0.014	35.802	35.959
7602 Tromsø	-0.218 ±0.230 (1600)	-0.023	31.226	30.985
7834 Wettzell	0.294 ±0.10 (1440)	-0.025	46.405	46.674
7835 Grasse	-0.180 ±0.18 (1280)	-0.113	51.383	51.090
7840 Herstmonceux	-0.223 ±0.06 (1480)	0.011	45.583	45.371

Table 4.12 Computation of Gravimetric Undulation at Australian Space Geodetic Stations Using LSC Technique. Units are in meters.

Station	Residual undulation 6'x6' Δg in 4°x4° area	Ellipsoidal correction + Indirect effect	Ref. undulation (OSU91A)	Total undulation N_{LSC}
7090 Yarragadee	-0.072 ±0.06 (1600)	-0.004	-24.833	-24.909
7943 Orraral Valley, Canberra	-0.230 ±0.13 (1600)	-0.035	18.957	18.692

4.3.3 Inter Comparison Between Different Techniques

Tables 4.13 to 4.15 show the comparison between the gravimetric undulation/ height anomaly computed using the Modified Stokes' technique and the LSC technique with the geometric undulations computed at the stations for the ideal reference ellipsoid (a = 6378136.3 m). The undulations interpolated from regional geoid height models GEOID93 (in the United States), NKG-89 (in Scandinavian countries) and AUSGEOID93 in Australia received from Jim Steed (personal communication, Dec. 93) are also included in the tables for better comparison. The values of the observation vector Y in equations (3.41) and (3.63) are also shown for the different techniques with corresponding statistical information such as

$$\text{mean} = \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\text{r.m.s. value} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}} \qquad (4.13)$$

$$\text{standard deviation} = \sqrt{\frac{\sum_{i=1}^{n} \left(x_i - \overline{x}\right)^2}{(n-1)}}$$

Table 4.13 Comparison of Geoid Undulation Computed Using Different Techniques at the U.S. Space Geodetic Stations. Units are in meters.

Stations	Gravimetric Undulation			Geometric Undulation	$Y = h - H^i - N_1^i$		
	LSC N_{LSC}	Stokes N_S	GEOID93 N_{93}	N_{88}	$N_{88} - N_{LSC}$	$N_{88} - N_S$	$N_{88} - N_{93}$
7051 Quincy	-22.958	-23.082	-23.303	-23.734	-0.776	-0.652	-0.431
7082 Bear Lake	-13.900	-13.268	-13.944	-14.095	-0.195	-0.827	-0.151
7091 Westford	-28.735	-28.589	-28.166	-28.296	0.439	0.293	-0.130
7086 Ft. Davis	-21.673	-21.758	-21.465	-21.642	0.031	0.116	-0.177
7105 GSFC 5	-33.025	-33.079	-32.740	-32.940	0.085	0.139	-0.200
7204 Green Bank	-31.832	-31.255	-31.124	-31.442	0.390	-0.187	-0.318
7110 Monu. Peak	-31.989	-31.594	-31.685	-31.816	0.173	-0.222	-0.131
7234 Pie Town	-21.240	-21.069	-21.043	-21.569	-0.329	-0.500	-0.526
7069 Patrick AF	-29.838	-29.546	-29.157	-29.030	0.808	0.516	0.344
				Mean	0.070	-0.147	-0.191
				Std. deviation	0.440	0.426	0.231
				r.m.s. value	0.445	0.450	0.300

Table 4.14(a) Comparison of Height Anomaly Computed Using Different Techniques at the European Space Geodetic Stations (except 7840 Herstmoneux). Units are in meters.

Stations (1)	Gravimetric height anomaly			Geometric Ht. anomaly	$Y = h - H_i^* - \zeta_i^i$	
	LSC ζ_{LSC} (2)	Stokes ζ_S (3)	NKG-89 ζ_{89} (4)	ζ_{GEOM} (5)	$\zeta_{GEOM} - \zeta_{LSC}$ (6)	$\zeta_{GEOM} - \zeta_S$ (7)
7601 Metsahovi	19.578	19.374	18.431	19.751	0.173	0.377
1001 Onsala (GPS)	35.959	35.866	35.359	37.223	1.264	1.357
7602 Tromsø	30.985	30.786	31.484	32.389	1.404	1.603
7834 Wettzell	46.674	46.761	---	47.396	0.722	0.635
7835 Grasse	51.090	50.943	---	51.460	0.370	0.517
				Mean	0.787	0.898
				Std. deviation	0.482	0.489
				r.m.s. value	0.923	1.022

Table 4.14(b) Comparison of Gravimetric Undulations Computed Using Different Techniques at 7840 Herstmonceux, England Station. Units are in meters.

Station	Gravimetric undulation		Geometric undulation N_{GEOM}	$Y = h - H_i - N_i^i$	
	N_{LSC}	N_S		$N_{GEOM} - N_{LSC}$	$N_{GEOM} - N_S$
7840 Herstmonceux	45.371	45.359	45.405	0.034	0.046

Table 4.15 Comparison of Gravimetric Undulation Computed by Different Techiques at the Australian Stations. Units are in meters.

Stations	Gravimetric undulation			Geometric undulation N_{AHD71}	$Y = h - H_i - N_i^i$		
	N_{LSC}	N_S	AUSGEOID93		$N_{ADH71} - N_{LSC}$	$N_{ADH71} - N_S$	$N_{ADH71} - N_{93}$
7090 Yarragadee	-24.909	-25.304	-24.988	-24.517	0.392	0.787	0.466
7943 Orroral Canberra	18.692	17.996	18.644	20.105	1.413	2.109	1.441

4.3.4 Discussion of the Results

The results from Tables 4.13 and 4.14(a) and (b) show that the computation of gravimetric undulation/height anomaly using the two different techniques, i.e., Modified Stokes' and LSC techniques, are consistent at ±20 cm, except for the four stations 7082, 7204, 7086 and 7069 falling in the United States, where the difference between the two techniques varies from 29 cm to 54 cm. In Table 4.15 also, we see the undulation computed at the two Australian stations using the Modified Stokes' and LSC techniques differ by about 45 cm to 65 cm. One of the reasons for such a large difference between the computed gravimetric undulations from the two techniques may be that only a limited number of data points (1600) was used in the LSC technique for predicting the residual undulation due to computer limitations in memory and CPU time. The following Table 4.16 show the gravimetric undulations computed at the SLR station 7943 Orroral Valley, Canberra, Australia using Modified Stokes' technique with terrestrial gravity data in different cap sizes, the undulation value computed using the LSC technique with the data points limited to 1600 and also the geoid undulation value interpolated from AUSGEOID93 model received from Jim Steed (personal communication, Dec. 1993).

Table 4.16 Computation of Gravimetric Undulation Using Modified Stokes' Technique With Terrestrial Gravity Data in Different Cap Sizes. (Units are in meters)

Cap size	Cap contribution N_1	Remote zone contribution N_2	$N_S = N_1 + N_2$ + corrections	Undulation from LSC technique (*)	Undulation from AUSGEOID93 Model
0°.5	2.620	15.820	18.440	18.692	18.644
1°.0	5.019	13.242	18.261		
1°.5	6.744	11.409	18.153		
2°.0	7.447	10.584	17.996		

(*with data points for prediction limited to 1600 at Orroral Valley, Canberra)

From Table 4.16, we see that the undulations computed using the terrestrial gravity data in a cap size of 0°.5, can vary to an extent of about 40 cm, when compared with undulations computed with gravity data in a cap size of 2°.0. The geoidal undulations obtained from the AUSGEOID93 model which uses terrestrial gravity data in a cap size of 0°.5 and OSU91A potential coefficients for its computation (Kearsley et al., 1990) compares well with the undulations computed using the LSC technique with the number of data points used for prediction limited to 1600 and also with the undulation computed using the Modified Stokes' technique with cap size = 0°.5. This suggests that the differences in the computation of undulation between the Modified Stokes' and LSC techniques may be due to the extent of the terrestrial gravity data used in the computation/prediction.

In Table 4.15 we also notice a difference of about 1.0 meter between the computed Y values at the Yarragedee station in Western Australia and Orroral Valley station in eastern Australia. The difference observed is for both the Modified Stokes' and LSC techniques used for computing the gravimetric undulations, as well as for the AUSGEOID93 undulation values received from Jim Steed (personal communication, Dec. 93), though both the stations are assumed to lie in the same height system AHD71. Since as quoted by J. Steed (ibid.), AUSGEOID93 is the most accurate geoid model available in Australia at present and also ellipsoidal heights of the stations are obtained to an accuracy of ±5 cm, the large difference of 1.0 meter in computed Y value ($Y = h - H_{AHD71} - N_g$) between the two stations as observed from all the three techniques, causes serious concern. This can be due to one or more of the following factors:

(a) the long wavelength error in the global geopotential model used,
(b) bias caused by systematic errors in the gravity anomaly data used, (for example the gravity anomaly around the Australian stations are not corrected for terrain), and
(c) distortions in the defined height system.

To verify whether the difference in Y-value is mainly due to long wave length errors in the OSU91A global geopotential model used in the study, the computations of the height anomaly/undulation using Modified Stokes' technique were repeated with the latest JGM-2 (NASA model) from degree 2 to 70 and augmented with OSU91A model from degree 71 to 360 and the results are shown in Appendix 'A' of this report. Details regarding JGM-2 model development can be found in Nerem et al. (1993). From the results, we see the difference in the value of Y computed using the different potential models at the Australian space geodetic stations vary only by 5 to 9 cm, implying that the long wavelength error in the geopotential model used alone is not the contributor to the 1.0 m difference. Also the terrain corrections computed at the Australian stations using ETOPO5U 5'x5' elevation data set (as no other dense elevation data are available in this area), shows the terrain correction is on the order of 0.07 mgals accounting for an undulation correction of about 2 cm. The above discussions point out that the large difference in computed Y-values between the two stations may be mainly due to the distortions in the AHD71 height system. As discussed by Morgan (1992) and referred to in Section 4.1.5, there is reason to believe that substantial errors must exist in the original leveling data used for developing the AHD71 height system.

4.4 Estimation of Parameters Defining the Global Vertical Datum

The estimated values of parameters defining the GVD, that is ΔW_0 and C_{Q_i0} (for i = 1, 2,...n, n being the number of regional vertical datums considered) and their accuracy estimates were computed using equations (3.87) and (3.88) after setting up the observation equations as explained in Chapter III at each of the 17 stations considered in this study. The elements of the A matrix (of size 17x7) used in these equations are shown in Table 4.17, where its first column corresponds to the ΔW_0 term and the rest six columns to the C_{Q_i0} values. The elements of the K vector were chosen to be equal to the number of space geodetic stations used in the corresponding regional vertical datum. As stated in Section 3.4, the value in K corresponding to the ΔW_0 parameter was chosen to be equal to zero.

4.4.1 Setting up the Variance-Covariance Matrix (Σ_y)

The variance-covariance matrix of observations Σ_y was set up assuming that the three observations types: ellipsoidal height (h), orthometric height (H) and height anomaly/undulation (N) are independently determined such that:

$$\Sigma_y = \Sigma_h + \Sigma_H + \Sigma_N \tag{4.4}$$

where Σ_h, Σ_H and Σ_N are the variances of the observed/computed quantities h, H and N respectively.

Determination of orthometric height standard deviation is rather a difficult task. Conceptually, one would take the diagonal elements of the variance-covariance matrix of the vertical network adjustment for a specific datum. Since such information is not available for all the datums considered in this study, based on the general information regarding the regional leveling networks, certain reasonable standard deviations can be assumed. For the NAVD88 datum heights obtained by applying corrections to NGVD29 heights, a standard deviation of ±5 cm has been assumed at all the stations in the United States. Similarly, at the Australian stations, an accuracy of ±10 cm has been assumed, since AHD71 is based on only a third order leveling network (Morgan, 1992). For the European stations, the standard deviations of the normal height has been interpolated from the figures of the adjusted UELN73 leveling networks given in Ehrnsperger et al. (1982).

Determination of ellipsoidal height standard deviation at the space geodetic stations can be easily done, since Table T6 in IERS Technical Note 12 (Boucher et al., 1992a) gives the standard deviations of the rectangular coordinates X, Y, Z also, in addition to providing station coordinates in ITRF91 reference frame. Assuming that the rectangular coordinates are uncorrelated, by error propagation we can compute the error in the corresponding ellipsoidal coordinates.

The determination of geoid undulation accuracy can be done based on the discussion in Chapter 5.4 of Rapp and Balasubramania (1992), when using the Modified Stokes' technique for computing the gravimetric height anomaly/undulation. With OSU91A standard deviations of the potential coefficients, assuming a correlation length of $0\overset{\circ}{.}1$ and using Gauss-Markov first order error covariance model (Pavlis, 1991, p. 107), the estimated standard deviations of the height anomaly/undulation are given in Table 4.18. For the LSC technique the standard deviations of the predicted undulations can be obtained from Tables 4.10 to 4.12. The diagonal elements of the Σ_y matrix for the Modified Stokes' technique and the LSC technique are shown in Tables 4.19 and 4.20, respectively.

Table 4.17 Elements of A Matrix With 17 Stations and 7 Datum Unknowns With Varying Cap Radius. Units are in cm.

-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	1.0978	0.0000	0.0000	0.0000	0.0000	0.0000
-1.0206	0.0000	1.0581	0.0000	0.0000	0.0000	0.0000
-1.0206	0.0000	1.0390	0.0000	0.0000	0.0000	0.0000
-1.0206	0.0000	1.0390	0.0000	0.0000	0.0000	0.0000
-1.0206	0.0000	0.0000	1.0978	0.0000	0.0000	0.0000
-1.0206	0.0000	0.0000	0.0000	1.0978	0.0000	0.0000
-1.0206	0.0000	0.0000	0.0000	0.0000	1.0978	0.0000
-1.0206	0.0000	0.0000	0.0000	0.0000	0.0000	1.0978
-1.0206	0.0000	0.0000	0.0000	0.0000	0.0000	1.0978

Table 4.18 Estimation of Standard Deviation of Geoid Undulation/Height Anomaly at the Space Geodetic Stations. Units are in cm. (for Modified Stokes' Technique)

Station/Datum	Cap size	Accuracy of gravity data	Sampling error	Propagated error	Commission error	Omission error	Total error estimate
All station in U.S. (NAVD88 Datum)	2° (3'x3')	±1.5 mgal	±0.37	±10.44	±14.26	±1.80	±18 cm
7061 Metsahovi, Finland	1° (2'x2')	±2 mgal	±0.16	±5.23	±18.37	±2.52	±19 cm
1001 Onsala (GPS), Sweden	0°.5 (2'x2')	±2 mgal	±0.16	±4.38	±24.99	±3.44	±26 cm
7062 Tromsø, Norway	0°.5 (2'x2')	±2 mgal	±0.16	±4.38	±24.99	±3.44	±26 cm
For stations in England, Germany and France	2° (6'x10')	±7 mgal	±3.82	±20.40	±14.26	±1.80	±25 cm
For stations in Australia	2° (2'x2')	±7 mgal	±0.16	±20.40	±14.26	±1.80	±25 cm

Table 4.19 Computation of Variance-Covariance Matrix of Observations Σ_y (Modified Stokes' Technique)

Station	Standard Deviations (meters)			$\Sigma_y = \sigma_h^2 + \sigma_H^2 + \sigma_N^2$	Weight matrix $P = \Sigma_y^{-1}$
	ellipsoidal ht. σ_h	ortho. ht. σ_H	undulation σ_N		
7051 Quincy	±0.01	±0.05	±0.18	0.035	28.57
7082 Bear Lake	±0.02	±0.05	±0.18	0.035	28.33
7091 Westford	±0.01	±0.05	±0.18	0.035	28.57
7086 Ft. Davis	±0.01	±0.05	±0.18	0.035	28.57
7105 GSFC 5	±0.01	±0.05	±0.18	0.035	28.57
7204 Green Bank	±0.01	±0.05	±0.18	0.035	28.57
7110 Monu. Peak	±0.01	±0.05	±0.18	0.035	28.57
7234 Pie Town	±0.01	±0.05	±0.18	0.035	28.57
7069 Patrick AF	±0.11	±0.05	±0.18	0.047	21.28
7601 Metsahovi	±0.03	±0.05	±0.19	0.040	25.32
1001 Onsala	±0.24	±0.04	±0.26	0.095	10.49
7602 Tromsø	±0.01	±0.05	±0.26	0.070	14.25
7834 Wettzell	±0.01	±0.04	±0.25	0.064	15.58
7835 Grasse	±0.01	±0.04	±0.25	0.064	15.58
7840 Herstmonceux	±0.01	±0.06	±0.25	0.070	15.11
7090 Yarragadee	±0.01	±0.10	±0.25	0.073	13.77
7943 Orroral Valley, Canberra	±0.01	±0.10	±0.25	0.073	13.77

Table 4.20 Computation of Variance-Covariance Matrix of Observations Σ_y (Least-Squares Collocation Technique)

Station/Datum	Standard Deviations (meters)			$\Sigma_y = \sigma_h^2 + \sigma_H^2 + \sigma_N^2$	Weight matrix $P = \Sigma_y^{-1}$
	ellipsoidal ht. σ_h	ortho. ht. σ_H	undulation σ_N		
7051 Quincy	±0.01	±0.05	±0.19	0.387	25.84
7082 Bear Lake	±0.02	±0.05	±0.19	0.039	25.64
7091 Westford	±0.01	±0.05	±0.06	0.006	161.29
7086 Ft. Davis	±0.01	±0.05	±0.10	0.013	79.37
7105 GSFC 5	±0.01	±0.05	±0.05	0.005	196.08
7204 Green Bank	±0.01	±0.05	±0.05	0.005	196.08
7110 Monu. Peak	±0.01	±0.05	±0.18	0.035	28.57
7234 Pie Town	±0.01	±0.05	±0.18	0.035	28.57
7069 Patrick AF	±0.11	±0.05	±0.05	0.017	58.48
7601 Metsahovi	±0.03	±0.05	±0.08	0.010	102.05
1001 Onsala	±0.24	±0.04	±0.07	0.064	15.60
7602 Tromsø	±0.01	±0.05	±0.24	0.060	16.61
7834 Wettzell	±0.01	±0.04	±0.10	0.012	85.47
7835 Grasse	±0.01	±0.04	±0.18	0.034	29.33
7840 Herstmonceux	±0.01	±0.06	±0.06	0.007	136.99
7090 Yarragadee	±0.01	±0.10	±0.06	0.014	72.99
7943 Orroral Valley, Canberra	±0.01	±0.10	±0.13	0.027	37.04

4.4.2 Estimation of Parameters Using Different Techniques

The parameters defining the GVD estimated separately for Modified Stokes' technique and LSC technique are shown in Figures 4.5 to 4.7. The estimated parameters are the vertical separation between the defined Global Vertical Datum (GVD) and the adopted reference ellipsoid, $\Delta W_0/\gamma$, and the vertical separation between the GVD and regional vertical datum, C_{Q_i0}/γ, for i = 1, 2,...n where n is the number of regional vertical datums considered in the study.

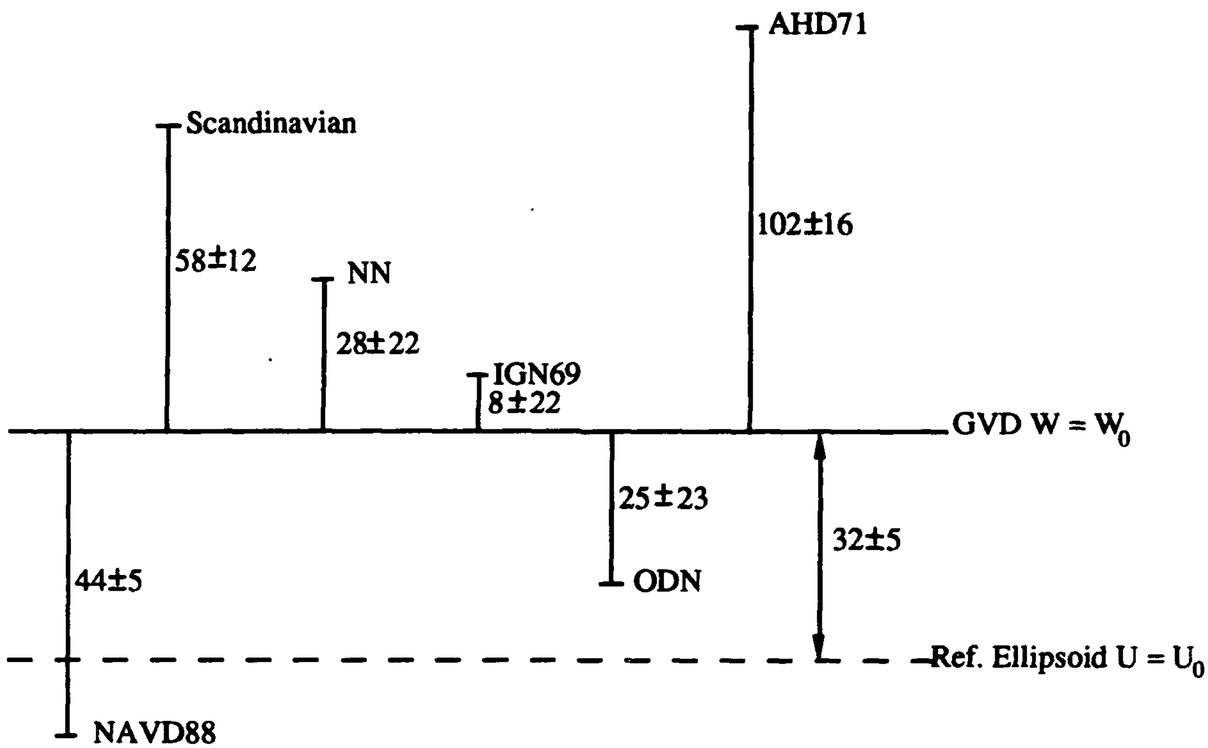


Figure 4.5 Regional Vertical Datum Separation From Defined Global Vertical Datum (Using Modified Stokes' Technique to Compute N or ζ); Units are in cm

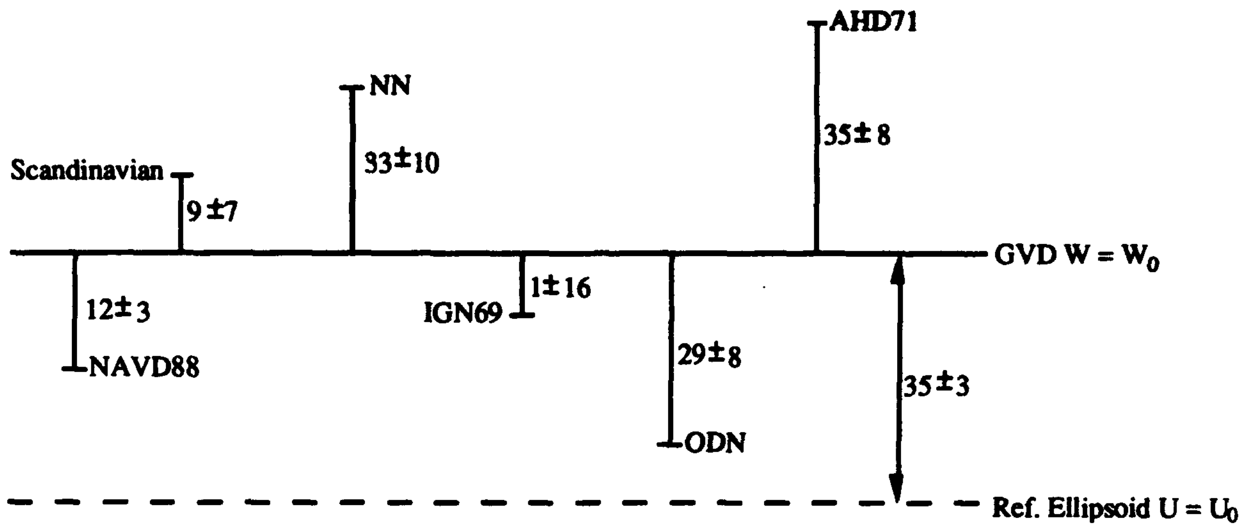


Figure 4.6 Regional Vertical Datum separation from defined Global Vertical Datum (using LSC technique for computation of N or ζ); units are in cm

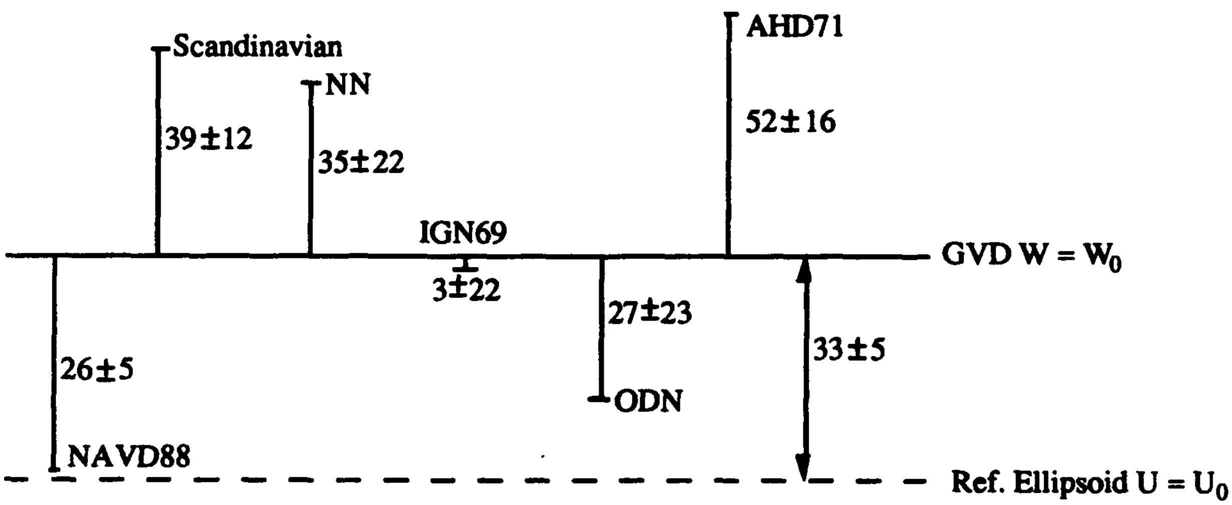


Figure 4.7 Regional Vertical Datum Separation From Defined Global Vertical Datum (From LSC Technique but Using the Weight Matrix of Modified Stokes' Technique); Units are in cm

4.4.3 Discussion of the Results

Figures 4.5 and 4.6 show the estimated values of the separation between the regional vertical datums considered in the study, as well as the adopted reference ellipsoid from the defined Global Vertical Datum (GVD) surface computed using Modified Stokes' and LSC techniques. Though from these figures it is clear that the different regional vertical datums used in the GVD definition, show the same pattern of their relative positions (except Scandinavian datum), their numerical values of separation are quite different. One of the reasons for their numerical difference is the used weight matrix, Σ_y^{-1}, for estimating the parameters. As can be derived from Tables 4.19 and 4.20, the estimated errors in the predicted height anomaly/undulation using LSC technique is unbelievably over-optimistic at certain stations, leading to a larger weight matrix than the one used for Modified Stokes' technique. Even though the computed Y values using Stokes' and LSC techniques agree within ±20 cm for most stations, say particularly in Europe, the standard deviations of the computed height anomaly/undulation using Stokes' technique is 2 to 4 times larger than for the predicted undulation using LSC technique. This is mainly because, the ergodic assumption on which the LSC technique is based, does not differentiate between probabilistic mean and spatial mean (which are very different when real world data is considered) and hence generally result in optimistic accuracy estimation.

Since the two different techniques are mainly employed to compute the gravimetric height anomaly/undulation and the results compare very well also (at least at U.S. and European stations) it should be possible to use the same variance-covariance matrix for both the cases. The main difference in setting up the weight matrix for both the cases, as can be inferred from Tables 4.19 and 4.20, comes from the large standard deviations in predicted/computed undulation, as we have assumed the same standard deviations in orthometric heights and ellipsoidal heights for both techniques.

Assuming the weight matrix computed for Modified Stokes' is acceptable to LSC technique also, the parameters defining the GVD were estimated again using the LSC technique with the new weight matrix. The results are shown in Figure 4.7.

Figures 4.5 and 4.7 compare well except for certain numerical differences. Even the Scandinavian datum which came out to be below the German datum in Figure 4.6, is placed above in Figure 4.7 agreeing with the results in Figure 4.5. The Scandinavian datum which is 58 cm above the defined GVD in Figure 4.5 is about 39 cm above the same in Figure 4.7. The Australian height datum AHD71 which is about 102 cm above the GVD in Figure 4.5 is only 52 cm in Figure 4.7. The reason as can be seen from Table 4.15 is that the gravimetric undulations computed at the Australian stations using Modified Stokes' technique and LSC technique differ by about 50 to 70 cm. The probable reasons for such a difference are discussed in Section 4.3.4. The datum differences between the Scandinavian datum (assumed to coincide with Swedish Height System) and German Height System, varies from 4 cm ± 12 cm in Figure 4.7 to 30 cm ± 12 cm in Figure 4.5. The estimates of difference between the same datums as given by Pan and Sjöberg (1993) is about 16 cm ± 9 cm. The difference of 33 cm between the IGN69 and ODN height systems agrees well with the estimate of 30 cm ± 8 cm given by Willis et al. (1989). Similarly, the difference of 38 cm ±22 cm in Figure 4.7 and 20 cm ± 22 cm in Figure 4.5 between the German height datum (NN) and French height system (IGN69) compares well with the estimate of 50 cm given by Boucher (1993, personal communication) based on the comparison of heights at stations along the French-German border. Though there is a conflict in the Australian Height datum (AHD71) which is shown as 72 cm below the German height system (NN) in Rapp (1993) and it is 74 cm above the German height datum in Figure 4.5, the difference between the German height datum (NN) and the NAVD88 datum shown in Figure 4.5 agrees well with the estimates given by Rapp (1993) to ±4 cm.

Also the difference of 32 cm (or 33 cm, depending on the solution) between the defined Global Vertical Datum surface and the adopted reference ellipsoid suggests that the exact value of the semi-major axis 'a' should be about 6378136.62 m or 6378136.63 m which agrees well with the recent estimates of 'a' from TOPEX/Poseidon data (Rapp, 1993, personal communication).

CHAPTER 5

GLOBAL VERTICAL DATUM DEFINITION - A SIMPLIFIED APPROACH

In Chapter 3 we have considered an ideal solution for defining a Global Vertical Datum (GVD), assuming that the required data are available everywhere and also to required precision to achieve an accuracy of ±10 cm in its definition. In fact reliable and accurate gravimetric data are available only in small portions of the world like in the United States, western Europe, etc. and may require substantial effort and coordination in the part of an International organization to come up with implementation of GVD that can meet the scientific and practical needs of the geodetic community. A simplified approach in realizing a GVD has been discussed in Rapp and Balasubramania (1992,Sections 3 and 7). As pointed out in the report (ibid.) the realized Vertical Datum will not have the sophistication of the ideal GVD but could meet the various practical needs when the highest accuracy is not required. Theory behind such a simplified approach, development of Height Bias models, and certain numerical tests are presented in this chapter and the discussions closely follow the concepts explained in Rapp and Balasubramania (ibid.).

5.1 Basic Concepts

This approach is mainly dependent on the widely available space geodetic techniques such as Global Positioning System (GPS) and DORIS tracking networks which are becoming, and promise to remain for sometime, some of the most important and readily accessible geodetic measurement systems, in particular GPS. The concept underlying this approach is to let ITRF 92 or any other geocentric reference frame realized by the International Earth Rotation Service (IERS) define a fundamental reference system. Knowing the transformation parameters that will allow the conversion from the individual coordinate system that is associated with the tracking type used, such as GPS, DORIS, etc. to the ITRF system, the rectangular coordinates of the stations will be known in a center-of-mass, properly scaled coordinate system. Now assuming an ideal reference ellipsoid, after defining its equatorial radius and flattening (refer to equation 4.1), the ellipsoidal height h at the stations can be computed using standard techniques. The ellipsoidal heights or height differences are then used in conjunction with the geoid undulation information to compute orthometric heights using equation 2.1. The orthometric heights thus computed refer to a conceptual surface (the geoid) which is not physically realizable and also not associated with any specific mean sea level. The geoid undulation information required at the stations can either be obtained from regional geoid height models like, GEOID93 (in USA), GSD91 (in Canada), NKG-89 (in Scandinavian countries), etc. if available, or can be computed through Stokes' technique/LSC technique by combining potential coefficient information from high degree (360) model with surface gravity data (discussed in Chapter 3). In areas where either of the above two methods is not feasible, the global geopotential models available today that are complete to degree 360 can be used. One such model giving moderately good results is the OSU91A model described by Rapp, Wang, Pavlis (1991). The use and accuracy of the OSU91A model for geoid determination are described by Rapp (1992) and Rapp and Balasubramania (1992).

The limitations of this method relate to the precision in which the ellipsoidal heights or height differences can be determined in geocentric reference frames with GPS and DORIS tracking networks and also on the accuracy of available geoidal information. Widely available GPS observations (height determination specifically), precise undulation values and a Height Bias model are essential for implementing this approach.

5.2 Development of a Height Bias Model

Since the regional vertical datums that are realized and used today do not have the ideal geoid as reference surface, the orthometric heights computed on a regional vertical datum will be different from the orthometric height that is computed with respect to an ideal geoid. The difference may simply be a constant or it may be a difference driven by a number of factors including distortions in a local vertical network (Rapp, 1992). Let H_I be the orthometric height computed using the relation,

$$H_I = h - N \tag{5.1}$$

where h is the ellipsoidal height of the station measured along the ellipsoidal normal passing through the station and N is the geoidal undulation which is the separation between the defined ellipsoid and the ideal geoid Let H_D be the orthometric height computed in a local vertical datum. Assume a linear relation between H_D and H_I as follows:

$$H_D = H_I + c(\phi,\lambda) \tag{5.2}$$

where $c(\phi,\lambda)$ can be considered as a correction term or height bias term, dependent on position in the datum to correct orthometric heights referred to an ideal reference surface to the regional vertical reference datum. If sufficient points are available, the value of 'c' can be mapped and represented in some functional form depending on the size of the area and the behavior of 'c'. Once this bias function is determined for a region, the ideal orthometric height computed based on discussion in the previous section can be converted into the orthometric height in the local datum.

Development of a Height Bias model or setting procedures for mapping the Bias function 'c' in some functional form requires GPS observations (height determination specifically), precise undulation data and consistent orthometric heights in a local vertical datum. Currently such information is available in the United States, Canada and in some parts of Europe. On our request, the Geodetic Survey of Canada (A. Mainville, personal communication, April 1992) provided the GPS data for four traverses: South Alberta (106 stations), North Alberta (51 stations), Central Alberta (52 stations), and the Great Slave Lake Area (91 stations). In addition, the orthometric heights for all the GPS stations in the Canadian Geodetic Vertical Datum-28 (CGVD-28) and the high resolution geoid model GSD91 were also obtained. For the United States, the National Geodetic Survey (D.Milbert, personal communication, Oct. 1992) provided the data for GPS traverses in Florida (52 stations), Virginia G105 (62 stations), and in Oregon (44 stations). They also supplied the orthometric heights of all the traverse stations in National Geodetic Vertical Datum 29 (NGVD-29) and the high resolution Geoid Height Models GEOID90 and the latest GEOID93. GPS traverse data for 49 stations in the Tennessee area along with their orthometric heights received from Zeigler, State of Tennessee, Department of Transportation, Nashville, Tennessee were also used for developing Height Bias model in the Tennessee area. The VERTCON (Version 1.00,1992) software (mentioned in Chapter 4) received from Vertical Network Branch, NGS was used to convert NGVD-29 normal orthometric heights to NAVD88 Helmert orthometric heights.

Based on the data received, the following procedure was followed to develop a suitable Height Bias model:

(a) Since the GPS station coordinates in the Canadian GPS traverses as well as Tennessee GPS traverse were given in the WGS84 coordinate system, they were transformed to ITRF90 geocentric coordinate system, using the transformation parameters given in Boucher and

Altamimi (1991), to make the origin of the coordinate system compatible with the geopotential model used. In the case of GPS traverses received from NGS, the station coordinates were already in the ITRF90 geocentric system as reported by Milbert (1991b) and no transformation was required.The geoidal undulations were then interpolated using the corresponding geoid height models.

(b) The orthometric height bias 'c' was then computed using the relation

$$c(\phi,\lambda) = (h_{GPS} - N_{GEOID\ MODEL}) - H_{LOCAL\ DATUM} \tag{5.3}$$

Examples of orthometric height bias tables are given in Table 5.1 for the Oregon area in the USA and in Table 5.2 for the Great Slave Lake Area in Canada. For the U.S traverses, both the currently used NGVD-29 heights as well as the newly established NAVD88 heights were used in developing the Height Bias functions. Statistical analysis of the height bias values for U.S and Canadian traverses are given in Table 5.3.

Table 5.3 Statistics of the Difference in Orthometric Heights Derived From GPS/GSD91 and CGVD28 Heights in Case of Canadian Traverses and From GPS/GEOID93 and NAVD88 Heights for GPS Traverses in the United States. Units are in cm.

Traverse	ORE	FL	VA	TN	NA	SA	CA	GSL
No. of stations	44	52	62	49	51	52	106	91
Min. Ht. Bias value	-101	-129	-31	69	-27	-89	-27	-31
Max. Ht. Bias value	-20	32	66	120	62	-45	84	48
Mean	-64	-55	30	102	-1	-59	23	-3
Standard deviation*	20	46	19	11	11	8	21	16

*derived from spatial distribution

Table 5.1 Computation of Orthometric Height Bias Function (c), in Oregon GPS Traverse Area, Oregon, United States. Heights are in meters.

Lat (deg)	Lon (deg)	Ell.ht (h)	Ortho.ht (H) (NGVD29)	Ortho.ht (H) (NAVD88)	Geoid.ht (N) (GEOID93)	c = (h-N)-H Ht.bias value (NAVD88)	(NGVD29) (units in cm)
45.47	239.26	557.101	577.160	578.180	-20.573	-50.6	51.4
42.22	237.29	533.994	556.686	557.736	-22.988	-75.4	29.6
44.83	235.93	-6.483	16.121	17.121	-22.817	-78.7	21.3
43.98	242.76	665.492	682.364	683.344	-17.630	-22.2	75.8
43.13	238.20	1392.601	1412.875	1414.165	-20.648	-91.6	37.4
43.81	239.40	1394.157	1412.739	1413.949	-19.044	-74.8	46.2
45.52	237.01	36.439	58.039	59.089	-21.098	-74.2	30.8
46.11	236.79	-16.333	3.562	4.542	-20.297	-57.8	40.2
45.30	242.18	845.869	862.066	863.176	-17.049	-25.8	85.2
44.39	236.71	59.025	80.570	81.590	-21.900	-66.5	35.5
45.24	239.82	863.124	882.423	883.513	-19.975	-41.4	67.6
45.00	239.78	784.779	803.820	804.930	-19.616	-53.5	57.5
43.74	241.92	882.872	899.963	901.063	-17.773	-41.3	68.7
42.09	241.43	1255.053	1273.403	1274.553	-19.139	-36.1	78.9
45.86	237.20	1.696	22.843	23.783	-21.548	-53.9	40.1
42.98	242.95	1323.322	1339.038	1340.108	-16.443	-33.8	73.2
45.35	242.75	1246.949	1261.640	1262.940	-15.771	-22.0	108.0
42.32	237.12	400.721	423.829	424.849	-23.335	-79.3	22.7
45.61	238.81	17.243	37.637	38.637	-20.810	-58.4	41.6
43.23	242.74	1971.917	1988.030	1989.070	-16.749	-40.4	63.6
45.13	236.03	12.240	33.989	35.029	-22.019	-77.0	27.0
44.93	236.77	41.651	62.907	63.927	-21.539	-73.7	28.3
46.30	235.92	53.461	76.690	77.720	-23.670	-58.9	44.1
43.12	235.82	3.745	27.942	29.012	-24.483	-78.4	28.6
42.70	239.45	1334.608	1354.065	1355.265	-19.834	-82.3	37.7
44.17	235.88	-15.986	7.050	8.100	-23.219	-86.7	18.3
45.71	238.45	81.190	101.612	102.632	-20.773	-66.9	35.1
44.44	241.30	1099.198	1115.895	1117.105	-17.553	-35.4	85.6
43.70	235.90	-21.098	2.492	3.562	-23.925	-73.5	33.5
45.80	236.54	416.104	435.452	436.522	-19.856	-56.2	50.8
45.47	236.16	-15.598	5.530	6.580	-21.365	-81.3	23.7
43.25	236.64	140.826	163.290	164.350	-22.788	-73.6	32.4
44.43	238.05	1116.756	1136.953	1138.153	-20.494	-90.3	29.7
45.48	236.30	17.992	38.079	39.139	-20.419	-72.8	33.2
44.31	238.44	947.275	967.450	968.600	-20.483	-84.2	30.8
43.59	240.93	1256.144	1273.921	1275.111	-18.452	-51.5	67.5
44.55	239.96	1301.683	1319.711	1320.941	-18.798	-46.0	77.0
44.15	237.43	221.158	242.724	243.854	-21.958	-73.8	39.2
42.19	239.64	1425.684	1445.488	1446.678	-20.060	-93.4	25.6
43.01	239.23	1328.545	1347.868	1349.088	-19.742	-80.1	41.9
43.01	236.30	302.623	325.620	326.680	-23.320	-73.7	32.3
43.64	236.42	21.257	43.451	44.541	-22.538	-74.6	34.4
42.47	235.84	1126.284	1151.520	1152.590	-25.296	-101.0	6.0
42.01	242.28	1343.711	1361.257	1362.397	-18.209	-47.7	66.3

Table 5.2 Computation of Orthometric Height Bias Function in Great Slave Lake Area, Canada. Heights are in meters.

Lat (deg)	Lon (deg)	Ellip.ht (h)	geoid.ht (N) (from GSD91)	(h-N)	Ortho.ht (H) (in CGVD28)	c=(h-N)-H Ht.error (in cm)
62.475	245.559	183.350	-25.905	209.255	209.229	2.5
60.038	243.117	279.969	-19.840	299.809	299.902	-9.2
60.426	243.647	263.951	-21.571	285.522	285.730	-20.7
60.800	243.408	244.827	-21.935	266.762	267.028	-26.5
60.709	244.995	216.481	-24.809	241.290	241.269	2.0
62.479	245.273	155.184	-25.623	180.807	180.706	10.1
62.544	245.010	141.715	-25.083	166.798	166.700	9.8
62.593	244.813	142.511	-24.582	167.093	167.061	3.2
62.660	244.705	152.975	-24.274	177.249	177.221	2.8
62.686	244.535	148.808	-23.980	172.788	172.753	3.4
62.709	244.411	147.611	-23.776	171.387	171.339	4.8
62.784	244.007	140.531	-22.935	163.466	163.455	1.0
62.641	243.743	210.678	-22.284	232.962	232.996	-3.3
62.556	243.591	233.449	-22.033	255.482	255.507	-2.5
62.382	243.503	250.968	-21.915	272.883	272.957	-7.3
62.302	243.571	221.520	-22.063	243.583	243.699	-11.6
62.204	243.683	206.345	-22.334	228.679	228.722	-4.3
62.109	243.704	191.012	-22.422	213.434	213.524	-9.0
62.024	243.687	188.108	-22.451	210.559	210.650	-9.1
61.979	243.563	196.285	-22.232	218.517	218.608	-9.1
61.837	243.329	203.860	-21.826	225.686	225.825	-13.8
61.781	243.228	201.425	-21.666	223.091	223.148	-5.7
61.711	243.103	197.065	-21.465	218.530	218.533	-0.3
61.677	242.975	190.166	-21.261	211.427	211.571	-14.4
61.601	242.859	175.645	-21.128	196.773	196.921	-14.7
61.514	242.754	154.450	-21.025	175.475	175.646	-17.1
61.428	242.601	138.736	-20.823	159.559	159.752	-19.2
61.366	242.499	136.108	-20.732	156.840	157.021	-18.0
61.095	242.500	174.119	-20.585	194.704	194.958	-25.4
60.870	243.269	206.432	-21.749	228.181	228.496	-31.4
60.717	243.528	231.687	-22.030	253.717	253.948	-23.0
60.629	243.659	240.712	-22.095	262.807	262.961	-15.3
60.784	245.342	211.651	-25.497	237.148	237.128	2.0
60.611	245.510	245.756	-25.469	271.225	271.277	-5.2
60.536	245.610	222.953	-25.550	248.503	248.527	-2.4
60.407	245.733	242.665	-25.609	268.274	268.213	6.1
60.228	246.153	247.940	-25.958	273.898	273.778	11.9
60.174	246.295	239.781	-26.070	265.851	265.705	14.6
60.149	246.444	241.794	-26.275	268.069	267.889	18.0
60.120	246.619	232.888	-26.527	259.415	259.224	19.0
60.035	246.872	226.431	-26.810	253.241	253.042	19.9
60.026	247.027	219.796	-27.054	246.850	246.597	25.3
60.014	247.753	161.315	-28.010	189.325	188.848	47.7
60.013	247.955	172.707	-28.241	200.948	200.560	38.7
60.005	248.046	178.882	-28.335	207.217	206.868	34.8
59.999	248.162	183.599	-28.439	212.038	211.711	32.7
62.458	245.413	165.830	-25.813	191.643	191.579	6.3

Table 5.2 (cont.)

62.519	245.101	160.654	-25.285	185.939	185.810	12.9
62.759	244.193	158.507	-23.318	181.825	181.830	-0.4
60.456	245.675	229.981	-25.583	255.564	255.514	4.9
60.283	245.955	245.985	-25.714	271.699	271.624	7.4
61.185	246.301	132.292	-27.566	159.858	159.663	19.4
61.313	242.400	138.238	-20.580	158.818	158.955	-13.6
60.106	243.242	269.264	-20.182	289.446	289.596	-14.9
60.173	243.308	267.289	-20.420	287.709	287.873	-16.3
60.263	243.424	269.261	-20.821	290.082	290.243	-16.0
60.343	243.555	269.673	-21.223	290.896	291.060	-16.3
60.506	243.747	251.176	-21.990	273.166	273.324	-15.8
60.625	243.939	190.459	-22.742	213.201	213.289	-8.7
60.684	244.067	170.230	-23.100	193.330	193.446	-11.6
60.739	244.482	165.580	-23.981	189.561	189.673	-11.1
60.725	244.645	167.320	-24.253	191.573	191.611	-3.8
60.740	245.179	204.241	-25.163	229.404	229.384	2.0
60.804	245.465	206.889	-25.756	232.645	232.681	-3.5
61.039	246.312	135.807	-27.525	163.332	163.270	6.1
60.838	244.220	138.667	-23.710	162.377	162.663	-28.6
60.000	247.395	197.123	-27.534	224.657	224.418	23.8
60.047	247.190	203.371	-27.345	230.716	230.463	25.2
60.046	246.745	242.739	-26.620	269.359	269.141	21.7
60.329	245.851	241.294	-25.623	266.917	266.865	5.2
60.670	245.404	232.053	-25.382	257.435	257.380	5.5
60.698	245.256	230.009	-25.197	255.206	255.157	4.8
60.825	245.629	190.050	-26.062	216.112	216.110	0.2
60.865	245.759	179.217	-26.381	205.598	205.594	0.4
60.928	245.883	165.257	-26.669	191.926	191.923	0.2
60.986	246.180	139.620	-27.235	166.855	166.851	0.4
61.132	246.370	131.929	-27.653	159.582	159.459	12.3
60.965	246.045	137.190	-27.013	164.203	164.180	2.346
60.731	244.844	178.432	-24.599	203.031	203.107	-7.581
60.742	244.260	151.468	-23.571	175.039	175.221	-18.233
60.766	244.148	148.883	-23.434	172.317	172.526	-20.886
60.559	243.860	236.930	-22.382	259.312	259.488	-17.575
60.001	243.018	276.084	-19.586	295.670	295.814	-14.376
60.985	242.753	171.685	-20.924	192.609	192.884	-27.462
60.931	243.082	226.397	-21.475	247.872	248.167	-29.489
60.936	242.933	214.140	-21.195	235.335	235.559	-22.448
61.048	242.599	187.730	-20.697	208.427	208.678	-25.071
61.173	242.469	152.332	-20.634	172.966	173.093	-12.733
61.255	242.473	134.933	-20.709	155.642	155.604	-16.164
61.731	243.139	208.827	-21.516	230.343	230.492	-14.921
60.654	243.634	243.395	-22.101	265.496	265.708	-21.137

Though several procedures exist to map the bias function 'c', the following three are the generally used techniques for interpolating a function value at any point, given the function values at certain known points in that area. A few other techniques are also discussed in Shrestha et al (1993).

(a) Global surface fitting using deterministic polynomial functions : This technique provides a simple way of modelling the bias function value. Different types of polynomial functions can be used to model the Height Bias function 'c', for example - bilinear polynomials, quadratic polynomials, cubic polynomials, etc. Though it is a very simple technique, these polynomial functions provide unpredictable errors in the interpolated values away from the known station points, making it unsuitable for our purpose.

(b) Minimum curvature fitting using spline functions : Cubic spline functions provide a continuous and smooth composite function fitting the known data more accurately without the undue oscillations caused by polynomial functions away from the known stations. Due to the versatility of this technique, it has been implemented in the NADCON software of the NGS (Dewherst, 1990) for the transformation of positional data from the NAD27 to NAD83 coordinate system. The main disadvantage of this technique is that the accuracy of the predicted value of the function cannot be estimated in a simple way. Though the height bias plots made from this method for Canadian GPS traverse areas showed a good agreement with plots made using least squares collocation technique, it was not chosen to model the height bias function for the main reason that the accuracy of the predicted bias function value couldn't be determined in a simpler way as in the case of the LSC technique.

(c) Height Bias modelling using least squares collocation technique : As explained in Chapter 3, this technique, in addition to providing the Best Linear Prediction of height bias values, also gives the accuracy of the predicted value as given in equation (3.76) though sometimes too optimistic. This technique was chosen to develop Height Bias model functions in the test areas where GPS traverse data and geoid undulation data were made available.

5.3 Numerical Results

Assuming a correlation length of 40 km and data noise of ±5 cm, with a second-order Markov model covariance function fitted to the known height bias data (after removing the mean) at the GPS traverse stations, the Height Bias value along with its accuracy was predicted for any other location in the area using least squares collocation technique. Contour plots showing the predicted Height Bias function in the area and its accuracy, one each for U.S traverse (in Oregon) and Canadian traverse (in South Alberta) are shown in Figures 5.1, 5.2 and 5.4, 5.5, respectively, with a contour interval of 5 cm. The perspective view of the Height Bias functions are also shown as 5.3 and 5.6 for those areas. To compare the Height Bias function values with different geoid height models as well as height systems (NAVD88 and NGVD-29) two other plots are also included as Figures 5.7 and 5.8. An orthometric height bias function plot in case of the Scandinavian GPS traverse computed based on normal heights of the stations derived from UELN-73 geopotential numbers is also given as Figure 5.9.

5.4 Discussion of Results

From Figures 5.1 and 5.4, we see the advantage of developing such Height Bias models. By knowing the orthometric height with respect to an ideal geoid using Geoid/GPS observations using equation (5.1), its orthometric height in a local vertical datum can be computed using the interpolated bias function 'c'. Also the plot showing the accuracy of predicted 'c' values serves as a reliability diagram intimating the user with the accuracy of the predicted 'c' value and hence the orthometric height in the local vertical datum. From Figures 5.2 and 5.5, we see the accuracy of predicted height bias function is of the order of ±20 cm if we have well distributed GPS stations in the area. Considering Figure 5.9, one can see the Height Bias function modelled for the

Scandinavian GPS traverse, which runs through four different countries, namely, Germany, Denmark, Sweden, and Norway cannot find its intended use for the following reasons:

- Height Bias function which was computed based on normal heights of the stations derived from UELN-73 geopotential numbers is not compatible with the National height systems used in different countries involved in the GPS traverse,
- The GPS traverse is almost a straight line traverse running from South-West to North-East direction and covering no area as such.

One can conclude that this plot indicates how a GPS traverse should not be used if we want to develop a Height Bias model for an area with available undulation data. Comparing the contour plots in Figures 5.1 and 5.7, the clear signature in the eastern part of Oregon due to the large bias terms in using NGVD-29 heights, can be seen to be minimized when we use NAVD88 heights. But we see the large bias terms in the western part of Oregon exist for both the height systems. The possible reason for this may be the error in leveling data caused by the network design in mountainous areas or may be due to the sea level slope observed along the Pacific Coast. As pointed out by Zilkoski et al (1992), there are long north-south leveling lines through the valleys, and only a few east-west lines that cross the mountains to create loops to help control and check accumulation of errors.

5.5 Summary and Conclusions

1. The definition of a Global Vertical Datum (GVD) using the procedures described in this Chapter eliminate a direct reference to mean sea level. This means that countries that do not have access to the oceans can also use such a GVD if precise geocentric coordinates for one or more points can be determined in the country (Rapp,1983).

2. The accuracy limitation of this approach relates to the precision in which the ellipsoidal heights or height differences can determined in the geocentric reference frame with GPS or any other space geodetic techniques and also on the accuracy of geoidal information available.

3. For getting the undulation information we have assumed that the terrestrial gravity has had a common height system in the calculation of gravity anomalies. This assumption is clearly false and hinders the fundamental goal at the ±10 cm level of accuracy (Rapp and Balasubramania,1992).

4. The simplified approach discussed in this Chapter does provide a means for determining the Global Vertical Datum since the geoid forms the reference surface. Unfortunately, we lack sufficiently precise knowledge of the geoid undulations all over the globe to make the procedure universally applicable.

5. The least squares collocation (LSC) technique described in Chapter 3, provides an accurate procedure for modelling the Height Bias function and also to compute the accuracy of such Height Bias function.

6. The modelled Height Bias function refers only to the particular geoid model and local height system used in its development.

7. Well distributed control points in the local height datum help in developing a meaningful Height Bias function that would enable the orthometric heights computed with respect to the geoid be converted to the specific vertical datum system.

HEIGHT BIAS FUNCTION PLOT

Contour Interval 5 cm

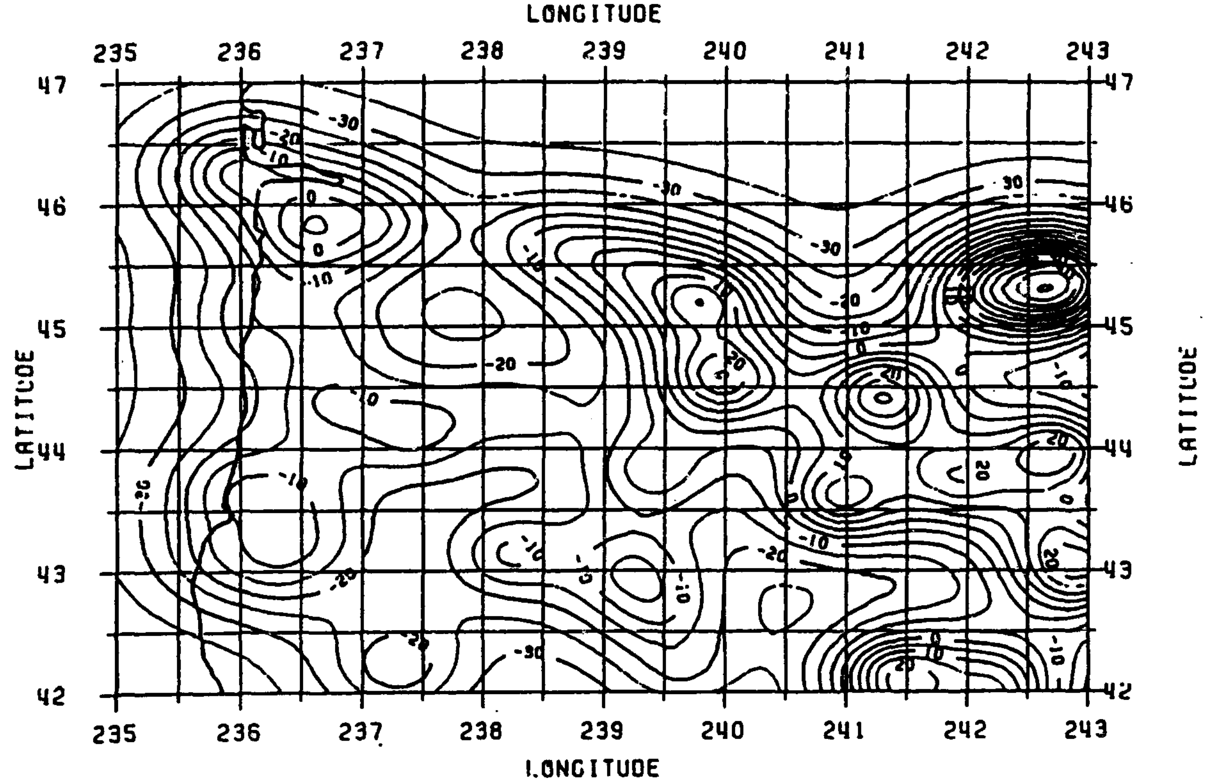


Figure 5.1 Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with Oregon GPS Traverse Data Set (GEOID93/NAVD88)

ACCURACY ESTIMATE OF BIAS FUNCTION PLOT

Contour Interval 5 cm

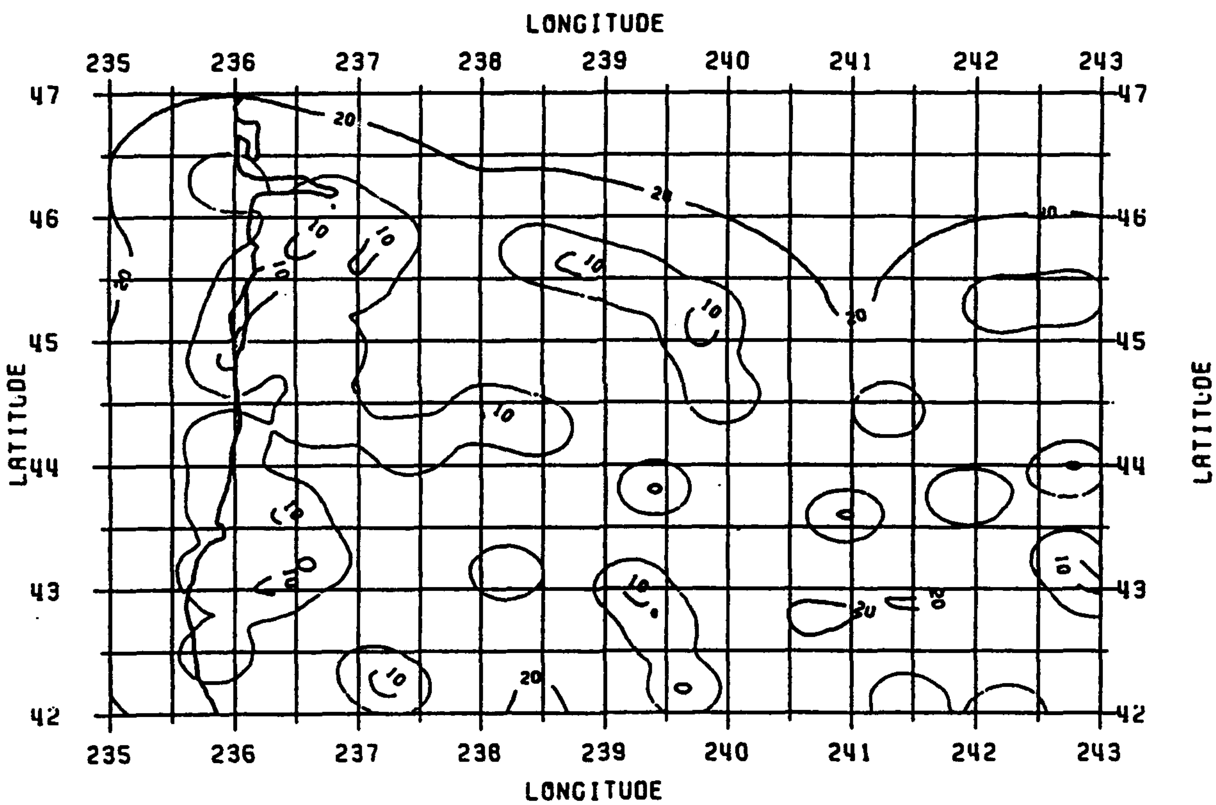


Figure 5.2 Accuracy Plot of Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with Oregon GPS Traverse Data Set (GEOID93/NAVD88)

HEIGHT BIAS FUNCTION - A PERSPECTIVE VIEW

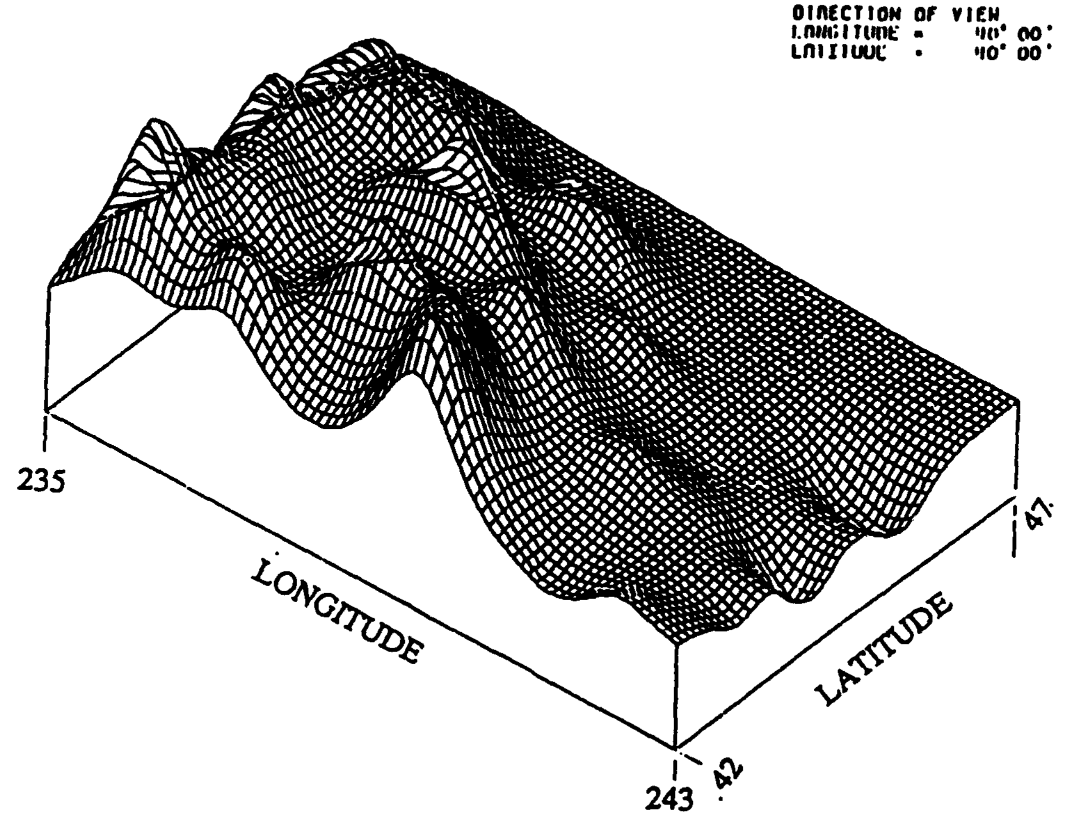


Figure 5.3 Perspective View of the Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with Oregon GPS Traverse Data Set (GEOID93/NAVD88)

HEIGHT BIAS FUNCTION PLOT

Contour Interval 5 cm

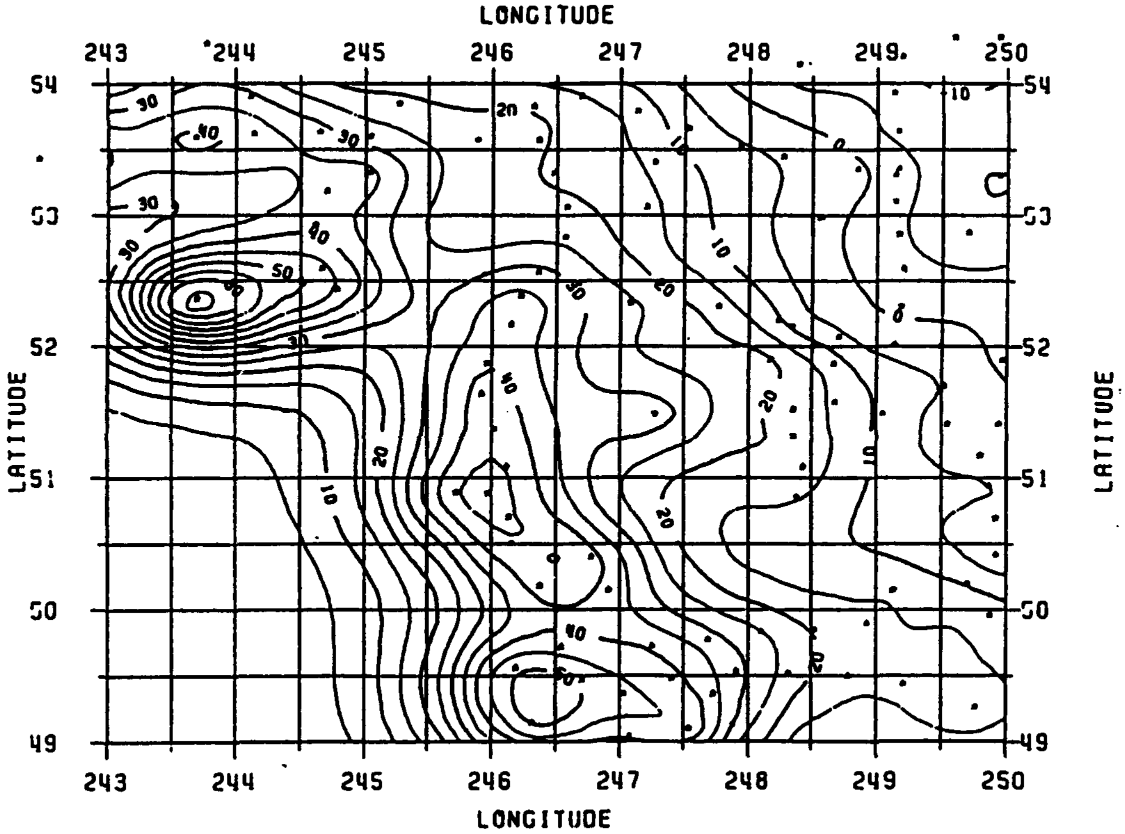


Figure 5.4 Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with South Alberta (Canada) GPS Traverse Data Set (GSD91/CGVD28)

ACCURACY OF HEIGHT BIAS FUNCTION PLOT

Contour Interval 5 cm

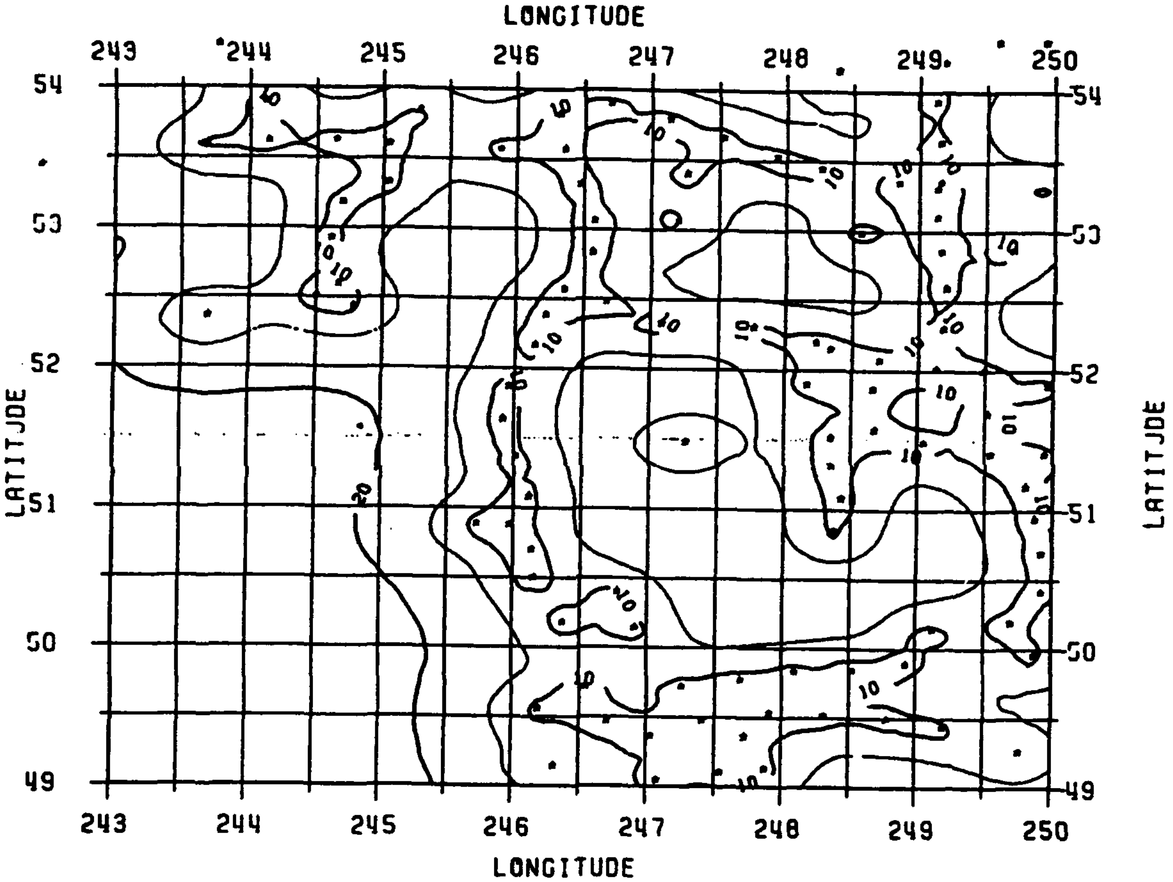


Figure 5.5 Accuracy Plot of Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with South Alberta (Canada) GPS Traverse Data Set (GSD91/CGVD28)

HEIGHT BIAS FUNCTION - A PERSPECTIVE VIEW

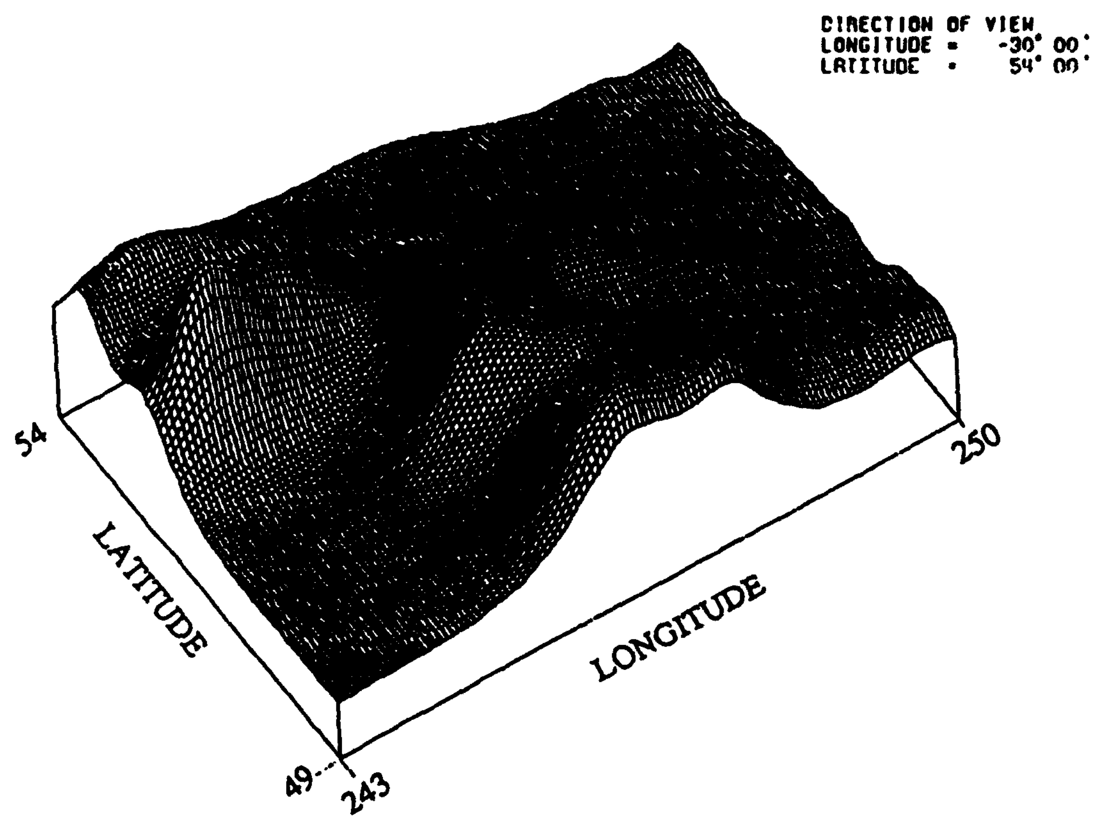


Figure 5.6 Perspective View of the Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with South Alberta (Canada) GPS Traverse Data Set (GSD91/CGVD28)

HEIGHT BIAS FUNCTION PLOT

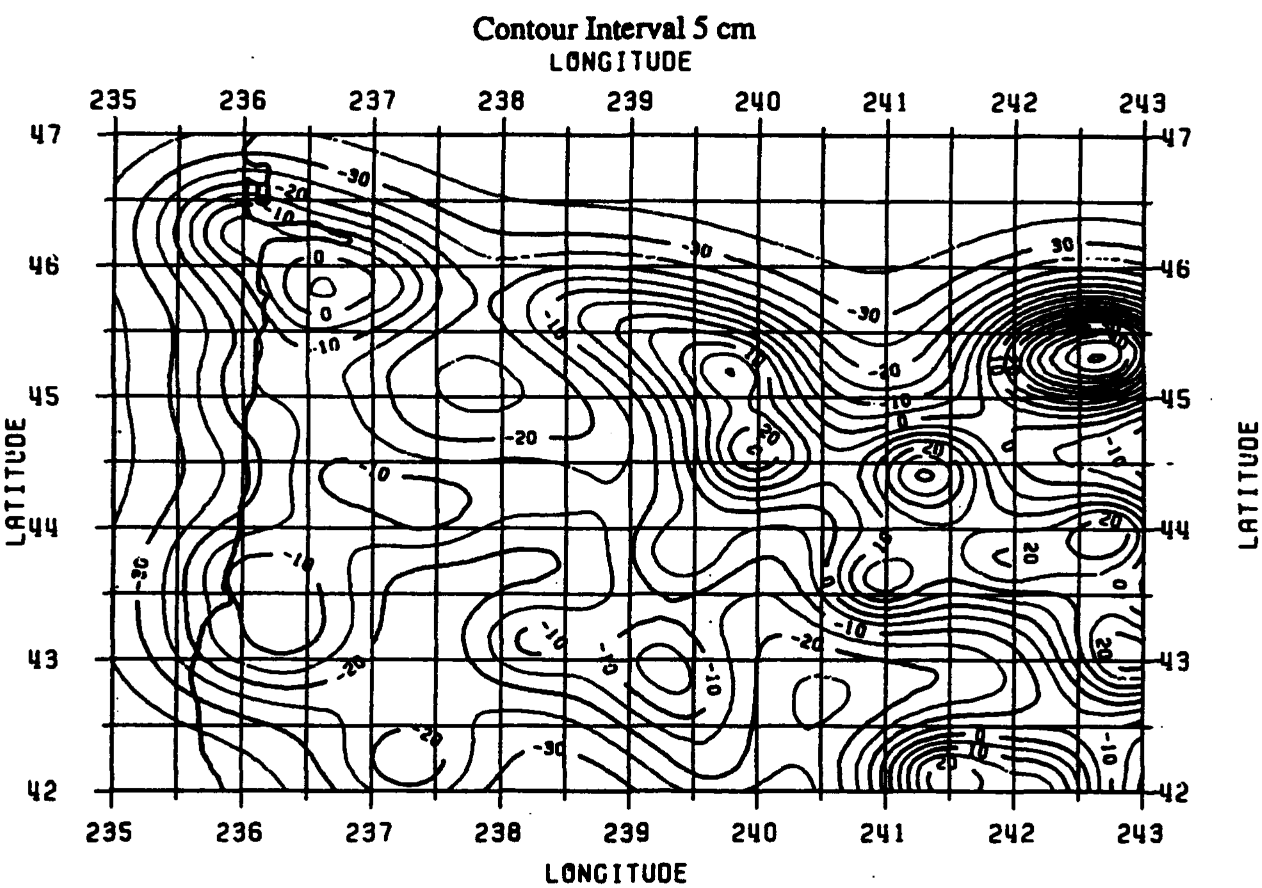


Figure 5.7 Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with Oregon GPS Traverse Data Set (GEOID93/NGVD29)

HEIGHT BIAS FUNCTION PLOT

Contour Interval 5 cm

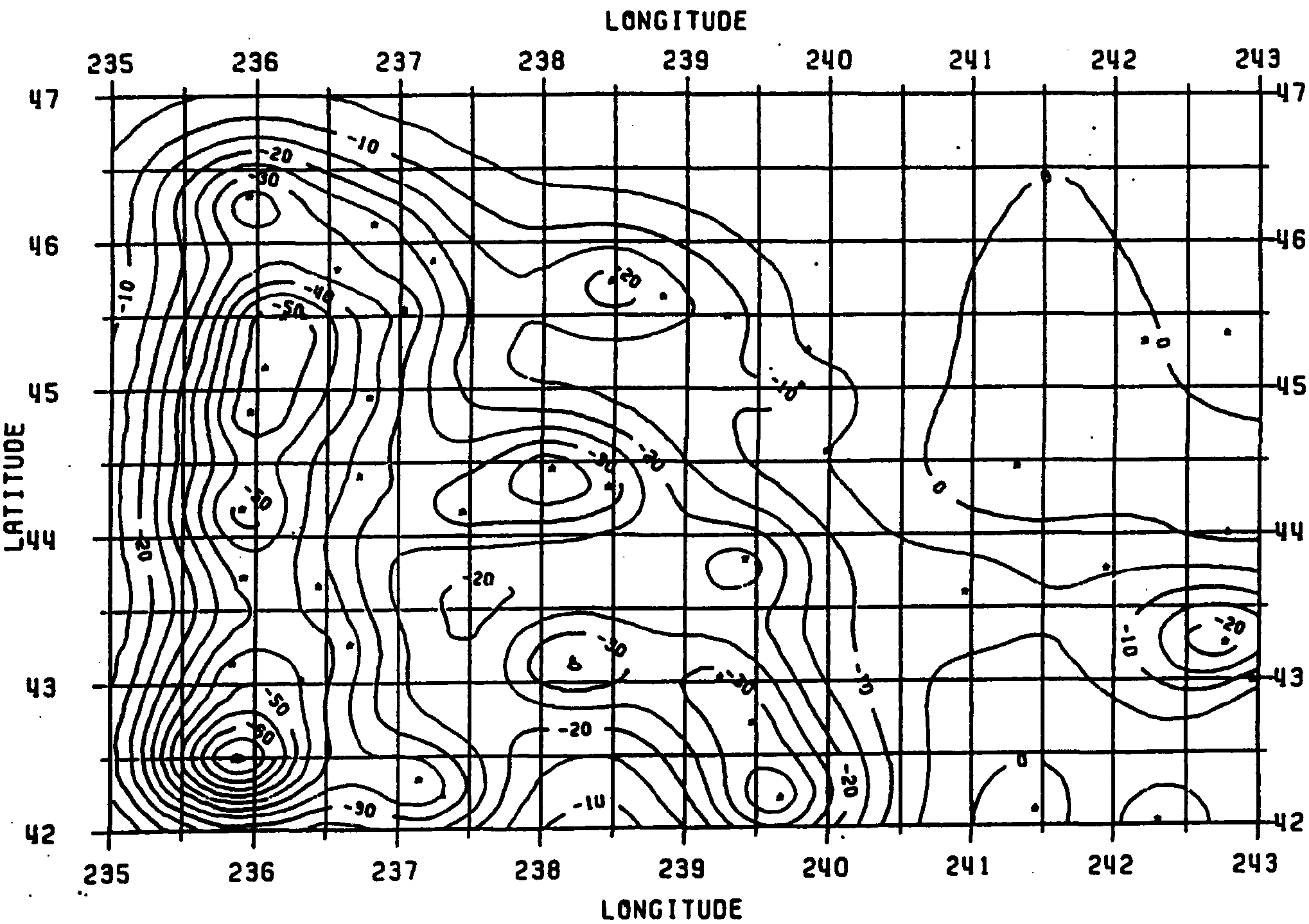


Figure 5.8 Height Bias Function 'c' Interpolated Using Least Squares Collocation Technique with Oregon GPS Traverse Data Set (GEOID90/NGVD29)

HEIGHT BIAS FUNCTION PLOT

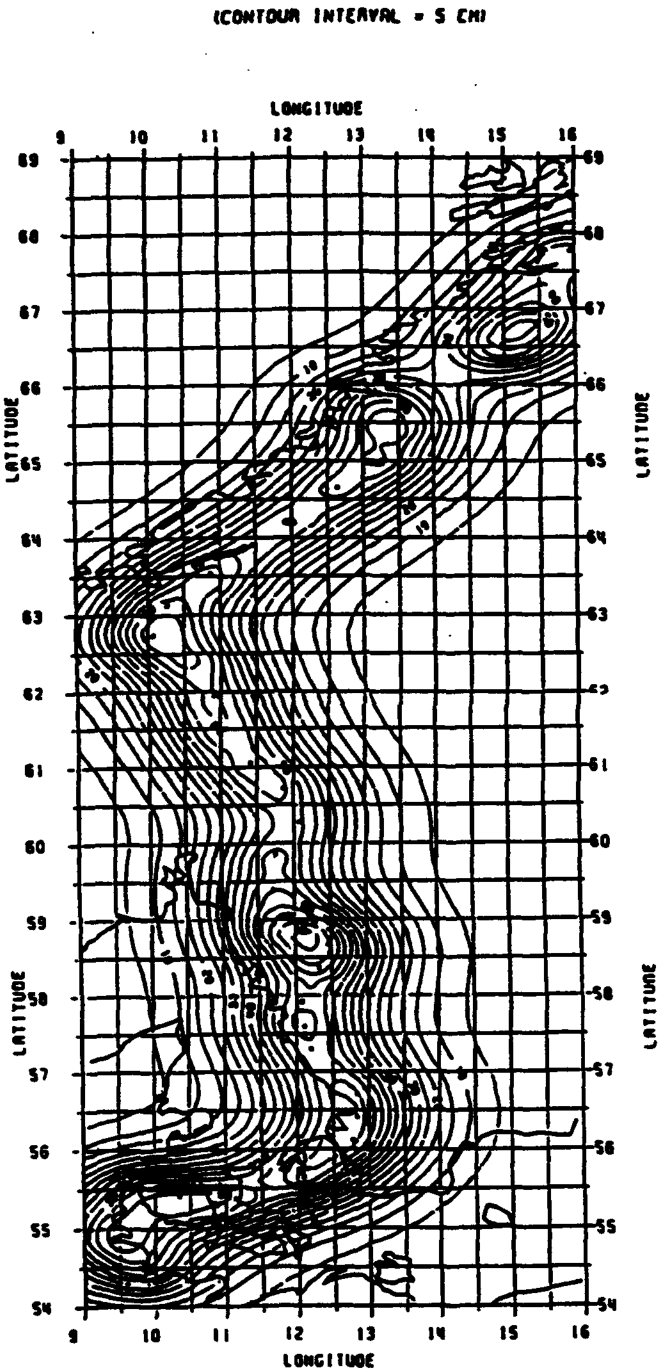


Figure 5.9 Height Bias Function Plot using Scandinavian GPS Traverse Data/UELN Heights

CHAPTER 6

SUMMARY AND CONCLUSIONS

Extending the concepts proposed by Rapp and Balasubramania (1992) for formulating the World Height System, this study has considered two different approaches for defining and realizing a Global Vertical Datum (GVD). The first approach is an idealistic approach in which accurate information from various regional vertical datums are brought together in an adjustment leading to the definition of GVD and the other one is a simplified approach based on space positioning and geoid undulation information with geoid as reference surface. While the geoid in the latter approach has no direct relationship with a mean sea level, the regional vertical datum information used in the idealistic approach will be dependent on the mean sea level observations at one or more tide gauges for their definition and establishment.

After a brief review of the different heights and height systems practically used in different parts of the world along with their intercomparisons, the current procedures used to define and establish a regional vertical datum using the long term mean sea level observations at one or more primary tide gauge stations were presented. It was emphasized here that since the mean sea levels computed at the different tide gauges do not coincide with the geoid due to the existence of Sea Surface Topography (SST) caused by ocean currents and various other factors, the resulting adjustment of leveling networks to fit these mean sea levels will result in distortions in the realized regional vertical datums. Several researchers have come up with ways and means to remove such distortions including attempts to model the SST at the tide gauges as an absolute value and also as difference between adjacent tide gauges. After listing the efforts so far made by numerous researchers, a mathematical model for the combined adjustment solution to define an ideal Regional Vertical Datum was proposed. This proposed solution was based on the assumption that realistic models to estimate the SST values at the tide gauges are available. But since we realize that perfect modelling of SST at the tide gauges is not possible, at least in the near future, and there will be inconsistencies in the estimated SST values at the tide gauges, adopting the robust sequential procedure put forward by Schaffrin (1986, 1990) was recommended for realizing an ideal regional vertical datum.

Mathematical modelling required to relate the primary observables to the parameters to be established was studied for different data scenarios. Separate modelling procedures were developed for setting up the observation equations at the space geodetic stations, depending on the type of free-air anomaly data and heights used in a regional vertical datum. Special emphasis was also placed on identifying different error sources affecting the terrestrial gravity anomalies used for defining and realizing a GVD to an accuracy of ±10 cm. The effect of horizontal datum inconsistencies in the used gravity data (δg_H) and the corresponding correction needed to the computed gravimetric height anomaly/ undulation were also determined. The maximum δg_H correction amounted to -0.097 mgal at the Yarragadee station in Australia with a resulting undulation correction of about -2.3 cm. Though case studies have shown that for a highly accurate computation of height anomaly using equation (3.24), consideration of Molodensky's correction terms up to g_2 was essential, as dense grid of gravity data as required for such computations may not be available all over the globe in the near future, correction terms only up to g_1 were considered sufficient in the modelling procedure.

A simplified approach for defining the GVD was also proposed in which the geoid forms the ideal reference surface. Widely available GPS observations (height determination specifically), precise undulation values, and a Height Bias model were considered essential for this approach. Procedures for developing a Height Bias function which could enable the orthometric heights computed with respect to the geoid be converted to the specific vertical datum system were also discussed. Height Bias functions developed for certain areas along with their predicted accuracy were also presented in the form of contour plots.

Two different techniques, namely modified Stokes' technique and least squares collocation technique (LSC), were considered for computing the gravimetric height anomaly/ undulation at the space geodetic stations using the surface gravity data in a small cap around the computation point and potential coefficient information from a global geopotential model. To avoid time consuming and unstable numerical inversion, the number of points considered for predicting gravimetric height anomaly/undulation using LSC technique was restricted to 1600 points.

For numerical analysis OSU91A model (to degree 360) was considered as reference model. The recently made available JGM-2 (NASA model) from degree 2 to 70, and augmented by OSU91A model coefficients from degree 71 to 360 was also used to estimate the remote zone contribution (beyond the cap radius in which surface gravity data were used) for the computation of gravimetric height anomaly/undulation using modified Stokes' technique. The height anomaly/undulations computed using the OSU91A reference model differed from the one computed using JGM-2 (augmented) model by 2 to 17 cm at the space geodetic stations in the United States, 4 to 25 cm at the European and Scandinavian stations and about 5 to 9 cm at the Australian stations. These differences are consistent with the accuracy estimate of height anomaly/undulations shown in Table 4.19 for the Modified Stokes' technique.

A first iteration attempt to estimate the parameters defining the GVD was also carried out using the currently available data in six regional vertical datums. Data used in the numerical studies had the following shortcomings:

- Limited gravity data were available in the Scandinavian datum, sufficient enough to use the gravity data in a cap radius of $0^\circ.5$ to $1^\circ.0$ only, though capsize of 2° was considered optimal.

- Required data were available only at very few stations in the various regional datums considered. While the number of stations used in the Australian datum was two, it was limited to just one in Germany, France and England regional vertical datums. Also the gravity data that were used at these stations was not corrected for terrain.

- In the United States, the NAVD88 heights used did not result from direct precise leveling measurements from an adjusted Bench Mark. They were interpolated using the VERTCON (Version 1.00) software, which gives the corrections to convert NGVD-29 normal orthometric heights to NAVD88 Helmert orthometric heights, only up to cm accuracy.

The following conclusions were arrived at based on the numerical investigations carried out to realize a GVD, using currently available data with the mathematical modelling procedures developed for an idealistic approach:

1. The different regional vertical surfaces used in the definition of GVD show the same pattern of their relative positions to one another, in spite of using two different techniques, such as the Modified Stokes' and the LSC technique, in their estimation.

2. The GVD can be realized to an accuracy of ± 5 cm with the accuracies of regional vertical datum

connections to GVD ranging from ±5 to ±23 cm. It should be emphasized that the error estimates reported here correspond to a 'worst-case' scenario, where ±7 mgal was the accuracy assumed for the terrestrial gravity data available at the European and Australian space geodetic stations and the variance-covariance matrix set up for modified Stokes' technique was used in the computations.

3. The results (refer to Figure 4.5) obtained from this study show a major conflict with the results reported by Rapp (1993). The Australian height datum AHD71 which is shown as 74 cm above the German height datum NN in Figure 4.5 of this report, is shown as 72 cm below the German height datum in Rapp (1993, fig 1).

4. Other than the discrepancy stated in (3) above, the estimated separation between the regional vertical surfaces compared mostly well with the results reported by various geodesists and oceanographers based on their detailed regional studies. Some of the compared results are :

- Rossiter (1967, p 295) based on his analysis of annual sea level variations in European waters, has reported that the average value of the mean sea level in the northern Mediterranean (where the Marseille tide gauge is located) is 20 cm below the N.A.P (on which German height datum is based). This suggests that IGN 69 (France) is about 20 cm below NN (Germany). Boucher (personal communication, 1993) has indicated that IGN 69 is about 50 cm below NN (Germany). The result from this study shows that IGN 69 is 20 cm ±22 cm below NN datum.

- Also based on oceanographic studies, Lisitzin (1974) shows (refer figure 30, p 149, ibid) that the mean sea level at southern England (where the Newlyn tide gauge is located) is about 20 cm below the mean sea level in southern France suggesting that the ODN (England) may be 20 cm lower than the IGN 69 (France). Willis et al (1989) based on geodetic observations using GPS techniques have reported that ODN is 30cm ±8cm below IGN 69. The result from this study also shows that ODN is 33 cm below IGN 69.

- The difference between German height datum NN and NAVD88, reported as 76 cm by Rapp (1993), agrees well with 72 cm from this study.

5. The value of the semi-major axis 'a' of the adopted ideal reference ellipsoid obtained as a=6378136.60 ±0.05 m, for both the Modified Stokes' and the LSC technique, as a by-product to this study agrees well with the recent estimates from TOPEX/Poseidon data (Rapp, 1993, personal communication).

6. Estimation of the separation between various regional vertical surfaces from the realized GVD was done on the assumption that the regional vertical datums were established to the required accuracy, which in turn can help in defining the GVD to ±10 accuracy level. In many of the regional vertical datums only a single station has been used assuming that it represents the local height system in entirety, which may not be true. Also bias caused by using gravity anomalies with certain systematic errors (for example, gravity anomalies not corrected for terrain) should also be kept in mind when looking at the estimated values of separation between regional surfaces.

7. The estimates of the parameters defining the GVD computed both from modified Stokes' and LSC techniques are generally in agreement, except for some numerical differences. These numerical differences are mainly due to the differences in computed gravimetric height anomaly/undulation between the two techniques which varied from 20 to 50 cm at certain stations. Since we use homogeneous and dense gravity data around the computation point, one can consider that for practical computation of height anomaly/undulation, modified Stokes' technique provides a better solution. As also expressed by Moritz (1976, p. 38), even if the data distribution is not ideal, it is better to use collocation for interpolation, in order to densify

the data distribution, and subsequently apply Stokes' method, rather than using LSC technique for a straight one step solution.

8. Finally, the results reported here are promising enough to warrant actual implementation of a Global Vertical Datum that can help one go from one regional vertical datum to another knowing the single bias function that separates the two datums.

CHAPTER 7

RECOMMENDATIONS FOR THE ACQUISITION AND USE OF DATA FOR DEFINING AND REALIZING A GLOBAL VERTICAL DATUM

In Section 3.1, the different kinds of data that are essential for defining and realizing a Global Vertical Datum (GVD) are discussed in detail. A recommended strategy for acquiring and use of such data for the implementation of a GVD are discussed in this chapter. The strategy recommended is conceptually simple and built on ongoing International activities.

The definition and realization of the GVD should be developed around the IERS terrestrial reference frame using the precise positioning data given by the SLR/VLBI observations all over the world. There are about 171 IERS sites (refer to figure 7.1) carrying out mobile and fixed SLR/VLBI observations distributed in over 42 countries (Boucher et al., 1993, Table 1). In addition there are 85 operational/proposed IGS stations (refer to figure 7.2) and several DORIS tracking stations (refer to figure 7.3), adequately complementing the SLR/VLBI stations forming a dense mesh of space geodetic stations which can provide the required precise ellipsoidal heights for the realization of GVD. Global geopotential models to degree 360 are currently available and efforts are also on to improve the accuracy of such models.

As the problem of implementing a Global Vertical Datum, acceptable on a global basis, is of longterm character requiring close cooperation between specialists from various countries, a special study group similar to IAG SSG 1.75 (Rapp, 1987) under aegis of IAG can be recreated or the objective of the currently operational IAG SSG 5.149 on 'Studies on Vertical datums' can be enhanced with more members from participating countries. This SSG in addition to identifying a suitable International organization to coordinate the activities of the various scientific organizations in different countries, will also lay down the procedure for data collection, storage, manipulation and transfer of data and results by the participating countries.

The individual participating countries should be made responsible for the accurate computation of gravimetric geoid undulation using a dense grid of anomaly and terrain data around the computation point and also the same geopotential model considered to be accurate and used by other participating countries. They will also provide the International organizations coordinating this collective endeavor with the precise height of the space geodetic stations, referring to the same point to which ellipsoidal height is referred, following the 1st order leveling procedures from an adjusted Bench Mark in the regional vertical reference frame.

Finally, by building on existing structures, adding to the existing networks, and adapting the existing infrastructure to the expanded needs associated with the Global Vertical Datum concept, we can achieve a gradual and flexible implementation. For this we need to begin the operation of realizing a GVD immediately, using existing data sources. The development of required mathematical models and the first iteration attempt in realizing a GVD discussed in this dissertation report are the first few steps in that direction.

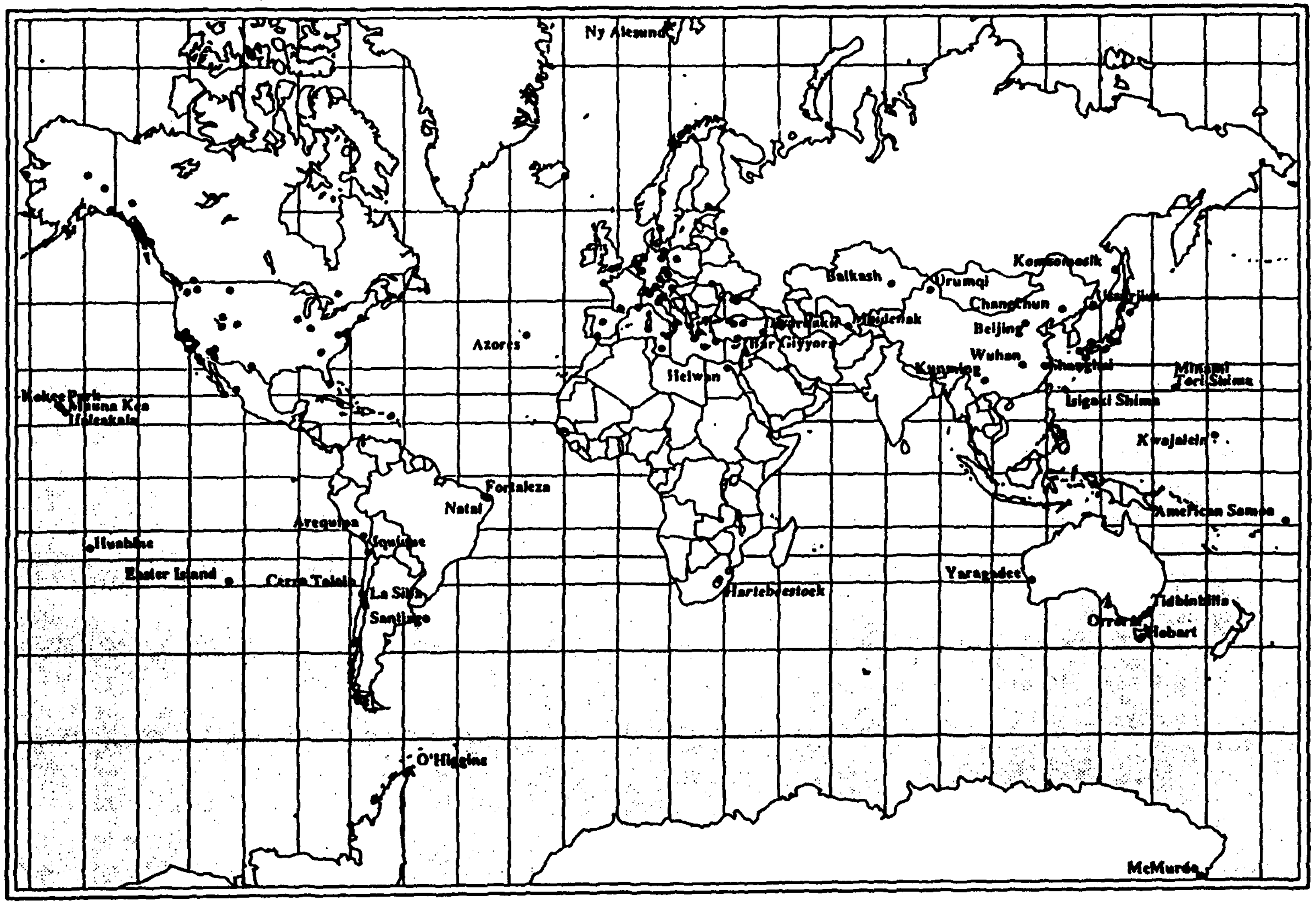


Figure 7.1 Map of Geodetic Sites - Global (NASA Tech. Memo 4482)

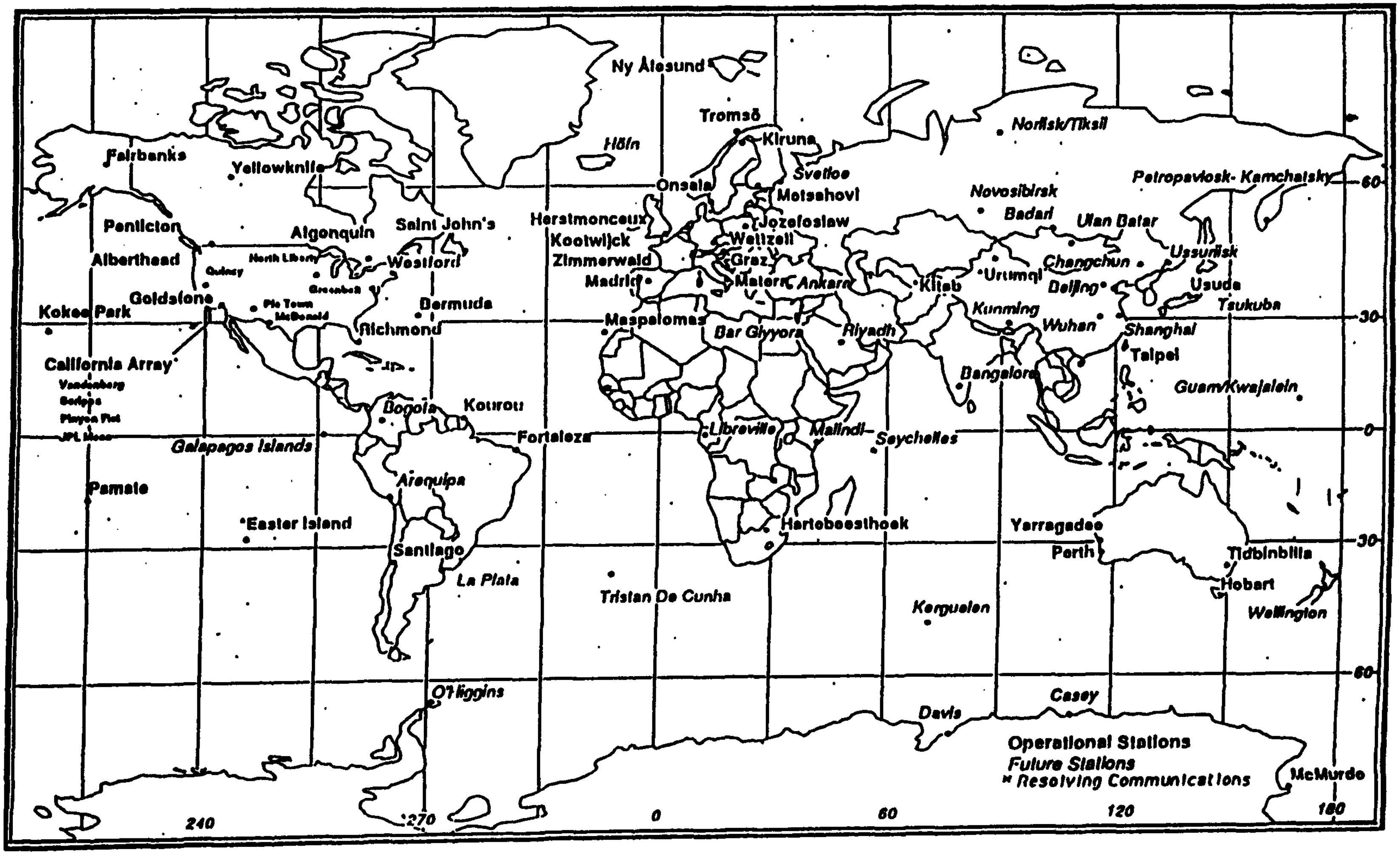
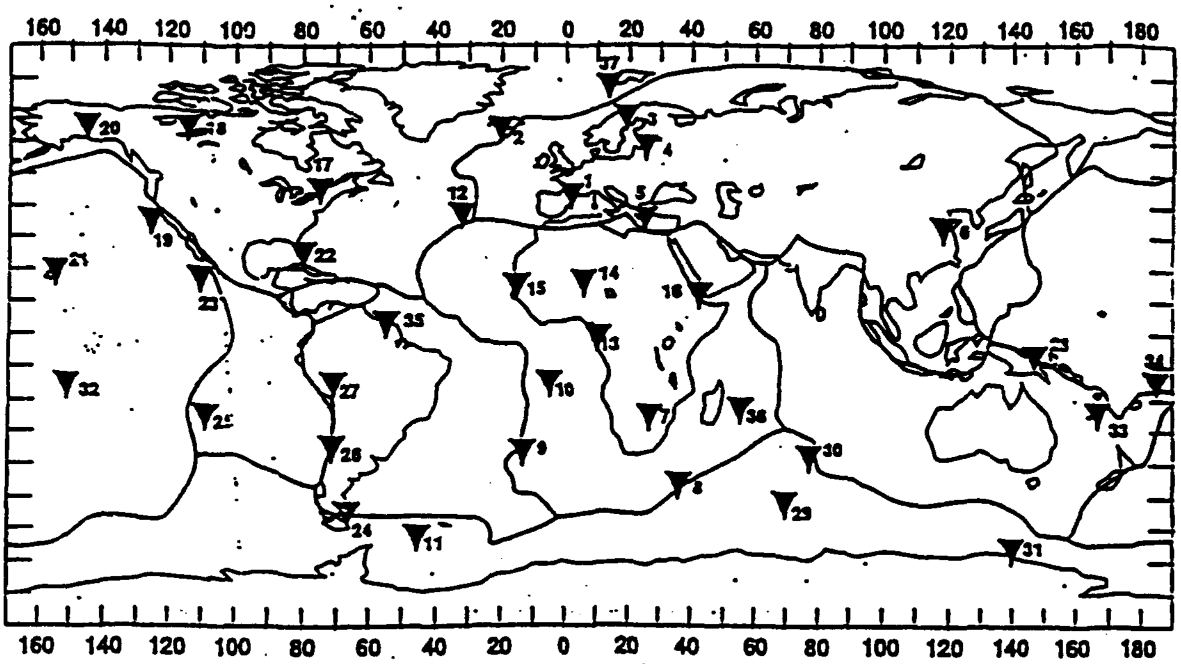


Figure 7.2 GPS Tracking Network of the International GPS Service for Geodynamics - Operational and Proposed Stations



1	Toulouse	10003	TLS A	4002
	Toulouse	10093	TLT A	4029
2	Reykjavik	10202	REK A	
3	Tromso (R)	10302	TRO A	4054
4	Metzahovi (R+L)	10503	MET A	4006
5	Dionysos (L)	12602	DIO A	4047
6	Purple Mountain	21604	PUR A	4045
7	Hartebeesthoek (R)	30302	HBK A	4019
8	Marion Island	30313	MAR A	4022
9	Tristan da Cunha	30604	TRI A	4005
10	Sainte Helene	30606	HEL A	4043
12	Flores	31901	FLO A	4053
13	Libreville	32809	LIB A	4013
14	Arlit	33710	ARL A	4035
15	Dakar	34101	DAK A	4018
16	Djibouti	39901	DJI A	4025
	Djibouti	39914	DJC B	4103
	Djibouti	39916	DJM B	4102
17	Ottawa	40102	OTT A	4048
18	Yellowknife (R)	40127	YEL A	4051
19	Goldstone (R+L)	40405	GOL A	4010
20	Fairbanks (R)			
21	Kauai (R+L)			
22	Richmond (R)	40499	RIC A	4023
23	Socorro	40503	SOC A	4040
24	Rio Grande	41507	RIO A	4017
25	Easter Island (L)	41703	EAS A	4041
26	Santiago	41705	SAN A	4038
27	Arequipa (L)	42202	ARE A	4046
28	Port Moresby	51001	MOR A	4055
29	Kerguelen	91201	KER A	4009
30	Amsterdam	91401	AMS A	4006
31	Terre Adelie	91501	ADE A	4042
32	Huahine (L)	92202	HUA A	4027
33	Noumea	92701	NOU A	4036
34	Wallis	92901	WAL A	4037
35	Kourou	97301	KRU A	4016
36	La Reunion	97401	REU A	401[illegible]
37	Spitzberg	10317	SPI A	402[illegible]

(L): SLR
(R): VLBI

A: Orbitography Beacon
B: Localization Beacon

Figure 7.3 DORIS Orbitography Beacon Network (Dec. 1990)

APPENDIX A

COMPUTATION OF GRAVIMETRIC HEIGHT ANOMALY/UNDULATION USING JGM-2 (AUGMENTED) GEOPOTENTIAL MODEL (TO DEGREE 360)

The OSU91A geopotential model which was used as the reference model in the numerical investigations reported in Chapter 4 was replaced by the newly made available JGM-2 (NASA model) (Nerem et al., 1993) from degree 2 to 70 and augmented by OSU91A model coefficients from degree 71 to 360, and the computation of gravimetric height anomaly ζ_1^i and undulation N_1^i from equations (3.71) and (3.72) was repeated for all space geodetic stations used in the GVD definition study. Table A1 shows the newly computed values of remote zone contribution to gravimetric height anomaly/undulation computation using JGM-2 (augmented model) in column (2), with cap contribution in column (1) and other corrections in column (3) reproduced from Tables (4.7) to (4.9b) along with the other components reproduced from Tables (4.7), (4.8), (4.9a) and (4.9b). Total height anomaly/undulation as well as the Y values computed using OSU91A model and JGM-2 (augmented) model are also shown for better comparison of the variation in ζ_1^i/N_1^i computation using different global geopotential model coefficients.

From Table A1 we see that the height anomaly/undulations computed using the OSU91A reference model differ from the one computed using JGM-2 (augmented) model by about 2 to 17 cm at the space geodetic stations in the United States, 4 to 25 cm at the European and Scandinavian stations, and about 5 to 9 cm at the Australian stations. Since in the JGM-2 (augmented) model the lower degree harmonics have been computed more accurately than the OSU91A model, the difference noted in the computed undulations can be considered to be due to long wavelength error in the OSU91A geopotential model used as the reference model in the numerical studies. The differences between the height anomaly/undulations computed using the OSU91A reference model and the JGM-2 (augmented) model are consistent with the accuracy estimate given in Table 4.19 for the Modified Stokes' technique. The estimated parameters defining the Global Vertical Datum computed using the Y-values from JGM-2 (augmented) model are shown in Figure A1. Comparison of Figures 4.5 and A1 show that the estimated parameters agree to an extent of ±10 cm in all cases except for ODN datum where the disagreement is about 19 cm.

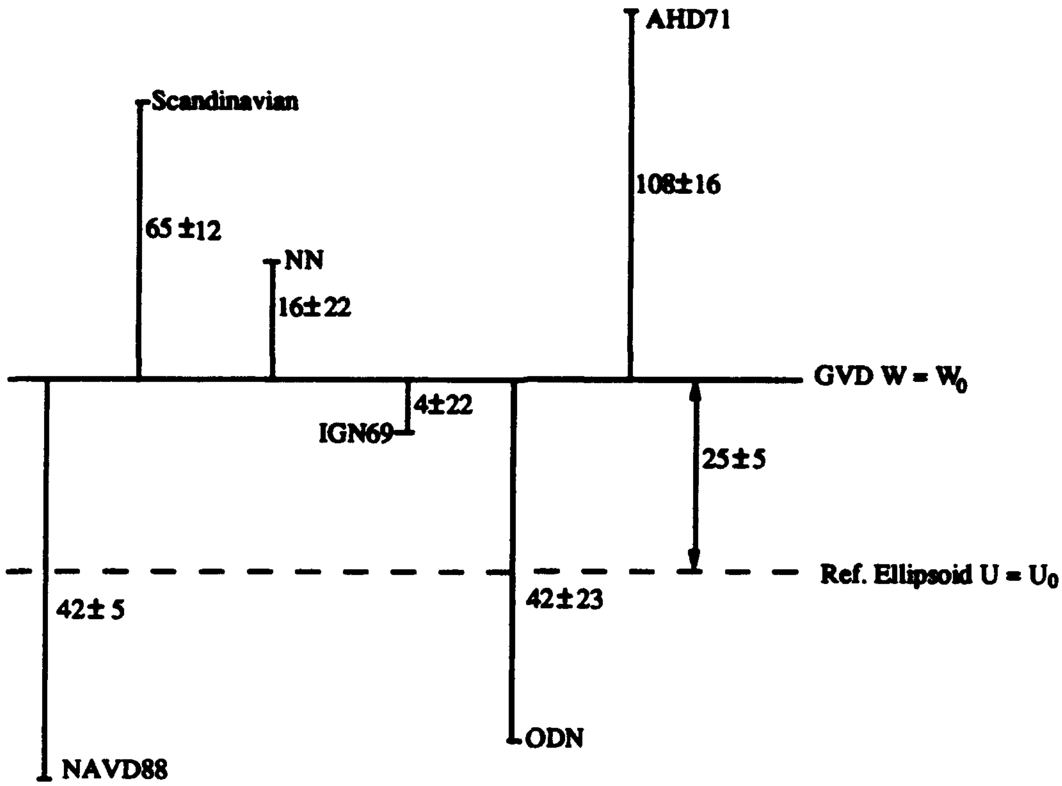


Figure A1 Regional Vertical Datum Separation From Defined Global Vertical Datum (computed using JGM-2 (augmented) Model and the Modified Stokes' Technique). Units are in cm.

Table A1: Comparison of Y-values Computed Using Different Global Geopotential Models. Units are in meters.

Station name	Various components of gravimetric ht anomaly/ undulation			Total Ht anomaly/ undulation	Geometric undulation	Y values computed using:	
NAVD88 Datum	(1)	(2)	(3)	(5)	(6)	JGM-2 (aug) (7)	OSU91A (8)
7051 Quincy	3.423	-26.445	-0.077	-23.099	-23.734	-0.635	-0.652
7082 Bear Lake	4.581	-17.460	-0.221	-13.100	-14.095	-0.995	-0.827
7091 Westford	0.873	-29.449	-0.017	-28.593	-28.296	0.297	0.293
7086 Ft. Davis	2.411	-24.064	-0.215	-21.868	-21.642	0.226	0.116
7105 GSFC5	0.263	-33.192	-0.014	-32.943	-32.940	0.003	0.139
7110 Monu. Peak	0.878	-32.290	-0.206	-31.618	-31.816	-0.198	-0.222
7234 Pie Town	3.440	-24.191	-0.331	-21.082	-21.569	-0.487	-0.500
7069 Patrick AF	0.910	-30.384	0.008	-29.466	-28.813	0.436	0.516
7204 Green Bank	1.750	-32.839	-0.050	-31.139	-31.442	-0.303	-0.187
European datums 7834 Wettzell (NN)	4.533	42.410	0.020	46.963	47.396	0.433	0.590
7835 Grasse (CGDV69)	6.106	45.128	0.014	51.248	51.460	0.212	0.390
7840 Herstmonceux (ODN)	-2.433	48.032	0.011	45.610	45.405	-0.205	0.046
Scandinavian datums 7601 Metsahovi	-0.698	20.042	-0.010	19.334	19.751	0.417	0.377
1001 Onsala	-0.214	35.995	-0.014	35.767	37.223	1.456	1.357
7602 Tromsø	-1.241	32.269	-0.023	31.005	32.289	1.503	1.284
Australian datums 7090 Yarragadee	-0.990	-24.291	0.000	-25.281	-24.517	0.764	0.813
7943 Orroral Valley, Canberra	7.447	10.520	0.014	17.981	20.105	2.124	2.030

REFERENCES

Bašić, T. and Rapp, R.H., Oceanwide Prediction of Gravity Anomalies and Sea Surface Heights Using Geos-3, Seasat, and Geosat Altimeter Data and ETOPO5U Bathymetric Data, Report No. 416, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, February 1992.

Bomford, G., Geodesy, Clarendon Press, Oxford, 1980.

Boucher, C. and Z. Altamimi, ITRF90 and Other Realizations of the Terrestrial Reference System for 1990, IERS Technical Note 9, Observatoire de Paris, Dec. 1991.

Boucher, C. and Z. Altamimi, The IERS Terrestrial Reference System, in Proc. Sixth Int. Geodetic Symp. on Satellite Positioning, The Ohio State University, pp. 47-55, March 1992.

Boucher, C., Z. Altamimi and Duhem, L., IERS Technical Note 12, ITRF91 and its Associated Velocity Field, Observatoire de Paris, October 1992a.

Boucher, C., Z. Altamimi and Duhem, L., IERS Technical Note 15, ITRF92 and its Associated Velocity Field, Observations de Paris, October 1993.

Bracewell, R.N., The Fourier Transform and Its Applications, 2nd edition, New York: McGraw-Hill Book Co., 1986.

Carter, W.E. and 11 others, Geodetic Fixing of Tide Gauge Bench Marks, Technical Report, CRC-89-5, Coastal Research Center, Woods Hold Oceanographic Institution, August, 1989.

Cartwright, D.E., A Practical Oceanographic Approach to the Problem of Defining a World Vertical Datum, in Proc. of the 3rd International Symposium on the North American Vertical Datum, NOAA, Rockville, MD, pp. 155-159, 1985.

Collier, P.A. and Leahy, F.J., The General Model of Least Squares Collocation - A Versatile Geodetic Computational Tool, Australian Journal of Geod. Photogram. Surv., No. 57, Dec. 1992, pp. 37-61.

Colombo, O.L., A World Vertical Network, Report No. 296, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1980. AD-A086011/4

Cross, P.A. and 10 others, Four-Dimensional Geodetic Positioning, Report of the IAG SSG 4.96, Manuscripta Geodaetica, Vol. 12, No. 3, pp. 147-222, 1987.

Cruz, J., Ellipsoidal Corrections to Potential Coefficients Obtained from Gravity Anomaly Data on the Ellipsoid, Dept. Report No. 371, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1986. AFGL-TR-86-0178, ADA176901

Despotakis, V.K., Geoid Undulation Computation at Laser Tracking Stations, Report No. 383, Geodetic Science and Surveying, The Ohio State University, Columbus, 1987.

Despotakis, V.K., Geoid Undulation Computations at Laser Tracking Stations, Bulletin Géodésique, 63, 342-358, 1989.

Dewhurst, W.T., NADCON - The Application of Minimum-Curvature-Derived Surfaces in the Transformation of Positional Data from the North American Datum of 1927 to the North American Datum of 1983, NOAA Tech. Memo No. 5, NGS-50, Rockville, MD, January 1990.

DMA Technical Report, Part 1 - Methods, Techniques and Data used in WGS84 Development, December 1987 (Second Printing).

Ehrnsperger, W., J.J. Koch, and J. Van Mierlo, Stations and Provisional Results of the 1981 Adjustment of the United European Leveling Network - UELN-73, in the Proc. Int. Symp. on Geodetic Networks and Computations of the International Association of Geodesy, Publication of the German Geodetic Commission, Series B, Vol. 25/II, pp. 25-50, Munich 1982.

Ekman, M. and Mäkinen, J., Land uplift and gravity change in Fennoscandia 1966-1989, Bureau Gravimétrique International, Bulletin d'Information, 67, pp. 114-117, 1990.

Ekman, M. and Mäkinen, J., The deviation of mean sea level from the mean geoid in Baltic Sea, Bulletin Géodésique, 65, pp. 83-91, 1991.

Erker, E. and Sünkel, H., Control and Refinement of the Austrian Geoid Using GPS, Sea Surface Topography and the Geoid, IAG Symposium No. 104, Springer: Berlin/ Heidelberg/New York, pp. 80-86, 1989.

Farelly, B., The Geodetic Approximations in the Conversion of Geoid Height to Gravity Anomaly by Fourier Transform, Bulletin Géodésique, 65, pp. 92-101, 1991.

Featherstone, W.E., A GPS Controlled Gravimetric Determination of the Geoid of the British Isles, Ph.D. Thesis, University of Oxford, 1992.

Forsberg, R., A New High-Resolution Geoid of the Nordic Area, Determination of the Geoid - Present and Future, IAG Symposium 106, Springer: Berlin/Heidelberg/New York, pp. 241-250, 1990.

Groten, E. and Müller, T., Combined Sea Surface Determination Based on Various Geometric and Dynamic Techniques, Marine Geodesy, Vol. 14, pp. 185-195, 1990.

Hajela, D.P., Accuracy Estimates of Intercontinental Vertical Datum Connections, in Proc. 3rd International Symposium of the North American Vertical Datum, NOAA, Rockville, MD, pp. 145-154, 1985.

Heck, B. and Rummel, R., Strategies for Solving the Vertical Datum Problem Using Terrestrial and Satellite Geodetic Data, Sea Surface Topography and the Geoid, International Association of Geodesy Symposium, Springer: Berlin/Heidelberg/New York, pp. 116-127, 1990.

Heck, B., An Evaluation of Some Systematic Error Sources Affecting Terrestrial Gravity Anomalies, Bulletin Géodésique, 64, pp. 88-108, 1990.

Hein, G.W. and Eissfeller, B., Vertical Datum Definition by Integrated Geodesy Adjustment, in Proc. 3rd International Symposium on the North American Vertical Datum, NOAA, Rockville, MD, pp. 121-136, 1985.

Heiskanen, W. and H. Moritz, Physical Geodesy, W.H. Freeman and Co., San Francisco, 1967.

Hwang, C., High Precision Gravity Anomaly and Sea Surface Height Estimation from Geos-3/Seasat Altimeter Data, Report No. 399, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1989.

Jekeli, C., An Investigation of Two Models for the Degree Variances of Global Covariance Functions, Report No. 275, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1978. AFGL-TR-78-0235, ADA063196

Kearsley, A.W.H. and Govind, R., Geoid Evaluation in Australia Status and Problems, Determination of the Geoid: Present and Future, IAG Symposium No. 106, Springer: Berlin/Heidelberg, New York, pp. 251-260, 1990.

Krakiwsky, E.J., Heights, M.S. Thesis submitted to the Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1965.

Lambert, W. and F. Darling, Tables for Determining the Form of the Geoid and its Indirect Effect on Gravity, Special Publication No. 199, U.S. Coast and Geodetic Survey, 1936.

Laskowski, P., The Effect of Vertical Datum Inconsistencies on the Determination of Gravity Related Quantities, Report No. 349, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1983.

Lisitzin, E., Sea-level Changes, Elsevier Oceanography Series, 8, Elservier Scientific Publishing Co., Amsterdam, 1974.

Mainville, A. and Véronneau, M., Orthometric Heights Using GPS, Proceedings of the XIX FIG Congress, Helskinki, Vol. 5, 1990.

Mather, R.S., C. Rizos and T. Morrison, On the Unification of Geodetic Levelling Datums using Satellite Altimetry, NASA Technical Memo 79533, April 1978.

Merry, C.L. and Vanicek, P., Investigation of Local Variations of Sea-Surface Topography, Marine Geodesy, Vol. 7, No. 1-4, pp. 101-126, 1983.

Merry, C.L., Recent Variations in Mean Sea Level in Southern Africa, Sea Surface Topography and the Geoid, IAG Symposium No. 104, pp. 149-157, Springer-Verlag: Berlin/Heidelberg, New York, pp. 149-157, 1989.

Milbert, D.G. and Shultz, D.G., GEOID93: A New Geoid Model for the United States, AGU 1993 Spring Meeting, Baltimore, 1993, (abstract in EOS, April 20, 1993).

Milbert, D.G., GEOID90: A High-Resolution Geoid for the United States, EOS, Trans. American Geophysical Union, Vol. 72, No. 49, pp. 545-554, 1991a.

Milbert, D.G., An Accuracy Assessment of the GEOID90 Geoid Height Model for the commonwealth of Virginia, Manuscript, NOAA, Rockville, MD, June 1991b.

Morgan, P., An Analysis of the Australian Height Datum: 1971, The Australian Surveyor, Vol. 34, No. 1, pp. 46-63, March 1992.

Moritz, H., Integral Formulas vs Collocation, Manuscripta Geodaetica, Vol. 1, pp. 1-40, 1976.

Moritz, H., Advanced Physical Geodesy, Wichmann, Karlsruhe, 1980.

Moritz, H., Geodetic Reference System 1980, Bulletin Géodésique, 66, No. 2, pp. 187-192, 1992.

NASA Tech. Memo 4482, NASA Space Geodesy Program: Catalogue of Site Information, March 1993.

National Mapping Council of Australia, Special Publication 8, The Australian Height Datum (AHD), Dept. of National Development, Canberra, A.C.T., Australia, 1979.

Nerem, R.S. and 18 others, Gravity Model Development for TOPEX/Poseidon: Joint Gravity Models 1 and 2, submitted November 8, 1993 to JGR (Oceans), TOPEX/Poseidon Special Issue.

Niemeier, W., Observation Techniques for Height Determination and Their Relation to Usual Height Systems, Determination of Heights and Height Changes, Contributions to the Symposium on Height Determination and Recent Vertical Crustal Movements in Western Europe, Hannover, Germany, pp. 85-108, 1987.

Pan, M. and L. Sjöberg, Baltic Sea Level Project with GPS, Bulletin Géodésique, 67, 51-59, 1993.

Pavlis, N.K., Estimation of Geopotential Difference over Intercontinental Locations Using Satellite Terrestrial Measurements, Report No. 409, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1991.

Pavlis, N.K., Modelling and Estimation of Low Degree Geopotential Model from Terrestrial Gravity Data, Report No. 386, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1988.

Rapp, R.H., Separation Between Reference Surfaces of Selected Vertical Datums, Bulletin Géodésique, 68, 1994 (to be published).

Rapp, R.H., Geoid Undulation Computations with Cap Integrations Involving Terrain Correction, Internal note, 1993(a).

Rapp, R.H., Computation and Accuracy of Global Geoid Undulation Models, in Proceedings of Sixth International Geodetic Symposium on Satellite Positioning, pp. 865-872, March 17-20, The Ohio State University, Columbus, 1992.

Rapp, R.H., The Earth's Gravity Field from Satellite Geodesy - A 30 Year Adventure, Proceedings of the Workshop on Solid Earth Mission ARISTOTELES, Anacapri, Italy, pp. 29-32, Sept. 1991.

Rapp, R.H., Report of SSG No. 1.75 on World Vertical Reference System presented at the XIX General Assembly of IUGG, International Association of Geodesy, Vancouver, B.C., Canada, June 1987.

Rapp, R.H., The Need and Prospects of a World Vertical Datum, presented at the Symposium on the Future of Terrestrial and Space Methods for Positioning, XVII General Assembly, IAG/IUGG, Hamburg, August 1983.

Rapp, R.H., Ellipsoidal Corrections for Geoid Undulation Computations Using Gravity Anomalies in a Cap, Journal of Geophysical Research, Vol. 86, No. B11, pp. 10843-10848, November 10, 1981.

Rapp, R.H., Precise Definition of the Geoid and its Realization for Vertical Datum Applications, in Proceedings of the 2nd International Symposium on Problems Related to the Redefinition of the North American Vertical Geodetic Network, Ottawa, pp. 73-86, 1980.

Rapp, R.H., The Orthometric Height, MS. Thesis submitted to the Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1961.

Rapp, R.H. and Balasubramania, N., A Conceptual Formulation of a World Height System, Report No. 421, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, August 1992.

Rapp, R.H. and Pavlis, N.K., The Development and Analysis of Geopotential Models to Spherical Harmonic Degree 360, Journal of Geophysical Research, Vol. 95 No. B13, pp. 21885-21911, 1990.

Rapp, R.H., Y.M. Wang and N.K. Pavlis, The Ohio State 1991 Geopotential and Sea Surface Topography Harmonic Coefficient Models, Report No. 410, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, Dec. 1991.

Rizos, C., R. Coleman, N. Ananga, The Bass Strait GPS Survey: Preliminary Studies of an Experiment to Connect Australian Height Datums, Aust. J. Geod. Photogram. Surv. 55, pp. 1-25, Dec. 1991.

Rossiter, J.R., An Analysis of Annual Sea Level Variations in European Waters, Geophysical J.R. Astr. Soc. (1967), Vol. 12, pp. 259-299, 1967.

Rummel, R. and Teunissen, P., Height Datum Definition, Height Datum Connection, and the Role of the Geodetic Boundary Value Problem, Bulletin Géodésique, 62, pp. 477-498, 1988.

Schaffrin, B., Course materials for GS762, Advanced Adjustment Computations, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1990.

Schaffrin, B., An Alternative Approach to Robust Collocation, Bulletin Géodésique, 63, pp. 395-404, 1989.

Schaffrin, B., Merging Gauge Registrations of Minor Accuracy into a First Order Levelling Network, Paper presented at the Symposium on Height Determination and Recent Vertical Crustal Movements in Western Europe, Hannover, Germany, pp. 397-401, 1987.

Schwarz, K.P., Sideris, M.G. and Forsberg, R., The Use of FFT Techniques in Physical Geodesy, Geophys. J. Intl., 100, pp. 485-514, 1990.

Shrestha, R., Nazir, A., Dewitt, B. and Smith, S., Surface Interpolation Techniques to Convert GPS Ellipsoid Heights to Elevations, Surveying and Land Information Systems, Vol. 53, No. 2, pp. 133-144, 1993.

Sideris, M.G. and Schwarz, K.P., Advances in the Numerical Solution of the Linear Molodensky Problem, Bulletin Géodésique, 62, pp. 59-69, 1988.

Sideris, M.G. and Schwarz, K.P., Solving Molodensky's Series by Fast Fourier Transform Techniques, Bulletin Géodésique, 60, pp. 51-63, 1986.

Sjöberg, L.E., Refined Least Squares Modification of Stokes' Formula, Manuscripta Geodaetica, 16, pp. 367-375, 1991.

Strang van Hees, G., Stokes' Formula Using Fast Fourier Techniques, Manuscripta Geodaetica, 15, pp. 235-239, 1990.

Sturges, W., Sea Level Slope Along Continental Boundaries, Journal of Geophysical Research, Vol. 79, No. 6, pp. 825-830, 1974.

Torge, W., Geodesy, 2nd Edition, de Gruyter, Berlin, New York, 1991.

Torge, W. et al., Long Range Geoid control through the European GPS Traverse, Reihe B, Heft Nr. 290, Deutsche Geodätische Kommission, bei der Bayerischen Akademie der Wissenschaften, München 1989.

Torge, W., Accuracy and Stability of the Height Reference Surface, Determination of Heights and Height Changes, Contributions to the Symposium on Height Determination and Recent Vertical Crustal Movements in Western Europe, Hannover, Germany, pp. 69-82, 1987.

Vanicek, P. and Krakiwsky, E.J., Geodesy: The Concepts, Second Edition, North - Holland Publishing Co: Amsterdam/New York/Oxford/Tokyo, 1986.

Vanicek, P., Vertical Datum and NAVD88, Surveying and Land Information Systems, Vol. 51, No. 2, pp. 83-86, 1991.

Vignal, J., Nivellement et Gravité, Troisième Rapport préliminaire de la Section des Nivellements et Section de Gravimétrie présenté au groupe d'études No. 4, IAG, Paris, 1954.

Wang, Y.M., Comments on Proper Use of the Terrain Correction of Height Anomalies, Manuscript, Dept. of Geodetic Science and Survying, The Ohio State University, Columbus, 1993.

Wichiencharoen, C., The Indirect Effects on the Computation of Geoid Undulations, Report No. 336, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, December 1982.

Willis, P. et al., Connection of the two levelling system datum IGN69 and ODN through the channel by using GPS and other techniques, presented at the First International workshop on Geodesy for the Europe-Africa fixed link feasibility in the Strait of Gibraltar, Madrid, 8-10 March 1989.

Xu, P. and Rummel, R., A Quality Investigation of Global Vertical Datum Connection, Netherlands Geodetic Commission, New Series, No. 34, Delft, 1991.

Zhao, S., The Computation of Detailed Geoids Using the Fast Fourier Transform Method, Report No. 400, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1989.

Zilkoski, D.B. and G.M.Young, Status of NGS' North American Vertical Datum (NAVD) Project, Proceedings of Third Int. Symp. on the North American Vertical Datum, NOAA, Rockville, Maryland, pp. 21-36, 1985.

Zilkoski, D.B., Richards, J.H. and Young, G.M., Special Report on Results of the General Adjustment of the North American Vertical Datum of 1988, Surveying and Land Information Systems, Vol. 52, No. 3, pp. 133-149, 1992.