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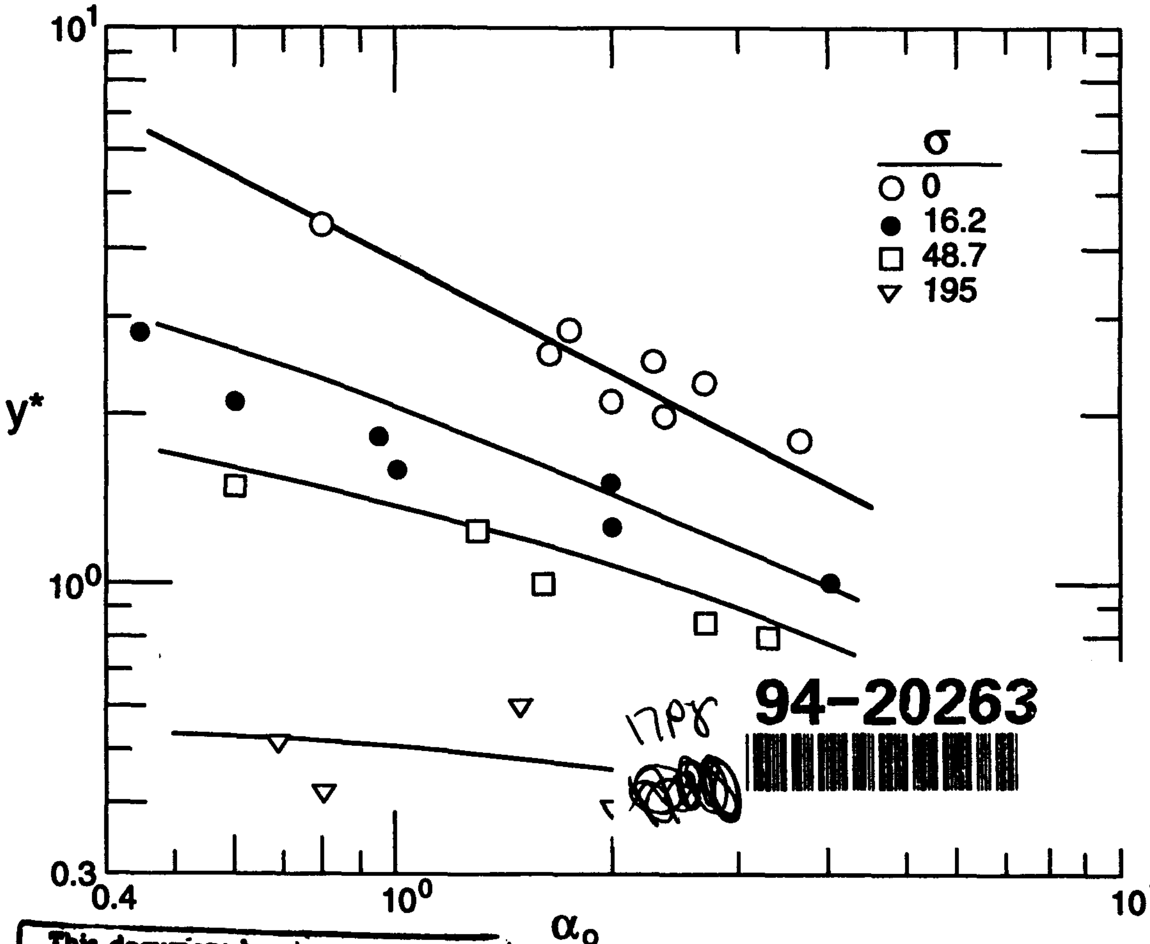
Dependence of Segregation Potential on the Thermal and Hydraulic Conditions Predicted by Model M_1

Yoshisuke Nakano

April 1994



Abstract

The segregation potential (SP) model is semiempirical in nature. An accurate mathematical model is needed that provides the functional dependence of SP on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) des-bing the properties of a given soil. In response to such a need a mathemati model called M_1 was introduced and efforts have been made to validate M_1 with empirical findings and experimental data. In this report we will show that the functional dependence of SP on pertinent variables predicted by M_1 is consistent with empirical findings that were used to build the SP model.

Cover: See Figure 5.

For conversion of SI metric units to U.S./British customary units of measurement consult *Standard Practice for Use of the International System of Units (SI)*, ASTM Standard E380-89a, published by the American Society for Testing and Materials, 1916 Race St., Philadelphia, Pa. 19103.

CRREL Report 94-4

US Army Corps
of Engineers
Cold Regions Research &
Engineering Laboratory

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April 1994

Prepared for
OFFICE OF THE CHIEF OF ENGINEERS

PREFACE

This report was prepared by Dr. Yoshisuke Nakano, Chemical Engineer, of the Applied Research Branch, Experimental Engineering Division, U.S. Army Cold Regions Research and Engineering Laboratory. Funding was provided by DA Project 4A161102AT24, *Research in Snow, Ice and Frozen Ground,* Task SC, Work Unit F01, *Physical Processes in Frozen Soil.*

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CONTENTS

ILLUSTRATIONS

NOMENCLATURE

- a — correction factor
- A — small negative number
- A_0 — positive number
- b — correction factor
- b_i — positive number where $i = 1, 2$
- B — positive number
- c_2 — heat capacity of R_2
- C_i — constant where $i = 0, 1, ..., 4$
- d — unit of time, day
- f — mass flux of water
- F — function defined by eq A1
- I — function defined by eq 12b
- J — function defined by eq 23c, 23d, 25b and 25c
- J_i — function where $i = 1, 2, 3$
- k — thermal conductivity of a frozen fringe
- k_o — thermal conductivity of the unfrozen part of the soil
- k_1 — thermal conductivity of an ice layer
- K_0 — hydraulic conductivity in the unfrozen part of the soil
- K_i — empirical function defined by eq 4a where $i = 1,2$
- K_{i1} — limiting value of K_i as x approaches n_1 while x is in R_1, $i = 1,2$
- K_{io} — limiting value of K_i as x approaches n_0 while x is in R_1, $i = 1,2$
- L — latent heat of fusion of water, 334 J g^{-1}
- m — location of the free end of the column
- n — boundary in R_0.
- n_i — boundary with $i = 0, 1$ where n_0 denotes the boundary where $T = 0°C$ and n_1 the interface between an ice layer and a frozen fringe.
- n_{10} — boundary between R_{10} and R_{11}
- $n_i^+(n_i^-)$ — neighborhood of n_i where $x < n_i$ $(x > n_i)$
- P — pressure of water
- P_a — overburden pressure
- P_0 — value of P at n_0
- P_n — value of P at n
- ΔP — defined by eq 10
- q — heat flux
- R_0 — unfrozen part of the soil
- R_1 — frozen fringe
- R_2 — ice layer
- R_{10} — region where $0 \geq T > T_1^{**}$
- R_{11} — region where $T_1^{**} > T \geq T_1$
- R_m — region in the diagram of temperature gradients where an ice layer melts
- R_s — region in the diagram of temperature gradients where the steady growth of an ice layer occurs
- R_s^* — boundary between R_s and R_u
- R_s^{**} — boundary between R_m and R_s
- R_u — region in the diagram of temperature gradients where the steady growth of an ice layer does not occur
- T — temperature
- T_0 — temperature at $n_0 = 0°C$.
- T_1 — temperature at n_1
- T_1^* — temperature at n_1 at the formation of the final ice lens
- T_1^{**} — temperature at n_1 at the phase equilibrium of water
- T_f' — average temperature gradient in R_1
- U — defined by eq 29a
- U_0 — defined by eq 29b
- V — defined by eq 33a
- V_0 — defined by eq 33b
- x — spatial coordinate
- y — segregation potential function
- y^{**} — value of y at $T_1 = T_1^{**}$
- z — defined by eq 24c
- α_o — absolute value of the temperature gradient at n_0
- α_1 — absolute value of the limiting temperature gradient as x approaches n_1 while x is in R_2, defined by eq 9
- γ — constant, 1.12 MPa $°C^{-1}$
- δ — thickness of a frozen fringe
- δ_0 — distance between n and n_o
- λ_0 — constant defined by eq 24e
- λ_1 — constant defined by eq 24f
- σ — effective pressure defined by eq 13a
- ϕ_{01} — empirical function of T_1 defined by eq 13b
- ϕ_{11} — empirical function of T_1 defined by eq 13c
- $*$ — superscript used to indicate the value of any variable evaluated at the formation of the final ice lens
- $**$ — superscript used to indicate the value of any variable evaluated at the phase equilibrium of water

Dependence of Segregation Potential on the Thermal and Hydraulic Conditions Predicted by Model M_1

YOSHISUKE NAKANO

INTRODUCTION

We will consider the one-directional steady growth of an ice layer under an overburden pressure P_a (≥ 0). Let the freezing process advance from the top down and the coordinate x be positive upwards, with its origin fixed at some point in the unfrozen part of the soil. A freezing soil in this problem may be considered to consist of three parts: the unfrozen part R_0, the frozen fringe R_1 and the ice layer R_2, as shown in *Figure 1* where n_1 is the interface between R_1 and R_2, n_0 is the 0°C isotherm and n is the reference boundary. We will assume that the pressure of water is kept constant, P_n at n. The physical properties of parts R_0, and R_2 are well understood but our knowledge of the physical properties and the dynamic behavior of part R_1 does not appear sufficient for engineering applications.

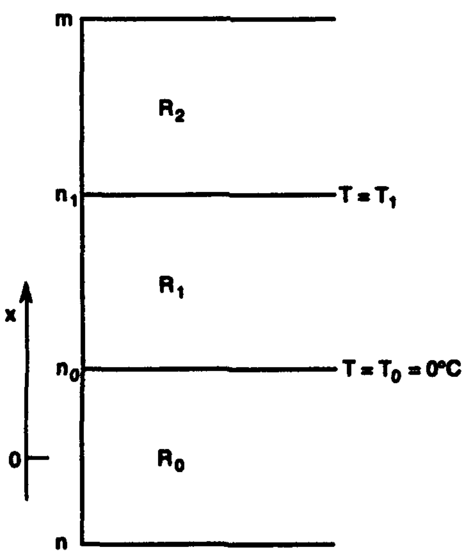

Figure 1. Schematic drawing of a steadily growing ice layer in a freezing soil.

Konrad and Morgenstern (1980, 1981) empirically found that the mass flux of water f at the formation of the so-called final ice lens is proportional to the average temperature gradient T_f' in R_1. This may be written as

$$SP = -f/T_f' \tag{1}$$

where a prime denotes differentiation with respect to x. The positive proportionality factor SP is termed the segregation potential, which is a property of a given soil.

Konrad and Morgenstern (1982b) also found empirically that SP depends on the applied pressure P_a and the pressure of water P_0 at n_0 and that SP is a decreasing function of P_a and the suction $-P_0$. This may be written as

$$SP = y\,(P_0, P_a) \tag{2}$$

$$\frac{\partial}{\partial P_0} y > 0 \;, \quad \frac{\partial}{\partial P_a} y < 0. \tag{3}$$

Extending the concept of segregation potential, Konrad and Morgenstern (1982a) introduced a semiempirical model (SP model) of soil freezing.

Several researchers have used the SP model to analyze frost heave data from field and laboratory tests (Nixon 1982, Knutsson et al. 1985, Jessberger and Jagow 1989). Their results suggest that the accuracy of the SP model suffices engineering needs. However, the SP model is not immune to criticism. Some limitations or shortcomings of the SP model have appeared (Ishizaki and Nishio 1985, Van Gassen and Sego 1989, Nixon 1991).

Nevertheless, it is important to point out that an accurate model of soil freezing, if it exists, should be able to explain important empirical findings such as eq 1, 2 and 3.

Recently the steady growth condition of an ice layer was studied mathematically and experimentally under a negligible applied pressure (Nakano 1990, Takeda and Nakano 1990, Nakano and Takeda 1991) and under an applied pressure (Takeda and Nakano 1993, Nakano and Takeda 1994). They have shown that a model called M_1 accurately describes the experimentally determined condition of steady growth.

For the problem of a steadily growing ice layer, the M_1 model is defined as a frozen fringe where ice may exist but does not grow, and where the mass flux of water f is given as

$$f = -K_1 \frac{\partial P}{\partial x} - K_2 \frac{\partial T}{\partial x} \qquad \text{for } x \text{ in } R_1 \tag{4a}$$

$$K_2/K_1 \to \gamma \qquad \text{as} \qquad f \to 0 \tag{4b}$$

$$\lim_{\substack{x \to n_1 \\ x \text{ in } R_1}} P(x) = P_a \tag{4c}$$

where γ is a constant, P is the pressure of unfrozen water and K_i ($i = 1, 2$) is the transport property of a given soil that generally depends on the temperature and the composition of the soil. The M_1 model is a generalization of somewhat simpler models (Derjaguin and Churaev 1978, Ratkje et al. 1982, Kuroda 1985, Horiguchi 1987) in which the ratio K_2/K_1 is equal to γ regardless of f.

Experimental methods were proposed to determine K_1 (Williams and Burt 1974, Horiguchi and Miller 1983) and K_2 (Perfect and Williams 1980). According to Horiguchi and Miller (1983) K_1 of several frozen porous media is described in the general form given as

$$K_1 = A_0 |T|^{-B} \tag{5}$$

where A_0 and B are positive material constants. Nakano and Takeda (1994) experimentally determined K_2 of Kanto loam in the form given as

$$K_2 = \begin{cases} K_{20} & A \le T < 0 \\ K_{20} |A/T|^B & A > T \end{cases} \tag{6}$$

where K_{20} and B are positive constants and A is a small negative constant. Since empirically determined functions K_1 and K_2 are known to be bounded, the functional form of eq 6 is a better approximation to their actual behaviors in a neighborhood of $T = 0°C$.

The SP model is semiempirical in nature. An accurate mathematical model is needed that provides the functional dependence of *SP* on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) describing the properties of a given soil. In this report we will present the results of our study on the problem of a steadily growing ice layer by using the M_1 model. We will show that the M_1 model provides the functional dependence of SP that is consistent with empirical findings (eq 1, 2 and 3).

PROPERTIES OF M_1

An analytical solution was derived to the problem of a steadily growing ice layer under the M_1 model (Nakano 1990). A schematic drawing of a typical temperature profile is shown in Figure 2, where the temperature is continuous but the temperature gradient is discontinuous at n_1 where the ice layer grows. Since the amount of heat transported by convection is generally much less than that by conduction, the temperature profiles in R_0, and R_2 in the vicinity of R_1 are nearly linear (Nakano and Takeda 1991).

The temperature T_1 and the limiting value of the temperature gradient at n_1 are given approximately (Nakano 1990) as

$$T_1 = -a\, \alpha_0\, \delta \tag{7a}$$

$$T'(n_1^+) = -b\, \alpha_0 \tag{7b}$$

where α_0 is the absolute value of the temperature gradient at n_0, δ is the thickness of R_1 and n_1^+ denotes the limiting value as x approaches n_1 while x is in R_1. The a and b in eq 7a and 7b are correction factors that take account of effects of the convective heat transport and the variable thermal conductivity in R_1 (Nakano and Takeda 1991). When these effects are negligible, a and b are equal to one. We will neglect these effects hereafter.

When an ice layer is steadily growing, according to the M_1 model, ice does not grow in R_1; the growth of ice occurs only at n_1. The balance of heat for R_1 is given as

$$k_1 \alpha_1 = k_0 \alpha_0 + (L - c_2 T_1) f \tag{8}$$

where k_1 and k_0 are the thermal conductivities of R_2

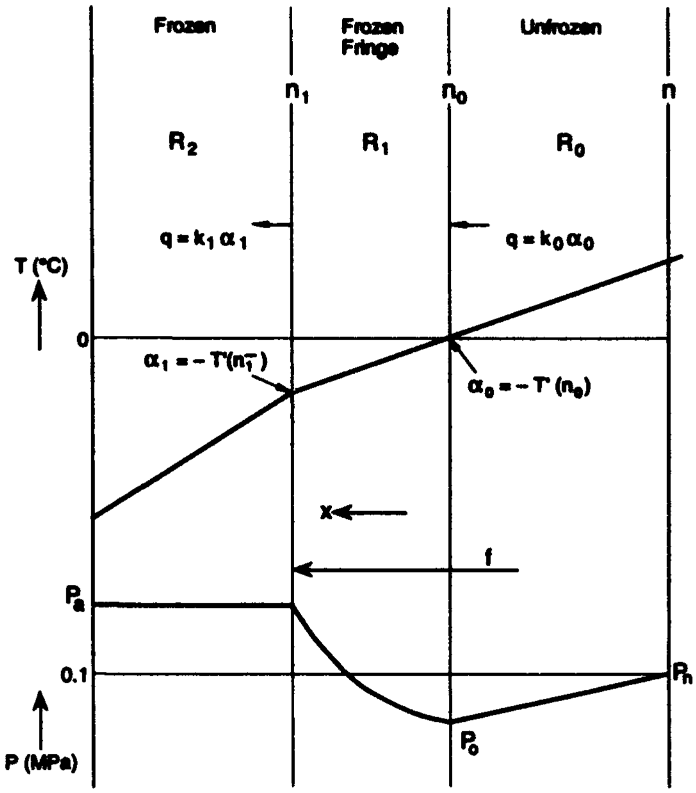

Figure 2. Schematic drawing of typical temperature and pressure profiles according to the M_1 model.

and R_0, L is the latent heat of fusion of water, and c_2 is the heat capacity of R_2. α_1 is the absolute value of the limiting temperature gradient at n_1 defined as

$$\alpha_1 = -\lim_{\substack{x \to n_1 \\ x \text{ in } R_2}} T'(x) = -T'\left(n_1^-\right) \tag{9}$$

where n_1^- denotes the limiting value as x approaches n_1 while x is in R_2.

A schematic drawing of a typical pressure profile is given also in Figure 2 where P_n is kept at 0.1 MPa. The pressure of water P is continuous in the combined region of R_0, and R_1 but the gradient of P may be discontinuous at n_0. Darcy's law clearly holds true in R_0. Neglecting the gravity effect, we obtain

$$\Delta P = P_n - P_0 = \delta_0 K_0^{-1} f \tag{10}$$

where $\delta_0 = n_0 - n > 0$.

It has been shown (Nakano 1992) that according to the M_1 model, the segregation potential function y is given as

$$y = -f^* / T'\left(n_1^+\right) = f^* \alpha_0^{-1} = K_2\left(T_1^*\right) \tag{11a}$$

where an asterisk denotes the value of a variable at the formation of the final ice lens. For given σ and δ_0, the necessary and sufficient condition for the steady growth of an ice layer is given (Nakano and Takeda 1991, Nakano and Takeda 1994) as

$$(k_1/k_0)\alpha_1 > \alpha_0 \geq k_1(k_0 + Ly)^{-1}\alpha_1. \tag{11b}$$

The condition of eq 11b is explained in the diagram of temperature gradients (Fig. 3). R_s^{**} in Figure 3 is given as

$$\alpha_0 = (k_1/k_0)\alpha_1 \quad \text{on} \quad R_s^{**}. \tag{11c}$$

If an existing ice layer neither grows nor melts, f vanishes on R_s^{**}. We will refer to the region as R_m where $\alpha_0 > (k_1/k_0)\alpha_1$, $f < 0$ and an ice layer is melting. R_s^* is given as

$$\alpha_0 = k_1(k_0 + \mathrm{L}y)^{-1}\alpha_1 \quad \text{on} \quad R_s^*. \tag{11d}$$

The boundary R_s^* is a curve stemming from the origin because y depends on α_0. The steady growth of an ice layer occurs in the region R_s. The formation of the final ice lens occurs on R_s^* where eq 11a holds true.

The temperature T_1 is given (Nakano 1990, Nakano and Takeda 1994) as

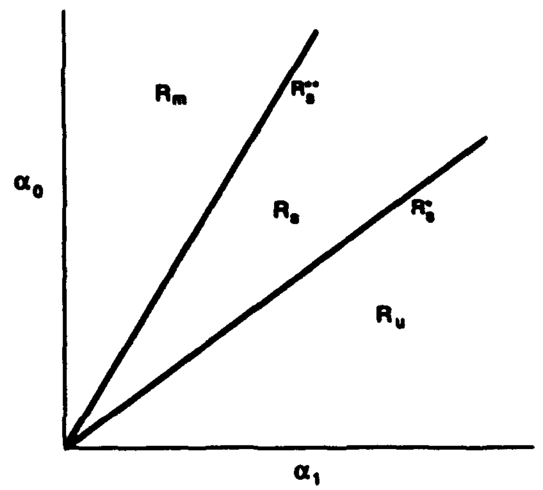

Figure 3. Diagram of temperature gradients α_1 and α_0.

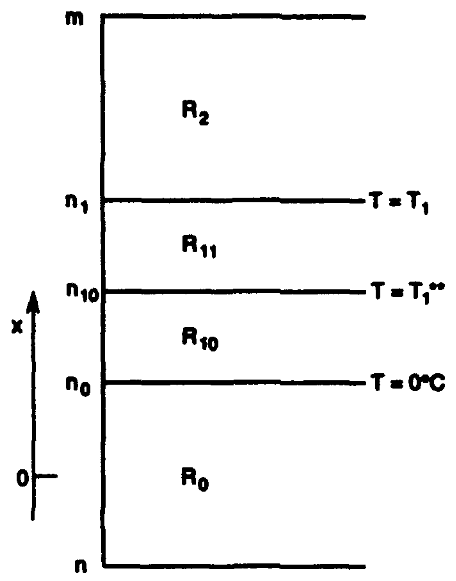

Figure 4. Schematic drawing of a steadily growing ice layer under an applied load.

$$-T_1 = \left(\sigma + \delta_0 K_0^{-1} f\right)/I \tag{12a}$$

$$I = \gamma\,\phi_{01} - K_0^{-1} f\,\alpha_0^{-1}\,\phi_{11} \tag{12b}$$

where σ, ϕ_{01} and ϕ_{11} are defined as

$$\sigma = P_a - P_n \tag{13a}$$

$$\phi_{01}(T_1) = T_1^{-1}\int_0^{T_1}(K_0/K_1)(K_2/K_{20})\,dT \tag{13b}$$

$$\phi_{11}(T_1) = T_1^{-1}\int_0^{T_1}(K_0/K_1)\,dT. \tag{13c}$$

Using eq 11a, 12a and 12b, the mass flux f^* at the formation of the final ice lens is given as

$$f^* = (K_0/\delta_0)\,[-T_1^*\,I - \sigma] \tag{14a}$$

$$I(T_1^*) = \gamma\,\phi_{01}(T_1^*) - K_0^{-1}K_2(T_1^*)\,\phi_{11}(T_1^*). \tag{14b}$$

Equations 14a and 14b imply that f^* is uniquely determined by T_1^* when δ_0 and σ are given. Using eq 11a, we will reduce eq 14a and 14b to:

$$K_2(T_1^*) = K_0(\alpha_0\delta_0)^{-1}\,[-T_1^*\,I - \sigma]. \tag{15}$$

Since K_2 is an increasing function of T_1^*, K_2 and T_1^* are one-to-one and T_1^* may be considered to be a function of K_2. Therefore, eq 15 implies that y is a function of α_0, δ_0 and σ.

$$y = y(\alpha_0, \delta_0, \sigma). \tag{16}$$

It has been shown empirically (Radd and Oertle 1973, Takashi et al. 1981) that there is a unique temperature T_1^{**} at n_1 for a given $\sigma > 0$ when an existing ice layer neither grows nor melts and f vanishes. This temperature T_1^{**} at the phase equilibrium of water is given as

$$\sigma = -\,\gamma T_1^{**}, \quad \text{if} \quad f = 0. \tag{17}$$

When an ice layer is steadily growing under a positive σ, T_1 is less than T_1^{**} (Nakano and Takeda 1994) and the frozen fringe R_1 may be divided into two regions $R_{10}(0 \geq T \geq T_1^{**})$ and $R_{11}(T_1^{**} > T > T_1)$ as shown in Figure 4. At the phase equilibrium f vanishes and R_{11} also vanishes. From eq 4b we obtain

$$K_2/K_1 = \gamma, \quad \text{in } R_{10} \quad \text{if } f = 0. \tag{18}$$

It is easy to see that eq 12a is reduced to eq 17 at the phase equilibrium because of eq 18. An important question arises whether or not eq 18 holds true when f does not vanish. Since K_i ($i = 1, 2$) mainly depends on T and the composition of R_{10}, it is probable that eq 18 holds true for $f > 0$ unless f significantly affects the composition. We will assume the validity of eq 18 regardless of f. With this assumption eq 4a becomes one step closer to the simpler models described earlier.

FUNCTION y

The functional form of y depends on those of K_1 and K_2. Based on available experimental findings (Horiguchi and Miller 1983, Nakano and Takeda 1994), we will assume that K_1 and K_2 are given as

$$K_1 = \begin{cases} K_0 & A \le T < 0 \\ K_0(A/T)^{b_1} & A > T \end{cases} \tag{19}$$

$$K_2 = \begin{cases} K_{20} & A \le T < 0 \\ K_{20}(A/T)^{b_2} & A > T \end{cases} \tag{20}$$

Because of eq 4b K_0 and K_{20} are related as

$$K_{20} / K_0 = \gamma. \tag{21}$$

The values of parameters in eq 19 and 20 determined empirically (Nakano and Takeda 1994) for Kanto loam are

$K_0 = 1.77 \times 10^3$ g/(cm d MPa)
$K_{20} = 1.98 \times 10^3$ g/(cm d °C)
$b_1 = 0.520$
$b_2 = 1.04$
$A = -1.5 \times 10^{-4}$ °C

where d denotes day.

Using eq 19, 20 and 21 and eliminating T_1^{**}, we reduce eq 15 to

$$J_1 - J(\sigma) = \alpha_0 \delta_0 C_0 (1 + J_2) y \tag{22}$$

where

$$J_1 = C_1 y^{-\lambda_0} \tag{23a}$$

$$C_1 = \lambda_1^{-1} K_{20}^{\lambda_0} b_2/(b_1 + 1) \tag{23b}$$

$$J(0) = \lambda_1^{-1} (b_2 - b_1) \tag{23c}$$

$$J(\sigma) = \begin{cases} \lambda_1^{-1} (b_2 - b_1) + z & A < T_1^{**} \\ \lambda_1^{-1} z^{\lambda_1} & A \ge T_1^{**} \end{cases} \tag{23d}$$

$$C_0 = -(A K_{20})^{-1} > 0 \tag{23e}$$

$$J_2 = C_2 (\alpha_0 \delta_0)^{-1} > 0 \tag{24a}$$

$$C_2 = -A b_1 (b_1 + 1)^{-1} > 0 \tag{24b}$$

$$z = C_3 \sigma \tag{24c}$$

$$C_3 = -(\gamma A)^{-1} > 0 \tag{24d}$$

$$\lambda_0 = \lambda_1 / b_2 \tag{24e}$$

$$\lambda_1 = b_1 - b_2 + 1. \tag{24f}$$

For a special case where $\lambda_1 = 0$, J_1 and J are given as

$$J_1 = -b_2^{-1} \ln(y/K_{20}) \tag{25a}$$

$$J(0) = -b_1 b_2^{-1} \tag{25b}$$

$$J(\sigma) = \begin{cases} -b_1 b_2^{-1} + z & A < T_1^{**} \\ -b_2^{-1} + \ln z & A \ge T_1^{**} \end{cases} \tag{25c}$$

It is clear from eq 10 and 11 that α_0 and δ_0 must be positive because the mass flux f^* is positive at the formation of the final ice lens. When $\alpha_0\delta_0$ is very small, eq 22 is reduced to

$$J_1 - J(\sigma) = C_0 C_2 y \tag{26}$$

It follows from eq 26 that y no longer depends on α_0 or δ_0 when $\alpha_0\delta_0$ is very small. As shown in Appendix A, eq 22 generally possesses a unique positive root y. However, under certain conditions this unique root may not be physically meaningful. This implies that the formation of the final ice lens may not take place under such conditions.

When eq 22 possesses as a meaningful root, we will study the dependence of y on parameters, α_0, δ_0, ΔP and σ below. We will write eq 22 as

$$J_1 - J(\sigma) = \alpha_0 \delta_0 C_0 y + C_0 C_2 y. \tag{27}$$

Differentiating eq 27 with respect to α_0 and δ_0, we obtain

$$\frac{\partial y}{\partial \alpha_0} = \begin{cases} -\delta_0 C_0 y U^{-1} & \lambda_1 \ne 0 \\ -\delta_0 C_0 y U_0^{-1} & \lambda_1 = 0 \end{cases} \tag{28a}$$

$$\frac{\partial y}{\partial \delta_0} = \begin{cases} -\alpha_0 C_0 y U^{-1} & \lambda_1 \ne 0 \\ -\alpha_0 C_0 y U_0^{-1} & \lambda_1 = 0 \end{cases} \tag{28b}$$

where

$$U = \alpha_0 \delta_0 C_0 + C_0 C_2 + C_1 \lambda_0 y^{-(\lambda_0 + 1)} \tag{29a}$$

$$U_0 = \alpha_0 \delta_0 C_0 + C_0 C_2 + b_2^{-1} y^{-1} \tag{29b}$$

$$C_1 \lambda_0 = K_{20}^{\lambda_0} / (b_1 + 1) > 0. \tag{29c}$$

It follows from eq 28a and 28b that y is a decreasing function of both α_0 and δ_0.

From eq 10 and 11, we obtain

$$\alpha_0 \delta_0 = K_0 \Delta P y^{-1}. \tag{30}$$

Using eq 30, we write eq 22 as

$$J_1 - J(\sigma) = K_0 C_0 \Delta P + C_0 C_2 y. \tag{31}$$

Differentiating eq 31 with respect to ΔP, we obtain

$$\frac{\partial y}{\partial \Delta P} = \begin{cases} -K_0 C_0 V^{-1} & \lambda_1 \neq 0 \\ -K_0 C_0 V_0^{-1} & \lambda_1 = 0 \end{cases} \tag{32}$$

where

$$V = C_0 C_2 + C_1 \lambda_0 y^{-(\lambda_0 + 1)} \tag{33a}$$

$$V_0 = C_0 C_2 + b_2^{-1} y^{-1}. \tag{33b}$$

It follows from eq 32 that y is a decreasing function of ΔP. This property of y is consistent with the empirical finding (the first inequality of eq 3).

Differentiating eq 27 with respect to σ, we obtain for $\lambda_1 \neq 0$:

$$\frac{\partial y}{\partial \sigma} = \begin{cases} -C_3 U^{-1} & A < T_1^{**} \\ -C_3^{\lambda_1} \sigma^{\lambda_1 - 1} U^{-1} & A \geq T_1^{**} \end{cases}. \tag{34a}$$

When $\lambda_1 = 0$, we obtain

$$\frac{\partial y}{\partial \sigma} = \begin{cases} -C_3 U_0^{-1} & A < T_1^{**} \\ -\sigma^{-1} U_0^{-1} & A \geq T_1^{**} \end{cases}. \tag{34b}$$

It is easy to find from eq 34a and 34b that y is a decreasing function of σ. This property of y is consistent with the empirical finding (the second inequality of eq 3).

Using experimentally determined parameters of Kanto loam (Nakano and Takeda 1994), we will show the behavior of y of Kanto loam below. For Kanto loam the values of λ_0 and λ_1 are 0.462 and 0.480, respectively. We will present the actual values of each term in eq 27:

$$J_1 = 4.75 \times 10\, y^{-0.462} \tag{35a}$$

$$J(0) = 1.08 \tag{35b}$$

$$z = 5.95 \times 10^3 \sigma \tag{35c}$$

$$J(\sigma) = \begin{cases} 1.08 + 5.95 \times 10^3 \sigma & A < T_1^* \\ 6.48 \times 10\, \sigma^{0.480} & A \geq T_1^* \end{cases} \tag{35d}$$

$$\alpha_0 \delta_0 C_0 y = 3.37 \alpha_0 \delta_0 y \tag{35e}$$

$$C_0 C_2 y = 1.75 \times 10^{-4} y \tag{35f}$$

where units of variables are y [g(cm °C d)$^{-1}$], σ [MPa], α_0 [°C cm^{-1}] and δ_0 [cm].

It is easy to find from eq 35e and 35f that the term C_0C_2y is much less than the term $\alpha_0\delta_0C_0y$ unless $\alpha_0\delta_0$ is very small. Neglecting this small term, we reduce eq 27 to

$$J_1 - J(\sigma) = \alpha_0 \delta_0 C_0 y. \tag{36a}$$

Using ΔP, we write eq 36a as

$$J_1 - J(\sigma) = K_0 C_0 \Delta P \tag{36b}$$

When $\sigma = 0$, the term $J(0)$ is much less than the other two terms of eq 36a or 36b. When $J(0)$ is neglected, eq 36a and 36b are reduced to

$$y = (C_1 / C_0)^{1/(\lambda_0 + 1)} (\alpha_0 \delta_0)^{-1/(\lambda_0 + 1)} \tag{37a}$$

$$y = [C_1 / (K_0 C_0)]^{1/\lambda_0} (\Delta P)^{-1/\lambda_0}. \tag{37b}$$

Suppose that we measure a set of values either (y, $\alpha_0\delta_0$) or (y, ΔP) and that ln y and ln $\alpha_0\delta_0$ (or ln ΔP) are linear. If we know one of three parameters, b_1, b_2 and A, the remaining two parameters can be determined by eq 37a (or eq 37b). When $\sigma \neq 0$, the term $J(\sigma)$ is not generally negligible and we must use eq 36a (or eq 36b) to determine an accurate dependence of y on $\alpha_0\delta_0$ (or ΔP).

In Figure 5 we plotted experimental data, y vs. α_0 with $\delta_0 = 2$ cm under $\sigma = 0$, 16.2, 48.7 and 195 kPa (Nakano and Takeda 1994) together with predicted y calculated by eq 36a. It is easy to see that ln y and ln α_0 are nearly linear when $\sigma = 0$. As σ increases the relationship between ln y and ln α_0 becomes nonlinear as anticipated.

CONCLUDING REMARKS

Many models of ice segregation have been proposed in the past. However, the *SP* model proposed by Konrad and Morgenstern (1980, 1981) is one of few that were built on an empirical base. The SP model has been conveniently used to solve engineering problems. However, we need an accurate mathematical model that provides the functional dependence of *SP* on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) describing the properties of a given soil.

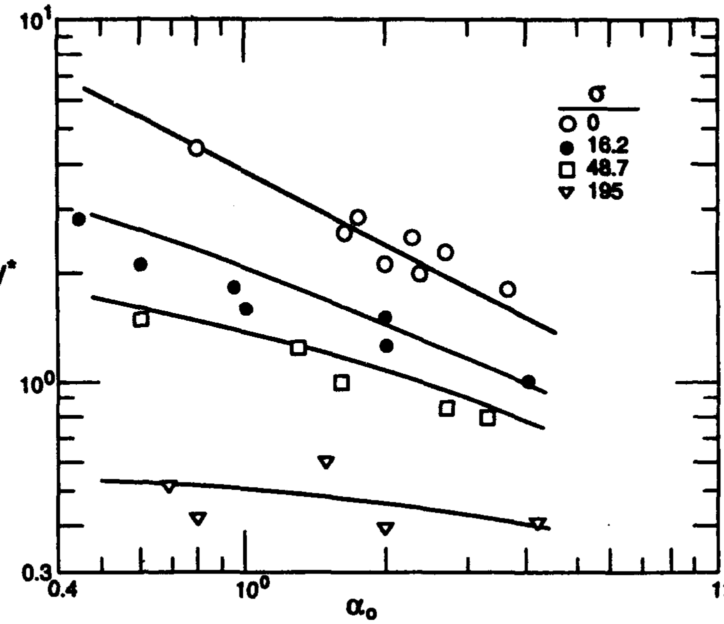

Figure 5. Experimental data of y [g(cm °C d)$^{-1}$] vs. α_0 [°C/cm] together with predicted y calculated by eq 36a.

In response to such a need Nakano (1990) proposed a new mathematical model called M_1. Efforts have been made to validate the M_1 model by empirical findings and experimental data (Takeda and Nakano 1990, Nakano and Takeda 1991, Takeda and Nakano 1993, Nakano and Takeda 1994). In this report we presented the result of our efforts to validate the M_1 model by using empirical findings that were used to build the *SP* model. It is important to mention that eq 1, 2 and 3 are empirical findings, not assumptions. If the M_1 model is accurate, it must be able to explain these important empirical relationships. We have shown that the M_1 model can explain these relationships.

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APPENDIX A: POSITIVE ROOT OF EQUATION 22

We will show that eq 22 possesses a unique positive root $y > 0$. Using eq 27 that is equivalent to eq 22, we will introduce a function $F(y)$ defined as:

$$F(y) = J_1 - \alpha_0 \delta_0 C_0\, y - C_0 C_2\, y - J(\sigma). \tag{A1}$$

It is clear that a root of eq 22 satifies

$$F(y) = 0 \tag{A2}$$

Differentiating $F(y)$ with respect to y, we obtain

$$F'(y) = J_1'(y) - (\alpha_0 \delta_0 C_0 + C_0 C_2) \tag{A3}$$

where a prime denotes the differentiation with respect to y.

From eq 23a and 25a we obtain

$$J'(y) = \begin{cases} -\lambda_0 C_1 y^{-(\lambda_0+1)} & \lambda_1 \neq 0 \\ -b_2^{-1}\, y^{-1} & \lambda_1 = 0 \end{cases} \tag{A4}$$

Since $\lambda_0 C_1$ and b_2 are positive, $J'(y)$ is negative. Hence $F'(y)$ is also negative, in other words, $F(y)$ is a strictly decreasing function of y. We will examine the behavior of $F(y)$ for two cases: Case 1, $\lambda_1 \geq 0$, and Case 2, $\lambda_1 < 0$.

For Case 1, $F(y)$ approaches $+\infty$ as y approaches 0 while $F(y)$ approaches $-\infty$ as y approaches $+\infty$. Since $F(y)$ is a strictly decreasing function, eq 22 possesses a unique positive root. For Case 2, $F(y)$ approaches $-J(\sigma)$ as y approaches 0 while $F(y)$ approaches $-\infty$ as y approaches $+\infty$. When $\lambda_1 < 0$, $b_2 > b_1 + 1$. Hence $J(\sigma)$ is negative and eq 22 possesses a unique positive root. It may be noted parenthetically that $A < T_1^{**}$ implies $z < 1$ because of eq 17

We have shown that eq 22 possesses a unique positive root regardless of λ_1. When an ice layer is growing, the temperature T_1 must be less than T_1^{**}. This implies that y must be less than $K_2(T_1^{**})$, namely:

$$y < y^{**} = K_2(T_1^{**}) = K_{20}\, z^{-b_2}. \tag{A5}$$

It should be noted that $y^{**} = K_{20}$ when $\sigma = 0$. Equation A5 implies that $F(y^{**})$ must be negative; that is

$$F(y^{**}) = J_3(y^{**}, \sigma) - C_4 z^{-b_2} < 0 \tag{A6}$$

where

$$J_3 = J_1(y^{**}) - J(\sigma) \tag{A7}$$

$$C_4 = K_{20}\, C_0 (\alpha_0 \delta_0 + C_2) > 0. \tag{A8}$$

The function J_3 is given as:

$$J_3 = \begin{cases} b_1 b_2^{-1} > 0, & \lambda_1 = 0\,, \sigma = 0 \\ \ln z + b_1 b_2^{-1} - z & \lambda_1 = 0\,, A < T_1^{**} \\ b_2^{-1} > 0, & \lambda_1 = 0\,, A \geq T_1^{**} \end{cases} \quad . \tag{A9}$$

$$J_3 = \begin{cases} b_1 b_2^{-1} > 0, & \lambda_1 = 0\,, \sigma = 0 \\ \ln z + b_1 b_2^{-1} - z & \lambda_1 = 0\,, A < T_1^{**} \\ b_2^{-1} > 0, & \lambda_1 = 0\,, A \geq T_1^{**} \end{cases} \quad . \tag{A10}$$

It follows from eq A6, A9 and A10 that the condition of eq A6 is always satisfied when $\lambda_1 \neq 0$ and $A \geq T_1^{**}$. However, for other cases it is not certain that eq A6 is satisfied.

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The segregation potential (SP) model is semiempirical in nature. An accurate mathematical model is needed that provides the functional dependence of SP on pertinent variables specifying given thermal and hydraulic conditions in terms of well-defined functions (or parameters) describing the properties of a given soil. In response to such a need a mathematical model called M_1 was introduced and efforts have been made to validate M_1 with empirical findings and experimental data. In this report we will show that the functional dependence of SP on pertinent variables predicted by M_1 is consistent with empirical findings that were used to build the SP model.

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