Clutters that Pack and the Max Flow Min Cut Property: A Conjecture

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Clutters that Pack and the Max Flow Min Cut Property: A Conjecture

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A clutter $C$ is a collection $E(C)$ of subsets of a finite set $V(C)$ with the property that $A_1 \not\subseteq A_2$ for all $A_1, A_2 \in E(C)$. Let $M(C)$ be the 0,1 matrix with columns indexed by $V(C)$ whose rows are the incidence vectors of the members of $E(C)$.

Consider the two linear programs

$$\min \{wx : x \geq 0, \ M(C)x \geq 1\} \quad (1)$$
$$\max \{y1 : y \geq 0, \ yM(C) \leq w\}. \quad (2)$$

The clutter $C$ has the Max Flow Min Cut property if (1) and (2) have integral optimum solutions $x$ and $y$ for all nonnegative integral vectors $w$. $C$ is ideal if (1) has an integral optimum solution $x$ for all nonnegative (integral) vectors $w$. $C$ packs if (1) and (2) have integral optimum solutions $x$ and $y$ when $w = 1$.

For a clutter $C$, the deletion $C\setminus j$ and contraction $C/j$ of an element $j \in V(C)$ are clutters defined as follows: $V(C\setminus j) = V(C/j) = V(C) - \{j\}$, $E(C\setminus j) = \{A \in E(C) : j \not\in A\}$ and $E(C/j)$ are the minimal members of

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\{A - \{j\} : A \in E(C)\}. Any clutter obtained from \(C\) by repeated application of the contraction and deletion operations is called a minor of \(C\).

We propose the following conjecture.

**Conjecture 1** A clutter \(C\) has the Max Flow Min Cut property if and only if \(C\) and all its minors pack.

**Remark 1** The "only if" part of the above statement is obvious since saying that a minor of \(C\) packs is equivalent to stating that (1) and (2) have integral solution vectors \(x\) and \(y\) for the objective function \(w_j = 0\) for a deleted element \(j\), \(w_j = \infty\) for a contracted element \(j\) and \(w_j = 1\) for the remaining elements.

**Remark 2** A weakening of the "if" condition follows from Lehman's characterization of ideal clutters [2] which implies the following: If \(C\) is not ideal, it contains a minor \(C'\) such that the linear program (1) associated with \(C'\) has a unique optimum solution \(x\) whose components are all fractional when \(w = 1\). It follows that if \(C\) and all its minors pack, then \(C\) is ideal.

**Remark 3** A special case where the conjecture holds is when \(C\) is a binary clutter. Seymour [4] proved that a binary clutter has the Max Flow Min Cut property if and only if it does have a \(Q_6\) minor, where \(V(Q_6) = \{1,2,3,4,5,6\}\) and \(E(Q_6) = \{(1,3,5), (1,4,6), (2,3,6), (2,4,5)\}\). It is easy to check that the linear program (2) associated with \(Q_6\) has a unique optimum solution vector \(y = \frac{1}{2}\) when \(w = 1\).

**Remark 4** This conjecture is the exact analog of the replication lemma used in the proof of the Fulkerson-Lovász [1] [3] pluperfect graph theorem. Let

\[
\begin{align*}
\max \{wx : x \geq 0, M(C)x \leq 1\} & \quad (3) \\
\min \{y1 : y \geq 0, yM(C) \geq w\} & \quad (4)
\end{align*}
\]

The replication lemma shows that (3) and (4) have integral optimum solution vectors \(x\) and \(y\) for every 0,1 vector \(w\) if and only if they have integral optimum solution vectors for every (nonnegative) integral vector \(w\).

The pluperfect graph theorem states that if (4) has an integral optimum solution \(y\) for every 0,1 vector \(w\), then (3) and (4) have integral optimum solutions \(x\) and \(y\) for every (nonnegative) integral vector \(w\).
Remark 5 To prove the "if" part of Conjecture 1, it is sufficient to show the following.

Conjecture 2 (Replication Conjecture) If $C$ and all its minors pack, then the clutter $C_j$ defined below packs. For $j \in V(C)$, let

$$V(C_j) = V(C) \cup \{j'\}$$

$$E(C_j) = E(C) \cup \left( \bigcup_{A \neq j} A \setminus \{j\} \cup \{j'\} \right).$$

Indeed, observe that the linear programs (1) and (2) have integral optimum solutions for the vector $w$ such that $w_j = 2$ and $w_i = 1$ for $i \neq j$ if and only if the clutter $C_j$ packs. Therefore, using the replication conjecture recursively, it follows that $C$ has the Max Flow Min Cut property.

References


