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   Angular Intensity Correlations in the Double Passage of Waves through a Random Phase Screen

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13. **ABSTRACT** (Maximum 200 words)
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The problem of light scattering in folded-path or double-passage configurations is studied theoretically. Assuming as the random medium a deep phase screen that introduces Gaussian distributed phase fluctuations, we study the motion of the speckle as the source is moved. Some attention is also given to the phenomenon of backscattering enhancement. Our analysis is based on a novel expression for the complex amplitude that has a simple physical interpretation. For simplicity, only the one-dimensional case is considered, but an extension of the analysis to two-dimensional screens is not difficult.

Using the factorization properties of the moments of a complex Gaussian process, we are able to derive analytical expressions for the mean intensity and the intensity correlation of the backscattered radiation. We find that, in most cases, the speckle field decorrelates rapidly as one moves the angle of incidence and shifts toward the direction of specular reflection with a rate of motion that is different from that of the angle of incidence. We also find conditions under which, when the angle of incidence is modified, the speckle pattern produced in the region of observation tracks the backscattering direction.

Key words: Enhanced backscattering, double-passage analysis, speckle motion, multiple scattering, random phase screen.

1. Introduction

The problem of light scattering in folded-path or double-passage configurations has been studied in the last three decades by several authors.\textsuperscript{1-6} The system is known to exhibit the phenomenon of backscattering enhancement. This phenomenon can be explained by considering the coherent addition of waves that follow identical paths in opposite directions on interaction with the random medium. Most of these studies have dealt with the propagation through an extended random medium, and Ref. 6 provides several references for this case. More recently, the scattering system consisting of a plane mirror in the diffraction field of a deep random phase screen has also been studied by several authors.\textsuperscript{4,5,7-10} To our knowledge, however, only the mean intensity and the first-order spatial coherence function of the scattered light have been studied.

In this paper we study theoretically the deep phase screen double-passage configuration [see Fig. 1(a)] with particular interest in the correlation of the intensity fluctuations that describes the motion of the speckle as the source is moved, but some attention is also given to the angular distribution of the mean intensity around the backscattering direction. Our analysis is based on an expression for the scattered complex amplitude that can be obtained by manipulating the diffraction integrals that describe the propagation to the observation point. Assuming that a Gaussian speckle pattern\textsuperscript{11} is formed in the plane of the mirror and that the speckle grain on the mirror is much smaller than the mirror aperture, we derive exact (within the model) analytical expressions for the mean intensity and the intensity correlation of the backscattered radiation by using the factorization properties of the moments of a complex Gaussian process. In this paper we deal with a one-dimensional random phase screen that introduces smoothly varying Gaussian-distributed phase fluctuations, but the calculations can be easily extended to the case of two-dimensional phase screens.

We find that, under certain circumstances, when the angle of incidence is modified, the speckle pattern produced in the region of observation tracks the backscattering direction. This is in contrast with the well-known laws of speckle motion with random surfaces\textsuperscript{12,13} and with the type of motion found for volume scatterers.\textsuperscript{14-17} Another interesting feature present in our calculations is that, in some cases, the
The complex amplitude \(A(x, 0_i)\) is given by

\[
A(x, 0_i) = \frac{1}{K} \exp \left( -\frac{x^2}{W_1^2} \right) \exp \left[ i k x_1 \sin \theta_i \right] \exp \left[ -\frac{ik}{2d} (x_1 - x)^2 \right] dx_1. \tag{2}
\]

where \(\phi(x_1)\) is the random phase introduced by the screen, \(d\) is the distance from the screen to the mirror, and \(K\) is another complex constant. The aperture function in the mirror plane is given by

\[
T(x) = \exp \left( -\frac{x^2}{W_2^2} \right), \tag{3}
\]

where \(W_2\) is the aperture radius of the mirror.

Using these results, we propose a modification of the original scattering geometry. With this new configuration, the backscattering enhancement, the symmetry of the speckle pattern, and the tracking of the speckle pattern of the backscattering direction are present under less restrictive conditions.

### 2. Formulation of the Problem

The double-passage configuration is shown in Fig. 1(a). A plane mirror lies in the Fresnel region of the phase screen. Finite apertures are located in the plane of the mirror and of the screen. The angles of incidence \(\theta_i\) and of scattering \(\theta_f\) are taken as positive in the sense indicated by the arrows in the figure. It is clear that the unfolded geometry of Fig. 1(b) is equivalent to that shown in Fig. 1(a). The aperture \(W_1\) refers to the width of the incoming beam.

#### A. Scattered Complex Amplitude

With reference to Fig. 1(b), we see that a nearly collimated monochromatic Gaussian beam of \(1/e^2\) amplitude radius \(W_1\) is incident upon the first phase screen, which is located on the plane \(x_1\), at an angle \(\theta_i\) with respect to the \(z\) axis. This incident complex amplitude \(A_i(x_1)\) may be written as

\[
A_i(x_1) = K_0 \exp \left( -ik x_1 \sin \theta_i \right) \exp \left( -\frac{x^2}{W_1^2} \right). \tag{1}
\]

where \(K_0\) is a complex constant and \(k = 2\pi/\lambda\), \(\lambda\) being the wavelength of the incident light. The complex amplitude of the scattered light in the plane of the mirror, which is denoted by \(A(x, \theta_f)\), can be found by using the Kirchhoff diffraction theory in the Fresnel approximation. We have

\[
A(x, \theta_f) = K_1 \int_{-\infty}^{\infty} A_i(x_1) \exp \left[ i \phi(x_1) \right] \exp \left[ i k \frac{1}{2d} (x_1 - x)^2 \right] dx_1, \tag{2}
\]

where \(\phi(x_1)\) is the random phase introduced by the screen, \(d\) is the distance from the screen to the mirror, and \(K_1\) is another complex constant. The aperture function in the mirror plane is given by

\[
T(x) = \exp \left( -\frac{x^2}{W_2^2} \right), \tag{3}
\]

where \(W_2\) is the aperture radius of the mirror. The complex amplitude \(A(x_2)\), which is incident upon the second phase screen located in the plane \(x_2\), can be written as

\[
A(x_2) = K_2 \int_{-\infty}^{\infty} A(x, \theta_f) T(x) \exp \left[ i k \frac{1}{2d} (x_2 - x)^2 \right] dx, \tag{4}
\]

where \(K_2\) is a complex constant, and we have again made use of the Fresnel approximation. At an observation point in the far field of the second screen, the scattered complex amplitude can be written as

\[
A(\theta_f, \theta_i) = K \int_{-\infty}^{\infty} A(x_2) \exp \left( -\frac{x^2}{W_2^2} \right) \exp \left[ i \phi(x_2) \right] \exp \left[ -ik x_2 \sin \theta_i \right] dx_2, \tag{5}
\]

where \(\theta_i\) denotes the direction of observation and \(K\) is another complex constant; we have included a Gaussian aperture function of radius \(W_2\) in the plane of the second screen. From Eqs. (3)-(5) we may write, after interchanging the orders of integration,

\[
A(\theta_f, \theta_i) = K \int_{-\infty}^{\infty} \exp \left( -\frac{x^2}{W_2^2} \right) A(x, \theta_f) A(x, \theta_i) dx, \tag{6}
\]

where

\[
A(x, \theta_f) = K_2 \int_{-\infty}^{\infty} \exp \left( -\frac{x^2}{W_2^2} \right) \exp \left[ i \phi(x_2) \right] \exp \left[ -ik x_2 \sin \theta_f \right] dx_2. \tag{7}
\]

The quantity \(A(x, \theta_f)\) is the complex amplitude that a point source located at the observation point in the far field would produce in the plane of the mirror. Substitution of Eq. (1) into Eq. (2) shows that, except for an irrelevant constant factor, Eqs. (2) and (7) have exactly the same form. Our approach is based on Eq. (6) and a few additional assumptions that are described in Subsection 2.B. Equation (6) has a simple physical interpretation: mathematically it is the product of two speckle patterns integrated over
the mirror aperture. This expression has some close parallels with the equation describing image formation in confocal scanning microscopy.

B. Correlation of the Scattered Intensity

At this point it is necessary to introduce a statistical model for the phase fluctuations \( \phi(x) \). We assume that \( \phi(x) \) is a stationary zero-mean Gaussian random process with a Gaussian correlation coefficient \( \rho(x-x') \): 

\[
\rho(x-x') = \frac{\langle \phi(x)\phi(x') \rangle}{\sigma^2} = \exp \left( -\frac{|x-x'|^2}{\xi^2} \right), \tag{8}
\]

where \( \sigma^2 \) is the variance of the random phase fluctuations and \( \xi \) is the characteristic correlation length of the process.

Now, for a deep phase screen, \( \sigma^2 \gg 1 \), and if, in addition, \( W_r^2 \gg \xi^2 \) and \( (k\xi^2/2d) \ll 1 \), then the complex amplitude \( A(x, \theta) \) corresponds to that of a fully developed speckle pattern. That is, \( A(x, \theta) \) is a zero-mean circular complex Gaussian random variable (CCGRV). The condition \( (k\xi^2/2d) \ll 1 \) ensures that the mirror is far from the focusing region of the screen where strong departures from Gaussian statistics take place. Likewise, if \( W_r^2 \gg \xi^2 \), \( A(x, \theta) \) is also a zero-mean CCGRV. In addition, if \( W_r \) is much greater than the speckle size in the plane of the mirror, the scattered complex amplitude \( A(\theta_1, \theta_2) \) is also a zero-mean CCGRV.

Consider now the intensity \( I(\theta_1, \theta_2) \) at detection angle \( \theta_1 \), which is due to an incoming beam incident at angle \( \theta_2 \), and the intensity \( I(\theta_2', \theta_1') \) at detection angle \( \theta_2' \), which is due to an incoming beam at incident angle \( \theta_1' \). With the statements made above regarding the statistics of \( A(\theta_1, \theta_2) \), the intensity correlation \( C_1(\theta_1, \theta_2; \theta_1', \theta_2') \) is given by \( \langle I(\theta_1, \theta_2)I(\theta_1', \theta_2') \rangle \) can be written as:

\[
C_1(\theta_1, \theta_2; \theta_1', \theta_2') = \langle I(\theta_1, \theta_2)I(\theta_1', \theta_2') \rangle = \langle I(\theta_1, \theta_2) \rangle \langle I(\theta_1', \theta_2') \rangle + \langle I(\theta_1, \theta_2) \rangle \langle I(\theta_1', \theta_2') \rangle = \langle I(\theta_1, \theta_2) \rangle \langle I(\theta_1', \theta_2') \rangle, \tag{9}
\]

where

\[
C_A(\theta_1, \theta_2; \theta_1', \theta_2') = \langle A(\theta_1, \theta_2)A^*(\theta_1', \theta_2') \rangle. \tag{10}
\]

Using Eq. (6), we see that the problem reduces to the calculation of the function:

\[
C_A(\theta_1, \theta_2; \theta_1', \theta_2') = \langle A(\theta_1, \theta_2)A^*(\theta_1', \theta_2') \rangle = \langle A(\theta_1, \theta_2)A^*(\theta_1', \theta_2') \rangle + \langle A(\theta_1, \theta_2)A^*(\theta_1', \theta_2') \rangle = \langle A(\theta_1, \theta_2) \rangle \langle A^*(\theta_1', \theta_2') \rangle, \tag{11}
\]

Recalling again the statistical considerations made above, we also find that the fourth-order moment in the integrand of Eq. (11) can be expressed as a sum of products of first-order correlation functions.

![Image](https://via.placeholder.com/150)

Thus we can write

\[
C_A(\theta_1, \theta_2; \theta_1', \theta_2') = C_1(\theta_1, \theta_2; \theta_1', \theta_2') + C_2(\theta_1, \theta_2; \theta_1', \theta_2'), \tag{12}
\]

where

\[
C_1(\theta_1, \theta_2; \theta_1', \theta_2') = |K|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{x^2 + x'^2}{W_m^2} \right) \langle A(x, \theta)A^*(x', \theta') \rangle \langle A(x', \theta') \rangle \, dx \, dx', \tag{13}
\]

and

\[
C_2(\theta_1, \theta_2; \theta_1', \theta_2') = |K|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{x^2 + x'^2}{W_m^2} \right) \langle A(x, \theta)A^*(x', \theta') \rangle \langle A^*(x', \theta') \rangle \, dx \, dx'. \tag{14}
\]

The last two expressions involve only the calculation of second-order moments or correlations of the amplitude, and are much simpler to calculate than the fourth-order moment of Eq. (11). The amplitude correlations in Eqs. (13) and (14) were evaluated in the limit \( \sigma^2 \gg 1 \) by using a steepest-descent method (see Appendix A). Using these results, we can obtain analytical expressions for \( C_1(\theta_1, \theta_2; \theta_1', \theta_2') \) and \( C_2(\theta_1, \theta_2; \theta_1', \theta_2') \) without further approximations. The result involves some rather long algebraic expressions that are difficult to reduce and thus are given in Appendix A. A simple formula for a particular case is given in Section 3.

The mean intensity is then calculated from

\[
\langle I(\theta_1, \theta_2) \rangle = C_A(\theta_1, \theta_2; \theta_1, \theta_2), \tag{15}
\]

while the normalized intensity correlation \( \gamma(\theta_1, \theta_2; \theta_1', \theta_2') \) is given by

\[
\gamma(\theta_1, \theta_2; \theta_1', \theta_2') = 1 + 1 = 1 + |C_A(\theta_1, \theta_2; \theta_1, \theta_2')|^2. \tag{16}
\]

It is pertinent to mention that, recently, a compact formula for the mean intensity produced by the scattering of a Gaussian beam by the double passage through a deep phase screen has been derived that also uses the factorization properties of the moments of a Gaussian random process.

3. Results

We now turn our attention to the study of the behavior of the angular distribution of the mean intensity around the backscattering direction. Figure 2 shows the evolution of the enhancement factor \( E_f \) as the distance from the phase screen to the
mirror increases for the case of infinite apertures at the mirror and the second screen. We define the enhancement factor $E_f$ as the ratio of the mean intensity in the backscattering direction to the mean background intensity. In Fig. 2, the circles represent the values obtained from the evaluation of the general formulas given in Appendix A, while the solid curve is a plot of the expression

$$E_f = 1 + \frac{1}{[1 + (d/L)^2]^{1/2}},$$  \hspace{1cm} (17)$$

where the normalization distance $L$ is given by

$$L = \frac{k\xi W_1}{2\sigma}.$$  \hspace{1cm} (18)$$

For the numerical calculations presented we have set $\lambda = 0.6328 \, \mu\text{m}$, $\sigma = 8 \, \text{rad}$, $\xi = 5.0 \, \mu\text{m}$, and $W_1 = 2$ mm. So the parameter $L$ was fixed at the value $L = 6.206 \, \text{mm}$. Equation (17) can be derived from the results given in Ref. 4. The agreement between the two plots is excellent. It is seen that the height of the peak diminishes as the screen to the mirror distance increases. This is because, as this distance increases, more of the scattered light reflected by the mirror onto the screen lies outside the region illuminated in the first passage. It is only the region of overlap that contributes to the backscattering enhancement. The light reflected outside this region contributes to only the background scattered intensity.

The relative enhancement can also be increased by reducing the area of the screen illuminated in the second passage by placing a physical aperture at either the mirror or at the screen. This is illustrated in Fig. 3 for a normalized distance $d/L = 6.5$, for which expression (17) predicts a small enhancement (see Fig. 2). In Fig. 3(a) the aperture radius of the mirror has been set to one half of the radius of the incoming Gaussian beam. A high value is obtained for the relative enhancement because, after reflection, the illuminated region of the second screen almost coincides with the area illuminated in the first passage. Figure 3(b), on the other hand, shows that the maximum theoretical enhancement factor of 2 (see Refs. 21–23) is achieved when the apertures at the two screens have equal sizes, as this maximizes the proportion of reversible paths. The mirror, in this case, has been assumed to be infinite.

In Fig. 4 we study the motion of the speckle pattern as the direction of the incoming beam changes. We proceed by first fixing the original angles of incidence $\theta_o$ and detection $\theta_d$. We then choose a new angle of incidence $\theta_o'$ and find the maximum value of the correlation $\chi(\theta_o, \theta_o'; \theta_d, \theta_d')$ as we scan the second angle of observation $\theta_d'$. In these figures, the vertical dashed-dotted line indicates the original direction of observation $\theta_d$, while the dotted line indicates the original direction of incidence $\theta_o$. Figure 4(a) is a plot of the maximum value of the intensity correlation...
Fig. 4(b) then represents (for given angles of incidence $\theta'_i$, $\theta'_s$), the correlation envelope shown in Fig. 4(a) in terms of the direction of incidence $\theta'_i$. It can be seen that, when the new direction of incidence $\theta'_i$ does not move at the same rate as the angle of incidence, the original directions of incidence and observation are practically the same. A similar result holds for scattering from dense media, where it has been shown that the shape of the envelope of correlation is equal to the square of the backscattering enhancement peak. We also mention that the situation shown in Fig. 4(b) does not always hold. For instance, if the mirror is moved closer to the phase screen, one eventually reaches a point beyond which the shape of the envelope correlation and that of the backscattering peak are different.

Figure 5 shows two aspects of the behavior of the intensity correlation function as the direction of incidence changes from $\theta'_i$ to $\theta'_s$. The geometry of the arrangement is the same as in Fig. 4, but we have now set the distance $d = 0.8 L$. Also, as in Fig. 4, the vertical dashed-dotted line in the upper half of these figures indicates the original direction of observation $\theta'_s$, while the dotted lines indicate the original direction of incidence $\theta'_i$. The long-dashed vertical lines in the bottom halves of the figures indicate the new direction of incidence $\theta'_i$, while the solid-vertical lines indicate the new direction of observation $\theta'_s$ for which maximum correlation is achieved. In Fig. 5(a), the thick solid curve corresponds to the intensity correlation function, while the thin curve shows the envelope of the intensity correlation function. For the value of $d/L$ employed in Fig. 5(a), the envelope is much wider than that in Fig. 4(a). We also see from Fig. 5(a) that, as the direction of incidence changes from $\theta'_i = 0.0$ mrad to $\theta'_i = 0.15$ mrad, the peak of the intensity correlation function, and thus the speckle pattern, moves in the direction of the specular reflection from the original angle of observation $\theta'_s = -1$ mrad to the position $\theta'_s = -1.04$ mrad. In Fig. 5(b), the original directions of incidence and observation are, respectively, $\theta'_i = 0.0$ mrad and $\theta'_s = -1.0$ mrad. It can be seen that, when the new direction of incidence $\theta'_s$ coincides with the original direction of observation $\theta'_s$, the peak of the intensity correlation function appears at the original direction of incidence.
So, when source and detector are interchanged, the corresponding intensities are perfectly correlated. This result shows that the formulas derived in Appendix A obey reciprocity. This kind of correlation was recently discussed for volume scatterers and has been termed the "time-reversed memory effect." 17

In Fig. 6 we consider the case of a small aperture at the mirror, much smaller than the cross section of the incoming beam. For these figures we assumed an infinitely large aperture at the second screen. Figure 6(a) shows the spatial intensity autocorrelation function. It can be seen that the speckle pattern is nearly symmetrical with respect to the backscattering direction. This is because [with reference to Fig. 1(b)] the light scattered from the first screen that reaches the second screen goes through a small aperture on the axis of the system so that an inverted image of the area illuminated on the first screen is projected onto the second. Figure 6(b) shows that when the incident direction changes from $\theta_i = 0$ to $\theta_i' = 1$ mrad, the correlation peaks move in a direction opposite to that of the specular direction and at the same rate as the incident direction. That is, in this case, the speckle pattern tracks the backscattering direction.

These striking results are due to the fact that, for the complex amplitudes $A(x, \theta_i)$ and $A(x, \theta_i')$ [see Eq. (6)], the phase curvatures within the mirror aperture are negligible. This suggests a modification of the scattering geometry that will produce these effects under less restrictive conditions. The new geometry is shown in Fig. 7. The simple formula referred to in...
Subsection 2.2B was derived for the case $W_1 = W_2 = W$ and for this specialized geometry, including a spheri-
cal mirror, with an aperture of radius $W_m$, situated in the 
far field of the screen. The radius of curvature of the 
mirror is $d$ and is equal to the distance from the 
screen to the mirror.

It is found that up to a constant factor, the 
amplitude correlation is given by the expression

$$\gamma_{A}(\theta_i, \theta_i'; \theta_s, \theta_s') = \exp \left[ -\frac{1}{8} \frac{(\theta_i^+ + \theta_i^-)^2}{\varphi_x^2 + \varphi_y^2} - \frac{1}{8} \frac{(\theta_s^+ + \theta_s^-)^2}{\varphi_m^2} \right]$$

$$\times \left[ \exp \left[ -\frac{1}{8} \frac{(\theta_i^+ - \theta_i^-)^2}{\varphi_x^2} - \frac{1}{4} \frac{(\theta_s^+ - \theta_s^-)^2}{\varphi_m^2} \right] + \exp \left[ -\frac{1}{8} \frac{(\theta_i^- - \theta_i^+)^2}{\varphi_x^2} - \frac{1}{4} \frac{(\theta_s^- - \theta_s^+)^2}{\varphi_m^2} \right] \right], \quad (19)$$

where

$$\begin{align*}
\theta_i^+ &= \sin \theta_i + \sin \theta_i', \quad (20) \\
\theta_i^- &= \sin \theta_i - \sin \theta_i', \quad (21) \\
\theta_s^+ &= \sin \theta_s + \sin \theta_s', \quad (22) \\
\theta_s^- &= \sin \theta_s - \sin \theta_s', \quad (23) \\
\varphi_x &= \frac{k \varphi_m}{W}, \quad (24) \\
\varphi_y &= \frac{2\sigma}{k \xi}, \quad (25) \\
\varphi_m &= \frac{W_m}{d}. \quad (26)
\end{align*}$$

These last three quantities represent, respectively, the 
angular widths of the speckle, of the speckle 
envelope, and of the mirror. To derive expression 
(19) we assumed that $\varphi_m, \varphi_x \gg \varphi_y$.

Also, it is straightforward to show that, with 
approximations consistent with those employed in the 
derivation of Eq. (19), the mean intensity pre-
dicted by this equation is given by

$$\langle I(\theta_i, \theta_i') \rangle = \exp \left[ -\frac{1}{2} \frac{\varphi_x^2}{\varphi_y^2} \left( \sin \theta_i + \sin \theta_i' \right)^2 \\
+ \left( \sin \theta_i - \sin \theta_i' \right)^2 \right] \times \left[ 1 + \exp \left( -\frac{\left( \sin \theta_i - \sin \theta_i' \right)^2}{\varphi_y^2} \right) \right]. \quad (27)$$

which predicts a broad envelope with a narrow peak 
(width $\varphi_y$) in the backscattering direction ($\theta_i = \theta_i'$).

Figure 8 corresponds to the geometry depicted in 
Fig. 7, in which the phase screen is at the center of curvature of the spherical mirror. Figure 8(a) shows 
the behavior of the mean intensity. It is seen that 
the relative value of the backscattering enhancement 
reaches a value of 2. The reason for this is that, in 
the present case, an image of the region of the screen

![Figure 8](image_url)
illuminated by the incoming beam is formed on the screen by the mirror. This image is inverted and is of unit magnification. The intensity correlation function in Fig. 6(b) shows the same features observed in Fig. 6. The autocorrelation function shows two peaks, indicating the symmetry of the speckle pattern with respect to the backscattering direction. Also, as the angle of incidence of the incoming beam is changed, the speckle pattern tracks the backscattering direction and not the specular direction, as would be the case with a single-scattering rough surface. These features can be explained from Eq. (19): The term \( \theta_i' = \theta_i + \phi \) is the one responsible for the tracking of the backscattering direction, as it predicts a peak when \( \sin \theta_i' = \sin \theta_i \). On the other hand, the term \( \theta_i' - \theta_i \) is responsible for the symmetry of the speckle pattern about the backscattering direction (i.e., \( \theta_i' = \theta_i \)), as it predicts a peak when \( \sin \theta_i' = \sin \theta_i - \sin \theta_i \).

4. Summary and Conclusions

We have studied the scattering of light resulting from double passage through a deep random phase screen for an incoming Gaussian beam. Our analysis is for a one-dimensional random phase screen but the extension to two dimensions is straightforward. Assuming that the complex amplitude of the speckle formed on the mirror follows Gaussian statistics, and that the speckle grain on the mirror is much smaller than the mirror aperture, we have derived formulas for the mean intensity of the backscattered light and for the correlation of the intensities for different directions of incidence and observation. Using the factorization properties of the moments of a complex Gaussian process, we have derived exact (within the model) analytical expressions for these two quantities. The formulas so obtained are long and difficult to interpret but we have studied a variety of double-passage configurations from the numerical evaluation of these expressions.

We have studied the backscattering enhancement factor as well as the motion of the speckle pattern as the angle of incidence is changed. We have found that, in general, the rate of change in the position of the speckle pattern is not the same as that of the angle of incidence. In normal circumstances, the speckle moves in the direction of the specular reflection at a rate slower than that of the incident beam and decorrelates with an envelope described approximately by the shape of the backscattering enhancement peak. Conditions have also been found under which the speckle pattern is symmetric about the backscattering direction and tracks the backscattering direction as the source is moved. The conditions for the observation of this effect are, however, fairly restrictive. The new geometry proposed, with a spherical mirror, exhibits enhanced backscattering, the symmetry of the speckle pattern, and the tracking of the backscattering direction under less restrictive conditions.

Appendix A

Apart from some constant factors, the amplitude correlations appearing in Eqs. (13) and (14) are of the general form

\[
\langle A(x, \theta)A^*(x', \theta') \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[- \frac{x_1^2}{W_1^2} - \frac{x_2^2}{W_2^2} \right] |\exp[i\phi(x_i)]\exp[-i\phi(x_j)]| \times \exp \left[ \frac{ik}{2d} [2(x_1 - x_1') - 2(x_2 - x_2')] \right] \times \exp[-iK(x, \sin \theta - x_1' \sin \theta')] dx_1 dx_2 dx_3 dx_4. \tag{A1}
\]

The steepest-descent method that we employed for the calculation of these expressions basically consists of making the approximation

\[
\exp[i\phi(x)] \exp[-i\phi(x')] \approx \exp\left(-\sigma^2 \frac{|x - x'|^2}{\xi^2} \right) \tag{A2}
\]

in the evaluation of the amplitude correlations given by Eq. (A1). Using these results, we find that Eq. (A3) gives

\[
C_i(\theta_i, \theta_i'; \theta_i', \theta_i') = |K|^2 \frac{\sigma^4 d^4}{k^2} \frac{1}{(\lambda A_1 A_2 R)} \exp \left[ X - B \right], \tag{A3}
\]

where

\[
X = \frac{k^2}{4} \frac{1}{R} \left[ Z e^2 + 2 \sigma^2 \frac{1}{\xi^2} \left( \frac{1}{A_1} + \frac{1}{A_2} \right) \eta e + Z^* \eta^2 \right], \tag{A4}
\]

\[
B = \frac{k^2}{4} \frac{1}{A_1} \left[ z_1 \sin^2 \theta_i + z_1^* \sin^2 \theta_i' - 2 \sigma^2 \frac{1}{\xi^2} \sin \theta_i \sin \theta_i' \right] \tag{A5}
\]

\[
+ \frac{k^2}{4} \frac{1}{A_2} \left[ z_2 \sin^2 \theta_i + z_2^* \sin^2 \theta_i' - 2 \sigma^2 \frac{1}{\xi^2} \sin \theta_i \sin \theta_i' \right],
\]

\[
R = |Z|^2 - \frac{\sigma^4}{\xi^4} \left[ \frac{1}{A_1} + \frac{1}{A_2} \right]^2, \tag{A6}
\]

\[
A_1 = |z_1|^2 - \frac{\sigma^4}{\xi^4}, \tag{A7}
\]

\[
A_2 = |z_2|^2 - \frac{\sigma^4}{\xi^4}, \tag{A8}
\]

with

\[
\eta = \frac{1}{A_1} \left( z_1 \sin \theta_i - \sigma^2 \frac{1}{\xi^2} \sin \theta_i' \right) \tag{A9}
\]

\[
+ \frac{1}{A_2} \left( z_2 \sin \theta_i - \sigma^2 \frac{1}{\xi^2} \sin \theta_i' \right).
\]

20 May 1993 / Vol. 32, No. 15 / APPLIED OPTICS 2741
\[ \epsilon = \frac{1}{A_1} \left( z_1^* \sin \theta_1' - \frac{\sigma^2}{\xi^2} \sin \theta_1 \right) \]
\[ + \frac{1}{A_2} \left( z_2^* \sin \theta_2' - \frac{\sigma^2}{\xi^2} \sin \theta_2 \right), \quad (A10) \]
\[
Z = \frac{z_1}{A_1} + \frac{z_2}{A_2} + \frac{4d^2}{k^2} \left( \frac{1}{W_n^2} - \frac{i}{d} k \right), \quad (A11) \]
\[
z_1 = \frac{1}{W_1^2} + \frac{\sigma^2}{\xi^2} i \frac{k}{2d}, \quad (A12) \]
\[
z_2 = \frac{1}{W_2^2} + \frac{\sigma^2}{\xi^2} i \frac{k}{2d}. \quad (A13) \]

Similarly, it is found that Eq. (14) gives the following result:
\[
C_2' (\theta_1'; \theta_2', \theta_1) = |K|^2 \pi^3 4d^2 \frac{1}{k^2} \frac{1}{|A| D^2} z \exp(Y - V), \quad (A14) \]
where
\[
Y = \frac{k^2}{4D} \left( S^2 + 2 \frac{\sigma^2}{\xi^2} \left( \frac{1}{A} + \frac{1}{A^*} \right) \mu \nu + S^* \mu^2 \right), \quad (A15) \]
\[
V = \frac{k^2}{4A} \left( z_1 \sin^2 \theta_1 + z_2^* \sin^2 \theta_1' - 2 \frac{\sigma^2}{\xi^2} \sin \theta_1 \sin \theta_1' \right) \]
\[
+ \frac{k^2}{4A^*} \left( z_2 \sin^2 \theta_1 + z_1^* \sin^2 \theta_1' - 2 \frac{\sigma^2}{\xi^2} \sin \theta_1 \sin \theta_1' \right), \quad (A16) \]
\[
D = |S|^2 - \frac{\sigma^2}{\xi^2} \left( \frac{1}{A} + \frac{1}{A^*} \right)^2, \quad (A17) \]
\[
A = z_1 z_2^* - \frac{\sigma^4}{\xi^4}, \quad (A18) \]

with
\[
\mu = \frac{1}{A} \left( z_1 \sin \theta_1 - \frac{\sigma^2}{\xi^2} \sin \theta_1' \right) \]
\[ + \frac{1}{A^*} \left( z_2 \sin \theta_1' - \frac{\sigma^2}{\xi^2} \sin \theta_1 \right), \quad (A19) \]
\[
\nu = \frac{1}{A} \left( z_2^* \sin \theta_1' - \frac{\sigma^2}{\xi^2} \sin \theta_1 \right) \]
\[ + \frac{1}{A^*} \left( z_1 \sin \theta_1 - \frac{\sigma^2}{\xi^2} \sin \theta_1' \right), \quad (A20) \]
\[
S = \frac{z_1}{A} + \frac{z_2}{A^*} + \frac{4d^2}{k^2} \left( \frac{1}{W_n^2} - \frac{i}{d} k \right). \quad (A21) \]

The expressions corresponding to the geometry depicted in Fig. 7 are obtained by dropping the imaginary terms inside parentheses in Eqs. (A11) and (A21). Equation (19) was obtained by dropping, in addition, the imaginary terms in Eqs. (A12) and (A13) and by putting \( W_1 = W_2 = W \).

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