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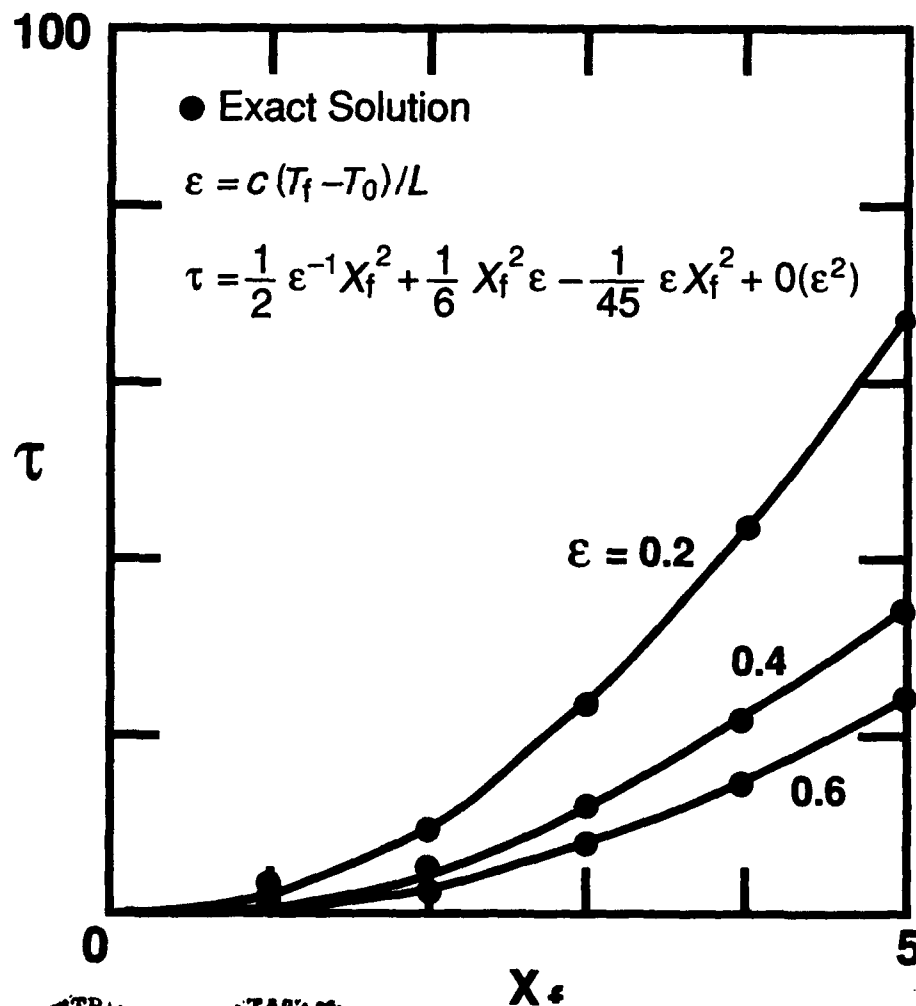
Perturbation Techniques in Conduction-Controlled Freeze–Thaw Heat Transfer

Virgil J. Lunardini and Abdul Aziz

June 1993

MONOGRAPH 93-1

- Exact Solution

$$\varepsilon = c\,(T_f - T_0)/L$$

$$\tau = \frac{1}{2}\,\varepsilon^{-1}X_f^2 + \frac{1}{6}\,X_f^2\varepsilon - \frac{1}{45}\,\varepsilon X_f^2 + 0(\varepsilon^2)$$

$\varepsilon = 0.2$

0.4

0.6

τ

X_f

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Abstract

Heat transfer with change of phase (freezing or melting) is important in numerous scientific and engineering applications. Since the pioneering works of Lamé and Clapeyron, Neumann and Stefan, a number of analytical and numerical techniques have been developed to deal with freezing and melting problems. One such analytical tool is the method of perturbation expansions, which is the main focus of this work. The report begins with a review of the perturbation theory and outlines the regular perturbation method, the method of strained coordinates, the method of matched asymptotic expansions, and the recently developed method of extended perturbation series. Next, the applications of these techniques to phase change problems in Cartesian, cylindrical, and spherical systems are discussed in detail. Although the bulk of the discussion is confined to one-dimensional situations, the report also includes two- and three-dimensional cases where admittedly the success of these techniques has so far been limited. The presentation is sufficiently detailed that even the reader who is unfamiliar with the perturbation theory can understand the material. However, at the same time, the discussion covers the latest literature on the subject and therefore should serve as a state-of-the-art review.

Cover: Perturbation solution for Stefan problem.

Monograph 93-1

US Army Corps of Engineers
Cold Regions Research & Engineering Laboratory

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NOMENCLATURE

a	amplitude of temperature oscillation, °C (°F) or a constant
$a_{i,j}$	binomial coefficients
a_n	series coefficients
A	amplitude of temperature oscillation, dimensionless or area of cross section, m^2 (ft^2).
A_o	constant
A_m	series coefficients
b	a constant
b_0	a function of Bi and ψ
b_1	a function of Bi and ψ or a constant
b_3	a function of r_f
b_n	series coefficients
B_1	a constant
Bi	convective Biot number
Bi_r	radiative Biot number
c	specific heat, kJ/kg K (Btu/lbm°F) or a constant
$\bar{c}$	average specific heat, kJ/kg K (Btu/lbm°F)
c_f	specific heat at temperature T_f, kJ/kg K (Btu/lbm°F)
c_n	series coefficients
C,C_1,C_2,C_3	constants
e	surface emissivity for radiation, dimensionless
e_1	first Shanks transformation
E	freezing front location, dimensionless
f	function of ε
$f(\theta)$	function of θ
$f_{j,m}$	functions of x_f
F	temperature or overall radiation shape factor, dimensionless
F_n	temperature series coefficients
g	velocity of freezing front, dimensionless
g_i	coefficients of series for g
h	convective heat transfer coefficient, W/(m^2 K) (Btu/ft^2hr°F) or an integer
i,j	integers
k,k_s	thermal conductivity of solid phase, W/m K (Btu/ft hr°F)
k_f	thermal conductivity at temperature T_f
k_ℓ	thermal conductivity of liquid phase, W/m K (Btu/ft hr°F)
$\bar{k}$	average thermal conductivity, W/m K (Btu/ft hr°F)
K	ratio k_s/k_ℓ or a constant
K_1	a constant
ℓ	slab thickness, m (ft)
L	latent heat of fusion, kJ/kg (Btu/lbm)
L_1	a constant
m,n	integers
P	perimeter of cross section, m (ft)
q	heat flux, W/m^2 (Btu/ft^2hr)
Q	heat flux, dimensionless
r	radial coordinate, dimensionless
r_f	radial location of freezing front, dimensionless

R	radial coordinate, m (ft)
R_f	radial location of freezing front, m (ft)
R_w	wall radius, m (ft)
s	Laplace variable
S	Stefan number, dimensionless
S_{n-1}, S_n, S_{n+1}	partial sums
t	time, sec
T	temperature, °C (°F)
T_a	ambient temperature, °C (°F)
T_f	freezing temperature, °C (°F)
T_i	initial temperature, °C (°F)
T_o	coolant temperature or mean coolant temperature or wall temperature or fin base temperature, °C (°F)
T_v	vaporization temperature, °C (°F)
u	temperature distribution in the solid phase, dimensionless
u_i, u_n	coefficients of series for u
v	temperature distribution in the liquid phase or steady-state velocity of vaporizing front, dimensionless
v_n	coefficients of series for v
x	distance from wall, m (ft)
x^*	stretched distance, m (ft)
x_f	location of freezing front, m (ft)
x_s	reference of characteristic distance, m (ft)
x_v	location of vaporizing front, m (ft)
X	distance from wall, dimensionless
X_f	location of freezing front, dimensionless
X_{fn}	coefficients of series for X_f
z	time, dimensionless

Greek

α	thermal diffusivity, m²/s (ft²/s) or slope of the specific heat-temperature curve divided by c_f, °C⁻¹ (°F⁻¹) or ambient temperature, dimensionless or slope of Domb-Sykes plot or $\alpha = \sqrt{\alpha_s / \alpha_\ell}$
α_s	thermal diffusivity of solid phase, m²/s (ft²/s)
α_ℓ	thermal diffusivity of liquid phase, m²/s (ft²/s)
β	inverse of Stefan number S or the ratio of convective and radiative Biot numbers or the slope of thermal conductivity-temperature curve divided by k_f, (°C⁻¹ or °F⁻¹)
$\varepsilon, \varepsilon_1, \varepsilon_2$	perturbation parameters or coordinates
ε_o	radius of convergence of series
ε^*	Eulerized perturbation parameter
η	distance or similarity variable or inverse of r_f or vaporization velocity, all dimensionless
η_n	coefficients of series for η
η^*	η for inner expansion
η_n^*	coefficients of series for η^*
η_ℓ	similarity variable based on the thermal diffusifity of liquid phase
η_s	similarity variable based on the thermal diffusivity of solid phase
θ	temperature, dimensionless

θ_a	ambient teperature, dimensionless
θ_f	freezing temprature, dimensionless
θ_i	initial temperature, dimensionless
θ_ℓ	temperature of the liquid phase, dimensionless
θ_s	temperature of the solid phase, dimensionless
θ^*	θ for the inner expansion
θ_0^*	coefficients for series for θ^*
λ	freezing or vaporizing front location, dimensionless
λ_m	eigenvalues
λ_n	coefficients of series for λ
λ_∞	location of stationary freezing front, dimensionless
λ^*	λ for inner expansion
μ	fin parameter, m^{-1} (ft^{-1})
ρ	density, kg/m^3 (lbm/ft^3)
σ	Stefan-Boltzmann constant or location of vaporization front, dimensionless
σ_i	straining functions
σ_n	coefficients of series for σ (the location of vaporization front)
σ^*	inner expansion variable for σ (the location of vaporization front)
σ_n^*	coefficient of series σ^*
τ	time, dimensionless
τ_p	preheating time, dimensionless
τ_n	coefficients of series for τ
τ_v	time to complete vaporization, dimensionless
τ^*	τ for inner expansion or τ after Euler transformation
Ω	frequency of temperature oscillation, sec^{-1}
ω	frequency of temperature oscillation, dimensionless
ϕ	strained variable, or temperature (outer expansion), dimensionless
ϕ_*^*	temperature (inner expansion), dimensionless
ϕ_n	coefficients of series for ϕ^*
ψ	strained variable

Perturbation Techniques in Conduction-Controlled Freeze–Thaw Heat Transfer

VIRGIL J. LUNARDINI AND ABDUL AZIZ

1. INTRODUCTION

Heat transfer with freezing or melting occurs in a number of applications such as ice formation, permafrost melting, metal casting, food preservation, storage of latent energy, automatic welding, etc. The vast literature that exists on the subject has been codified recently into a monograph by Lunardini (1991). In the simplest models, the two phases in a typical freezing or melting problem are separated either by a sharp boundary or a mushy zone that moves with time. Since the location of this moving boundary or zone is not known a priori, the solution is difficult to achieve even when the heat transfer process is assumed to be conduction controlled. Most exact solutions rely on similarity transformations, introduced by Lamé and Clapeyron (1831), or Neumann (1860), and have been examined in some detail (Lunardini 1981, 1991). It is only under highly idealized conditions that one can obtain useful, exact, analytical solutions.

For more realistic situations, one must therefore think in terms of either an approximate analytical technique or a fully numerical approach based on finite differences or finite elements. The former strategy is particularly convenient for preliminary calculations. Over the years, several approximate techniques have been devised to solve freezing and melting problems, including perturbation methods, the heat balance integral technique, and variational methods.

This report is devoted to the application of perturbation techniques to freezing and melting problems. The coverage begins with an overview of the perturbation theory and includes a brief discussion of the basic concepts, regular perturbation method, singular perturbation techniques such as the method of strained coordinates and the method of matched asymptotic expansions, and the method of extended perturbation series. The applications of these methods will cover both one-dimensional and two-dimensional phase change problems in Cartesian, cylindrical and spherical systems. The accuracy of the perturbation solutions will be assessed in the light of other approximate solutions and also of exact and numerical results.

Asymptotic methods are excellent for the study of freeze–thaw problems. Singular perturbation techniques may be applied to treat the singularities that occur when phases appear or disappear. In principle, multidimensional problems can be handled and perturbation methods are often superior for dealing with nonlinear boundary conditions or convective effects. Perturbation also yields valuable insights into the basic physics of the problems. The main disadvantage of the method is the increasing difficulty of obtaining higher order terms.

2. REVIEW OF PERTURBATION THEORY

The method of perturbation expansion is a well established analytical tool that has found applications in many areas of engineering. The subject is covered in detail in several currently

available books. Those dealing with engineering applications include Nayfeh (1981), Van Dyke (1975) and Aziz and Na (1984). While the first two discuss applications in solid mechanics and fluid dynamics, the last one is exclusively devoted to problems in heat transfer.

In freezing and melting problems, the main difficulty is the presence of a moving boundary that separates the solid and liquid phases, but several other difficulties can arise. These additional difficulties may be due to nonlinear or time-dependent boundary conditions, finite phase-change domain, an intermediate mushy zone, domain geometry, etc. Some of these difficulties can be handled by the perturbation method, as will be demonstrated later in the report.

Perturbation theory is based on the concept of an asymptotic solution. If the basic equations describing a phase-change problem can be expressed such that one of the parameters or variables is small (or very large) then the full equations can be approximated by letting the perturbation quantity approach its limit and an approximate solution can be found in terms of this perturbation quantity. Such a solution approaches a limit as the perturbation quantity approaches zero (or infinity) and is thus an asymptotic solution. The result can often be improved by expanding in a series of successive approximations, the first term of which is the limiting solution. One then has an asymptotic series or expansion. Thus we perturb the limiting solution by parameter or coordinates.

One is then concerned with the asymptotic expansion, generally for a small parameter such as the Stefan number, of the solutions of the conduction equation with solidification.

The first step in a perturbation analysis is to identify the perturbation quantity. This is done by expressing the mathematical model in a dimensionless form, assessing the order of magnitude of different terms and identifying the term that is small compared to others. The coefficient of this term, which could be a dimensionless parameter or a dimensionless variable, is then chosen as a perturbation quantity and designated by the symbol ε. Once ε is identified, the solution is assumed as an asymptotic series of ε. Next, this series solution is substituted into the governing equations for the problem. By equating the coefficients of each power of ε to zero, one can generate a sequence of subproblems. These problems are solved in succession to obtain the unknown coefficients of the series solution.

The foregoing procedure is termed *parameter perturbation* or *coordinate perturbation* depending on whether ε is a parameter or a coordinate. In either case, a further distinction is made between *regular perturbation* if the expansion is uniformly valid and *singular perturbation* if the expansion fails in certain regions of the domain. When a singular perturbation expansion is encountered, the usefulness of the solution is limited unless it can be rendered uniformly valid. Note that the terms in the expansion need not be convergent for the results to be useful since its asymptotic nature assures that only a few terms may yield adequate accuracy for small values of ε.

The two main techniques for achieving uniform validity that have been used in freezing and melting problems are the method of strained coordinates and the method of matched asymptotic expansions. In the method of strained coordinates, both the dependent and the independent variables are expanded in terms of ε such that the coefficients of the two series are functions of new, unknown independent variables. The assumed series expansions are substituted into the governing equations, and the unknown coefficients are found by ensuring that higher approximations are no more singular than the first one. The procedure leads to an implicit but uniformly valid solution.

The method of matched asymptotic expansions achieves uniform validity by supplementing the regular perturbation expansion, which is now called the outer expansion, with an inner expansion in which the independent variable is stretched out such that it describes the behavior in the region where the outer expansion breaks down. A uniformly valid solution is finally derived by matching the outer and the inner expansion according to Van Dyke's or some other matching principle.

In most instances the perturbation expansions are terminated at the second or the third term. There are two reasons for such an early truncation. First, the values of ε of interest are often small compared to unity, and therefore the truncated expansion that converges rapidly for small values of ε is sufficiently accurate. The second reason is that the higher-order terms of the series are increasingly

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more difficult to calculate. Sometimes, however, it may be desirable to improve the solution so that it can be used for values of ε that are not too small. This can be achieved with the method of extended perturbation series. The approach consists of three steps. The first step is to write a computer program to solve the sequence of perturbation equations either in symbolic form or numerically and generate a large number of terms. The second step is to examine the coefficients generated in the first step to identify the singularities in the complex plane that restrict the convergence of the series. This information about the analytic structure of the solution is finally used to transform the series using one or a combination of techniques such as the Euler transformation, Padé approximants, Shanks transformation, series reversion, etc. The three-step procedure results in a series solution that is accurate over a wider range of ε than the original series.

3. REGULAR PERTURBATION EXPANSIONS

The regular perturbation method has proved to be an effective tool to solve a variety of freezing and melting. Most applications involve parameter perturbations but as shown later, the coordinate perturbation approach is also feasible in some cases. The method will be illustrated with the help of a number of examples.

3.1 The Stefan problem

The mathematical class of problems for which a moving boundary exists are often called Stefan problems. The name derives from Stefan's (1891) early work on sea ice formation. In this section the Stefan problem will be limited to the following case. The classical Stefan problem considers the freezing of a saturated liquid of semi-infinite extent as shown in Figure 1. The liquid is assumed to be initially at its freezing temperature T_f. At the time $t = 0^+$, the surface temperature at $x = 0$ is suddenly reduced to a subfreezing value $T_0 < T_f$. The lowering of the surface temperature causes the liquid to freeze. With time, the freezing front progresses in the direction of increasing x values. Let x_f be the thickness of the solid phase at any instant of time. The unfrozen liquid

Figure 1. One-dimensional freezing of a semi-infinite region.

continues to remain at T_f throughout the solidification process. With appropriate thermal property values, the Stefan problem can also describe the melting of a solid of semi-infinite extent initially at its melting temperature and heated by a higher wall temperature.

The Stefan problem is a highly idealized model for the actual freezing or melting process. The mathematical model to be described is based on the following assumptions:

1. Both solid and liquid phases are homogeneous and isotropic.

2. The phase change occurs at a discrete temperature and consequently there is no mushy zone containing a mixture of two phases.

3. The two phases are separated by a sharply defined interface (front) instead of whisker-like (dendritic) growth observed experimentally (Sparrow et al. 1979).

4. Conduction is the only heat transfer mode.

For the case of freezing, the temperature distribution in the solid phase is described by the following mathematical model:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

$$T(0,t) = T_o, \quad T(x_f,t) = T_f \tag{2}$$

$$k\left.\frac{\partial T}{\partial x}\right|_{X=X_f} = \rho L \frac{dx_f}{dt} \tag{3}$$

where k, ρ and α represent the thermal conductivity, density, and thermal diffusivity, respectively of the solid phase. The quantity L is the latent heat of solidification.

To prepare the foregoing model for a perturbation analysis, we recognize that the Stefan number, which signifies the importance of sensible heat to the latent heat, is small in some phase change applications and therefore it can serve as a perturbation quantity. For example, water/ice systems or soil systems typically have Stefan numbers less than 1/2, unless the boundary temperatures are very high. With this in mind, we introduce the following dimensionless quantities:

$$\tau = kt/\rho c x_s^2, \varepsilon = c\left(T_f - T_o\right)/L \tag{4}$$

where c is the specific heat of the solid phase and x_s is a reference distance. Using eq 4 to nondimensionalize eq 1, 2, 3 we obtain

$$\frac{\partial^2 \theta}{\partial \lambda^2} = \frac{\partial \theta}{\partial \tau} \tag{5}$$

$$\theta(0,\tau) = 1, \quad \theta(X=X_f,\tau) = 0, \quad \left.\frac{\partial \theta}{\partial X}\right|_{X=X_f} = -\frac{1}{\varepsilon}\frac{dX_f}{d\tau} \tag{6}$$

The perturbation quantity ε which appears in the boundary condition at the interface can be brought into the governing equation (eq 5) by noting that the transient storage term, $\partial\theta/\partial\tau$, will be related to the transient motion of the interface, $dX_f/d\tau$. Thus we can think of θ as a function of two new independent variables, X, the nondimensional distance as before, and X_f, the instantaneous location of solid/liquid interface. For the time derivative, then, we obtain

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial \theta}{\partial X_f}\frac{dX_f}{d\tau} \tag{7}$$

From eq 6 we have

$$\frac{dX_f}{d\tau} = -\varepsilon\left.\frac{\partial \theta}{\partial X}\right|_{X=X_f} \tag{8}$$

Using eq 8 in eq 7 and the result in eq 5, the transient heat conduction equation becomes

$$\frac{\partial^2 \theta}{\partial X^2} = -\varepsilon\frac{\partial \theta}{\partial X_f}\left.\frac{\partial \theta}{\partial X}\right|_{X=X_f} \tag{9}$$

The first two boundary conditions (eq 6) can now be written in terms of X and X_f as

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$$\theta(X=0, \ X_f) = 1, \quad \theta(X=X_f, \ X_f) = 0 \tag{10}$$

Let us derive a perturbation solution by assuming an asymptotic series solution of the form

$$\theta = \sum_{n=0}^{\infty} \theta_n \, \varepsilon^n \tag{11}$$

Retaining only the first three-terms of the series (eq 11), we have

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon_2\theta_2 \tag{12}$$

Substituting eq 12 into eq 9 and eq 10, we get

$$\frac{\partial^2\theta_0}{\partial X^2} + \varepsilon\frac{\partial^2\theta_1}{\partial X^2} + \varepsilon^2\frac{\partial^2\theta_2}{\partial X^2} =$$

$$-\varepsilon\left[\frac{\partial\theta_0}{\partial X_f} + \varepsilon\frac{\partial\theta_1}{\partial X_f} + \varepsilon^2\frac{\partial\theta_2}{\partial X_f}\right]\left[\frac{\partial\theta_0}{\partial X}\bigg|_{X=X_f} + \varepsilon\frac{\partial\theta_1}{\partial X}\bigg|_{X=X_f} + \varepsilon^2\frac{\partial\theta_2}{\partial X}\bigg|_{X=X_f}\right] \tag{13}$$

$$\theta_0(X=0, X_f) + \varepsilon\theta_1(X=0, X_f) + \varepsilon^2\theta_2(X=0, X_f) = 1 \tag{14}$$

$$\theta_0(X=X_f, X_f) + \varepsilon_1\theta_1(X=X_f, X_f) + \varepsilon^2\theta_2(X=X_f, X_f) = 0 \tag{15}$$

By equating coefficients of like powers of ε in eq 13, 14 and 15 we obtain the following sub-problems.

Zero-order, ε^0

$$\frac{\partial^2\theta_0}{\partial X^2} = 0 \tag{16}$$

$$\theta_0(X=0, X_f) = 1, \quad \theta_0(X=X_f, X_f) = 0 \tag{17}$$

First-order, ε^1

$$\frac{\partial^2\theta_1}{\partial X^2} = -\frac{\partial\theta_0}{\partial X_f}\frac{\partial\theta_0}{\partial X}\bigg|_{X=X_f} \tag{18}$$

$$\theta_1(X=0, \ X_f) = 0, \quad \theta_1(X=X_f, X_f) = 0 \tag{19}$$

Second-order, ε^2

$$\frac{\partial^2\theta_2}{\partial X^2} = -\left[\frac{\partial\theta_0}{\partial X_f}\frac{\partial\theta_1}{\partial X}\bigg|_{X=X_f} + \frac{\partial\theta_0}{\partial X}\bigg|_{X=X_f}\frac{\partial\theta_1}{\partial X_f}\right] \tag{20}$$

$$\theta_2(X=0, \ X_f) = 0, \quad \theta_2(X=X_f, X_f) = 0 \tag{21}$$

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It may be noted that the zeroth-order problem represents the quasi-steady approximation to the problem which was used by Stefan (Lunardini 1981).

Integrating eq 16 twice and applying the boundary conditions (eq 17), the solution for θ_0 is found to be

$$\theta_0 = 1 - \frac{X}{X_f} \tag{22}$$

Using eq 22 on the right-hand side of eq 18 gives

$$\frac{\partial^2 \theta_1}{\partial X^2} = \frac{X}{X_f^3} \tag{23}$$

whose solution, subject to eq 19, is

$$\theta_1 = - \frac{1}{6} \left(\frac{X}{X_f}\right) \left[1 - \left(\frac{X}{X_f}\right)^2 \right] \tag{24}$$

Now we use eq 22 and eq 24 on the right hand-side of eq 20 to obtain

$$\frac{\partial^2 \theta_2}{\partial X^2} = - \left[\frac{1}{6} \left(\frac{X}{X_f^3}\right) + \frac{1}{2} \left(\frac{X^3}{X_f^5}\right) \right] \tag{25}$$

which can be integrated twice using eq 21 to give

$$\theta_2 = \frac{1}{360} \left(\frac{X}{X_f}\right) \left[19 - 10 \left(\frac{X}{X_f}\right)^2 - 9 \left(\frac{X}{X_f}\right)^4 \right] \tag{26}$$

Using eq 22, 24, and 26 in eq 12, the three-term perturbation solution takes the form

$$\theta = \left(1 - \frac{X}{X_f}\right) - \frac{1}{6} \, \varepsilon \left(\frac{X}{X_f}\right) \left[1 - \left(\frac{X}{X_f}\right)^2 \right] + \frac{1}{360} \varepsilon^2 \left(\frac{X}{X_f}\right) \left[19 - 10 \left(\frac{X}{X_f}\right)^2 - 9 \left(\frac{X}{X_f}\right)^4 \right] \tag{27}$$

The equation for X_f can now be derived by evaluating $\left. \frac{\partial \theta}{\partial X} \right|_{X = X_f}$ from eq 27 and substituting the result on the right-hand side of eq 8. The resulting equation is

$$\frac{dX_f}{d\tau} = \frac{1}{X_f} \left(\varepsilon - \frac{1}{3} \, \varepsilon^2 + \frac{7}{45} \, \varepsilon^3 \right) + 0 \left(\varepsilon^4\right) \tag{28}$$

Integrating eq 28 subject to the initial condition, $\tau = 0$, $X_f = 0$, we get

$$X_f^2 = 2\tau \left(\varepsilon - \frac{\varepsilon^2}{3} + \frac{7}{45} \, \varepsilon^3 \right) \tag{29}$$

If we wish to express τ as a function of X_f, we write eq 29 as

$$\tau = \frac{1}{2} \, X_f^2 \, \varepsilon^{-1} \left(1 - \frac{1}{3} \, \varepsilon + \frac{7}{45} \, \varepsilon^2 \right)^{-1} \tag{30a}$$

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Using binomial expansion

$$\left(1 - \frac{1}{3}\varepsilon + \frac{7}{45}\varepsilon^2\right)^{-1} = 1 + \frac{1}{3}\varepsilon - \frac{2}{45}\varepsilon^2 + 0\left(\varepsilon^3\right)$$

leads to

$$\tau = \frac{1}{2}\varepsilon^{-1} X_f^2 + \frac{1}{6} X_f^2 \varepsilon - \frac{1}{45}\varepsilon X_f^2 + 0\left(\varepsilon^2\right) \tag{30b}$$

By using the similarity technique, the exact analytical solution of eq 5, 6 can be obtained as [Lunardini 1991, Ozisik 1980]

$$\theta = 1 - \frac{\mathrm{erf}\left(\lambda X/X_f\right)}{\mathrm{erf}\lambda} \tag{31}$$

where $\lambda = X_f/2\sqrt{\tau}$ is the root of the equation

$$\sqrt{\pi}\ \lambda e^{\lambda^2}\ \mathrm{erf}\lambda = \varepsilon \tag{32}$$

It is interesting to demonstrate that the perturbation solution eq 29 reproduces the exact solution (eq 32) for sufficiently small values of λ. Consider the series expansions for e^{λ^2} and $\mathrm{erf}\,\lambda$ valid for small values of λ:

$$e^{\lambda^2} = 1 + \lambda^2 + \frac{\lambda^4}{2} + \frac{\lambda^6}{6} + \dots \tag{33}$$

$$\mathrm{erf}\lambda = \frac{2}{\sqrt{\pi}}\left(\lambda - \frac{\lambda^3}{3} + \frac{\lambda^5}{10} - \frac{\lambda^7}{42} + \dots\right) \tag{34}$$

Substituting eq 33 and eq 34 into eq 32 and simplifying, we obtain

$$\left(2\lambda^2 + \frac{4}{3}\lambda^4 + \frac{8}{15}\lambda^6 + \frac{16}{105}\lambda^8 + \dots\right) = \varepsilon \tag{35}$$

A general series $\displaystyle\sum_{n=1}^{\infty} a_n\left(\lambda^2\right)^n = \varepsilon$ can be reversed and written as

$$\lambda^2 = \sum_{n=1}^{\infty} b_n\,\varepsilon^n \tag{36}$$

where the first three coefficients b_n, for example, are

$$b_1 = \frac{1}{a_1},\ b_2 = -\frac{a_2}{a_1^3},\ b_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5} \tag{37}$$

Noting from eq 35 that $a_1 = 2$, $a_2 = 4/3$, $a_3 = 8/15$, we find from eq 37 that $b_1 = 1/2$, $b_2 = -1/6$, and $b_3 = 7/90$. Using these values in eq 36, we find

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$$\lambda^2 = \frac{1}{2}\varepsilon - \frac{1}{6}\varepsilon^2 + \frac{7}{90}\varepsilon^3 \qquad (38)$$

Now $\lambda^2 = X_f^2/4\tau$ and hence eq 38 can be written as

$$\frac{X_f^2}{4\tau} = \frac{1}{2}\varepsilon - \frac{1}{6}\varepsilon^2 + \frac{7}{90}\varepsilon^3$$

or

$$X_f^2 = 2\tau\left(\varepsilon - \frac{1}{3}\varepsilon^2 + \frac{7}{45}\varepsilon^3\right) \qquad (39)$$

which matches exactly with the perturbation solution eq 29.

In Figure 2, the three-term perturbation solution for the freezing time τ, eq 30 is compared with the exact analytical solution eq 32 for $\varepsilon = 0.2$, 0.4 and 0.6. The agreement is excellent even at $\varepsilon = 0.6$.

3.2. Planar freezing of a saturated liquid with convective cooling

The problem to be considered here is a variation of the Stefan problem of the previous section. The constant temperature boundary condition at $x = 0$ is now replaced with a convective boundary condition as shown in Figure 3. The coolant temperature is T_0 and the convective heat transfer coefficient is h. The problem is described by eq 1, 2, 3 except that the first boundary condition in eq 2 now becomes

Figure 2. Perturbation solution for Stefan problem.

Figure 3. Freezing of semi-infinite saturated medium with surface convection.

$$k\frac{\partial T}{\partial x}(0,t) = h\left[T(0,t) - T_0\right] \qquad (40)$$

In this case, it is more convenient to introduce the following dimensionless quantities:

$$\theta = \frac{T_f - T}{T_f - T_0}, \quad X = \frac{hx}{k}, \quad \tau = \frac{h^2 t}{\rho ck}, \quad \varepsilon = \frac{c(T_f - T_0)}{L} \qquad (41)$$

The resulting dimensionless model is

$$\frac{\partial^2\theta}{\partial X^2} = -\varepsilon\frac{\partial\theta}{\partial X_f}\left.\frac{\partial\theta}{\partial X}\right|_{X=X_f} \qquad (42)$$

$$\frac{\partial\theta}{\partial X}(0,X_f) = \theta(0,X_f) - 1, \quad \theta(X = X_f, X_f) = 0. \qquad (43)$$

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Assuming a three-term solution of the form of eq 12, and following the procedure of section 3.1, we find that the governing equations for θ_0, θ_1, and θ_2 are still given by eq 16–21 except that the wall boundary conditions are now given by

$$\frac{\partial \theta_o}{\partial X}(0, X_f) = \theta_o(0, X_f) - 1 \tag{44}$$

$$\frac{\partial \theta_1}{\partial X}(0, X_f) = \theta_1(0, X_f) \tag{45}$$

$$\frac{\partial \theta_2}{\partial X}(0, X_f) = \theta_2(0, X_f) \tag{46}$$

The solution of eq 16 subject to eq 44 and eq 17 is

$$\theta_o = \frac{1}{1 + X_f}(X_f - X) \tag{47}$$

Similarly, the solution of eq 18 subject to eq 45 and eq 19b is

$$\theta_1 = \frac{1}{6(1 + X_f)^4}\left[(1 + X_f)X^3 + 3(1 + X_f)X^2 - (3 + X_f)X_f^2 X - (3 + X_f)X_f^2\right] \tag{48}$$

Finally the solution for θ_2 subject to eq 46 and eq 21b can be obtained as

$$\theta_2 = \frac{1}{360(1 + X_f)^7}\left[\left(19 X_f^6 + 114 X_f^5 + 225 X_f^4 + 360 X_f^3\right)(1 + X)\right.$$

$$- 10 X_f(1 + X_f)\left(X_f^2 + 3 X_f + 12\right)(3 + X)X^2$$

$$\left. - 9 (1 + X_f)^2 (5 + X)X^4\right] \tag{49}$$

Following the procedure of section 3.1, the solution for freezing time τ can be obtained as

$$\tau = \frac{1}{2} \varepsilon^{-1}\left[(1 + X_f)^2 - 1\right] + \frac{1}{6(1 + X_f)}\left[(1 + X_f)^3 - 3(1 + X_f) + 2\right]$$

$$- \frac{1}{45} \varepsilon \frac{1}{(1 + X_f)^4}\left[(1 + X_f)^6 - 5(1 + X_f)^3 + 9(1 + X_f) - 5\right] \tag{50}$$

The perturbation analysis presented in this section was first discussed by Pedroso and Domoto (1973a) who presented a scheme for programming the perturbation calculations so that a digital computer can be used to generate as many terms as desirable. The scheme circumvents the mounting algebraic labor entailed in manual calculations as one goes to higher order terms. They present their results for τ as

$$\varepsilon \tau = \sum_{n=0}^{\infty} c_n \varepsilon^n \tag{51}$$

where the coefficients c_n are functions of X_f.

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Table 1. Numerical values of c_n.

x_f\\n	1	2	3	4	5	6	7	8	9
0.2	0.2200	0.01778	–0.001564	0.0003945	–0.0001383	0.00005842	–0.00002793	0.00001462	–0.000008213
0.4	0.4800	0.06476	–0.008154	0.002846	–0.001335	0.0007332	–0.0004438	0.0002869	–0.0001940
0.6	0.7800	0.1350	–0.01932	0.00748	–0.003804	0.002211	–0.001389	0.0009138	0.0006180
0.8	1.120	0.2252	–0.03398	0.01368	–0.007083	0.004134	–0.002573	0.001661	–0.001096
1	1.500	0.3333	–0.05139	0.02092	–0.01083	0.006253	–0.003828	0.002422	–0.001576
1.4	2.380	0.5980	–0.09282	0.02768	–0.01918	0.01082	–0.006461	0.004001	–0.002643
1.8	3.420	0.9257	–0.1418	0.05692	–0.02852	0.01585	–0.009353	0.005750	–0.004005
2.2	4.620	1.311	–0.1979	0.07854	–0.03893	0.02145	–0.01259	0.007719	–0.005770
2.6	5.980	1.753	–0.2608	0.1025	–0.05048	0.02767	–0.01620	0.009923	–0.007997
3.0	7.500	2.250	–0.3305	0.1292	–0.06323	0.03456	–0.02021	0.01237	–0.01071
4	12.00	3.733	–0.5347	0.2068	–0.1006	0.05479	–0.03199	0.01956	–0.01963
5	17.50	5.555	–0.7823	0.3008	–0.1459	0.07939	–0.04632	0.02832	–0.03153

Table 1 gives the first nine values of c_n for a range of values of X_f. The first three values agree with those given by eq 50.

The present problem was also solved by Huang and Shih (1975) by first immobilizing the interface position using the Landau transformation and then using a regular perturbation analysis. Their three-term solutions for the temperature distribution and freezing time match exactly with the present results. The use of the Landau transformation makes the nonlinearity due to the moving interface explicit and facilitates the subsequent use of the perturbation method. However, the approach adopted here achieves the same simplicity without a formal introduction of the Landau transformation.

The present problem is a special case of a more general problem treated by Westphal (1967) who considered the freezing of a semi-infinite medium cooled by the coolant having a time-dependent temperature, $T_0(t)$. Westphal's exact solution is in the form of integrals and infinite series, which make it extremely awkward for numerical computations. Another exact solution has been presented by Lozano and Reemsten (1981), but again the solution is tedious if numerical information is to be derived from it. In view of the complexity of these exact solutions, the perturbation solution is best compared with the heat balance integral solution of Goodman (1958), analog solution of Krieth and Romie (1955), and the local similarity solution of Aziz and Lunardini (1991). The comparison shown in

Figure 4. Approximate solution for freezing with surface convection, $S = c\Delta T/L$.

Figure 4 demonstrates that the prediction of the perturbation solution is consistent with the other approximate solutions.

3.3 Planar freezing of a saturated liquid due to sinusoidal surface temperature variation

This problem is of considerable importance in modeling the cyclical behavior of the active layer in permafrost regions because the actual annual ground surface temperature variation often approximates a sine curve (Kane et al. 1991, Aziz and Lunardini 1992). The problem is a variation of the Stefan problem; the mathematical description is given by eq 1–3 except that the constant temperature boundary condition $T(0, t) = T_0$ is now replaced by the following sinusoidal condition

$$T(0, t) = T_0 + a \sin \Omega t < T_f \tag{52}$$

Introducing the following dimensionless quantities,

$$\theta = (T - T_0)/(T_f - T_0), \quad X = x(\Omega/\alpha)^{1/2}, \quad \tau = \Omega t \tag{53}$$

$$A = a/(T_f - T_0), \quad \varepsilon = c(T_f - T_0)/L$$

into eq 1, 2, 3, 52 leads to

$$\frac{\partial^2 \theta}{\partial X^2} = \varepsilon \frac{\partial \theta}{\partial X_f} \left. \frac{\partial \theta}{\partial X} \right|_{X = x_f} \tag{54}$$

$$\theta(X = 0, X_f) = A \sin \tau, \quad \theta(X = X_f, X_f) = 1 \tag{55}$$

Assuming a perturbation expansion in the form of eq 12 and considering the first two terms only, we find that θ_0 is governed by eq 16 while θ_1 is governed by eq 18 without the negative sign on the right-hand side. However, the boundary conditions change to

$$\theta_0(X = 0, X_f) = A \sin \tau, \quad \theta_0(X = X_f, X_f) = 1 \tag{56}$$

$$\theta_1(X = 0, X_f) = 0, \quad \theta_1(X = X_f, X_f) = 0 \tag{57}$$

The solution of eq 16 subject to eq 56 is

$$\theta_0 = \frac{X}{X_f} + \left(1 - \frac{X}{X_f}\right) A \sin \tau \tag{58}$$

Similarly, the solution of eq 18 subject to eq 57 is

$$\theta_1 = \frac{1}{6}(1 - A \sin \tau)^2 \frac{X}{X_f}\left[1 - \left(\frac{X}{X_f}\right)^2\right] \tag{59}$$

Thus the two-term perturbation solution becomes

$$\theta = \frac{X}{X_f} + \left(1 - \frac{X}{X_f}\right) A \sin \tau + \frac{1}{6}\varepsilon(1 - A \sin \tau)^2 \frac{X}{X_f}\left[1 - \left(\frac{X}{X_f}\right)^2\right] \tag{60}$$

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Using eq 60 to evaluate, substituting it in eq 6, and noting that the negative sign on the right-hand side is to be ignored, we obtain the differential equation for X_f as follows:

$$\frac{dX_f}{dt} = \varepsilon \left(\frac{1 - A \sin \tau}{X_f}\right) \left[1 - \frac{1}{3} \varepsilon \left(1 - A \sin \tau\right)\right]$$

(61)

Integrating eq 61 subject to the initial condition, $t = 0$, $X_f = 0$, we obtain

$$X_f^2 = 2\varepsilon \left\{\left(\tau + A \cos \tau - A\right) - \frac{1}{3} \varepsilon \left[\left(\tau + 2 A \cos \tau - 2A\right) + \frac{1}{2} A^2 \left(\tau - \frac{1}{2} \sin 2\tau\right)\right]\right\}$$

(62)

If $A = 0$, the present problem reduces to the Stefan problem discussed in section 3.1. The perturbation approach adopted for this problem was first introduced by Lock et al. (1969) and Lock (1969, 1971), and they reported that the agreement between the perturbation and finite difference solutions was excellent.

3.4. Variable-property Stefan problem

The variable-property Stefan problem in which the thermal conductivity and specific heat of the solid phase are temperature-dependent has been studied by Aziz (1978) and Pedroso and Domoto (1973b) using a regular perturbation method. While Pedroso and Domoto use the Stefan number S as a perturbation parameter, Aziz identifies a new perturbation parameter that is a measure of the variation of a thermal property with temperature which makes the solution valid for all values of S. Both approaches will be illustrated here.

3.4.1. Pedroso and Domoto's solution

Consider a semi-infinite region of liquid initially at its freezing temperature T_f. At time $t > 0$ the face at $x = 0$ is maintained at constant temperature $T_0 < T_f$. The analysis takes into consideration the property variation in the solid phase, the unfrozen liquid being assumed to remain at the freezing temperature. The governing equations are

$$\rho c(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[k(T) \frac{\partial T}{\partial x}\right]$$

(63)

$$T(0, t) = T_0, \quad T(x_f, t) = T_f, \quad \left(k \frac{\partial T}{\partial x}\right)_{x=x_f} = \rho L \frac{dx_f}{dt}$$

(64)

By introducing the following quantities:

$$\bar{k} = \frac{\int_{T_0}^{T_f} k(T) dt}{T_f - T_0}, \quad \bar{c} = \frac{\int_{T_0}^{T_f} k(T) dt}{T_f - T_0}$$

$$\theta = \frac{\int_{T_0}^{T} k(T) dt}{\bar{k}(T_f - T_0)}, \quad \varepsilon = \frac{\bar{c}(T_f - T_0)}{L}$$

$$\tau = \frac{\bar{k}(T_f - T_o)t}{\rho L}, \quad f(\theta) = \frac{\bar{c}(\theta)/\bar{c}}{k(\theta)/\bar{k}}, \quad \eta = \frac{x}{x_f} \tag{65}$$

it can be shown that eq 63, 64 reduce to the following boundary-value problem:

$$\frac{d^2\theta}{d\eta^2} + \varepsilon \eta f(\theta) \frac{d\theta}{d\eta} \left. \frac{d\theta}{d\eta} \right|_{\eta=1} \tag{66}$$

$$\theta(0) = 0, \quad \theta(1)1 = 1, \quad x_f^2 = 2\tau \left. \frac{d\theta}{d\eta} \right|_{\eta=1} \tag{67}$$

We solve eq 66 and 67 assuming $f(\theta) = a + b\theta$, where a and b are constants. For a three-term expansion of the form $\theta = \theta_o + \varepsilon\theta_1 + \varepsilon^2\theta_2$, the governing equations for θ_o, θ_1 and θ_2 are

$$\varepsilon^0: \quad \frac{d^2\theta_o}{d\eta^2} = 0 \tag{68}$$

$$\theta_o(0) = 0, \quad \theta_o(1) = 1 \tag{68a}$$

$$\varepsilon^1: \quad \frac{d^2\theta_1}{d\eta^2} + \eta(a + b\theta_o) \frac{d\theta_o}{d\eta} \cdot \left. \frac{d\theta_o}{d\eta} \right|_{\eta=1} \tag{69}$$

$$\theta_1(0) = 0, \quad \theta_1(1) = 0 \tag{69a}$$

$$\varepsilon^2: \quad \frac{d^2\theta_2}{d\eta^2} + \eta \left\{ \left[\left. \frac{d\theta_1}{d\eta} \cdot \frac{d\theta_o}{d\eta} \right|_{\eta=1} + \left. \frac{d\theta_o}{d\eta} \cdot \frac{d\theta_1}{d\eta} \right|_{\eta=1} \right] (a + b\theta_o) \right.$$

$$\left. + b\theta_1 \left. \frac{d\theta_o}{d\eta} \frac{d\theta_o}{d\eta} \right|_{\eta=1} \right\} \tag{70}$$

$$\theta_2(0) = 0, \quad \theta_2(1) = 0. \tag{71}$$

The above sequence of equations can be integrated successively to give

$$\theta_o = \eta \tag{72}$$

$$\theta_1 = \frac{1}{6} \eta \left[\left(a + \frac{1}{2} b \right) - a\eta^2 - \frac{1}{2} b\eta^3 \right] \tag{73}$$

$$\theta_2 = -\frac{1}{45,360} \left(2394 a^2 + 765 b^2 + 2772 ab \right)\eta$$

$$+ \frac{1}{36} a(a + b)\eta^3 + \frac{1}{144} b^2\eta^4 + \frac{1}{40} a^2\eta^5$$

$$+ \frac{1}{30} ab\eta^6 + \frac{5}{504} b^2\eta^7. \tag{74}$$

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Thus the complete three-term perturbation solution is

$$\theta = \eta + \frac{1}{6}\,\varepsilon\,\eta\left[\left(a + \frac{1}{2}\,b\right) - a\eta^2 - \frac{1}{2}\,b\eta^3\right]$$

$$- \varepsilon^2\left[\frac{1}{45{,}360}\left(2394\,a^2 + 765\,b^2 + 2772\,ab\right)\eta\right.$$

$$+ \frac{1}{36}\,a(a + b)\eta^3 + \frac{1}{144}\,b^2\eta^4 + \frac{1}{40}\,a^2\eta^5$$

$$\left.+ \frac{1}{30}\,ab\eta^6 + \frac{5}{504}\,b^2\eta^7\right]. \tag{75}$$

Utilizing eq 75 in eq 67, the freezing front location is given by

$$x_f^2 = 2\tau\left[1 - \varepsilon\left(\frac{1}{3}\,a + \frac{3}{2}\,b\right) - \varepsilon^2\left(\frac{774}{2835}\,a^2 + \frac{345}{3024}\,b^2 + \frac{217}{630}\,ab\right)\right] \tag{76}$$

The accuracy of eq 75 and 76 has been checked against the direct numerical solution of eq 66 and 67 by Pedroso and Domoto (1973b) and found to be good. Furthermore, these authors give the solutions for any arbitrary form of the function $f(\theta)$.

3.4.2 Aziz's Solution

Aziz (1978) considered the solution of eq 63 and 64 for two cases of thermal property variation: 1) a linear specific heat-temperature variation with constant thermal conductivity; 2) a linear thermal conductivity–temperature variation with constant specific heat. Mathematically these variations are expressed as

$$c = c_f\left[1 + \alpha\left(T_f - T\right)\right] \tag{77}$$

$$k = k_f\left[1 + \beta\left(T_f - T\right)\right] \tag{78}$$

Introducing eq 77 and 78 into eq 63 and 64 and using the well known similarity transformation, the governing equations in dimensionless form become

$$F'' + 2\eta\left(1 + \varepsilon F\right)F' = 0 \tag{79}$$

for the temperature-dependent specific heat and

$$\left(1 + \varepsilon F\right)F'' + \varepsilon\left(F'\right)^2 + 2\eta F' = 0 \tag{80}$$

for the temperature-dependent thermal conductivity. The boundary conditions common to eq 79 and 80 are

$$F(0) = 1, \quad F(\lambda) = 0, \quad F'(\lambda) = -2\lambda/S \tag{81}$$

where

$$F = (T_f - T)/(T_f - T_o), \quad \eta = \frac{x}{2\sqrt{\alpha_f t}}, \quad \lambda = \frac{x_f}{2\sqrt{\alpha_f t}}, \quad \alpha_f = \frac{k_f}{\rho c_f}, \quad S = \frac{c_f}{L}\left(T_f - T_o\right), \quad \varepsilon = \alpha(T_f - T_o)$$

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for eq 79, $\varepsilon = \beta(T_f - T_o)$ for eq 80. The primes denote differentiation with respect to η.

Since ε is small for most applications (Aziz and Benzies 1976), F and λ are expanded in the form

$$F = \sum_{n=0}^{\infty} \varepsilon^n F_n \tag{82}$$

$$\lambda = \sum_{n=0}^{\infty} \varepsilon^n \lambda_n \tag{83}$$

Substituting eq 82 and 83 into eq 79 and 80 and equating coefficients of like powers of ε, a set of equations for F_0, F_1 etc. can be generated. The condition $F(0) = 1$ in eq 81 becomes

$$F_o(0) = 1 \qquad F_i(0) = 0 \qquad i = 1, 2, \cdots, n \tag{84}$$

However, the freezing front conditions in eq 81 contain ε implicitly. To remove the implicitness, we expand $F_n(\lambda)$ and $F'_n(\lambda)$ in a Taylor's series about λ_o to recast these in explicit form as

$$\sum_{n=0}^{\infty} \varepsilon^n \sum_{i=0}^{\infty} \frac{\left(\sum_{j=1}^{\infty} \varepsilon^j \lambda_j\right)^i}{i!} \frac{d^i F_n(\lambda_o)}{d\eta^i} = 0 \tag{85}$$

$$\sum_{n=0}^{\infty} \varepsilon^n \sum_{i=0}^{\infty} \frac{\left(\sum_{j=1}^{\infty} \varepsilon^j \lambda_j\right)^i}{i!} \frac{d^{i+1} F_n(\lambda_o)}{d\eta^i} = -\frac{2}{S} \sum_{n=0}^{\infty} \varepsilon^n \lambda_n . \tag{86}$$

Since, as shown later, the two-term perturbation expansion gives accurate results for a wide range of parameters, we formulate only the zero- and first-order problems and present their solutions.

Temperature-dependent specific heat. The zero-order problem, which is the classical Neumann problem, has the solution

$$F_o = 1 - \text{erf}\eta / \text{erf}\lambda_o \tag{87}$$

and

$$\sqrt{\pi} \lambda_o \exp(\lambda_o^2) \text{erf}\lambda_o = S \tag{88}$$

The first-order problem is

$$F_1'' + 2\eta F_1' = -2\eta F_o F_1' \tag{89}$$

$$F_1(0) = 0$$

$$F_1(\lambda_o) = -\lambda_1 F_o'(\lambda_o)$$

$$F_o'(\lambda_o) = -\lambda_1 F_o''(\lambda_o) - 2\lambda_1/S \tag{90}$$

Using the method of variation of parameters, the solution of eq 89 and 90 is finally obtained as

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$$F_1 = \frac{\lambda_0^2 \{\exp(2\lambda_0^2) - 1\} + 2S\lambda_0\lambda_1}{S^2 \operatorname{erf}\lambda_0} \operatorname{erf}\eta$$

$$- \frac{\eta\exp(-\eta^2)}{\sqrt{\pi}\operatorname{erf}\lambda_0}\left(1 - \frac{\operatorname{erf}\eta}{\operatorname{erf}\lambda_0}\right) - \frac{1}{\pi(\operatorname{erf}\lambda_0)^2}\{1 - \exp(-2\eta^2)\} \tag{91}$$

and

$$\lambda_1 = \frac{\lambda_0^3\{S + 1 - \exp(2\lambda_0^2)\}}{S\{S + 2(1 + S)\lambda_0^2\}} \tag{92}$$

Temperature-dependent thermal conductivity. The solution of the zero-order problem is given by eq 87 and 88. The first-order problem is

$$F_1'' + 2\eta F_1' = -F_0 F_0'' - (F_0')^2 \tag{93}$$

with boundary conditions given by eq 90. The method of variation of parameters gives the solution as

$$F_1 = \frac{\lambda_0^2\{1 - \exp(2\lambda_0^2)\} + 2S\lambda_0\lambda_1 + \frac{1}{2}S^2}{S^2\operatorname{erf}\lambda_0}\operatorname{erf}\eta + \frac{\eta\exp(-\eta^2)}{\sqrt{\pi}\operatorname{erf}\lambda_0}\left(1 - \frac{\operatorname{erf}\eta}{\operatorname{erf}\lambda_0}\right)$$

$$+ \frac{1}{\pi(\operatorname{erf}\lambda_0)^2}\{1 - \exp(-2\eta^2)\} - \frac{1}{2}\left(\frac{\operatorname{erf}\eta}{\operatorname{erf}\lambda_0}\right)^2 \tag{94}$$

and

$$\lambda_1 = \frac{\lambda_0[S^2 - 2S\lambda_0^2 + 2S\lambda_0^2\{\exp(2\lambda_0^2) - 1\}]}{2S\{S + 2(1 + S)\lambda_0^2\}}. \tag{95}$$

Aziz evaluated the above perturbation series for $S = 0.0822, 0.3564, 0.9205, 1.9956, 4.0601$ and 8.1720 ($\lambda_0 = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, respectively) with ε ranging in each case from -3 to $+3$.

First we discuss the results for a temperature-dependent specific heat. For $-1 \le \varepsilon < 1$ the perturbation solutions agree to within 0.5% with the numerical solutions. Indeed, the convergence is so rapid that even when ε is as large as ± 3 the error does not exceed 1.7%. A sample perturbation temperature distribution is given in Table 2. The results for the freezing front location appear in Table 3. These

Table 2. Temperature distribution for $S = 4.0601$ ($\lambda_0 = 1$): effect of a temperature-dependent specific heat.

η	$\varepsilon = 1.0$	0.5	0	-0.5	-1.0
0	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	0.7038	0.7198	0.7357	0.7517	0.7677
0.4	0.4436	0.4676	0.4916	0.5157	0.5397
0.6	0.2374	0.2604	0.2834	0.3064	0.3295
0.8	0.0858	0.1026	0.1194	0.1362	0.1529
0.9595	0				
0.9798		0			
1			0		
1.0202				0	
1.0404					0

Table 3. *Interface location parameter* λ *at different* S: *effect of a temperature-dependent specific heat.*

ε	S = 0.0822	0.3564	0.9205	1.9956	4.0601	8.1720
-1.0	0.1993	0.3953	0.5853	0.7735	0.9595	1.1472
-0.5	0.1997	0.3976	0.5932	0.7868	0.9798	1.1736
-0.0	0.2000	0.4000	0.6000	0.8000	1.0000	1.2000
-0.5	0.2003	0.4023	0.6068	0.8132	1.0202	1.2264
-1.0	0.2006	0.4047	0.6136	0.8264	1.0404	1.2528

results indicate that at low values of S the effect of a variation of specific heat on temperature and freezing front location is small but becomes progressively significant as S increases. The same conclusion holds for other types of boundary conditions, such as a wall heat flux varying linearly or exponentially with time (Chung and Yeh 1976) or the wall being cooled by combined convection and radiation (Yeh and Chung 1977).

In the case of a temperature-dependent thermal conductivity, the temperature series agree to within 2.2% with the numerical solutions in the range $-0.5 \le \varepsilon < 0.5$. The maximum error of 2.2% occurs in the neighborhood of the interface for $S = 8.1720$. Although not reported here, a second-order correction extends the range of validity of the solution to higher values of ε. Fortunately, the more important freezing front series converges rapidly and with first-order correction gives results within 1.8% in the range $-2 \le \varepsilon \le 2$. Sample results for temperature distribution and freezing front location are given in Tables 4 and 5, respectively. Compared with a variable specific heat, the effect of a variation in thermal conductivity is much more pronounced on both the temperature and the freezing front location at all values of S. This relatively strong effect of variable thermal conductivity is also evident in the results of Chung and Yeh (1976) who analyzed the case of a time-dependent (linear and exponential) wall heat flux using a heat balance integral. It can also be seen in another paper (Yeh and Chung 1977) in which they study the case of combined convective–radiative cooling of the wall using Biot's variational principle.

Table 4. **Temperature distribution for** $S = 4.0601$ ($\lambda_0 = 1$): **effect of a temperature-dependent thermal conductivity.**

η	$\varepsilon = 0.5$	0	-0.5
0	1.0000	1.0000	1.0000
0.2	0.8096	0.7357	0.6618
0.4	0.5960	0.4916	0.3872
0.6	0.3824	0.2834	0.1843
0.8	0.1935	0.1194	0.0453
0.9082			0
1.0		0	
1.0918	0		

Table 5. **Interface location parameter** λ **at different** S: **effect of a temperature-dependent thermal conductivity.**

ε	S = 0.0822	0.3564	0.9205	1.9956	4.0601	8.1720
1.0	0.2493	0.4949	0.7335	0.9634	1.1836	1.3946
0.5	0.2247	0.4474	0.6668	0.8817	1.0918	1.2973
0	0.2000	0.4000	0.6000	0.8000	1.0000	1.2000
0.5	0.1753	0.3526	0.5332	0.7183	0.9082	1.1027
1.0	0.1507	0.3051	0.4665	0.6366	0.8164	1.0054

3.5. Planar freezing with sinusoidal variation of coolant temperature

This problem is also a variation of the Stefan problem of section 3.1. The constant temperature boundary condition at $x = 0$ is now replaced by a boundary condition characterizing the sinusoidal variation of coolant temperature which can be expressed as

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$$T_a = T_o + a \sin \Omega t \tag{96}$$

where T_a is the coolant temperature fluctuating around a mean value of T_o, with an amplitude a and frequency Ω. The convective boundary condition is written as

$$k \frac{\partial T}{\partial x} (0,t) = h [T(0,t) - T_a(t)] \tag{97}$$

Equation 3 and the second condition of eq 2 also apply here.

The problem will be solved using a double series expansion involving two perturbation parameters, ε_1 and ε_2. As defined below, ε_1 represents the classical Stefan number, while ε_2 is a measure of coolant temperature fluctuations. The problem is recast into dimensionless form by defining the following quantities:

$$\theta = (T_f - T)/(T_f - T_o), \quad X = \frac{hx}{k}, \quad \varepsilon_1 = c(T_f - T_o)/L$$

$$\varepsilon_2 = a/(T_f - T_o), \quad \omega = \rho L k \Omega / h^2 (T_f - T_o) \tag{98}$$

$$\tau = h^2 (T_f - T_o) \, t/\rho L k$$

which give

$$\frac{\partial^2 \theta}{\partial X^2} = \varepsilon_1 \frac{\partial \theta}{\partial \tau} \tag{99}$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{X=0} = \theta(0,\tau) - 1 + \varepsilon_2 \sin \omega \tau \tag{100}$$

$$\theta(X_f, \tau) = 0 \tag{101}$$

$$\frac{dX_f}{d\tau} = - \left. \frac{\partial \theta}{\partial X} \right|_{X=X_f} \tag{102}$$

To solve eq 99–102, a double series expansion for θ and X_f is assumed. Retaining only the first three terms of the series, we have

$$\theta = \theta_o(X,\tau) + \varepsilon_1 \theta_1 (X,\tau) + \varepsilon_2 \theta_2 (X,\tau) \tag{103}$$

$$X_f = X_{fo}(\tau) + \varepsilon_1 X_{f1}(\tau) + \varepsilon_2 X_{f2}(\tau) \tag{104}$$

Substituting eq 103 and 104 into eq 99–102 and removing the implicitness of ε_1 and ε_2 in X_f by Taylor series expansions about X_{fo}, the following system of equations for $\theta_o, \theta_1, \theta_2, X_{fo}, X_{f1}$ and X_{f2} is obtained:

$$\frac{\partial^2 \theta_o}{\partial X^2} = 0 \tag{105}$$

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$$\left.\frac{\partial\theta_o}{\partial X}\right|_{X=0} = \theta_o(0,\tau) - 1; \;\; \theta_o(X_{fo}, \tau)) = 0; \;\; \frac{dX_{fo}}{dt} = -\left.\frac{\partial\theta_o}{\partial X}\right|_{X=X_f} \tag{106}$$

$$\frac{\partial^2\theta_1}{\partial X^2} = \frac{\partial\theta_o}{\partial\tau} \tag{107}$$

$$\left.\frac{\partial\theta_1}{\partial X}\right|_{X=0} = \theta_1(0,\tau); \;\; \theta_1(X_{fo}, \tau) = -X_{f1}\,\theta_o'(X_{fo}, \tau);$$

$$\frac{dX_{f1}}{dt} = -\left.\frac{\partial\theta_1}{\partial X}\right|_{X=X_{fo}} - X_{f1}\left.\frac{\partial\theta_o}{\partial X}\right|_{X=X_f} \tag{108}$$

$$\frac{\partial^2\theta_2}{\partial X^2} = 0 \tag{109}$$

$$\left.\frac{\partial\theta_2}{\partial X}\right|_{X=0} = \theta_2(0,\tau) + \sin\,\omega\tau; \;\; \theta_2(X_{fo}, \tau) = -X_{f2}\,\theta_o'(X_{fo}, \tau);$$

$$\frac{dX_{f2}}{dt} = -\left.\frac{\partial\theta_2}{\partial X}\right|_{X=X_{fo}} - X_{f2}\left.\frac{\partial\theta_o}{\partial X}\right|_{X=X_f} \tag{110}$$

where primes denote differentiation with respect to X.

The solutions of eq 105–110 are found to be as follows:

$$\theta_o = (X_{fo} - X)\,\dot{X}_{fo} \tag{111}$$

$$X_{fo} = (1 + 2\tau)^{1/2} - 1 \tag{112}$$

$$\theta_1 = -\left(\frac{X_{fo}^3\,\ddot{X}_{fo}}{1 + X_{fo}}\right)\left(\frac{1+X}{3}\right) + \frac{1}{2}\,X_{fo}\,\ddot{X}_{fo}\,X^2 - \frac{1}{6}\,\ddot{X}_{fo}\,X^3 \tag{113}$$

$$X_{f1} = -\frac{X_{fo}^2\,\ddot{X}_{fo}}{2(1 + X_{fo})}\left(1 + \frac{X_{fo}}{3}\right) \tag{114}$$

$$\theta_2 = \left[\frac{X + 1 - (1+2\tau)^{1/2}}{(1+2\tau)^{1/2}}\right]\sin\,\omega\tau - \frac{(1+X)}{\omega(1+2\tau)^{1/2}}(1-\cos\,\omega\tau) \tag{115}$$

$$X_{f2} = -(1-\cos\,\omega\tau)/\omega(1+2\tau)^{1/2} \tag{116}$$

The solutions given by eq 111–116 stem from the work of Gutman (1986) and have been corrected to account for the second term in both of the equations for $\dot{X}_{f1}$ and $\dot{X}_{f2}$. These terms were missed by Gutman.

3.6. Freezing of a semi-infinite region with convective and radiative cooling

Another variation of the Stefan problem arises when the boundary condition at $x = 0$ is that of simultaneous convection and radiation. A particular case was considered by Yan and Huang (1979) for a finite phase change region when the other end was insulated. Since the presence of radiation introduces a nonlinear boundary condition, they linearized it by using the approximation $T^4 = 4T_a^3 T - 3 T_a^4$ where T_a is the common ambient temperature for both convection and radiation. With this approximation, the problem was reduced to that of pure convective cooling, the case discussed in section 3.2. Yan and Huang developed three-term perturbation solutions for the temperature distribution and the freezing front location and found that the perturbation solutions agreed quite well with variational solutions.

Here we approach the semi-infinite region case using the double series expansion method of the previous section. Equations 1–3 apply except that the first condition in eq 2 is replaced by a convective-radiative condition

$$X = 0, \quad k \frac{\partial T}{\partial x} = h\left(T - T_a\right) + \sigma F\left(T^4 - T_a^4\right) \tag{117}$$

To normalize the equations, we introduce the following dimensionless quantities:

$$\theta = T/\left(T_f - T_a\right), \quad X = x/x_s, \quad \tau = k\left(T_f - T_a\right)t/\rho L x_s^2,$$

$$\varepsilon_1 = c\left(T_f - T_a\right)/L, \quad \varepsilon_2 = \sigma F\left(T_f - T_a\right)^3 x_s/k, \quad Bi = hx_s/k \tag{118}$$

into eq 1–3 and 117 and obtain

$$\frac{\partial^2 \theta}{\partial X^2} = \varepsilon_1 \frac{\partial \theta}{\partial X_f} \cdot \left. \frac{\partial \theta}{\partial X}\right|_{X=X_f} \tag{119}$$

$$X = 0, \frac{\partial \theta}{\partial X} = Bi\left(\theta - \theta_a\right) + \varepsilon_2\left(\theta^4 - \theta_a^4\right)$$

$$X = X_f, \quad \theta = \theta_f \tag{120}$$

$$\frac{dX_f}{dt} = \left. \frac{\partial \theta}{\partial X}\right|_{X=X_f} \tag{121}$$

Assuming an expansion for θ in the form of eq 103 and carrying out the procedure outlined there, we obtain

$$\frac{\partial^2 \theta_0}{\partial X^2} = 0 \tag{122}$$

$$X = 0, \frac{\partial \theta_0}{\partial X} = Bi\left(\theta_0 - \theta_a\right); \ X = X_f, \ \theta_0 = \theta_f \tag{123}$$

$$\frac{\partial^2 \theta_1}{\partial X^2} = \frac{\partial \theta_0}{\partial X_f} \cdot \left. \frac{\partial \theta_0}{\partial X}\right|_{X=X_f} \tag{124}$$

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$$X = 0, \frac{\partial \theta_1}{\partial X} = Bi\,\theta_1 ; \quad X = X_f, \; \theta_1 = 0 \tag{125}$$

$$\frac{\partial^2 \theta_2}{\partial X^2} = 0 \tag{126}$$

$$X = 0, \frac{\partial \theta_2}{\partial X} = Bi\,\theta_2 + \theta_o^4 - \theta_o^4; \quad X = X_f, \; \theta_2 = 0 \tag{127}$$

The solutions for θ_o, θ_1 and θ_2 are as follows:

$$\theta_o = \theta_a + \left(\theta_f - \theta_a\right)\left(\frac{X + Bi^{-1}}{X_f + Bi^{-1}}\right) \tag{128}$$

$$\theta_1 = \frac{1}{6}\frac{\theta_f - \theta_a}{\left(X_f + Bi^{-1}\right)^3}\left[\left(\frac{1 + BiX}{1 + BiX_f}\right)\left(X_f^3 + 3Bi^{-1}X_f^2\right) - \left(X^3 + 3Bi^{-1}X^2\right)\right] \tag{129}$$

$$\theta_2 = \left\{\left[\theta_a + \frac{\left(\theta_f - \theta_a\right)Bi^{-1}}{X_f + Bi^{-1}}\right]^4 - \theta_a^4\right\}\left(\frac{X - X_f}{1 + BiX_f}\right) \tag{130}$$

Utilizing eq 128–130 to evaluate in eq 121, the differential equation for X_f is obtained as

$$\frac{dX_f}{d\tau} = \frac{\theta_f - \theta_a}{X_f + Bi^{-1}} + \frac{1}{6}\,\varepsilon_1\frac{\left(\theta_f - \theta_a\right)}{\left(X_f + Bi^{-1}\right)^3}\left[\frac{Bi}{1 + BiX_f}\left(X_f^3 + 3Bi^{-1}X_f^2\right) - \left(3X_f^2 + 6Bi^{-1}X_f\right)\right]$$

$$+ \varepsilon_2\left\{\left[\theta_a + \frac{\left(\theta_f - \theta_a\right)Bi^{-1}}{X_f + Bi^{-1}}\right]^4 - \theta_a^4\right\}\frac{1}{1 + BiX_f} \tag{131}$$

For specified values of θ_f, θ_a, Bi, ε_1, and ε_2, eq 131 can be integrated numerically with the initial condition $\tau = 0$, $X_f = 0$ to give X_f as a function of τ. Figure 5 shows the variation of the freezing front
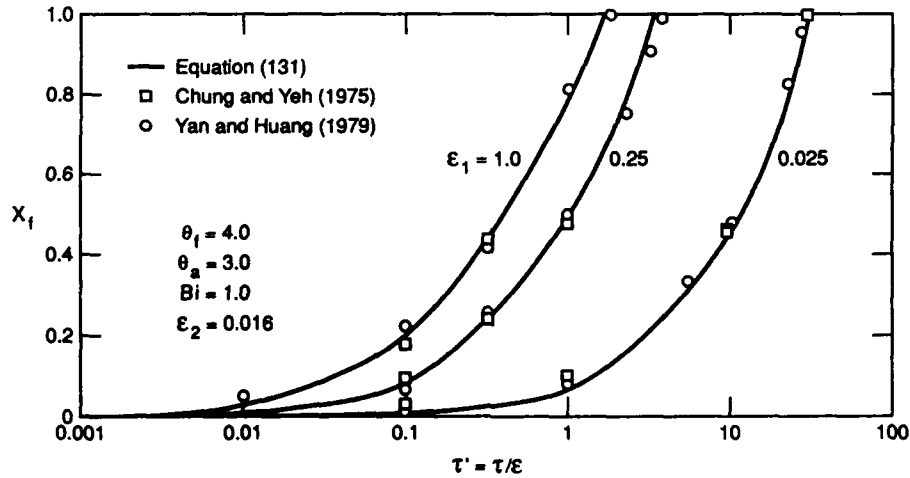

Figure 5. Perturbation solution for freezing with surface convection and radiation.

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X_f with $\tau' = \tau/\varepsilon_1 = \alpha t/X_s^2$ for $\theta_f = 4$, $\theta_a = 3$, $Bi = 1$, $\varepsilon_2 = 0.016$ and $\varepsilon_1 = 0.025$, 0.25 and 1.0. These values were chosen so that the present solution can be compared with the variational results of Chung and Yeh (1975) and a different perturbation solution reported by Yan and Huang (1979). The predictions of the present perturbation solution are quite close to the other reported solutions.

4. REGULAR COORDINATE PERTURBATION EXPANSIONS

Compared with the parameter perturbation expansion, the use of a coordinate perturbation approach to freezing and melting problem has been very limited. Two examples will be considered, one that has been developed by the present authors and is being reported for the first time, and a second one that has been presented by Rathjen and Jiji (1970).

4.1. Planar freezing of a saturated liquid with convective cooling

We revisit the problem of section 3.2 and develop a coordinate perturbation solution valid for short times when convection is strong or for long times when convection is weak. To solve the problem which is defined by eq 1, 3 and 40 and the second condition of eq 2, we introduce the similarity transformation as follows:

$$F = (T_f - T)/(T_f - T_o), \quad \eta = x/(2\sqrt{\alpha t}) \tag{132}$$

and define a dimensionless coordinate $\varepsilon = k/(2h\sqrt{\alpha t})$. It can be shown that the similarity transformation (eq 132) reduces eq 1 to an ordinary differential equation in F as

$$F'' + 2\eta F' = 0 . \tag{133}$$

The boundary conditions at $x = 0$ and $x = x_f$ reduce to

$$\eta = 0, \quad \varepsilon F' = F - 1; \quad \eta = \lambda = x_f/(2\sqrt{\alpha t}), \quad F = 0;$$

$$\eta = \lambda, \quad F' = -2\lambda/S \tag{134}$$

where $S = c(T_f - T_o)/L$. Note that the quantity $\varepsilon = k/(2h\sqrt{\alpha t})$ being a function of t rather than η, is the source of nonsimilarity. As $h \to \infty$, $\varepsilon \to 0$, and $T(0, t) \to T_o$, and the problem is reduced to the classical Stefan solution discussed in section 3.1, also called the single-phase Neumann solution. To solve the nonsimilar eq 133, 134, two-term perturbation expansions are assumed for F and λ as

$$F = F_0 = \varepsilon F_1 + 0(\varepsilon^2) \tag{135}$$

$$\lambda = \lambda_o = \varepsilon\lambda_1 + 0(\varepsilon^2) \tag{136}$$

Substituting eq 135 and 136 into 133 and 134, removing the implicitness due to λ by a Taylor series expansion about λ_o, and equating the coefficients of ε^0 and ε, we get

$$F_0'' + 2\eta F_0 = 0 \tag{137}$$

$$\eta = 0, \quad F_0 = 1; \quad \eta = \lambda_o, \quad F_0 = 0 \tag{138}$$

$$F_1'' + 2\eta F_1 = 0 \tag{139}$$

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$$\eta = 0, \quad F_1 = F_0'; \quad \eta = \lambda_o, \quad F_1 = -\lambda_1 F_0'(\lambda_o) \tag{140}$$

The solution of eq 137 and 138 is well known and can be written as

$$F_o = 1 - \left(\text{erf}\,\eta / \text{erf}\,\lambda_o\right) \tag{141}$$

where λ_o is given by the transcendental equation

$$\lambda_o \,\text{erf}\,\lambda_o \exp\left(\lambda^2\right) = S\sqrt{\pi} \tag{142}$$

The solution for F_1 can be obtained as

$$F_1 = \frac{2}{\sqrt{\pi}\,\text{erf}\,\lambda_o}\left[\left(1 + \lambda_1 e^{-\lambda_o^2}\right)\frac{\text{erf}\,\eta}{\text{erf}\,\lambda_o} - 1\right] \tag{143}$$

where λ_1 is given by the following equation:

$$\lambda_1 = \frac{4S}{2e^{\lambda_o^2}\pi\left(\text{erf}\,\lambda_o\right)^2 + 4S\left(\lambda_o\sqrt{\pi}\,\text{erf}\,\lambda_o - e^{\lambda_o^2}\right)}\;. \tag{144}$$

In order to compare the present solution with the solutions shown in Figure 4, we calculated $\lambda/\varepsilon = hX_f/k$ as a function of $1/4\varepsilon^2 = h^2t/(kpc)$ for $S = 0.1, 0.2, 0.5, 1.0, 2.0$ and 3.0. These values, shown in Figure 4, compare well with other approximate solutions.

4.2. Freezing of a semi-infinite strip (fin) of liquid not initially at freezing temperature

Figure 6 shows a semi-infinite strip of liquid in the form of a fin with cross-sectional area A and perimeter P. The liquid is initially at temperature T_i. At time $t \geq 0$ the base at $x = 0$ is brought and maintained at a constant temperature T_o that is lower than the freezing temperature T_f. Both the solid

Figure 6. Geometry for the freezing of a liquid fin.

and the liquid phases convect heat to the surroundings held at temperature T_a, the heat transfer coefficient being h. If we allow for the surface convective heat transfer, the transient heat conduction equations for the solid and liquid phases can be written in partially dimensionless form as

$$\frac{\partial^2\theta_s}{\partial x^2} - \frac{hP}{k_sA}\left(\theta_s - \theta_a\right) = \frac{1}{\alpha_s}\frac{\partial\theta_s}{\partial t}, \quad 0 < x < \lambda, \quad t > 0 \tag{145}$$

$$\frac{\partial^2\theta_\ell}{\partial x^2} - \frac{hP}{k_\ell A}\left(\theta_\ell - \frac{k_\ell}{k_s}\theta_a\right) = \frac{1}{\alpha_\ell}\frac{\partial\theta_\ell}{\partial t}, \quad \lambda < x < \infty, \quad t > 0 \tag{146}$$

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where $\theta_s = (T_s - T_f)/(T_f - T_o)$, $\theta_\ell = k_\ell (T_\ell - T_f)/(k_s(T_f - T_o))$, $\theta_a = (T_a - T_f)/(T_f - T_o)$

The initial and boundary conditions are

$$x_f(0) = 0, \ \theta_\ell(x,0) = \theta_i = k_\ell (T_i - T_f)/(k_s(T_f - T_o)) \ \Big\}$$

$$\theta_s(0,t) = -1, \ \theta_s(\lambda,t) = 0, \ \theta_\ell(\lambda,t) = 0 \ \Big\}$$

$$\frac{\partial\theta_s}{\partial x}(\lambda,t) - \frac{\partial\theta_\ell}{\partial x}(\lambda,t) = \frac{\rho L}{(T_f - T_o)k_s} \cdot \frac{d\lambda}{dt} \ \Big\}$$

(147)

Introducing the similarity variables η_s and η_ℓ as

$$\eta_s = x/\sqrt{4\alpha_s t}, \quad \eta_\ell = x/\sqrt{4\alpha_\ell t} \tag{148}$$

and defining a dimensionless coordinate (time) $\varepsilon = 4h \, P\alpha_s \, t/(k_s A)$, we seek two-term perturbation solutions for θ_s, θ_ℓ, and λ as

$$\theta_s(x,t) = u_0(\eta_s) + \varepsilon u_1(\eta_s) + 0(\varepsilon^2) \tag{149}$$

$$\theta_\ell(x,t) = v_0(\eta_\ell) + \varepsilon v_1(\eta_\ell) + 0(\varepsilon^2) \tag{150}$$

$$\lambda(x,t) = \sqrt{4\alpha_s t} \left[\lambda_0 + \varepsilon\lambda_1 + 0(\varepsilon^2) \right] \tag{151}$$

Substituting eq 149 into 145 and eq 150 into 146 and removing the implicitness in the interface boundary conditions by Taylor expansion about λ_0, we obtain the following equations for u_0, v_0

$$u_0'' + 2\eta_s u_0' = 0 \tag{152}$$

$$u_0(0) = -1, \ u_0(\lambda_0) = 0 \tag{153}$$

$$v_0'' + 2\eta_\ell v_0' = 0 \tag{154}$$

$$v_0(\alpha\lambda_0) = 0, \ v_0(\infty) = \theta_i \tag{155}$$

$$u_0'(\lambda_0) - \alpha v_0'(\alpha\lambda_0) = 2\beta\lambda_0 \tag{156}$$

where $\alpha = (\alpha_s/\alpha_\ell)^{1/2}$ and $\beta = L/(c_s(T_f - T_o))$. Similarly the equations for u_1 and v_1 take the form

$$u_1'' + 2\eta_s u_1' - 4u_1 = u_0 - \theta_a \tag{157}$$

$$u_1(0) = 0, \ u_1(\lambda_0) + \lambda_1 u_0'(\lambda_0) = 0 \tag{158}$$

$$v_1'' + 2\eta_\ell v_1' - 4v_1 = \gamma v_0 - \theta_a/\alpha^2 \tag{159}$$

$$v_1(\infty) \text{ is finite, } \ v_1(\alpha\lambda_0) + \alpha\lambda_1 v_0'(\alpha\lambda_0) = 0 \tag{160}$$

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$$\lambda_1 u_0''(\lambda_0) + u_1'(\lambda_0) - \alpha^2 \lambda_1 v_0''(\alpha\lambda_0) = 6\beta\lambda_1 \tag{161}$$

where $\gamma = c_\ell/c_s$.

Equations 152–156 constitute the classical two-region Neumann problem whose solution is

$$u_0 = \frac{\operatorname{erf}\eta_s}{\operatorname{erf}\lambda_0} - 1 \tag{162}$$

$$v_0 = \theta_i - \theta_i \frac{\operatorname{erfc}\eta_\ell}{\operatorname{erfc}(\alpha\lambda_0)} \tag{163}$$

where λ_0 is given by the transcendental equation

$$e^{-\lambda_0^2}/\operatorname{erf}\lambda_0 - \theta_i \alpha e^{-\alpha^2\lambda_0^2}/\operatorname{erfc}(\alpha\lambda_0) = \sqrt{\pi}\,\beta\lambda_0 \tag{164}$$

Churchill and Evans (1971) give the solution of eq 164 for a range of the parameters involved.
The solutions for u_1 and v_1 appear in the form of repeated integrals of the error function and are

$$u_1 = B_1 I_2(\eta_s) - (1 + \theta_a)\left[i^2 \operatorname{erfc}\eta_s - \frac{1}{4}\right] - (\operatorname{erfc}\eta_s)/4\operatorname{erf}\lambda_c \tag{165}$$

$$v_1 = C_1 i^2 \operatorname{erfc}\eta_\ell + \theta_a/(4\alpha^2) - \gamma\theta_i\left[1 - \operatorname{erf}\eta_\ell/\operatorname{erfc}(\alpha\lambda_0)\right]/4 \tag{166}$$

where

$$I_n(\eta_s) = i^n \operatorname{erfc}(-\eta_s) + (-1)^{n+1} i^n \operatorname{erfc}\eta_s,$$

$$B_1 = \left[(1+\theta_a)i^2 \operatorname{erfc}\lambda_0 - 2\lambda_0\lambda_1 - \theta_a/4\right]/I_2(\lambda_0),$$

$$C_1 = \left[-2(A_0 - \beta\lambda_0)\lambda_1 - \theta_a/(4\alpha^2)\right]/i^2 \operatorname{erfc}(\alpha\lambda_0)$$

The solution for λ_1 takes the form

$$\lambda_1 = \{[(1 + \theta_a)A_0]/[1 + 2\lambda_0(\lambda_0 + A_0)]$$

$$-\theta_a[J_1 + K_1/\alpha^2]/4 + \gamma(A_0 - \beta\lambda_0)/2 - A_0/2\}/L_1 \tag{167}$$

where

$$A_0 = e^{-\lambda_0^2}/(\sqrt{\pi}\,\operatorname{erf}\lambda_0), \quad J_1 = I_1(\lambda_0)/I_2(\lambda_0)$$

$$K_1 = \alpha i \operatorname{erfc}(\alpha\lambda_0)/i^2 \operatorname{erfc}(\alpha\lambda_0)$$

$$L_1 = 6\beta + 4(1-\alpha^2)\lambda_0 A_0 + 2A_0(J_1 + K_1) + 2(2\alpha^2\lambda_0 - K_1)\beta\lambda_0$$

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It is interesting to note that unlike the Neumann problem, the freezing front can become stationary if $T_\infty > T_f$. This would occur when the convective heat transport from the surroundings to the fin from the top and bottom faces matches the heat extracted through the base of the fin. Under the condition of a stationary freezing front, the temperature distributions through the solid and the liquid phases are given by

$$\theta_s = \theta_a - \left(1 + \theta_a\right)\cosh\mu x + \left[\left(1 + \theta_a\right)\cosh\mu\lambda_\infty - \theta_a\right]\left(\sinh\mu x\right)/\left(\sinh\mu\lambda\right) \tag{168}$$

$$\theta_\ell = \left[1 - e^{\mu K(\lambda_\infty - x)}\right]\theta_a / K^2 \tag{169}$$

where $\mu = \left(hP/k_sA\right)^{1/2}$, $K = k_s/k_\ell$ and λ_∞ (the stationary location of the freezing front) is given by the explicit relation

$$\lambda_\infty = \left(1/\mu\right)\ln\left[c + \left(c^2 - b\right)^{1/2}\right] \tag{170}$$

where $b = (K-1)/(K+1)$ and $c = K(1+\theta_a)/((1+K)\theta_a)$.

Figure 7 shows a typical result for the progress of the freezing front for $\alpha = 0.71$, $\beta = 2.5$ (Stefan number = 0.4), $K = 0.71$, $\theta_i = 1$, and $\theta_a = 0.5$. This case corresponds to a solder fin for which $T_0 = 82°C$, $T_f = 183°C$, and $T_i = T_a = 233.5°C$. Examination of Figure 7 reveals that the two-term perturbation solution virtually coincides with the finite difference solution up to $\varepsilon = 4$. However, as ε increases further, the perturbation solution begins to deviate significantly from the finite difference solution. At $\varepsilon = 10$, the error is about 13%. The zero-order solution (Neumann) is also shown for comparison and it is evident that the first-order correction improves the accuracy considerably. Figure 8 shows the corresponding temperature profiles. Again the perturbation solution is quite accurate, up to $\varepsilon = 3$, but the accuracy deteriorates considerably at $\varepsilon = 10$, particularly in the liquid region. Figures 7 and 8 have been adapted from Rathjen and Jiji (1970), who also provide a graph of base heat flux vs. ε.
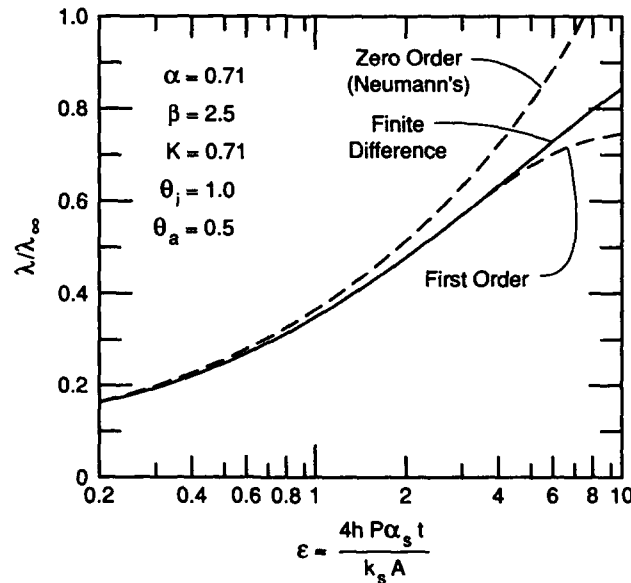

Figure 7. Comparison of perturbation and numerical solution for fin freezing.

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Figure 8. Temperature profiles during freezing of fins.

5. SINGULAR PERTURBATION EXPANSIONS

The term *singular perturbation expansion* is used to describe expansions that lack the feature of uniform validity that characterizes the regular perturbation expansion. In sections 3 and 4, the solutions obtained were valid throughout the domain of the independent variable. Now we study problems for which the expansion fails in certain regions, called "regions of nonuniformity" or "boundary layers."

The nonuniformity exhibits itself in several forms: 1) the solution becomes infinite at some value of the independent variable, 2) the solution has a discontinuity within the domain of interest, 3) the solution fails to satisfy some boundary condition, and 4) the solution contains an essential singularity. Two examples of nonuniformities arising in freezing problems are discussed below. Other examples will be discussed when the various techniques of handling singular expansions are presented.

5.1. Solidification of a saturated liquid in a spherical domain

Consider a saturated liquid of infinite extent outside a sphere of radius R_w as shown in Figure 9. At $t > 0$ the surface at R_w is suddenly lowered to a subfreezing temperature T_0. For the solid phase, we have

$$\frac{1}{R} \frac{\partial^2 (TR)}{\partial R^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{171}$$

Figure 9. Freezing exterior to a finite sphere.

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$$T(R_w, t) = T_o, \quad T(R_f, t) = T_f, \quad k \left. \frac{\partial T}{\partial R} \right|_{R=R_f} = \rho L \frac{dR_f}{dt} \tag{172}$$

Introducing the following dimensionless quantities

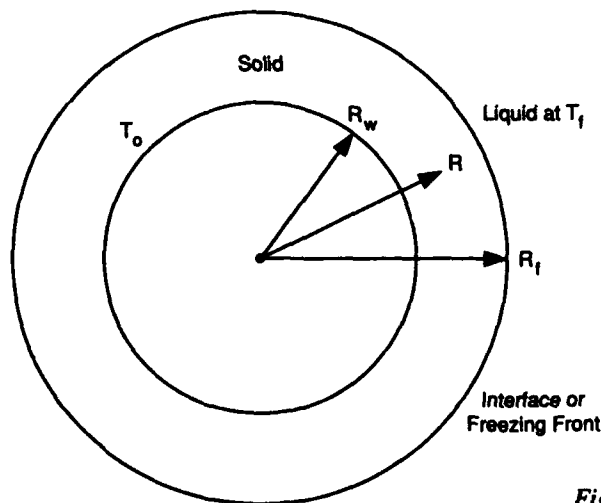$$u = (T - T_o)/(T_f - T_o), \quad r = R/R_w, \quad \tau = k (T_f - T_o) \, t/\rho L R_w^2 \tag{173}$$

into eq 171 and 172 and replacing the variable t by the variable R_f, we get

$$\frac{1}{r} \frac{\partial^2 (ur)}{\partial r^2} = \varepsilon \frac{\partial u}{\partial r_f} \cdot \left. \frac{\partial u}{\partial r} \right|_{r=r_f} \tag{174}$$

$$u(r=1, r_f) = 0, \quad u(r=r_f, r_f) = 1, \quad \frac{dr_f}{d\tau} = \left. \frac{\partial u}{\partial r} \right|_{r=r_f} \tag{175}$$

Assuming a two-term perturbation expansion of the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \tag{176}$$

and substituting into eq 174, 175, the governing equations for u_0, u_1 and u_2, can be obtained as

$$\frac{1}{r} \frac{\partial^2 (u_0 r)}{\partial r^2} = 0 \tag{177}$$

$$u_0(r=1, r_f) = 0, \quad u_0(r=r_f, r_f) = 1 \tag{178}$$

$$\frac{1}{r} \frac{\partial^2 (u_1 r)}{\partial r^2} = \frac{\partial u_0}{\partial r_f} \cdot \left. \frac{\partial u_0}{\partial r} \right|_{r=r_f} \tag{179}$$

$$u_1(r=1, r_f) = 0, \quad u_1(r=r_f, r_f) = 0 \tag{180}$$

$$\frac{1}{r} \frac{\partial^2 (u_2 r)}{\partial r^2} = \frac{\partial u_0}{\partial r_f} \cdot \left. \frac{\partial u_1}{\partial r} \right|_{r=r_f} + \frac{\partial u_1}{\partial r_f} \cdot \left. \frac{\partial u_0}{\partial r} \right|_{r=r_f} \tag{181}$$

$$u_2(r=1, r_f) = 0, \quad u_2(r=r_f, r_f) = 0 \tag{182}$$

Solving eq 177–182, the solution for u is obtained as

$$\frac{u}{u_0} = 1 + \frac{1}{6} \frac{\varepsilon}{r_f^2} \left[1 - \left(\frac{r}{r_f} \right)^2 u_0^2 \right] - \frac{\varepsilon^2}{r_f^4} \left\{ \frac{1}{36} \left[1 - \left(\frac{r}{r_f} \right)^2 u_0^2 \right] + \frac{4r_f - 1}{120} \left[1 - \left(\frac{r}{r_f} \right)^4 u_0^4 \right] \right\} \tag{183}$$

where $u_0 = \left(1 - \frac{1}{r} \right) / \left(1 - \frac{1}{r_f} \right)$.

Using eq 183 to evaluate and then integrating eq 175, the solution for τ can be obtained as

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$$\tau = \frac{2r_f^3 - 3r_f^2 + 1}{6} + \frac{1}{6} \varepsilon (r_f - 1)^2 - \frac{1}{45} \varepsilon^2 \frac{(r_f - 1)^2}{r_f} \tag{184}$$

where the initial condition $r_f = 1$, $\tau = 0$ has been utilized.

The solutions represented by eq 183 and 184 were first reported by Pedroso and Domoto (1973c). They are uniformly valid for outward spherical solidification. However, for inward spherical solidification, the second term in eq 183 becomes singular as the freezing front approaches the center, that is, as $r_f \rightarrow 0$. The divergence of the solution as $r_f \rightarrow 0$ is shown in Figure 10. This singularity grows further in the third term because of the presence of the term r_f^4 in the denominator. Examination of eq 184, however, shows that the freezing time solution is uniformly valid up to $O(\varepsilon)$ and the singularity first appears in the third term. Both eq 183 and 184 must be rendered uniformly valid if they are to be used for complete inward freezing. This will be achieved later using the method of strained coordinates.

Figure 10. Freezing radius vs time for inward spherical freezing.

5.2. Solidification of a saturated liquid in cylindrical domain

The problem of section 5.1 is now considered for cylindrical geometry. The temperature distribution in the solid phase is governed by

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T}{\partial R} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{185}$$

with boundary conditions dictated by eq 172. Using the dimensionless quantities eq 173, the problem is reduced to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \varepsilon \frac{\partial u}{\partial r_f} \cdot \left. \frac{\partial u}{\partial r} \right|_{r=r_f} \tag{186}$$

with boundary conditions established by eq 175. Repeating the procedure of the previous section, the two-term perturbation solution for u is obtained as

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$$u = \frac{\ln r}{\ln r_f} + \varepsilon \left\{ \left[\frac{1 + r_f^2 (\ln r_f - 1)}{4 r_f^2 (\ln r_f)^4} \right] \ln r - \frac{1 + r^2 (\ln r - 1)}{4 r_f^2 (\ln r_f)^3} \right\} \tag{187}$$

If we use eq 187 to calculate $\left. \dfrac{\partial u}{\partial r} \right|_{r = r_f}$ and then integrate eq 175, the solution for τ subject to the initial condition, $r_f = 1$, $\tau = 0$, is obtained as

$$\tau = \frac{1}{2} r_f^2 \ln r_f + \frac{1}{4} \left(1 - r_f^2 \right) + \frac{1}{4} \varepsilon \left(1 - r_f^2 \right) \left(1 + \frac{1}{\ln r_f} \right) \tag{188}$$

Equations 187 and 188 are uniformly valid for outward cylindrical solidification. These same equations are valid for the inward cylindrical solidification but they break down near the end of the solidification process because if $r_f \rightarrow 0$, the second term in eq 187 becomes singular. If one calculates the third term of the expansion, singular behavior as $r_f \rightarrow 0$ worsens because of the appearance of the higher powers of r_f in the denominator. The situation parallels the one discussed in the previous section for the inward spherical solidification. The two-term solution for τ given by eq 188, however, retains its validity for complete inward cylindrical solidification, although if the third term is added to eq 188, it exhibits singular behavior near the end of the solidification process. We will return to the present problem in the section on the method of strained coordinates where uniformly valid solutions are developed for the inward case.

6. MISCELLANEOUS EXAMPLES OF PERTURBATION EXPANSIONS

Besides the problems discussed so far, perturbation methods have also been used for other phase change problems. This section is a review of such studies with details in many cases left to the appropriate references. However, some of the problems will be examined in detail in the sections that follow.

For one-dimensional freezing in planar geometry, Huang and Shih (1975) developed a three-term perturbation solution for the freezing of a warm fluid over a flat plate cooled from below, and found that the growth of the solid phase predicted by the perturbation analysis agrees with the experimental results of Siegel and Savino (1966) for time $t > 100$ seconds, but for $t < 100$ seconds the perturbation solution overestimates the growth. A variation of the problem discussed in section 3.5 is the subject of a paper by Gutman (1986). He considered the freezing of a finite slab cooled on both sides by a coolant whose temperature changes sinusoidally with time. Because he assumed that the freezing is independent on each side of the slab, he was able to demonstrate that his solution for the case when the far end of the slab is insulated matches with the two-term expansion of the exact Stefan solution (section 3.1). Gutman also considered the case of a finite slab with constant heat flux at one face while the other face is insulated. The perturbation approach used in section 3.3 can also be used for other surface temperature-time variations. For example, Lock (1969) gives a three-term solution for the power law variation, $T(0,t) = Ct^n$, where C and n are constants. Indeed, he showed in another paper that his method holds for arbitrary variations of the surface temperature (Lock 1971).

The analysis for spherical and cylindrical solidification with constant surface temperature, presented in sections 5.1 and 5.2, has been extended to the case of convective cooling of the wall by Huang and Shih (1975). They found that for outward growth of the solid phase, the solutions are uniformly valid but the same solutions when applied to the inward freezing situation begin to diverge towards the end of the freezing process. The case of a more general boundary condition involving simultaneous convective and radiative cooling (or heating) has been considered by Seeniraj and Bose

(1982). However, the solutions for both cylindrical and spherical geometries are restricted to only two terms.

The difficulty associated with the singular behavior of the regular perturbation solutions for inward phase change problems has been addressed by several authors. For example, the singularity of eq 183 and 184 for inward spherical freezing has been remedied by Pedroso and Domoto (1973d) with the aid of the method of strained coordinates. The same method was used by Milanez and Boldrini (1988) to obtain a uniformly valid solution for inward freezing of a sphere with convective wall cooling. For inward cylindrical freezing with constant temperature, the singular solution given by eq 187 was rendered uniformly valid by Afsar et al. (1979) who developed an appropriate strained coordinates solution. In a more recent paper, Parang et al. (1990) have reported strained coordinate solutions for both inward cylindrical and spherical solidification when the freezing at the wall is accomplished by simultaneous convection and radiation. A unified approach for inward solidification that allows simultaneous treatment of the problem in plane, cylindrical, and spherical geometries with boundary conditions of constant temperature, constant heat flux, or pure convection was developed by Prud'homme et al. (1989). Again, the tool used to derive uniformly valid solutions was the method of strained coordinates.

Another technique that has been employed to treat singular perturbation expansions is the method of matched asymptotic expansions. Weinbaum and Jiji (1977) used this technique to study the freezing of a finite slab not initially at the freezing temperature. The boundary conditions chosen were those of a constant subfreezing temperature at one face while the other face was either insulated or kept at the initial temperature. Since even the first-order term of Weinbaum and Jiji's solution was complicated, Charach and Zoglin (1985) approached the same problem by first developing the heat balance integral formulation and then constructing a perturbation expansion. With this strategy they were able to determine higher-order terms more easily. Gutman (1987) attacked the same problem using the classical Neumann solution for the two-region problem as the inner solution, and a modified solution as the outer solution. The modification to the Stefan solution involved the replacement of the time variable t by $(t - C)$ where the constant C accounted for the additional time needed for complete freezing because of initial superheating of the liquid, that is, $T_i > T_f$. By matching the inner and the outer solutions using the overall energy balance as a criterion, Gutman was able to find the constant C. In another study, Jiji and Weinbaum (1978) also used the method of matched asymptotic expansions for freezing in an annular region with liquid not initially at the freezing temperature. A constant temperature was imposed at the outer surface while the inner surface was assumed to be either isothermal or adiabatic. However, unlike their earlier analysis for the plane geometry, the method failed to give a uniformly valid solution for the annular geometry. The solution behaved well initially but as the freezing front approached the inner surface, that is, as the last bit of liquid froze, a singularity appeared. Such behavior is to be anticipated because the liquid phase degenerates at this point.

The method of matched asymptotic expansions has also proved effective in dealing with the singular nature of the perturbation solution for inward spherical solidification. For example, Riley et al. (1974) corrected eq 183 and 184 in a multi-region structure using this method. However, Stewartson and Waechter (1976) later found that the inner expansion developed by Riley et al. also fails in a minute region just before the center freezes and must be supplemented by an additional expansion in that region. Thus a completely valid solution consists of three expansions rather than two. A similar approach, but unified for both spheres and cylinders, has also been presented by Soward (1980). Howarth (1987) adopted the Riley et al. (1974) method to study inward spherical freezing under the condition of constant heat flux and reached the conclusion that the breakdown of the outer solution for the constant heat flux condition occurs at 0 $(\varepsilon^{1/4})$ rather than 0 $(\varepsilon^{1/2})$ as is the case for the constant temperature condition.

Another class of phase change problems where the perturbation techniques have proved effective is the two-dimensional Stefan problems. The basic idea is to develop a two-dimensional solution as

a perturbation of a one-dimensional case. For example, Schulze et al. (1983) have shown how the two-region planar freezing of a liquid with slightly varying wall temperature or heat flux can be treated as a perturbation of the Neumann solution. Similarly, Howarth (1990) considered freezing on a slightly wavy wall as a linearized perturbation of the Neumann solution. The problem of two-dimensional solidification in a corner can also be analyzed as a perturbation of the Neumann solution (Howarth 1985).

When the liquid flows over the cold surface, the phase change process is controlled simultaneously by conduction and convection. Despite the complexity, Lock and Nyren (1971) found that a regular perturbation approach is still useful. In particular, they considered the fully developed flow of a freezing liquid in a long circular tube cooled by external convection. The perturbation parameter was chosen as $\varepsilon = SBi/(1+Bi)$, where S is the Stefan number and Bi is the Biot number representing the external convection. Closed form solutions are given for the zero-order and the first-order problems. The growth of solid phase due to sudden lowering of the wall temperature in fully developed laminar flow of a liquid in a circular tube has been studied by Cervantes et al. (1990) using $k_s/k_\ell \, (T_f - T_w)/(T_o - T_w)$ as a perturbation parameter, where T_o represents the initial temperature of the liquid and the other symbols have their usual meaning. They generated a two-term solution which is valid when the thickness of the frozen layer is small compared to the tube radius.

To conclude this section, we finally refer to the applications of perturbation methods to soldering and welding problems. Andrews and Atthey (1975) model the process of hole formation by a laser or an electron beam as a planar evaporating boundary heated by a constant power density source. The model was solved as a two-term perturbation expansion in which the ratio of sensible heat to raise the material to the evaporation temperature and the latent heat is taken as a perturbation parameter. The method of matched asymptotic expansion was used to obtain a uniformly valid solution for the velocity of the evaporation front. A similar technique has been used by Antaki (1990) to predict the laser-induced sublimation of a finite layer of solid material.

7. METHOD OF STRAINED COORDINATES

The basic idea underlying the technique is to expand both the independent and dependent variables in terms of ε with coefficients expressed as functions of a new independent variable. The coefficients of the independent variable series are called the *straining functions*. The expansions are next substituted into the original equations to generate the usual sequence of perturbation equations. It is at this stage that the choice of the straining functions is made such that higher approximations are no more singular than the first. This principle is often referred to as Lighthill's rule, and its application is the crucial step of the whole analysis. If successful, the result is an implicit but uniformly valid solution. Because of its first appearance in Lighthill's paper (1949), the method is also called Lighthill's technique.

The spirit of Lighthill's technique is also reflected in earlier works of Lindstedt (1882) and Poincaré (1892) where, instead of a coordinate, a parameter is strained to achieve uniform validity. When a parameter is strained, the technique may be appropriately termed the method of strained parameters. Giving credit to the contributions of Poincaré, Lighthill, and later works of Kuo (1953, 1956), the name PLK method was coined by Tsien (1956).

We illustrate the application of the technique to several problems. The first problem is discussed in full detail but, for the remaining ones, the essential steps are indicated with discussion of results.

7.1. Inward spherical solidification with
constant wall temperature

In section 5.1, a regular perturbation solution was developed for outward spherical solidification, and it was noted that the same solution applies to inward solidification, but the solution becomes

singular as the center freezes. This deficiency can be remedied by straining the coordinates r and r_f, together with the dependent variable u. Following Pedroso and Domoto (1973d), we introduce two new variables ϕ and ψ, and expand u, r, and r_f as follows:

$$u = u_0(\phi, \psi) + \varepsilon u_1(\phi, \psi) + \varepsilon^2 u_2(\phi, \psi) \tag{189}$$

$$r = \phi + \varepsilon \sigma_1(\phi, \psi) + \varepsilon^2 \sigma_2(\phi, \psi) \tag{190}$$

$$r_f = \psi + \varepsilon \sigma_1(\psi, \psi) + \varepsilon^2 \sigma_2(\psi, \psi) \tag{191}$$

where the straining functions $\sigma_1(\phi, y)$ and $\sigma_2(\phi, \psi)$ will be chosen to ensure uniform validity of the solution. Note that as $r \to r_f$, $\phi \to \psi$. Since the regular perturbation solution is satisfactory near the spherical boundary, we choose to make the straining vanish at the boundary so that

$$\lim_{\phi \to 1} \sigma_i(\phi, \psi) = 0 \qquad (r \to 1, \phi \to 1) \tag{192}$$

$$\lim_{\psi \to 1} \sigma_i(\psi, \psi) = 0 \qquad (r_f \to 1, \psi \to 1) \tag{193}$$

Also note that r is a function of both ϕ and ψ but r_f is a function of ψ alone.

To change the variables from (r, r_f) to (ϕ, ψ) in eq 174 and 175 we proceed as follows. For any function $f(r, r_f)$ where $r = f_1(\phi, \psi)$ and $r_f = f_2(\psi)$, we have

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial \phi} + \frac{\partial f}{\partial r_f} \frac{\partial r_f}{\partial \phi} \tag{194}$$

$$\frac{\partial f}{\partial \psi} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial \psi} + \frac{\partial f}{\partial r_f} \frac{\partial r_f}{\partial \psi} \tag{195}$$

Since $r_f = f_2(\psi)$, $r_f/\partial \phi = 0$; hence

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial \phi} \tag{196}$$

From eq 196 and 195 we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial \phi} \left(\frac{\partial r}{\partial \phi}\right)^{-1} \tag{197}$$

$$\frac{\partial f}{\partial r_f} = \left(\frac{\partial f}{\partial \psi} - \frac{\partial f}{\partial r} \frac{\partial r}{\partial \psi}\right) \left(\frac{\partial r_f}{\partial \psi}\right)^{-1}$$

or

$$\frac{\partial f}{\partial r_f} = \left[\frac{\partial f}{\partial \psi} - \frac{\partial f}{\partial \phi} \left(\frac{\partial r}{\partial \phi}\right)^{-1} \frac{\partial r}{\partial \psi}\right] \left(\frac{\partial r_f}{\partial \psi}\right)^{-1} \tag{198}$$

Since we need the second derivative with respect to r, we differentiate eq 197 to get

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$$\frac{\partial^2 f}{\partial r^2} = \left[\frac{\partial^2 f}{\partial \phi^2} - \frac{\partial f}{\partial \phi} \frac{\partial^2 r}{\partial \phi^2} \left(\frac{\partial r}{\partial \phi} \right)^{-1} \right] \left(\frac{\partial r}{\partial \phi} \right)^{-2} \tag{199}$$

For eq 174 the quantities required are $\partial u/\partial r$, $\partial u/\partial r_f$, and $\partial (ur)/\partial r^2$. Based on eq 197–199, we can write

$$\frac{\partial u}{\partial r} = \left(\frac{\partial u}{\partial \phi} \right) \left(\frac{\partial r}{\partial \phi} \right)^{-1} \tag{200}$$

$$\frac{\partial u}{\partial r_f} = \left[\frac{\partial u}{\partial \psi} - \frac{\partial u}{\partial \phi} \left(\frac{\partial r}{\partial \phi} \right)^{-1} \frac{\partial r}{\partial \psi} \right] \left(\frac{\partial r_f}{\partial \psi} \right)^{-1} \tag{201}$$

$$\frac{\partial^2 (ru)}{\partial r^2} = \left[\frac{\partial^2 (ru)}{\partial \phi^2} - \frac{\partial (ru)}{\partial \phi} \frac{\partial^2 r}{\partial \phi^2} \left(\frac{\partial r}{\partial \psi} \right)^{-1} \right] \left(\frac{\partial r}{\partial \phi} \right)^{-2} \tag{202}$$

Using eq 189–191, we derive the following derivatives

$$\frac{\partial u}{\partial \phi} = u_{0\phi} + \varepsilon u_{1\phi} + \varepsilon^2 u_{2\phi} \tag{203}$$

$$\frac{\partial u}{\partial \psi} = u_{0\psi} + \varepsilon u_{1\psi} + \varepsilon^2 u_{2\psi} \tag{204}$$

$$\frac{\partial r}{\partial \phi} = 1 + \varepsilon \sigma_{1\phi} + \varepsilon^2 \sigma_{2\phi} \tag{205}$$

$$\frac{\partial^2 r}{\partial \phi^2} = \varepsilon \sigma_{1\phi\phi} + \varepsilon^2 \sigma_{2\phi\phi} \tag{206}$$

$$\frac{\partial r}{\partial \psi} = \varepsilon \sigma_{1\psi} + \varepsilon^2 \sigma_{2\psi} \tag{207}$$

$$\frac{\partial r_f}{\partial \psi} = 1 + \varepsilon \widehat{\sigma}_{1\psi} + \varepsilon^2 \widehat{\sigma}_{2\psi} \tag{208}$$

where the subscripts ϕ and ψ are used from now on to indicate partial derivatives, and

$$\sigma_1 = \sigma_1(\phi, \psi)$$

$$\sigma_2 = \sigma_2(\phi, \psi)$$

$$\widehat{\sigma}_1 = \sigma_1(\psi, \psi) \tag{209}$$

$$\widehat{\sigma}_2 = \sigma_2(\psi, \psi)$$

Additionally, we need the derivatives $\partial (ru)/\partial \phi$ and $\partial^2 (ru)/\partial \phi^2$. From eq 189 and 190, we have

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$$ru = \left(\phi + \varepsilon\sigma_1 + \varepsilon^2\sigma_2\right)\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2\right)$$

or

$$ru = \phi u_0 + \varepsilon\left(u_0\sigma_1 + \phi u_1\right) + \varepsilon^2\left(u_0\sigma_2 + u_1\sigma_1 + \phi u_2\right)$$

Thus

$$\frac{\partial(ru)}{\partial\phi} = \left(\phi u_0\right)_\phi + \varepsilon\left(u_0\sigma_1 + \phi u_1\right)_\phi + \varepsilon^2\left(u_0\sigma_2 + u_1\sigma_1 + \phi u_2\right)_\phi \tag{210}$$

and

$$\frac{\partial^2(ru)}{\partial\phi^2} = \left(\phi u_0\right)_{\phi\phi} + \varepsilon\left(u_0\sigma_1 + \phi u_1\right)_{\phi\phi} + \varepsilon^2\left(u_0\sigma_2 + u_1\sigma_1 + \phi u_2\right)_{\phi\phi} \tag{211}$$

Returning to eq 200, we have

$$\frac{\partial u}{\partial r} = \left(u_{0\phi} + \varepsilon u_{1\phi} + \varepsilon^2 u_{2\phi}\right)\left(1 + \varepsilon u_{1\phi} + \varepsilon^2\sigma_{2\phi}\right)^{-1}$$

or

$$\frac{\partial u}{\partial r} = u_{0\phi} + \varepsilon\left(u_{1\phi} - u_{0\phi}\sigma_{1\phi}\right) \tag{212}$$

where we have retained only terms up to $O(\varepsilon)$. Since ε appears on the right-hand side of eq 174, we need to retain only two terms in eq 212 to obtain the solution up to $O(\varepsilon^2)$.

We now work out the expansion for $\partial u/\partial r_f$. Using eq 201, and retaining terms only up to $O(\varepsilon)$, we get

$$\frac{\partial u}{\partial r_f} = \left[\left(u_{0\psi} + \varepsilon u_{1\psi}\right) - \left(u_{0\phi} + \varepsilon u_{1\phi}\right)\left(1 - \varepsilon\sigma_{1\phi}\right)\varepsilon\sigma_{1\psi}\right]\left(1 - \varepsilon\widehat{\sigma}_{1\psi}\right)$$

or

$$\frac{\partial u}{\partial r_f} = u_{0\psi} + \varepsilon\left(u_{1\psi} - u_{0\phi}\sigma_{1\psi} - u_{0\psi}\widehat{\sigma}_{1\psi}\right) \tag{213}$$

Finally, coming to eq 202, we have

$$\frac{\partial^2(ru)}{\partial r^2} = \left\{\left(\phi u_0\right)_{\phi\phi} + \varepsilon\left(u_0\sigma_1 + \phi u_1\right)_{\phi\phi} + \varepsilon^2\left(u_0\sigma_2 + u_1\sigma_1 + \phi u_2\right)_{\phi\phi}\right.$$

$$-\left[\left(\phi u_0\right)_\phi + \varepsilon\left(u_0\sigma_1 + \phi u_1\right)_\phi + \varepsilon^2\left(u_0\sigma_2 + u_1\sigma_1 + \phi u_2\right)_\phi\right]$$

$$\left.\times\left(\varepsilon\sigma_{1\phi\phi} + \varepsilon^2\sigma_{2\phi\phi}\right)\left[1 - \varepsilon\sigma_{1\phi} + \varepsilon^2\left(\sigma_{1\phi}^2 - \sigma_{2\phi}\right)\right]\right\}$$

$$\times\left[1 - \varepsilon 2\sigma_{1\phi} + \varepsilon^2\left(3\sigma_{1\phi}^2 - 2\sigma_{2\phi}\right)\right]$$

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or

$$\frac{\partial^2 (ru)}{\partial r^2} = (\phi u_0)_{\phi\phi} + \varepsilon\left[(\phi u_1 + u_0\sigma_1)_{\phi\phi} - \sigma_{1\phi\phi}(\phi u_0)_\phi - 2\sigma_{1\phi}(\phi u_0)_{\phi\phi}\right]$$

$$+ \varepsilon^2\left[(\phi u_2 + u_1\sigma_1 + u_0\sigma_2)_{\phi\phi} - \sigma_{1\phi\phi}(\phi u_1 + u_0\sigma_1)_\phi\right.$$

$$- 2\sigma_{1\phi}(\phi u_1 + u_0\sigma_1)_{\phi\phi} - \left(2\sigma_{2\phi} - 3\,\sigma_{1\phi}^2\right)(\phi u_0)_{\phi\phi}$$

$$\left. - \left(\sigma_{2\phi\phi} - 3\sigma_{1\phi}\,\sigma_{1\phi\phi}\right)(\phi u_0)_\phi\right] \tag{213a}$$

The right-hand side of eq 174, when multiplied by r, can now be written as

$$\left.\varepsilon r\frac{\partial u}{\partial r}\right|_{r=r_f} \frac{\partial u}{\partial r_f} = \varepsilon(\phi + \varepsilon\sigma_1)\left[u_{0\phi}|_{\phi=\psi} + \varepsilon(u_{1\phi} - u_{0\phi}\sigma_{1\phi})\,|_{\phi=\psi}\right]$$

$$\times \left[u_{0\psi} + \varepsilon(u_{1\psi} - u_{0\phi}\sigma_{1\psi} - u_{0\psi}\widehat{\sigma}_{1\psi})\right]$$

$$= \varepsilon\left\{\phi u_{0\phi}|_{\phi=\psi}u_{0\psi} + \varepsilon\left[\phi u_{0\phi}|_{\phi=\psi}\left(u_{1\psi} - u_{0\phi}\sigma_{1\psi} - u_{0\psi}\widehat{\sigma}_{1\psi}\right)\right.\right.$$

$$\left.\left. + u_{0\psi}\left(\sigma_1 u_{0\phi}|_{\phi=\psi} + (u_{1\phi} - u_{0\phi}\sigma_{1\phi})\,|_{\phi=\psi}\right)\right]\right\} \tag{214}$$

We now use eq 213a and 214 to write the sequence of perturbation equations as follows:

$$\varepsilon^0: \qquad (\phi u_0)_{\phi\phi} = 0 \tag{215}$$

$$u_0(\psi, \phi = 1) = 0 \qquad u_0(\psi, \phi = \psi) = 1 \tag{216}$$

$$\varepsilon^1: \qquad (\phi u_1 + u_0\sigma_1)_{\phi\phi} - \sigma_{1\phi\phi}(\phi u_0)_\phi = \phi u_{0\phi}|_{\phi=\psi}u_{0\psi} \tag{217}$$

$$u_1(\psi, \phi = 1) = 0 \qquad u_1(\psi, \phi = \psi) = 0 \tag{218}$$

$$\varepsilon^2: \qquad (\phi u_2 + u_1\sigma_1 + u_0\sigma_2)_{\phi\phi} - \sigma_{1\phi\phi}(\phi u_1 + u_0\sigma_1)_\phi - 2\sigma_{1\phi}(\phi u_1 + u_0\sigma_1)_{\phi\phi}$$

$$- \left(\sigma_{2\phi\phi} - 3\sigma_{1\phi}\sigma_{1\phi\phi}\right)(\phi u_0)_\phi = \phi u_{0\phi}|_{\phi=\psi}\left(u_{1\psi} - u_{0\phi}\sigma_{1\psi}\right.$$

$$\left. - u_{0\psi}\widehat{\sigma}_{1\psi}\right) + u_{0\psi}\left[\sigma_1 u_{0\phi}|_{\phi=\psi} + \phi(u_{1\phi} - u_{0\phi}\sigma_{1\phi})_{\phi=\psi}\right] \tag{219}$$

$$u_2(\psi, \phi = 1) = 0 \qquad u_2(\psi, \phi = \psi) = 0 \tag{220}$$

Zero-order solution. Since eq 215 and 216 have the same form as eq 177 and 178, the solution is

$$u_0 = \frac{1 - 1/\phi}{1 - 1/\psi} \tag{221}$$

First-order solution. From eq 221 we have

$$u_{0\phi} = \frac{\psi}{\phi^2(\psi-1)}, \qquad u_{0\phi}|_{\phi=\psi} = -\frac{1}{\psi(1-\psi)}$$

$$\phi u_0 = \frac{\psi(\phi-1)}{\psi-1}, \qquad (\phi u_0)_\phi = -\frac{\psi}{1-\psi}, \qquad u_{0\psi} = \frac{1-\phi}{\phi(1-\psi)^2}. \tag{222}$$

Using eq 221 and 222, eq 217 can be simplified to

$$\left(\phi u_1 + \frac{\psi}{1-\psi}\frac{\sigma_1}{\phi}\right)_{\phi\phi} = -\frac{1-\phi}{\psi(1-\psi)^3} \tag{223}$$

Integrating eq 223, we have

$$\left(\phi u_1 + \frac{\psi}{1-\psi}\frac{\sigma_1}{\phi}\right)_{\phi\phi} = -\frac{1-\phi}{\psi(1-\psi)^3}$$

Let u_1 be *identically zero*. Then, imposing the condition $\lim_{\phi\to 1}(\sigma_1/\phi)_\phi = 0$, gives

$$f_1(\psi) = \frac{1}{2\psi(1-\psi)^3}$$

Integrating once again, we have

$$\frac{\psi}{1-\psi}\frac{\sigma_1}{\phi} = -\frac{\phi^2(3-\phi)}{6\psi(1-\psi)^3} + \frac{\phi}{2\psi(1-\psi)^3} + f_2(\psi)$$

Using the condition in eq 192 gives

$$f_2(\psi) = -\frac{1}{6\psi(1-\psi)^3}$$

Hence the solution for σ_1 becomes

$$\sigma_1 = -\frac{\phi(1-\phi)^3}{6\psi^2(1-\psi)^2} \tag{224}$$

Second-order solution. By making u_2 identically zero, eq 219 reduces to

$$(u_0\sigma_2)_{\phi\phi} - \sigma_{1\phi\phi}(u_0\sigma_1)_\phi - 2\sigma_{1\phi}(u_0\sigma_1)_{\phi\phi} - (\sigma_{2\phi\phi} - 3\sigma_{1\phi}\sigma_{1\phi\phi})(\phi u_0)_\phi$$

$$= -\phi u_{0\phi}|_{\phi=\psi}\left(u_{0\phi}\sigma_{1\psi} + u_{0\psi}\widehat{\sigma}_{1\psi}\right) + u_{0\psi}\left[\sigma_1 u_{0\phi}|_{\phi=\psi}\right.$$

$$\left. - \phi u_{0\phi}|_{\phi=\psi}\ \sigma_{1\phi}|_{\phi=\psi}\right] \tag{225}$$

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We now evaluate the various quantities appearing in eq 225. From eq 224, we have

$$\sigma_{1\phi} = \frac{(1-\phi)^2 (4\phi - 1)}{6\psi^2 (1-\psi)^2}; \quad \sigma_{1\phi\phi} = \frac{(1-\phi)(1-2\phi)}{\psi^2 (1-\psi)^2}.$$

From eq 221 and 224 it follows that

$$u_0 \sigma_1 = -\frac{(1-\phi)^4}{6\psi (1-\psi)^3}$$

and

$$(u_0 \sigma_1)_\phi = \frac{2(1-\phi)^3}{3\psi (1-\psi)^3}; \quad (u_0 \sigma_1)_{\phi\phi} = \frac{-6(1-\phi)^2}{\psi (1-\psi)^3}$$

Also, from eq 224 we obtain

$$\hat{\sigma}_1 = \sigma_1(\psi, \psi) = -\frac{1-\psi}{6\psi}$$

$$\sigma_{1\psi} = \frac{\phi(1-\phi)^3 (1-2\psi)}{3\psi^3 (1-\psi)^3}$$

and $\quad \hat{\sigma}_{1\psi} = \frac{1}{6\psi^2}$

Using the foregoing information we can evaluate the various terms appearing in eq 225 as follows

$$\phi_{1\phi\phi}(u_0 \sigma_1)_\phi = \frac{2(1-2\phi)(1-\phi)^4}{3\psi^3 (1-\psi)^5}$$

$$2\sigma_{1\phi}(u_0 \sigma_1)_{\phi\phi} = -\frac{2(4\phi - 1)(1-\phi)^4}{3\psi^3 (1-\psi)^5}$$

$$3\sigma_{1\phi} \sigma_{1\phi\phi} (\phi u_0)_\phi = \frac{(8\phi^2 - 6\phi + 1)(1-\phi)^3}{2\psi^3 (1-\psi)^5}$$

$$\phi u_0 \big|_{\phi=\psi} \left(u_{0\phi} \sigma_{1\psi} + u_{0\phi} \hat{\sigma}_{1\psi} \right) = \frac{(2-4\psi)(1-\phi)^3}{6\psi^3 (1-\psi)^5} - \frac{1-\phi}{6\psi^3 (1-\psi)^3}$$

$$u_{0\psi} \left(\sigma_1 u_{0\phi} \big|_{\phi=\psi} - \phi u_{0\phi} \big|_{\phi=\psi} \sigma_{1\phi} \big|_{\phi=\psi} \right) = \frac{(1-\phi)^4}{6\psi^3 (1-\psi)^5} + \frac{(1-\phi)(4\psi - 1)}{6\psi^3 (1-\psi)^3}.$$

Making use of the foregoing information and simplifying, eq 225 now appears as

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$$\frac{\psi}{1-\psi}\left(\frac{\sigma_2}{\phi}\right)_{\phi\phi} = \frac{(4+4\psi-15\phi)(1-\phi)^3}{6\psi^3(1-\psi)^5} + \frac{2(1-\phi)}{3\psi^2(1-\psi)^3}. \tag{226}$$

Integrating eq 226, we get

$$\frac{\psi}{1-\psi}\left(\frac{\sigma_2}{\phi}\right)_{\phi} = \frac{(1+\phi)^4(12\phi-4\psi-1)}{24\psi^3(1-\psi)^5} + \frac{\phi(2-\phi)}{3\psi^2(1-\psi)^3} + f_3(\psi).$$

Imposing the condition, $\lim_{\phi\to 1}(\sigma_2/\phi)_\phi = 0$, gives

$$f_3(\psi) = -\frac{1}{3\psi^2(2-\psi)^3}.$$

Integrating once again, we get

$$\frac{\psi}{1-\psi}\frac{\sigma_2}{\phi} = -\frac{(1-\phi)^5(1-4\psi+10\phi)}{120\psi^3(1-\psi)^5} + \frac{\phi^2(3-\phi)}{9\psi^2(1-\psi)^3} - \frac{\phi}{3\psi^2(1-\psi)^3} + f_4(\psi).$$

Using the condition eq 192 gives

$$f_4(\psi) = \frac{1}{9\psi^2(1-\psi)^3}.$$

Thus the solution for σ_2 becomes

$$\sigma_2 = \frac{\phi(1-\phi)^3}{\psi^3(1-\psi)^2}\left[\frac{1}{9} - \frac{(1-\phi)^2(1-4\psi+10\phi)}{120\psi(1-\psi)^2}\right]. \tag{227}$$

From eq 227 it follows immediately that

$$\widehat{\sigma}_2 = \sigma_2(\psi,\psi) = \frac{(22\psi-3)(1-\psi)}{360\psi^3}.$$

Also, for subsequent use, we deduce

$$\sigma_{2\phi}|_{\phi=\psi} = \frac{-22\psi^2+10\psi-3}{360\psi^4}; \qquad \widehat{\sigma}_{2\psi} = \frac{22\psi^2-50\psi+9}{360\psi^4}.$$

Having determined the straining functions, we can now write the final, formal solution from eq 189–191 as

$$u = \frac{1-1/\phi}{1-1/\psi} + 0\left(\varepsilon^3\right) \tag{228}$$

$$r = \phi - \varepsilon\frac{\phi(1-\phi)^3}{6\psi^2(1-\psi)^2} + \varepsilon^2\frac{\phi(1-\phi)^3}{\psi^3(1-\psi)^2}\left[\frac{1}{9} - \frac{(1-\phi)^2(1+4\psi+10\phi)}{120\psi(1-\psi)^2}\right] \tag{229}$$

$$r_f = \psi - \varepsilon \frac{1-\psi}{6\psi} + \varepsilon^2 \frac{(22\psi - 3)(1-\psi)}{360\psi^3}. \qquad (230)$$

Freezing time. To obtain the freezing time τ as a function of ψ, we consider the third condition in eq 175. Since

$$\frac{dr_f}{d\tau} = \frac{dr_f}{d\psi}\left(\frac{d\tau}{d\psi}\right)^{-1}$$

we have

$$\frac{d\tau}{d\psi} = \left(\frac{\partial u}{\partial r}_{r=r_f}\right)^{-1} \frac{dr_f}{d\psi} \qquad (231)$$

To obtain the expansion for $\left(\partial u/\partial r\big|_{r=r_f}\right)$ up to $O(\varepsilon^2)$, we return to eq 200. Noting that u_1 and u_2 are identically zero, we have

$$\frac{\partial u}{\partial r} = u_{0\phi}\left(1 + \varepsilon\sigma_{1\phi} + \varepsilon^2\sigma_{2\phi}\right)^{-1}$$

$$= u_{0\phi} - \varepsilon\sigma_{1\phi}u_{0\phi} - \varepsilon^2\left[(\sigma_{2\phi} - \sigma_{1\phi}^2)u_{0\phi}\right] + 0\left(\varepsilon^3\right) \qquad (232)$$

Hence

$$\left.\frac{\partial u}{\partial r}\right|_{r=r_f} = u_{0\phi}|_{\phi=\psi} - \varepsilon(\sigma_{1\phi}u_{0\phi})_{\phi=\psi} - \varepsilon^2\left[(\sigma_{2\phi} - \sigma_{1\phi}^2)u_{0\phi}\right]_{\phi=\psi} + 0\left(\varepsilon^3\right) \qquad (233)$$

Substituting eq 233 into eq 231 we have

$$\frac{d\tau}{d\psi} = \left\{u_{0\phi}|_{\phi=\psi} - \varepsilon(\sigma_{1\phi}u_{0\phi})_{\phi=\psi} - \varepsilon^2\left[(\sigma_{2\phi} - \sigma_{1\phi}^2)u_{0\phi}\right]_{\phi=\psi}\right\}^{-1}\left(1 + \varepsilon\widehat{\sigma}_{1\psi} + \varepsilon^2\widehat{\sigma}_{2\psi}\right)$$

which can be expanded and simplified to give

$$\frac{d\tau}{d\psi} = \left(u_{0\phi}|_{\phi=\psi}\right)^{-1}\left[1 + \varepsilon\left(\sigma_{1\phi}|_{\phi=\psi} + \widehat{\sigma}_{1\psi}\right)\right.$$

$$\left. + \varepsilon^2\left(\widehat{\sigma}_{2\phi} + \sigma_{1\phi}|_{\phi=\psi}\widehat{\sigma}_{1\psi} + \sigma_{2\phi}|_{\phi=\psi}\right)\right] \qquad (234)$$

Utilizing the appropriate expressions for quantities appearing in eq 234, it is found that

$$\frac{d\tau}{d\psi} = -\psi(1-\psi) - \varepsilon\frac{2}{3}(1-\psi) + \varepsilon^2\frac{1}{90}\left(\frac{1}{\psi^3} - \frac{1}{\psi^2}\right) \qquad (235)$$

Integrating eq 235 and imposing the condition $y = 1$, $\tau = 0$, the final result is

$$\tau = \frac{2\psi^3 - 3\psi^2 + 1}{6} + \varepsilon\frac{1}{3}(1-\psi)^2 - \varepsilon^2\frac{(1-\psi)^2}{180\psi^2} \qquad (236)$$
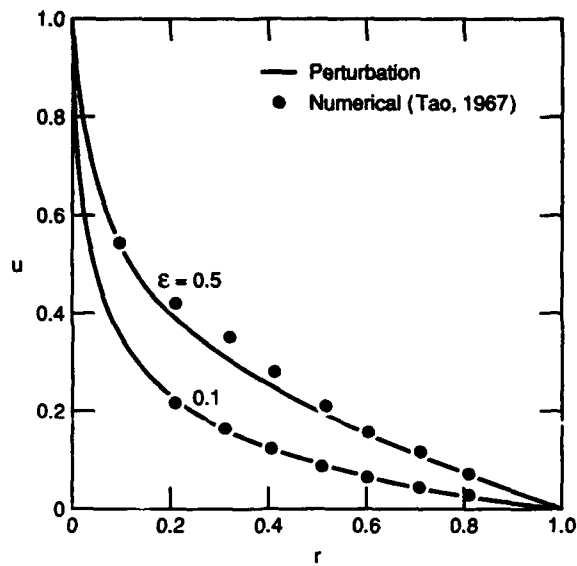
40

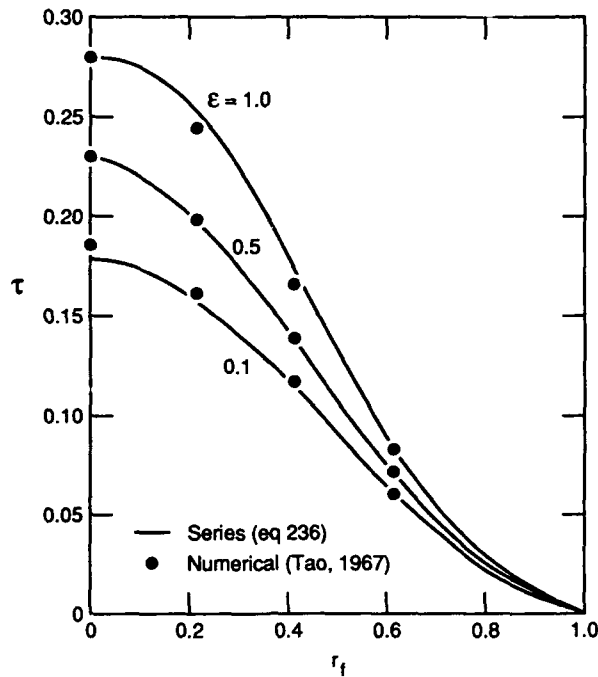Figure 11. Temperature distribution for inward spherical solidification.

Figure 12. Freezing radius vs time for inward spherical solidification.

Equations 228–230, and 236 constitute the complete uniformly valid solution. Since the solution is implicit, the computation proceeds as follows. First, values of ε and r_f are fixed. Next, eq 230 is solved by iteration to obtain the value of ψ. Choosing values of ϕ in the range ψ to 1, eq 229 and 228 are used to calculate the temperature distribution u as a function of r. The freezing time is calculated using eq 236.

Comparison with numerical solution. Sample results for the temperature at the instant of complete freezing are shown in Figure 11 for $\varepsilon = 0.1$ and 0.5. For comparison, the corresponding numerical results of Tao (1967) are indicated by crosses. Even at $\varepsilon = 0.1$, there exists some discrepancy between the perturbation and the numerical solutions. As discussed by Pedroso and Domoto (1973d), and Stephan and Holzknecht (1974), the error most probably lies in the numerical solution itself because for $\varepsilon = 0.1$, one would expect the perturbation solution to be accurate.

The results for freezing time appear in Figure 12, and compare well with the numerical predictions of Tao (1967). It must be kept in mind that Tao's freezing time results are believed to be more accurate than his temperature results.

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Pritulo's method. Since its original exposition in 1949, the spirit of Lighthill's technique has remained unchanged, but several approaches have been developed to simplify the procedure. One such simplification was introduced by Pritulo (1962). He proved that the straining expansion can be substituted directly into the ill-behaved perturbation expansion and straining functions can then be chosen in accordance with Lighthill's condition. In this way, one solves algebraic rather than differential equations to determine the straining functions. We illustrate the procedure by considering the inward spherical solidification example again.

From eq 183, the two-term perturbation solution for u is

$$u = \frac{1 - 1/r}{1 - 1/r_f} + \varepsilon \left[\frac{r_f^2 - 3r_f + 2}{6(1 - r_f)^4} \left(1 - \frac{1}{r}\right) + \frac{r^2 - 3r + 2}{6r_f(1 - r_f)^3} \right]$$

(237)

To render eq 237 uniformly valid, let us introduce the expansions, eq 190 and 191, directly into eq 237 and retain terms up to $O(\varepsilon)$. To this end, we need expansions for $1/r$ and $1/r_f$. These can be obtained using eq 190 and 191 and carrying out the binomial expansions. Thus

$$\frac{1}{r} = \frac{1}{\phi} - \varepsilon \frac{\sigma_1}{\phi^2} + O(\varepsilon^2) \; ; \quad \frac{1}{r_f} = \frac{1}{\psi} - \varepsilon \frac{\widehat{\sigma_1}}{\psi^2} + O(\varepsilon^2)$$

Consider now the zero-order term in eq 237. It can be written as

$$\left(1 - \frac{1}{r}\right)\left(1 - \frac{1}{r_f}\right)^{-1} = \left(1 - \frac{1}{\phi} + \varepsilon \frac{\sigma_1}{\phi^2}\right)\left(1 - \frac{1}{\psi} + \varepsilon \frac{\widehat{\sigma_1}}{\psi^2}\right)^{-1}$$

$$= \frac{1 - 1/\phi}{1 - 1/\psi} + \varepsilon \left[\frac{\sigma_1}{\phi^2(1 - 1/\psi)} - \frac{\widehat{\sigma_1}(1 - 1/\phi)}{\psi^2(1 - 1/\psi)^2}\right] + O(\varepsilon^2)$$

For the first-order term in eq 237, we simply need to replace r by ϕ and r_f by ψ. Thus the two-term solution for u is

$$u = \frac{1 - 1/\phi}{1 - 1/\psi} + \varepsilon \left[\frac{\sigma_1}{\phi^2(1 - 1/\psi)} - \frac{\widehat{\sigma_1}(1 - 1/\phi)}{\psi^2(1 - 1/\psi)^2} + \frac{\psi^2 - 3\psi + 2}{6(1 - \psi)^4}\left(1 - \frac{1}{\phi}\right)\right.$$

$$\left. + \frac{\phi^2 - 3\phi + 2}{6\psi(1 - \psi)^3}\right] + O(\varepsilon^2)$$

(238)

To determine σ_1 we set the term in square brackets in eq 238 to zero, giving

$$\frac{\sigma_1}{\phi^2(1 - 1/\psi)} - \frac{\widehat{\sigma_1}(1 - 1/\phi)}{\psi^2(1 - 1/\psi)^2} + \frac{\psi^2 - 3\psi + 2}{6(1 - \psi)^4}\left(1 - \frac{1}{\phi}\right) + \frac{\phi^2 - 3\phi + 2}{6\psi(1 - \psi)^3} = 0$$

(239)

which can be rearranged as

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$$\frac{\sigma_1}{\phi(\phi-1)} - \frac{\phi(\phi-2)}{6\psi^2(1-\psi)^2} \Bigg] = \frac{\psi^2-3\psi+2}{\psi(1-\psi)^3} - \widehat{\sigma}_1 \frac{1}{\psi(1-\psi)} \tag{240}$$

where it is noted that the right-hand side of eq 240 is a function of y alone.

Differentiating eq 240 with respect to ϕ, the differential equation for σ_1 is

$$\sigma_{1\phi} - \frac{2\phi-1}{\phi(\phi-1)} \sigma_1 = \frac{2\phi(\phi-1)^2}{6\psi^2(1-\psi)^2} \tag{241}$$

Integrating eq 241 we have

$$\frac{1}{\phi(\phi-1)} \sigma_1 = \frac{(\phi-1)^2}{6\psi^2(1-\psi)^2} + C$$

Imposing the condition, $\displaystyle\lim_{\phi \to 1} (\sigma_1/\phi)_\phi = 0$, makes C vanish. Thus

$$\sigma_1 = \frac{-\phi(1-\phi)^3}{6\psi^2(1-\psi)^2} \tag{242}$$

which agrees with eq 224.

In an analogous manner, it is possible to derive the solution for σ_2 but the algebra becomes lengthy. The reader who is not overwhelmed by the algebra can verify that the solution for σ_2 is in full agreement with eq 227.

7.2 Inward cylindrical solidification with constant wall temperature

As a second example, we consider the cylindrical solidification problem of section 5.2. Following the procedure of the previous section, we obtain the following results, which are taken from Asfar et al. (1979).

Zero-order

$$\frac{\partial}{\partial\phi}\left(\phi\frac{\partial u_0}{\partial\phi}\right) = 0, \quad u_0(\psi, \phi=1) = 0, \quad u_0(\psi, \phi=\psi) = 1 \tag{243}$$

with the solution

$$u_0 = \frac{\ln\phi}{\ln\psi} \tag{244}$$

First-order

Using the zero-order solution and making u_1 *identically zero*, the first-order problem reduces to

$$\frac{\partial}{\partial\phi}\left(\phi\frac{\partial}{\partial\phi}\left(\frac{\sigma_1}{\phi}\right)\right) = \frac{\phi\ln\phi}{\psi^2\ln^2\psi} \tag{245}$$

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One condition on σ_1 is provided by eq (192), that is, $\lim_{\phi \to 1} \sigma_1 = 0$. We impose the other condition by putting $\lim_{\phi \to 1} \frac{\partial}{\partial \phi} \left(\frac{\sigma_1}{\phi} \right) = 0$. The latter permits the simplest solution for σ_1. The solution is

$$\sigma_1 = \frac{\phi \left[\left(1 - \phi^2 \right) + \left(1 + \phi^2 \right) \ln \phi \right]}{4 \psi^2 \ln^2 \psi} \tag{246}$$

Second-order

Using eq 244 and 246 and making u_2 *identically zero*, the second-order problem, after considerable manipulation, finally leads to

$$\frac{\partial}{\partial \phi} \left(\phi \frac{\partial}{\partial \phi} \left(\frac{\sigma_2}{\phi} \right) \right) = \frac{1}{16 \phi \psi^4 \ln^4 \psi} \left(\left(1 - \phi^2 \right)\left(1 - 9\phi^2 \right) + 20\phi^2 \ln \phi \right.$$

$$\left. - 36\phi^4 \ln \phi + 12\phi^2 \ln^2 \phi + 40\phi^4 \ln^2 \phi \right) - \frac{1}{2\psi^4 \ln^3 \psi} \left(\phi + \phi \ln \phi \right.$$

$$\left. + \phi^3 \left(\ln \phi - 1 \right) \right) + \frac{1}{2\psi^4 \ln^5 \psi} \phi \ln \phi \left(1 - \psi^2 - 2\psi^2 \left(\ln \psi - 1 \right) \ln \psi \right). \tag{247}$$

Imposing, as for the first-order problem, the conditions $\lim_{\phi \to 1} \sigma_2 = 0$ and $\lim_{\phi \to 1} \frac{\partial}{\partial \phi} \left(\frac{\sigma_2}{\phi} \right) = 0$, the simplest solution for σ_2 is obtained as

$$\sigma_2 = \frac{1}{8 \psi^4 \ln^5 \psi} \left\{ \phi \left(1 - \phi^2 \right) + \phi \left(1 + \phi^2 \right) \ln \phi \right\} \left\{ 1 - \psi^2 + 2\psi^2 \left(1 - \ln \psi \right) \ln \psi \right\}$$

$$- \frac{1}{64 \psi^4 \ln^3 \psi} \left\{ 3\phi \left(1 - \phi^4 \right) + 2\phi \left(\phi^4 + 4\phi^2 + 1 \right) \ln \phi \right\}$$

$$+ \frac{1}{128 \psi^4 \ln^4 \psi} \left\{ \left(21\phi^5 - 24\phi^3 + 3\phi \right) \right.$$

$$\left. - \left(38\phi^4 + 8\phi^2 - 10 \right) \phi \ln \phi + \left(20\phi^4 + 24\phi^2 + 4 \right) \phi \ln^2 \phi \right\}. \tag{248}$$

The equation for the freezing time up to $O(\varepsilon^2)$ is

$$\frac{d\tau}{d\psi} = \frac{1}{\left. \frac{\partial u_0}{\partial \phi} \right|_{\phi = \psi}} \left(1 + \varepsilon \left(\frac{d\sigma_1(\psi, \psi)}{d\psi} + \left. \frac{\partial \sigma_1(\phi, \psi)}{\partial \phi} \right|_{\phi = \psi} \right) \right.$$

$$\left. + \varepsilon^2 \left(\frac{d\sigma_2(\psi, \psi)}{d\psi} + \left. \frac{\partial \sigma_2(\phi, \psi)}{\partial \phi} \right|_{\phi = \psi} + \frac{d\sigma_1(\psi, \psi)}{d\psi} \left. \frac{\partial \sigma_1(\phi, \psi)}{\partial \phi} \right|_{\phi = \psi} \right) \right). \tag{249}$$

Substituting eq 244, 246 and 248 into eq (249) and using the initial condition, $\psi = 1, \tau = 0$, the solution for τ is obtained as

$$\tau = \frac{1}{4}\left(1-\psi^2\right)+\frac{1}{2}\psi^2\ln\psi+\varepsilon\left(\frac{\left(1-\psi^2\right)+\left(1+\psi^2\right)\ln\psi}{2\ln\psi}\right)$$

$$+\frac{1}{96\psi^2\ln^4\psi}\varepsilon^2\left\{15\left(1-\psi^2\right)^2+21\left(1-\psi^4\right)\ln\psi\right.$$

$$+\left.12\left(1+\psi^4\right)\ln^2\psi+3\left(1-\psi^4\right)\ln^3\psi\right\} \tag{250}$$

Sample results for the temperature distribution at the instant of complete freezing ($r_f = 0$) are shown in Figure 13. Figure 14 shows the results for freezing time. For comparison the corresponding results of Tao (1967) are also shown. In Figure 13 even at $\varepsilon = 0.01$ there exists some discrepancy between the perturbation and numerical temperature distributions. However, the freezing time solutions are in good agreement. As discussed by Pedroso and Domoto (1973d) and Stephan and Holzknecht (1974) the error in the temperature distribution lies most probably in the numerical solution itself because for $\varepsilon = 0.01$, one would expect the perturbation solution to be accurate. Computational experiments with the perturbation solution show that a valid solution is obtained up to about $\varepsilon = 0.8$. Beyond this value, the values of ψ and ϕ obtained gave an unrealistic solution for u.
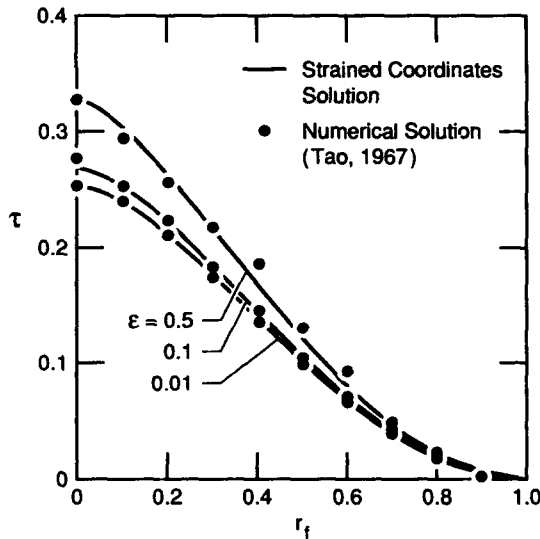

Figure 13. Temperature distribution for inward cylindrical solidification.
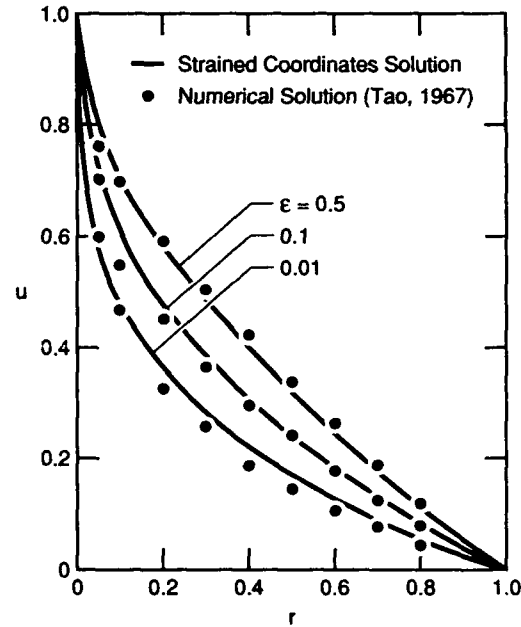
Figure 14. Inward freezing radius for cylindrical system.

7.3. Inward spherical solidification with convective cooling

The problem is described by eq 174 and 175 except that the convective boundary condition is

$$u\left(r=1,\ r_f\right)=-Bi^{-1}\left.\frac{\partial u}{\partial r}\right|_{r=1} \tag{251}$$

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where $Bi = hR_w/k$ and T_0, in the definitions of u, τ, and ε, is to be interpreted as the coolant temperature and not the wall temperature.

For the solution of this problem, Milanez and Boldrini (1988) followed the procedure detailed in section 7.1 and found the zero-order and the first-order problems and their solutions to be as follows:

Zero order

$$\frac{\partial^2(\phi u_0)}{\partial \phi^2} = 0 \tag{252}$$

$$u_0(\psi, \phi = 1) = -Bi^{-1} \frac{\partial u_0}{\partial \phi}\bigg|_{\phi = 1} ; \quad u_0(\phi = \psi, \psi) = 1 \tag{253}$$

$$u_0 = \frac{1}{Bi + (1 - Bi)\psi} \left[(1 - Bi)\psi + \frac{Bi\psi}{\phi}\right] \tag{254}$$

First order

$$\frac{1}{\phi} \frac{\partial^2(\phi u_1)}{\partial \phi^2} - \frac{2\sigma}{\phi^2} \frac{\partial u_0}{\partial \phi} - \frac{2}{\phi} \frac{\partial u_0}{\partial \phi} \frac{\partial \sigma_1}{\partial \phi} - 2 \frac{\partial \sigma_1}{\partial \phi} \frac{\partial^2 u_0}{\partial \phi^2}$$

$$- \frac{\partial u_0}{\partial \phi} \frac{\partial^2 \sigma_1}{\partial \phi^2} = \frac{\partial u_0}{\partial \psi} \frac{\partial u_0}{\partial \psi}\bigg|_{\phi = \psi} \tag{255}$$

$$u_1(\psi, \phi = 1) = -Bi^{-1}\left(\frac{\partial u_1}{\partial \phi} - \frac{\partial \sigma_1}{\partial \phi} \frac{\partial u_0}{\partial \phi}\right)_{\phi = 1} ; \quad u_1(\phi = \psi, \psi) = 0 \tag{256}$$

By choosing σ_1 such that $u_1 \equiv 0$, the equations for σ_1 and its boundary conditions are

$$-\frac{\partial u_0}{\partial \phi} \frac{\partial^2 \sigma_1}{\partial \phi^2} = \frac{\partial u_0}{\partial \psi} \frac{\partial u_0}{\partial \psi}\bigg|_{\phi = \psi} \quad \phi^2 \frac{\partial^2 \sigma_1}{\partial \phi^2} - 2\phi \frac{\partial \sigma_1}{\partial \phi} + 2\sigma_1 = -\frac{Bi\left[(1 - Bi)\phi^4 + Bi\phi^3\right]}{\psi^2\left[\psi(1 - Bi) + Bi\right]^2} \tag{257}$$

$$\sigma_1(\phi = 1, \psi) = 0 ; \quad \frac{\partial \sigma_1}{\partial \sigma}(\phi = 1, \psi) = 0 \tag{258}$$

and the solution for σ_1 is

$$\sigma_1 = \frac{Bi}{\psi^2\left[(1 - \psi)Bi + Bi\right]^2} \left[-\frac{1}{6}(2 + Bi)\phi + \frac{1}{2}(1 + Bi)\phi^2 - \frac{1}{2}Bi\phi^3 - \frac{1}{6}(1 - Bi)\phi^4\right] \tag{259}$$

Note that as $Bi \rightarrow \infty$, eq 254 and 259 reduce to 221 and 224, respectively.

The solution for the freezing time τ is given by

$$\tau = b_0(Bi, \psi) + \varepsilon b_1(Bi, \psi) + 0(\varepsilon^2) \tag{260}$$

where

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$$b_0(1, \psi) = \frac{1}{2}(1 - \psi^2); \quad b_1(1, \psi) = 1 - 2\psi + \psi^2 \tag{261}$$

and when $Bi \neq 1$,

$$b_0(Bi, \psi) = \frac{1}{2}(1 - \psi^2) + \frac{1 - Bi}{3Bi}(1 - \psi^3) \tag{262}$$

$$b_1(Bi, \psi) = \frac{Bi^2 - 3}{3(1 - Bi)^2} + \frac{2Bi}{3(1 - Bi)}\psi + \frac{2}{9(1 - Bi)^2[\psi(1 - Bi) + Bi]}\psi^2 \tag{263}$$

Now that the solutions for τ and r_f are known, the speed of the freezing front, $dr_f/d\tau = dr_f/d\psi \, (d\tau/d\psi)^{-1}$ can be evaluated.

Figure 15 shows a plot of $(1 - r_f)$ versus τ for $\varepsilon = 0.5$ and for $Bi = 1, 2, 5, \infty$. The dashed lines represent the two-term perturbation solution while the solid lines give the numerical solution of Milanez and Ismail (1984). The accuracy of the perturbation solution is excellent. Figure 16 show the results for $\varepsilon = 2$ and the comparison with the numerical solution indicates that the perturbation analysis predicts a slower growth of the solid phase. Figures 17 and 18 show the temperature profiles at various interface positions. Once again the agreement between the perturbation and numerical solutions is better at $\varepsilon = 0.5$ than at $\varepsilon = 1.0$.
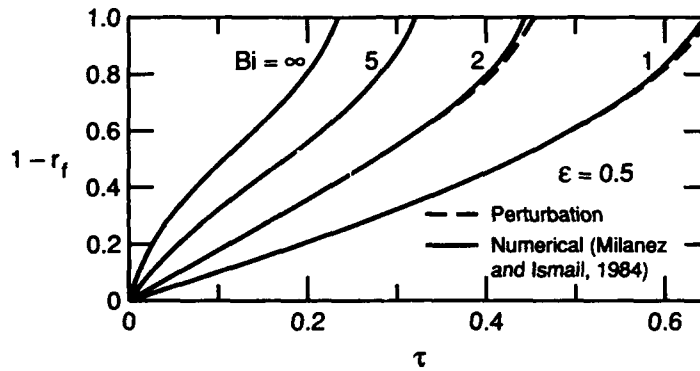

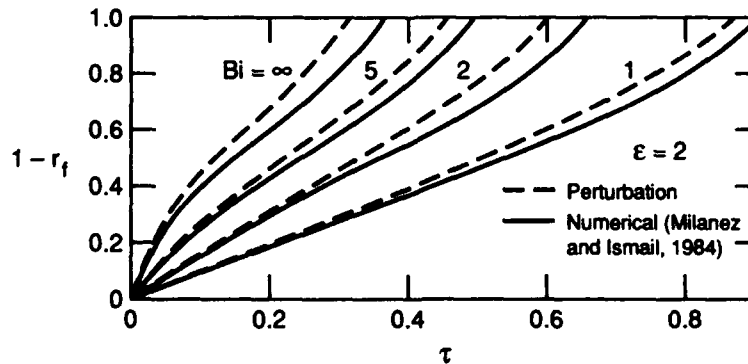Figure 15. Freezing time for inward spherical system with surface convection, $\varepsilon = 0.5$.

Figure 16. Freezing time for inward spherical system with surface convection, $\varepsilon = 2.0$.
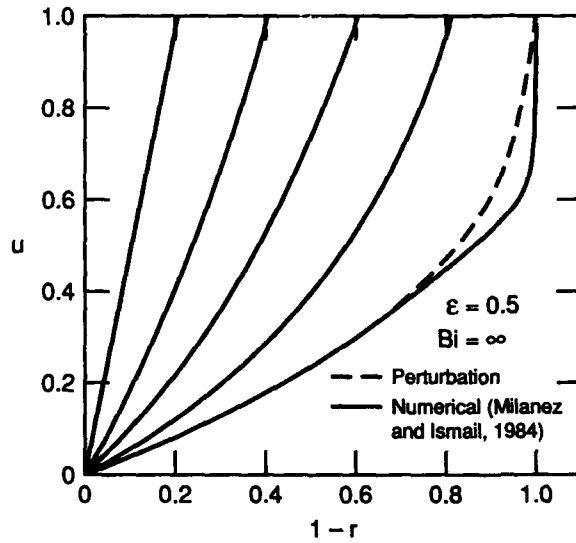
Figure 17. Temperature distribution, inward spherical solidification with convection, $\varepsilon = 0.5$, Bi $= \infty$.
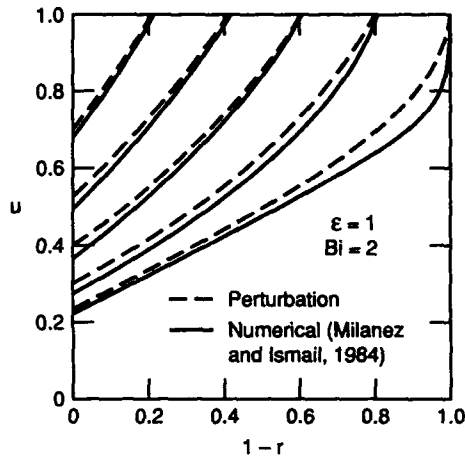

Figure 18. Temperature distribution, inward spherical solidification with convection, $\varepsilon = 1.0$, Bi $= 2.0$.

7.4. Inward spherical and cylindrical freezing
with combined convective and radiative cooling

Parang et al. (1990) have considered the general problem of inward freezing of spheres and cylinders due to combined convective and radiative cooling. They have shown that the governing equations for this problem are

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + j\frac{1}{r}\frac{\partial u}{\partial r} = \varepsilon\frac{\partial u}{\partial r_f}\cdot\frac{\partial u}{\partial r}\bigg|_{r=r_f} \ , \quad j = \begin{cases} 0 \text{ (cylinder)} \\ 1 \text{ (sphere)} \end{cases} \tag{264}$$

$$r = 1 , \ -\frac{1}{Bi_r}\frac{\partial u}{\partial r} = \beta(u - \alpha) + (u^4 - \alpha^4) ; \ u(r = r_f, r_f) = 1 \tag{265}$$

$$\frac{dr_f}{d\tau} = \frac{\partial u}{\partial r}\bigg|_{r=r_f} \tag{266}$$

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where $u = T/T_f$

$\quad\quad r = R/R_w$

$\quad\quad r_f = R_f/R_w$

$\quad\quad \alpha = T_a/T_f$

$\quad\quad Bi = hR_w/k$

$\quad\quad \varepsilon = cT_f/L$

$\quad\quad Bi_r = \sigma e T_f^3 R_w/k$

$\quad\quad \tau = kT_f t/\left(\rho L R_w^2\right)$

$\quad\quad \beta = Bi/Bi_r = h/(\sigma e T_f^3).$

Note that a common ambient at T_a has been assumed for both convection and radiation. The symbol e represents the emissivity of the surface. The perturbation parameter ε is equivalent to the Stefan number based on the absolute freezing temperature.

Following the procedure of section 7.1, the zero and the first-order problems and their solutions are obtained as

Zero-order

$$\frac{\partial}{\partial\phi}\left(\phi\,\frac{\partial u_0}{\partial\phi}\right) + j\,\frac{\partial u_0}{\partial\phi} = 0 \tag{267}$$

$$\phi = 1\,,\,\frac{-1}{Bi_r}\frac{\partial u_0}{\partial\phi} = \beta\left(u_0-\alpha\right) + \left(u_0^4-\alpha^4\right);\quad u_0\left(\phi = \psi,\,\psi\right) = 1 \tag{268}$$

For the sphere

$$u_0 = C_1 + C_2/\phi \tag{269}$$

where C_1 and C_2 are given by

$$C_1{}^4 + \frac{4\psi}{\left(1-\psi\right)}\,C_1{}^3 + \frac{6\psi^2}{\left(1-\psi\right)^2}\,C_1{}^2 + \left[\frac{4\psi^3+\beta}{\left(1-\psi\right)^3} + \frac{\psi}{Bi_r\left(1-\psi\right)^4}\right]C_1$$

$$+ \frac{\psi^4 - \alpha^4 + \beta\left(\psi-\alpha\right) - \left(\psi/Bi_r\right)}{\left(1-\psi\right)^4} = 0 \tag{270}$$

$$C_2 = \psi\,(1-C_1) \tag{271}$$

For the cylinder

$$u_0 = C_1\ln\phi + C_2 \tag{272}$$

where C_1 and C_2 are given by

$$C_1{}^4 - \frac{4C_1^3}{\ln\psi} + \frac{6C_1^2}{\left(\ln\psi\right)^2} - \frac{C_1\left(4+\beta\right)}{\left(\ln\psi\right)^3} + \frac{\left(1-\alpha^4+\beta\left(1-\alpha\right)+C_1/Bi_r\right)}{\left(\ln\psi\right)^4} = 0 \tag{273}$$

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$$C_2 = 1 - C_1 \ln \psi \qquad (274)$$

Equations 270 and 273 are polynomials for the constant C_1 and in general have to be solved numerically.

First-order

$$\phi \frac{\partial^2 u_1}{\partial \phi^2} + (1+j) \frac{\partial \sigma_1}{\partial \phi} \frac{\partial u_0}{\partial \phi} - \phi \frac{\partial^2 \sigma_1}{\partial \phi^2} \frac{\partial u_0}{\partial \phi} + \sigma_1 \frac{\partial^2 u_0}{\partial \phi^2} + (1+j) \frac{\partial u_1}{\partial \phi} = \phi \frac{\partial u_0}{\partial \psi} \frac{\partial u_0}{\partial \phi} \Big|_{\phi=\psi} \qquad (275)$$

$$\phi = 1 , \quad u_1 = - \frac{\left[\dfrac{\partial u_1}{\partial \phi} - \dfrac{\partial \sigma_1}{\partial \phi} \dfrac{\partial u_0}{\partial \phi} \right]}{Bi_r \left(4u_0^3 + \beta \right)} ; \quad u_1 \left(\phi = \psi, \psi \right) = 0 \qquad (276)$$

Choosing σ_1 such that u_1 is identically zero, the equation for σ_1 with its boundary conditions becomes

$$\phi^2 \frac{\partial^2 \sigma_1}{\partial \phi^2} - (1+j) \phi \frac{\partial \sigma_1}{\partial \phi} + (1+j) \sigma_1 = - \frac{\phi^3}{\psi^{1+j}} \left[(1-j) \ln \phi \frac{dC_1}{d\psi} + j\phi \frac{dC_1}{d\psi} + \frac{dC_2}{d\psi} \right] \qquad (277)$$

$$\sigma_1 \left(\phi = 1 , \psi \right) = 0 : \quad \frac{d\sigma_1}{d\psi} \left(\phi = 1 , \psi \right) = 0 \qquad (278)$$

The solution for σ_1 for the sphere is

$$\sigma_1 = - \frac{1}{\psi^2} \left[\frac{1}{6} \frac{dC_1}{d\psi} \phi^4 + \frac{1}{2} \frac{dC_2}{d\psi} \phi^3 - \frac{1}{2} \left(\frac{dC_1}{d\psi} + 2 \frac{dC_2}{d\psi} \right) \phi^2 + \frac{1}{6} \left(2 \frac{dC_1}{d\psi} + 3 \frac{dC_2}{d\psi} \right) \phi \right] \qquad (279)$$

and for the cylinder is

$$\sigma_1 = \frac{1}{4\psi} \left\{ \left[\frac{dC_2}{d\psi} - \frac{dC_1}{d\psi} + \left(2 \frac{dC_2}{d\psi} - \frac{dC_1}{d\psi} \right) \ln \phi \right] \phi + \left(\frac{dC_1}{d\psi} - \frac{dC_2}{d\psi} - \frac{dC_1}{d\psi} \ln \phi \right) \phi^3 \right\} \qquad (280)$$
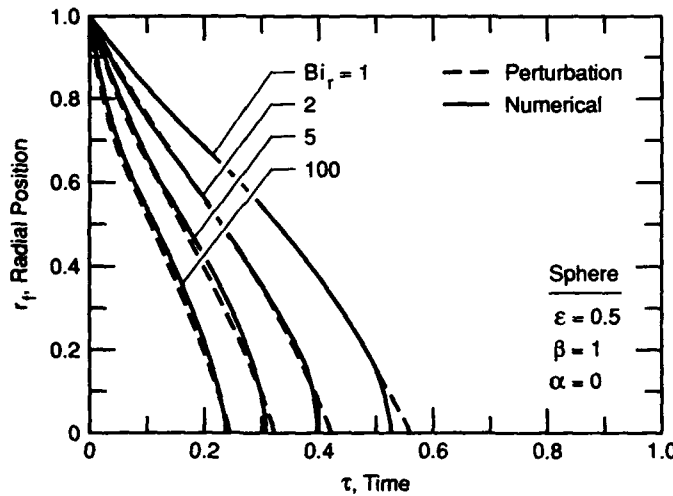

Figure 19. Freezing radius vs time for sphere with convection and radiation.

Sample results for the progress of the freezing front are shown in Figure 19 for the sphere and in Figure 20 for the cylinder. The dimensionless ambient temperature, α, is zero in both figures and ϵ is 0.5. The parameter β is taken as unity, indicating equal convective and radiative contributions to cooling. The perturbation results match closely with the numerical results derived using the enthalpy method. However, some divergence (about 7%) is noticeable near the end of the freezing process. Although not shown here, the discrepancy between the perturbation and the numerical predictions increases as ϵ increases.

Figure 21 shows how the time for complete freezing is affected by the increase in the ambient temperature. This figure is for the sphere and for $\epsilon = 0.5$ and $\beta = 1$. As expected, the total freezing time increases as α increases. Furthermore, the stronger the convective–radiative cooling, that is, the higher the value of Bi_r ($= Bi$), the lesser the freezing time. As Bi_r is decreases, the difference between the perturbation an the numerical solutions increases.

Finally, Figure 22 shows the temperature profiles at the instant of complete freeze for $\epsilon = 0.5$, 1.0, and 2.0. These curves pertain to spherical freezing and correspond to $\alpha = 0$, and $Bi = Bi_r = 1$. The maximum error between the perturbation and numerical solutions, which occurs in the vicinity of the center, is about 8%.
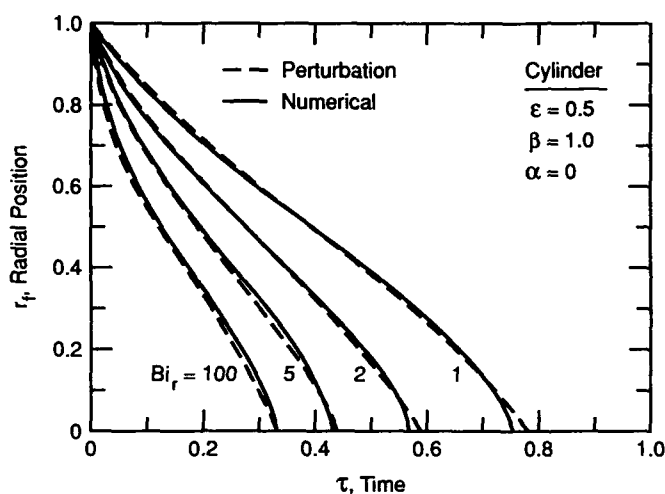

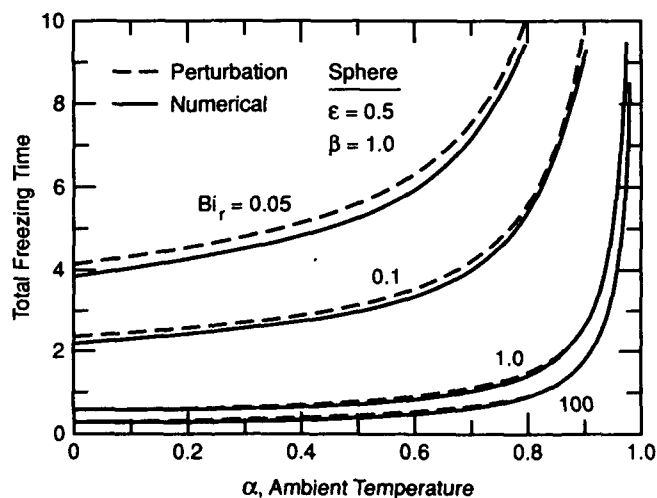*Figure 20. Freezing radius vs time for cylinder with convection and radiation.*

Figure 21. Final freezing time for sphere with convection and radiation.
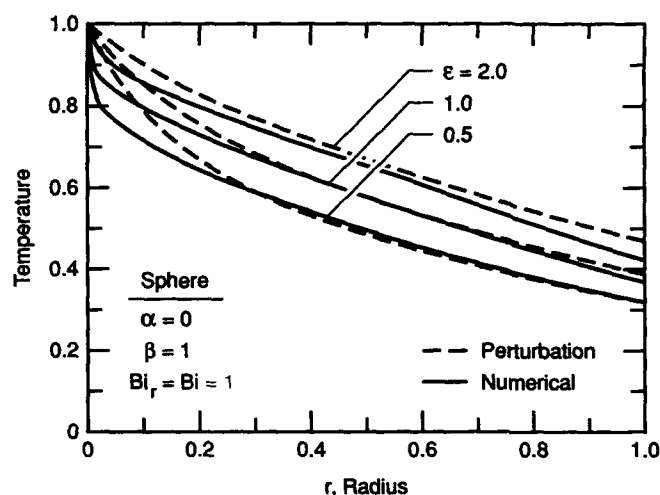
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Figure 22. Temperature profiles for sphere with convection and radiation.

8. METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

The most widely used singular perturbation technique is the method of matched asymptotic expansions. In this method the outer expansion, which is the regular perturbation expansion, is supplemented by an inner expansion in terms of a stretched variable, which then describes the behavior of the function in the region where the outer expansion breaks down. Finally the outer and the inner expansions are matched to derive a uniformly valid solution. A criterion which is commonly used is Van Dyke's matching principle. The method will be illustrated with the help of two examples.

8.1 Vaporization of a semi-infinite solid due to constant heat flux

The use of high power lasers and electron beams to induce evaporation (sublimation) of solid materials has become important in many materials processing applications. By ignoring the intermediate melting, the complexity of the problem is considerably reduced.

Consider a planar solid of semi-infinite extent heated by a constant heat flux q, applied at $x = 0$. In the simplest case one can envision that all the energy absorbed at the surface is used to vaporize the material and none is conducted into the solid. This vaporization-controlled limit can arise if the heat is applied too rapidly for conduction to occur. If the temperature distribution ahead of the vaporizing boundary approaches a steady state, the velocity will also attain a constant value. In this situation, the velocity of the vaporizing front v can be found as

$$v = \frac{q}{\rho\left[c\left(T_v - T_i\right) + L\right]} \tag{281}$$

where the extraction of the sensible heat has been accounted for.

However, a more realistic model must include the effect of transient heat conduction into the solid ahead of the vaporizing boundary. It is this model that is developed here and analyzed using the method of matched asymptotic expansions. The one-dimensional transient heat conduction equation and the associated conditions are

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{282}$$

$$T(x_v, t) = T_v; \quad k \left. \frac{\partial T}{\partial x} \right|_{x = x_v} = \rho L \frac{dx_v}{dt} - q \tag{283}$$

$$T(x, 0) = T_i; \quad T(\infty, t) = T_i$$

where x_v denotes the location of the vaporizing boundary and T_v is the vaporization temperature of the solid. It is assumed that the solid is initially at temperature T_i, and that at large distances, the temperature of the solid remains unaffected.

Introducing the following dimensionless quantities:

$$\theta = (T - T_i)/(T_v - T_i), \quad X = xv/\alpha, \quad \lambda = x_v v/\alpha \tag{284}$$

$$\tau = v^2 t/\alpha, \quad \varepsilon = c(T_v - T_i)/L$$

into eq 282 and 283 gives

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau} \tag{285}$$

$$\theta(\lambda, \tau) = 1; \quad \left(\frac{d\lambda}{d\tau} - 1 \right) - \varepsilon \left(\left. \frac{\partial \theta}{\partial X} \right|_{X = \lambda} + 1 \right) = 0 \tag{286}$$

$$\theta(X, 0) = 0; \quad \theta(\infty, \tau) = 0$$

Let us assume a regular perturbation solution for θ, λ, and $\eta = \frac{d\lambda}{dt}$ as follows:

$$\theta = \theta_0 + \varepsilon \theta_1 + 0(\varepsilon^2) \tag{287}$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + 0(\varepsilon^2) \tag{288}$$

$$\eta = \eta_0 + \varepsilon \eta_1 + 0(\varepsilon^2) \tag{289}$$

Substituting eq 287–289 into eq 285 and 286, we obtain the following equations for θ_0 and η_0:

$$\frac{\partial \theta_0}{\partial X^2} = \frac{\partial \theta_0}{\partial \tau} \tag{290}$$

$$\eta_0 = 1 \text{ or } \frac{d\lambda_0}{d\tau} = 1 \tag{291}$$

$$X = \lambda_0, \quad \theta_0 = 1. \tag{292}$$

Ignoring the preheating effects while the boundary is being raised to its evaporation temperature, since such effects are important only for $\tau = 0(\varepsilon^2)$ as seen later, we can integrate eq 291 with the initial condition $\tau = 0$, $\lambda_0 = 0$ to give

$$\lambda_0 = \tau. \tag{293}$$

To solve eq 290, we first make the transformation from the fixed coordinate system (X,τ) to the moving coordinate system (λ, τ) and rewrite eq 290 in moving coordinates (see Carslaw and Jaeger 1959, p. 13)

$$\frac{\partial^2\theta_0}{\partial\lambda^2} + \frac{\partial\theta_0}{\partial\lambda} = \frac{\partial\theta_0}{\partial\tau} \tag{294}$$

Taking the Laplace transform of eq 294 and noting that $\theta(\lambda,0) = 0$, we get

$$\frac{d^2\theta_0(s)}{d\lambda^2} + \frac{d\theta_0(s)}{d\lambda} - \cdots \theta_0(s) = 0 \tag{295}$$

where $\theta_0(s)$ is the Laplace transform of θ and s is the Laplace variable. The boundary conditions for eq 295 are

$$\lambda = 0, \ \theta_0 = 1 \ \text{ or } \ \theta_0(s) = \frac{1}{s}; \ \ \lambda = \infty, \ \theta_0 = 0 \text{ or } \theta_0(s) = 0 \ . \tag{296}$$

The solution of eq 295 subject to eq 296 is

$$\theta_0(s) = \frac{e^{-\lambda/2} \, e^{-\lambda\sqrt{s+1/4}}}{s} \tag{297}$$

From Carslaw and Jaeger (1959), p. 495, the inverse of eq 297 is obtained as

$$\theta_0(\lambda,\tau) = \frac{1}{2}\,\text{erfc}\left(\frac{\lambda+\tau}{2\sqrt{\tau}}\right) + \frac{1}{2}\,e^{-\lambda}\,\text{erfc}\left(\frac{\lambda+\tau}{2\sqrt{\tau}}\right) \tag{298}$$

Since the solution for λ to $O(\varepsilon)$ is given by eq 293, we can write $\lambda = X - \tau$, giving

$$\theta_0(X,\tau) = \frac{1}{2}\,\text{erfc}\left(\frac{X}{2\sqrt{\tau}}\right) + \frac{1}{2}\,e^{-(X-\tau)}\,\text{erfc}\left(\frac{\frac{1}{2}X-\tau}{2\sqrt{\tau}}\right). \tag{299}$$

We now consider the first-order problem for λ, which follows as

$$\eta_1 = \frac{d\lambda_1}{\partial\tau} = \frac{\partial\theta_0}{\partial X}\bigg|_{X=\lambda_0} + 1 \tag{300}$$

Differentiating eq 299, we have

$$\frac{\partial\theta_0}{\partial X} = \frac{1}{2}\left(-\frac{2}{\sqrt{\pi}}\,e^{-X^2/4\tau}\right)\frac{1}{2\sqrt{\tau}} + \frac{1}{2}\left[e^{-(X-\tau)}\left(-\frac{2}{\sqrt{\pi}}\,e^{-\left(\frac{X}{2}-\tau\right)^2/\tau}\,\frac{1}{2\sqrt{\tau}}\right)\right]$$

$$-\text{erfc}\,\frac{\frac{1}{2}X-\tau}{\sqrt{\tau}} \cdot \frac{1}{2}\,e^{-(X-\tau)} \tag{301}$$

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Evaluating eq 301 at $X = \lambda_0 = \tau$ and simplifying, we obtain

$$-\frac{\partial\theta_0}{\partial X}\bigg|_{X=\lambda_0} = -\frac{1}{2}\operatorname{erfc}\left(-\frac{\sqrt{\tau}}{2}\right) - \frac{e^{-\tau/4}}{\sqrt{\pi\tau}} \tag{301a}$$

Further noting that $\operatorname{erfc}(-X) = 2 - \operatorname{erfc} X$, eq 301a can be expressed as

$$\frac{\partial\theta_0}{\partial X}\bigg|_{X=\lambda_0} = -1 + \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2}\tau^{1/2}\right) - (\pi\tau)^{-1/2} e^{-\tau/4} \tag{302}$$

Using eq 302 in 300, the two-term solution for η becomes

$$\eta = 1 + \varepsilon\left[\frac{1}{2}\operatorname{erfc}\left(\frac{1}{2}\tau^{1/2}\right) - (\pi\tau)^{-1/2} e^{-\tau/4}\right] \tag{303}$$

Equation 303 is a good approximation for $\tau > 1$ but for $\tau = O(\varepsilon^2)$, the term $-\varepsilon(\pi\tau)^{-1/2} e^{-\tau/4}$ becomes $O(1)$ rather than $O(\varepsilon)$ which is an indication of eq 303 breaking down when $\tau = O(\varepsilon^2)$, that is, for small times. Thus the solution, eq 303, constitutes an outer solution valid for long times. For short times, an inner solution must be constructed.

Inner Solution

The preheating effect, which was ignored in developing the outer solution, becomes important in determining the motion of the vaporizing boundary for small times, $\tau = O(\varepsilon^2)$. During the preheating time, the heat conduction (eq 285) is subject to the following boundary and initial conditions:

$$\frac{\partial\theta}{\partial X}\bigg|_{X=0} = -1 - \frac{1}{\varepsilon}; \quad \theta(\infty, \tau) = 0 \tag{304}$$

$$\theta(X,0) = 0$$

where the first condition in eq 304 is obtained from the constant heat flux condition at $X = 0$, that is,

$$-k\frac{\partial T}{\partial X}\bigg|_{X=0} = q.$$

The solution to this problem can be adapted from Carslaw and Jaeger (1959) and expressed as

$$\theta = \left(1 + \frac{1}{\varepsilon}\right)\left[2(\tau/\pi)^{1/2}\exp\left(-\frac{X^2}{4\tau}\right) - X\operatorname{erfc}\left(\frac{X}{2\sqrt{\tau}}\right)\right]. \tag{305}$$

The preheating time can be obtained by using $\theta - 1$ and $X = 0$ and is

$$\tau_p = \frac{1}{4}\pi\varepsilon^2/(1 + \varepsilon)^2 \tag{306}$$

Equation 306 shows that $\tau_p = O(\varepsilon^2)$ and substantiates our earlier assumption about the preheating time. If we substitute eq 306 into eq 305, we obtain the temperature distribution in the solid at the end of the preheating period as

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$$\theta(X, \tau_p) = \exp\left[-\{X(1+\varepsilon)/(\pi^{1/2}\varepsilon)\}^2\right] - \{X(1+\varepsilon)/\varepsilon\}\operatorname{erfc}\{X(1+\varepsilon)/(\pi^{1/2}\varepsilon)\} \tag{307}$$

Examination of eq 306 and 307 shows that the appropriate stretching of the variables for small times is

$$X^* = X/\varepsilon, \quad \tau^* = (\tau - \tau_p)/\varepsilon^2, \quad \lambda^* = \frac{\lambda}{\varepsilon^2}, \quad \theta^* = \theta, \quad \eta^* = \eta \tag{308}$$

Substituting the transformation, eq 308, into eq 285 and 286, the equations for the inner region become

$$\frac{\partial^2\theta^*}{\partial X^{*2}} = \frac{\partial\theta^*}{\partial\tau^*} \tag{309}$$

At $X^* = \dfrac{\lambda}{\varepsilon} = \varepsilon\lambda^*$, $\quad \theta^* = 1$, $\quad \eta^* = 1 + \dfrac{\partial\theta^*}{\partial X^*} + \varepsilon$ $\tag{310}$

At $X^* = \infty$, $\quad \theta^* = 0$ $\tag{311}$

At $\tau^* = 0$, $\theta^* = \exp\left[-\{X^*(1+\varepsilon)/\pi^{1/2}\}^2\right] - \{X^*(1+\varepsilon)\}\operatorname{erfc}\left\{\dfrac{X^*(1+\varepsilon)}{\pi^{1/2}}\right\}$. $\tag{312}$

Again we seek a perturbation solution of eq 309–312 in the form

$$\theta^* = \theta_0^* + \varepsilon\theta_1^* \tag{313}$$

$$\eta^* = \eta_0^* + \varepsilon\eta_1^* \tag{314}$$

The zero-order problem is governed by

$$\frac{\partial^2\theta_0^*}{\partial X^{*2}} = \frac{\partial\theta_0^*}{\partial\tau^*} \tag{315}$$

At $X^* = 0$, $\quad \theta_0^* = 1$ $\quad \eta_0^* = 1 + \dfrac{\partial\theta_0^*}{\partial X^*}$ $\tag{316}$

$$X^* = \infty, \quad \theta_0^* = 0 \tag{317}$$

$$\tau^* = 0, \quad \theta_0^* = \exp\left(-X^{*2}/\pi\right) - X^*\operatorname{erfc}\left(X^*/\sqrt{\pi}\right) \tag{318}$$

Since the times involved are $0(\varepsilon^2)$, the motion of the vaporizing boundary during this time would be very slight. Thus one can ignore the last condition in eq 316 and treat the problem as a no-phase-change heat conduction problem. Taking the Laplace transform of eq 315 we get

$$\frac{d^2\theta_0^*(s)}{dX^{*2}} - s\,\theta_0^*(s) + \exp\left(-X^{*2}/\pi\right) - X^*\operatorname{erfc}\left(X^*/\sqrt{\pi}\right) = 0 \tag{319}$$

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where $\theta_0^*(s)$ is the Laplace transform of θ_0^* and s is the Laplace variable. The boundary conditions, eq 316 and 317, transform to

$$X^* = 0, \quad \theta_0^*(s) = \frac{1}{s}; \quad X^* = \infty, \quad \theta_0^*(s) = 0 \tag{320}$$

The solution of eq 319 and 320 using the variation of parameters method can be obtained as

$$\theta_0^*(s) = \frac{1}{2} \exp\left(\frac{1}{4}\pi s\right)\left[2\exp\left(-s^{1/2}X^*\right)\mathrm{erf}\left(\frac{1}{2}s\pi\right)^{1/2}\right.$$

$$+ \exp\left(s^{1/2}X^*\right)\mathrm{erfc}\left(X^*/\pi^{1/2} + \frac{1}{2}(s\pi)^{1/2}\right)$$

$$\left. - \exp\left(-s^{1/2}X^*\right)\mathrm{erfc}\left(X^*/\pi^{1/2} - \frac{1}{2}(s\pi)^{1/2}\right)\right]s^{-3/2}$$

$$+ \frac{1}{s}\exp\left(-X^{*2}/\pi\right) - \left(X^*/s\right)\mathrm{erfc}\left(X^*/\pi^{1/2}\right) \tag{321}$$

The Laplace transform of the last condition in eq 316 is

$$\eta_0^*(s) = \frac{1}{s} + \frac{d\theta_0^*(s)}{dX^*}\bigg|_{X^*=0} \tag{322}$$

Using eq 321 to evaluate and substituting in eq 322, the solution for $\eta_0^*(s)$ is given by

$$\eta_0^*(s) = \frac{1}{s}\exp\left(\frac{1}{4}\pi s\right)\mathrm{erfc}\left[\frac{1}{2}(s\pi)^{1/2}\right] \tag{323}$$

The inverse of eq 323 is given by

$$\eta_0^* = \frac{2}{\pi}\arcsin\left[\left(1 + \frac{1}{4}\pi/\tau^*\right)^{-1/2}\right] \tag{324}$$

Matching

Expressing the outer solution, eq 303, in terms of the inner variable τ^*, and expanding it to two terms and keeping τ^* constant, we find that this expansion matches with the one obtained by expressing the inner solution eq 324 in terms of the outer variable t and expanding it in two terms keeping τ constant. Thus the matching condition is automatically satisfied.

Following van Dyke (1975) we may now construct a uniformly valid solution by adding the outer expansion and the inner expansion and subtracting the solution in the domain of overlap. Thus

$$\eta = 1 + \varepsilon\left\{\frac{1}{2}\mathrm{erfc}\left(\frac{1}{2}\tau^{1/2}\right) - (\pi\tau)^{-1/2}e^{-1/4\tau}\right\}$$

$$+ \frac{2}{\pi}\arcsin\left[1 + \frac{\pi\varepsilon^2}{4(\tau - T_\rho)}\right]^{-1/2} - \left[1 - \frac{\varepsilon}{(\pi\tau)^{1/2}}\right], \quad \tau \geq \tau_\rho \tag{325}$$

Equation 325 is a uniformly valid solution having an error of $O(\varepsilon)$ when $\tau = O(\varepsilon^2)$ and of $O(\varepsilon^2)$ when $\tau = O(1)$. Table 6 and Figure 23 compare the perturbation solutions to a heat balance integral solution

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Table 6. Comparison perturbation, heat balance integral, and numerical solution, constant wall heat flux, $\varepsilon = 0.2257$.

$$\eta = \frac{1}{v}\frac{dx_v}{dt}$$

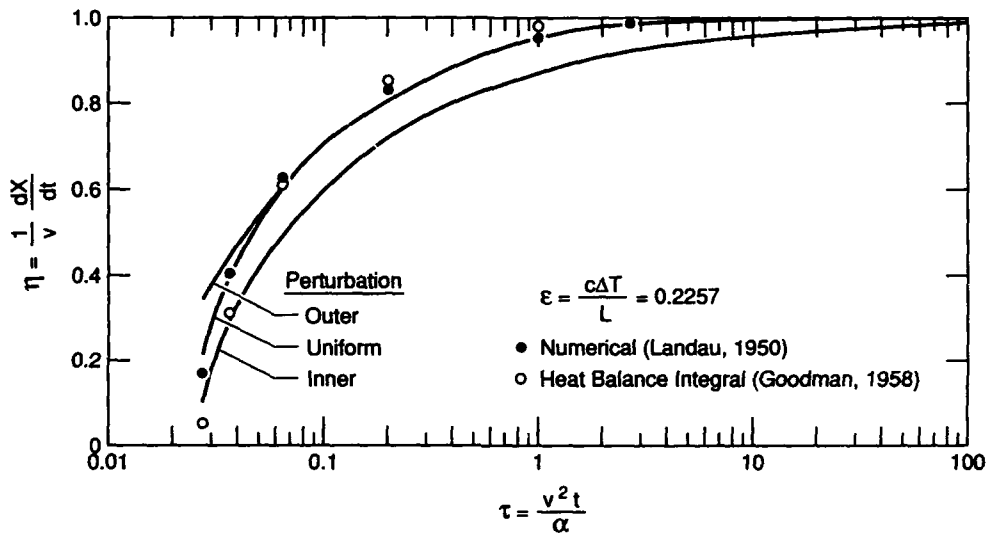| $\tau = \frac{v^2}{\alpha}t$	Perturbation			Numerical	Heat balance integral
	Inner	Outer	Uniformly valid	Landau (1950)	Goodman (1958)
0.0269	0.05183	0.3316	0.1090	0.0805	0.01423
0.02782	0.1076	0.3441	0.2162	0.1768	0.0584
0.03634	0.2913	0.4388	0.3981	0.4015	0.3178
0.0656	0.4959	0.6075	0.6005	0.6288	0.6129
0.20	0.7149	0.8135	0.8132	0.8357	0.8582
1.0	0.8727	0.9549	0.9549	0.9564	0.9845
9.0	0.9576	0.9994	0.9995	0.9995	—
100.0	0.9873	1.000	1.000	1.000	—

Figure 23. Melt velocity vs time for semi-infinite solid with constant surface heat flux.

and a numerical solution. The numerical results of Landau (1950) for $\varepsilon = 0.225$ are shown as circles for comparison. Figure 24 shows a plot of η versus τ obtained from eq 325 for $\varepsilon = 0$ (vaporization-controlled limit), 0.05 (graphite), 0.15 (tungsten) and 0.25 (lead). The horizontal intercepts correspond to the preheating time τ_p, eq 306. The integration (trapezoidal rule) of these curves gives the location of the vaporizing boundary as a function of τ and is shown in Figure 25. The curve marked $\varepsilon = 0$ is the linear relationship represented by eq 293. The agreement with the present results is good, and confirms the validity of eq 325.

8.2 Vaporization (sublimation) of a finite solid due to constant heat flux

Here we consider the problem of section 8.1 but now the solid has a finite thickness ℓ as shown in Figure 26. The face at $x = 0$ is assumed to be insulated. Let us first consider the preheating when the surface temperature is elevated from T_i to T_v. For this no-phase-change problem we have

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{326}$$
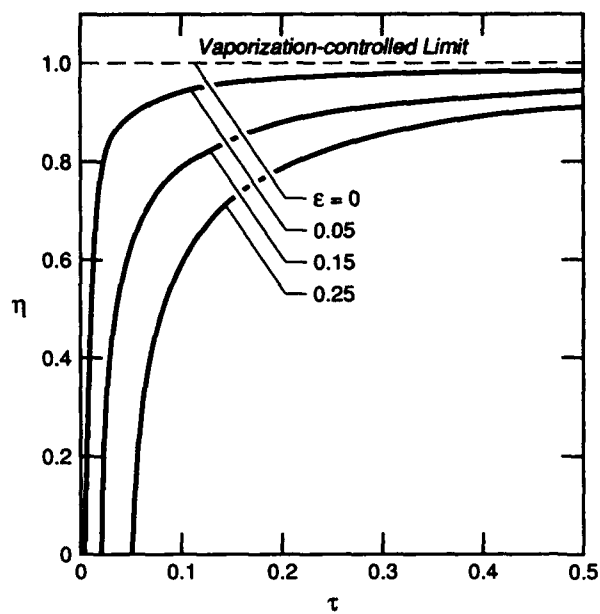
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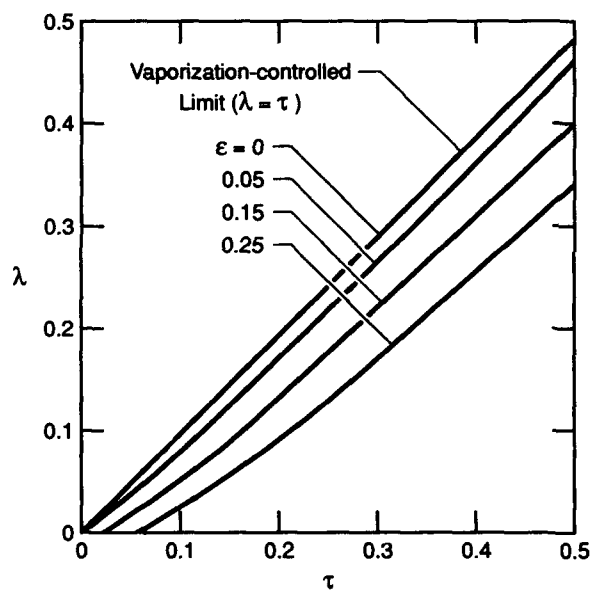Figure 24. Melt velocity vs. time, matched asymptotic solution.

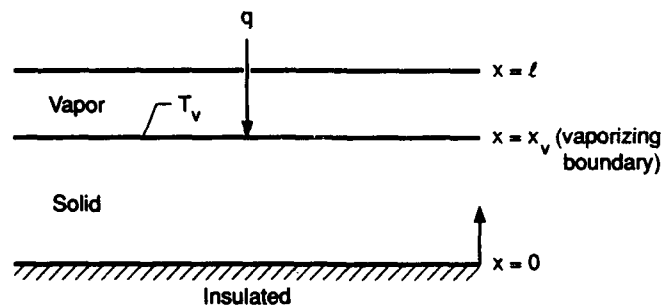Figure 25. Melt position vs. time, matched asymptotic solution.

Figure 26. Vaporization of finite solid due to constant surface heat flux.

$$x = 0, \quad \frac{\partial T}{\partial x} = 0; \quad x = \ell, \quad k\frac{\partial T}{\partial x} = q; \quad t = 0, \quad T = T_i. \tag{327}$$

The solution of eq 326 and 327 is given by Carslaw and Jaeger (1959) as

$$\theta = Q\tau + Q\left(\frac{3X^2 - 1}{6} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi X)\exp(-n^2\pi^2\tau)\right) \tag{328}$$

where $\theta = (T - T_i)/(T_v - T_i)$, $x = x/\ell$, $\tau = \alpha t/\ell^2$, and $Q = \dfrac{q\ell}{k(T_v - T_i)}$. The preheating time τ_p when $\theta = 1$, $X = 1$, is given by the following implicit relationship

$$\tau_p + \left\{\frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-n^2\pi^2\tau_p)\right\} = \frac{1}{Q} \tag{329}$$

Next, consider the phase change problem. The heat conduction into the solid is still governed by ∼q 326 but the conditions to be satisfied are

$$x = 0, \quad \frac{\partial T}{\partial x} = 0; \quad x = x_v, \quad T = T_v;$$

$$\tag{330}$$

$$x = x_v, \quad q - \frac{\partial T}{\partial x} = -\rho L\frac{dx_v}{dt}$$

The initial condition is given by eq 328 with $\tau = \tau_p$.

Introducing the following dimensionless variables $\phi = (T_v - T)/(T_v - T_i)$, $\eta = x/x_v$, $\sigma = x_v/\ell$, $\varepsilon = c(T_v - T_i)/L$, $z = \varepsilon\tau = \varepsilon\alpha t/\ell^2$ into eq 326 and 330, we obtain

$$\frac{\partial^2\phi}{\partial\eta^2} = \varepsilon\left[\sigma^2\frac{\partial\phi}{\partial z} - \eta\sigma\frac{d\sigma}{dz}\cdot\frac{\partial\phi}{\partial\eta}\right] \tag{331}$$

$$\eta = 1, \quad \phi = 0; \quad \eta = 0, \quad \frac{\partial\phi}{\partial\eta} = 0 \tag{332}$$

$$z = 0, \quad \phi = f(\eta)$$

where $f(\eta)$ is given by

$$f(\eta) = 1 - Q\tau_p - Q\left(\frac{3\eta^2 - 1}{6} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{\eta^2}\cos(n\pi\eta)\exp(-\eta^2\pi^2\tau_p)\right) \tag{333}$$

noting that $\tau = \tau_p$, $x_v = \ell$, $\sigma = 1$, $\eta = X$, and $\phi = 1 - \theta$.

The energy balance at the interface transforms to

$$Q\sigma + \frac{\partial\phi}{\partial\eta}(1,\tau) = -\frac{1}{\varepsilon}\sigma\frac{d\sigma}{d\tau} = -\sigma\frac{d\sigma}{dz} \tag{334}$$

Outer solution

The long time or the outer solution is assumed as

$$\phi = \phi_0 + \varepsilon\phi_1 \tag{335}$$

$$\sigma = \sigma_0 + \varepsilon\sigma_1 \tag{336}$$

Substituting eq 335 into eq 331 and 332 and equating coefficients of ε^0, we have

$$\frac{\partial^2\phi_0}{\partial\eta^2} = 0 \tag{337}$$

$$\eta = 0, \quad \frac{\partial\phi_0}{\partial\eta} = 0; \quad \eta = 1, \quad \phi_0 = 0 \tag{338}$$

The solution of eq 337 and 338 is

$$\phi_0 = 0 \tag{339}$$

Equation 339 shows that after a long time, the solid is at a uniform temperature T_v throughout. Since $\phi_0 = 0$, it is easy to see that $\phi_1 = \phi_2 = \ldots\phi_n = 0$.

Substituting eq 336 into eq 334 and equating coefficients of ε^0, the equation for σ_0 is obtained as

$$\frac{d\sigma_0}{dz} = -Q \tag{340}$$

Integrating eq 340 we have

$$\sigma_0 = -Qz + C_1 \tag{341}$$

where the constant C_1 would be determined by matching the outer and the inner solutions.

Equating coefficients of ε, gives the following equation for σ_1:

$$\sigma_0\frac{d\sigma_1}{dz} + \sigma_1\frac{\partial\sigma_0}{dz} = -Q\sigma_1 \tag{342}$$

Using eq 341 for σ_0 in eq 342, it can be seen that

$$\frac{d\sigma_1}{dz} = 0 \quad \text{or} \quad \sigma_1 = C_2 \tag{343}$$

where C_2 is a constant which would be determined by matching.

Inner solution

Examination of eq 331 reveals that for the inner solution, the appropriate time variable is $\tau = z/\varepsilon$. Thus the inner expansion is written as

$$\phi^* = \phi_0^* + \varepsilon\phi_1^* \tag{344}$$

$$\sigma^* = \sigma_0^* + \varepsilon\sigma_1^* \tag{345}$$

where $\phi^* \equiv \phi_i$ and $\sigma^* \equiv \sigma$.

Substituting eq 344 and 345 into eq 331 and 332 and equating coefficients of ϵ^0, we obtain

$$\frac{\partial^2 \phi_0^*}{\partial \eta^2} = \sigma_0^{*2} \frac{\partial \phi_0^*}{\partial \tau} - \eta \sigma_0^* \frac{d\sigma_0^*}{d\tau} \cdot \frac{\partial \phi_0^*}{\partial \eta} \tag{346}$$

$$\eta = 1 \, , \quad \phi_0^* = 0 \, , \quad \frac{\partial \phi_0^*}{\partial \eta} = 0 \, ; \quad \tau = 0 \, , \quad \phi_0^* = f(\eta) \tag{347}$$

Similarly, substituting eq 345 into 334 and equating coefficients of ϵ^{-1}, we get

$$\sigma_0^* \frac{d\sigma_0^*}{d\tau} = 0 \quad \text{or} \quad \frac{d\sigma_0^*}{d\tau} = 0 \tag{348}$$

Since at $\tau = 0$, $\sigma = 1$, the initial condition on σ_0^* is $\tau = 0$, $\sigma_0^* = 1$. Thus the solution of eq 348 is

$$\sigma_0^* = 1 \tag{349}$$

In view of eq 348 and 349, eq 346 reduces to

$$\frac{\partial^2 \phi_0^*}{\partial \eta^2} = \frac{\partial \phi_0^*}{\partial \tau} \tag{350}$$

The solution of eq 350 subject to eq 347 can be obtained using the method of separation of variables as

$$\phi_0^* = \sum_{m=0}^{\infty} A_m \cos\left(\lambda_m \eta\right) \exp\left(-\lambda_m^2 \tau\right) \tag{351}$$

where $\lambda_m = \left(m + \frac{1}{2}\right)\pi$ and

$$A_m = 2\frac{(-1)^m}{\lambda_m}\left[1 + Q\left(\frac{1}{\lambda_m^2} - \tau_p - \frac{1}{3}\right)\right]$$

$$+ \frac{2Q}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left\{ \frac{\sin\left(n\pi - \lambda_m\right)}{n\pi - \lambda_m} + \frac{\sin\left(n\pi + \lambda_m\right)}{n\pi + \lambda_m} \right\} \exp\left(-n^2 \pi^2 \tau_p\right)$$

To obtain the equation for σ_1^*, we substitute eq 344 and 345 into eq 334 and equate coefficients of ϵ^0 to give

$$Q\sigma_0^* + \frac{\partial \phi_0^*}{\partial \eta}(1, \tau) = -\left(\sigma_0^* \frac{d\sigma_1^*}{d\tau} + \sigma_1^* \frac{d\sigma_0^*}{d\tau}\right) \tag{352}$$

which, in view of eq 348 and 349, reduces to

$$Q + \frac{\partial \phi_0^*}{\partial \eta}(1, \tau) = -\frac{d\sigma_I^*}{d\tau} \tag{353}$$

Using eq 351 to evaluate $\dfrac{\partial \phi_0^*}{\partial \eta}(1, \tau)$ and substituting the result into eq 353, the solution for σ_I^*, subject to the initial condition, $\tau = 0$, $\sigma_I^* = 0$, is finally obtained as

$$\sigma_I^* = -Q\tau + \sum_{m=0}^{\infty} \frac{(-1)^m}{\lambda_m} A_m \left[1 - \exp\left(-\lambda_m^2 \tau\right)\right] \tag{354}$$

Matching

Writing the outer solution $\sigma = -Qz + C_1 + \varepsilon\, C_2$ in terms of the inner variable τ and matching it with the inner solution

$$\sigma^* = 1 + \varepsilon \left\{ -Q\tau + \sum_{m=0}^{\infty} \frac{(-1)^m}{\lambda_m} A_m \left[1 - \exp\left(-\lambda_m^2 \tau\right)\right] \right\}$$

expressed in terms of the outer variable z, we find that

$$C_1 = 1 \quad \text{and} \quad C_2 = \sum_{m=0}^{\infty} (-1)^m A_m / \lambda_m.$$

The outer and the inner expansions agree in the intermediate time zones. Eliminating the common parts of the two expansions, a uniformly valid solution is obtained as

$$\phi = \sum_{m=0}^{\infty} A_m \cos\left(\lambda_m \eta\right) \exp\left(-\lambda_m^2\, \tau\right) + 0\,(\varepsilon) \tag{355}$$

$$\sigma = 1 + \varepsilon \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{\lambda_m} A_m \left[1 - \exp\left(-\lambda_m^2 \tau\right)\right] - Q\tau \right\} + 0\!\left(\varepsilon^2\right) \tag{356}$$

Equation 355 shows that for long times, ϕ approaches zero, which in turn implies that T approaches T_v. For large times, eq 356 shows that $\sigma \rightarrow (1 - Q\tau)$. For complete vaporization, $\sigma = 0$ and assuming that at that time, the exponential term becomes negligible, the total time for complete vaporization, τ_v is given by

$$\tau_v = \frac{1}{\varepsilon Q} \left\{ 1 + \varepsilon \sum_{m=0}^{\infty} \frac{(-1)^m}{\lambda_m} A_m \right\} \tag{357}$$

where the second term in eq 357 represents the effect of transient heat conduction on the vaporization time. The first term, $1/(\varepsilon Q)$, represents the vaporization-controlled limit. The effect of transient conduction is to increase the time to complete vaporization because some of the energy supplied is used to heat the solid rather than being used for the vaporization of the solid.

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9. EXTENDED PERTURBATION SERIES METHOD

The perturbation expansions that we have discussed so far were all terminated at the second or third term. Such a truncated expansion is valid for a limited range of values of ε. If used beyond the range of applicability, the approximation fails to converge and gives erroneous answers. The method of extended perturbation series attempts to remove this limitation by combining analysis and numerical computation.

The method follows three steps. First, the set of perturbation equations is programmed for solution on a digital computer so that a large number of terms can be generated. Second, the coefficients of the series are utilized to identify the location and nature of singularities limiting the range of applicability of the series. With this knowledge, the final step is to recast the series using one or a combination of devices such as the Euler transformation, Shanks transformation, Padé approximants, extraction of singularities, and series reversion. The improved series generally has better accuracy and a wider range of applicability than the original series.

9.1 Extension of series

For simple problems, the addition of a few extra terms to a two- or three-term expansion can be achieved with hand computation. However, complex problems require the use of a digital computer. Depending upon the problem, there are two possible approaches. First, if the problem is such that a pattern can be established for the sequence of solutions, then one can write a program incorporating this pattern and calculating the terms in sequence. However, to establish the solution pattern it is essential to calculate the first few terms by hand. These hand computations also assist in debugging the program. Second, if the solution pattern is not discernible, one must adopt a fully numerical procedure to solve the sequence of perturbation equations.

An example of the former approach in phase change heat transfer is provided by Pedroso and Domoto (1973a), who give the details of how the terms of the perturbation series for eq 42 and 43 can be generated automatically. Using their nomenclature, these equations are written as

$$\frac{\partial^2 u}{\partial x^2} = \varepsilon \left. \frac{\partial u}{\partial x_f} \frac{\partial u}{\partial x} \right|_{x=x_f} \tag{358}$$

$$u(x=x_f, x_f)=1, \quad u\left(x=0, x_f\right)= \left.\frac{\partial u}{\partial x}\right|_{x=0}, \quad \frac{dx_f}{dt} = \left.\frac{\partial u}{\partial x}\right|_{x=x_f} \tag{359}$$

where $u = (T - T_0)/(T_f - T_0)$, $x = hX/k$, $x_f = hX_f/k$, $\tau = h^2(T_f - T_0)t/\rho Lk$, $\varepsilon = c(T_f - T_0)/L$ and X denotes the dimensional distance.

The solution of eq 358 is assumed to have the form

$$u(x, x_f; \varepsilon) = u_i(x, x_f)\varepsilon^{i-1}, \quad i = 1, 2, \cdots, N_t \tag{360}$$

where the summation convention over repeated indices is used in eq 360 and N_t is the number of terms in the perturbation series. Substituting eq 360 into eq 358 and 359 and equating coefficients of equal powers of ε, one obtains

$$\frac{\partial^2 u_i}{\partial x^2} = 0 \quad \text{for } i = 1$$

$$= \left. \frac{\partial u_i}{\partial x} \right|_{x = x_f} \cdot \frac{\partial u_{i-j}}{\partial x_f}, \quad j = 1, 2, \ldots, i-1 \quad \text{for} \quad i \geq 2$$

$$u_i(x = x_f, x_f) = \begin{cases} 1 \text{ for } i = 1 \\ 0 \text{ for } i \geq 2 \end{cases} ; \quad u_i(x = 0, \ x = x_f) = \left. \frac{\partial u_i}{\partial x} \right|_{x=0} \tag{361}$$

The solution to the differential equations in eq 361 can be obtained as

$$u_i = \lambda_{i,j,1} \, x^{j-1} \tag{362}$$

where $i = 1, 2, \ldots N_t, j = 1, 2, \ldots 2i$, and $\lambda_{i,j,1}$ are functions of x_f to be calculated from the boundary conditions in eq 361. Pedroso and Domoto (1973a) show that $\lambda_{i,j,1}$ can be evaluated from the following relationships.

$$\lambda_{1,1,1} = \lambda_{1,2,1} = \frac{1}{1 + x_f} \tag{363a}$$

$$\lambda_{1,1,m} = \lambda_{1,2,m} = (-1)^{m-1} \frac{(m-1)(m-2)\cdots(1)}{(1 + x_f)^m} \tag{363b}$$

where

$m = 2, 3, \ldots, N_t$, no sum on m.

$$\lambda_{1,l+2,1} = (k-1)\lambda_{j,k,l} x_f^{k-2} \frac{\lambda_{i-j,1,2}}{l(l+1)} \tag{363c}$$

where

$i = 2, 3, \ldots, N_t; \ l = 1, 2, \ldots, 2(i-1)$

$$j = 1, 2, \ldots, i - \frac{1}{2}\left[l + \frac{1 + (-1)^{l+1}}{2}\right], \ k = 2, 3, \ldots, 2i$$

no sum on l.

$$\lambda_{i,1,1} = \lambda_{i,2,1} = \frac{\lambda_{i,j,1} x_f^{i-1}}{1 + x_f} \tag{363d}$$

where

$i = 2, 3, \ldots, N_t; \quad j = 3, 4, \ldots, 2i$

$$\lambda_{i,l+2,m} = a_{m,m-n+1} \, a_{n,n-h+1} \, \lambda_{j,k,h} f_{k,n-h+2} \frac{\lambda_{i-j,l,m-n+2}}{l(l+1)} \tag{363e}$$

where

$i = 2, 3, \ldots, N_t - 1; \quad l = 1, 2, \ldots, 2(i-1)$

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$$i = 1,2, \cdots, i - \frac{1}{2}\left[l + \frac{1 + (-1)^{l+1}}{2}\right], \quad k = 2,3, \cdots, 2i$$

$$m = 2,3, \cdots, N_t - i + 1; \quad n = 1,2, \cdots, m$$

$$h = 1,2, \cdots, n, \qquad \text{no sum on } m$$

$$\lambda_{i,1,m} = \lambda_{i,2,m} = - a_{m,m-n+1} \, a_{n,n-h+1} \, \lambda_{i,j,h} \, f_{j,n-h+1} \, \lambda_{1,1,m-n+1} \tag{363f}$$

where

$$i = 2,3, \cdots, N_t - 1; \quad j = 3,4, \cdots, 2i$$

$$m = 2,3, \cdots, N_t - i + 1; \quad n = 1,2, \cdots, m$$

$$h = 1,2, \cdots, n, \qquad \text{no sum on } m.$$

The quantities a and f are defined as follows

$$a_{i,1} = a_{i,i} = 1 \text{ for } i = 2, 3, \cdots \qquad \text{no sum on } i$$

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j} \text{ for } i = 3,4, \cdots, \quad j = 2,3, \cdots, i-1 \tag{364}$$

$$f_{j,1} = x_f^{j-1} \tag{365}$$

and the $(m-1)$th derivative of the function f is given by

$$f_{j,m} = \begin{cases} (j-1)(j-2)\cdots(h-m+1)x_f^{j-m} \text{ for } m = 2,3,\cdots,j \\ 0 \text{ for } m \geq j+1 \qquad \text{no sum on } j \text{ or } m. \end{cases} \tag{366}$$

Combining eq 359c and 360, the equation for $d\tau/dx_f$ can be obtained as

$$\frac{d\tau}{dx_f} = \frac{d\tau_i}{dx_f} \varepsilon^{i-1}, \quad i = 1,2, \cdots, N_f$$

$$\frac{d\tau_i}{dx_f} = \begin{cases} \dfrac{1}{g_1} \text{ for } i = 1 \\[2ex] -\dfrac{g_2}{g_1^2} \text{ for } i = 2 \\[2ex] -\dfrac{1}{g_1^2}\left(g_i + g_{j+1} \dfrac{d\tau_{i-j}}{dx_f}\right) \text{ for } i = 3,4, \cdots N_t \\[2ex] \text{where } j = 1,2, \cdots i - 2 \end{cases} \tag{367}$$

$$g_i = (j-1) x_f^{j-2} \lambda_{i,j,1}$$

where $i = 1,2, \cdots N_t$, $j = 2,3, \cdots, 2i$.

For a given value of x_f, eq 367 can be evaluated once the values of $\lambda_{i,j,1}$ are known. By integrating the terms $d\tau_i/dx_f$, one can obtain the coefficients τ_i. In section 3.2, the coefficients τ_i were denoted by c_n and Table 1 contains the first nine values for a range of x_f values.

An example of the extension of the perturbation series using a fully numerical procedure is given by Beckett (1981). He considered the inward freezing of a cylinder and extended the series for the freezing front location E, in time ε, to 31 terms. For $\beta = 0.1$ (Stefan number $= 10$) his result is

$$E = 2.5139 \ \varepsilon^{1/2} + 0.5130 \ \varepsilon + 0.6540 \ \varepsilon^{3/2} + 1.2779 \ \varepsilon^2 + 2.2475 \ \varepsilon^{5/2}$$

$$+ 4.8757 \ \varepsilon^3 + 11.183 \ \varepsilon^{7/2} + 26.677 \ \varepsilon^4 + 65.528 \ \varepsilon^{9/2} + 164.64 \ \varepsilon^5$$

$$+ 421.22 \ \varepsilon^{11/2} + 1093.6 \ \varepsilon^6 + 2874.5 \ \varepsilon^{13/2} + 7643.2 \ \varepsilon^7 + \cdots$$

$$+ 8.651 \times 10^6 \varepsilon^{21/2} + \cdots + 2.759 \times 10^{11} \varepsilon^{31/2} .$$

$$(368)$$

9.2 Analysis of series

Once the extension of the series has been accomplished, the next step is to explore the analytic structure of the solution identifying the location and nature of singularities, if any exist. The final pattern of the sign prevailing in the series determines the location of the dominant singularity. If the signs are fixed as in eq 368, the singularity lies on the positive axis, but if the signs alternate, it lies on the negative axis. With random signs, the most likely possibility is that singularities occur as complex conjugate pairs.

When the nearest singularity lies on the negative axis, it usually carries no physical significance. On the other hand, a singularity appearing on the positive axis can often be interpreted physically. For example, the positive axis singularity associated with the series in eq 368 indicates the completion of the freezing process. In some cases, the positive axis singularity points to the limit of validity of the mathematical model itself. Another possibility with the positive axis singularity is that it is not real but simply an indication that the function is double valued.

Figure 27. Extended perturbation series, Domb-Sykes plot.

To establish the location and nature of singularities, the best approach is to calculate the ratio $|c_n/c_{n-1}|$ for the series $\sum\limits_{n=0}^{\infty} c_n \varepsilon^n$ and plot it against $1/n$, giving what is called a Domb-Sykes plot. For large n, the ratio c_n/c_{n-1} often becomes linear in $1/n$ and is of the form

$$\frac{c_n}{c_{n-1}} = \mp \frac{1}{\varepsilon_0} \left(1 - \frac{1+\alpha}{n} \right)$$

$$(369)$$

where $(1 + \alpha)$ is the slope and ε_0 is the radius of convergence of the series. By extrapolating the graph to $1/n = 0$, one can obtain the intercept $1/\varepsilon_0$. Figure 27 shows the Domb-Sykes plot for the series eq
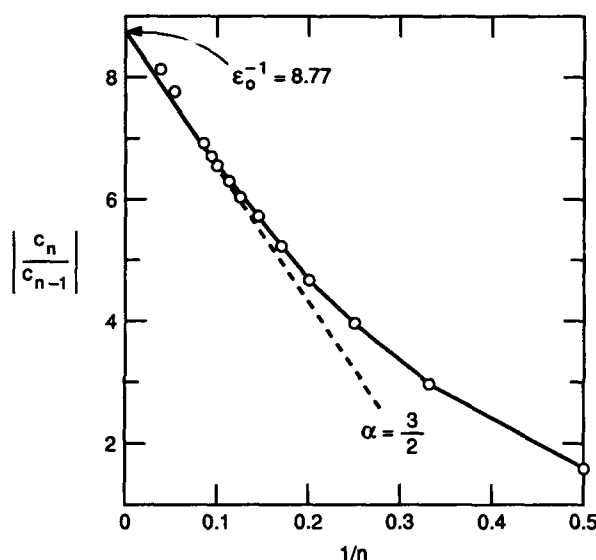
368 from which $a = 3/2$ and $\varepsilon_0^{-1} = 8.77$. Thus one can estimate the dimensionless time for complete freezing as $\varepsilon_0 = 0.1140$.

9.3 Improvement of series

As the final step, our knowledge about the leading singularity is used to improve the series. A number of improvement devices are available. The choice of a particular method depends on the direction, distance, and nature of the singularity as revealed by the Domb-Sykes plot. Often the problem is such that more than one device is applicable, and it is always enlightening to try different possibilities. Also, if a single technique is not adequate, one should consider using two or more of them in combination.

Euler transformation. When a perturbation series contains alternating signs, the singularity lies on the negative axis and carries no physical significance. In this case, the simplest device to use is an Euler transformation based on the estimate of ε_0. With this transformation the singularity is mapped away to infinity. The advantage of this device is that the exact nature of the singularity need not be known.

Let the perturbation series be

$$f = \sum_{n=0}^{\infty} \varepsilon^n a_n \tag{370}$$

and the nearest singularity be located at $\varepsilon = \varepsilon_0$ (estimated from a Domb-Sykes plot). The transformation envisages using a new variable ε^* such that

$$\varepsilon^* = \frac{\varepsilon}{\varepsilon + \varepsilon_0} \tag{371}$$

which gives the Eulerized series as

$$f = \sum_{n=0}^{\infty} b_n \varepsilon^{*n} \tag{372}$$

where the coefficients b_n are

$$b_0 = a_0 \tag{373}$$

$$b_n = \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!(j-1)!} a_j \varepsilon_0^j \tag{374}$$

Although the Euler transformation eq 371 has been written for the power series in ε, one can also use it for a power series in a variable. Consider, for example, the series in eq 183 which, for inward freezing, breaks down as $r_f \rightarrow 0$. Rewrite it as

$$\frac{u}{u_0} = 1 + \varepsilon a_1 \left(\frac{1}{r_f}\right) + \varepsilon^2 a_3 \left(\frac{1}{r_f}\right)^3 \tag{375}$$

where

$$a_1 = \frac{1}{6 r_f}\left[1 - \left(\frac{r}{r_f}\right)^2 u_0^2\right] \tag{376a}$$

$$a_3 = -\frac{1}{36r_f}\left\{1-\left(\frac{r}{r_f}\right)^2 u_0^2 + \frac{4r_f-1}{120}\left[1-\left(\frac{r}{r_f}\right)^4 u_0^4\right]\right\} \tag{376b}$$

The series in eq 375 can now be regarded as a series in $1/r_f$. Note that as $r_f \to 0$, $a_1 u_0$ and $a_3 u_0$ remain finite. For convenience, we can express eq 375 as

$$\frac{u}{u_0} = 1 + C_1\eta + C_3\eta^3 \tag{377}$$

where $C_1 = \varepsilon a_1$ and $C_3 = \varepsilon^2 a_3$ are functions of r, r_f, and ε. Applying the Euler transformation

$$\eta^* = \frac{\eta}{\eta+K} \tag{378}$$

to eq 377, we have the following Eulerized series:

$$\frac{u}{u_0} = 1 + Ka_1\,\varepsilon(1+\eta^*)\eta^* + K\varepsilon(a_1 + K^2 a_3\varepsilon)\eta^{*3} \tag{379}$$

The series in eq 379 remains valid as $r_f \to 0$ or $\eta \to \infty$ because as $\eta \to \infty$, $\eta^* \to 1$.

If one constructs a regular perturbation series for $g = \dfrac{dr_f}{d\tau}$, then it is found that

$$\frac{g}{g_0} = 1 + \varepsilon b_1\,\frac{1}{r_f} + \varepsilon^2 b_3\,\frac{1}{r_f^3} \tag{380}$$

where

$$b_1 = -\frac{1}{3}, \quad b_3 = \frac{1+6r_f}{45} \tag{381}$$

If the transformation eq 378 is now applied to eq 380, we obtain

$$\frac{g}{g_0} = 1 + Kb_1\,\varepsilon(1+\eta^*)\eta^* + K\varepsilon(b_1 + K^2 b_3\,\varepsilon)\eta^{*3} \tag{382}$$

If eq 379 and 382 are used in the overall energy balance for the entire duration of the freezing process, one can obtain an equation for $K = K(\varepsilon)$. It has been shown by Pedroso and Domoto (1973c) that if only terms in the first power of η^* are retained in eq 379 and 382, then the overall energy balance gives the following transcendental equation for K:

$$\frac{6-(3+2\varepsilon)K}{6K}\ln\left(1+\frac{3K}{3-\varepsilon K}\right) = 1 - \frac{7}{40}\frac{\varepsilon K^2}{(3-\varepsilon K)} \tag{383}$$

The solution of eq 383 for a range of values of ε is given below:

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	2	3
K	14.96	9.099	6.699	5.350	4.474	3.854	3.391	3.030	2.791	2.503	1.351	0.9287

Integrating eq 382 to obtain the freezing time as a function of r_f, the solution becomes

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$$\tau^*(\text{Euler}) = \frac{\varepsilon}{9K}\left[3 + (3 - \varepsilon)K\right]\left[1 - r_f - \frac{3 - \varepsilon K}{3K}\ln\frac{3 + (3 - \varepsilon)K}{3 + (3r_f - \varepsilon)K}\right]$$

$$+ \frac{3 - \varepsilon}{6}\left(1 - r_f^2\right) - \frac{1}{3}\left(1 - r_f^3\right) \tag{384}$$

The results from eq 384 are compared with the numerical results of Tao (1967) in Figure 28 for $\varepsilon = 0.1, 0.5$ and 1.0. While the regular series diverged as $r_f \to 0$ (see Fig. 10), the Eulerized series in eq 384 agrees quite well the numerical results.

Shank transformations. Shanks (1955) introduced a family of four nonlinear transformations to accelerate the convergence of slowly convergent and divergent series. The merit of these transformations is that they do not require any information about the analytic structure of the solution. The application is therefore rather blind, so the result should be viewed with skepticism. However, the pattern of convergence is often manifested so convincingly that it speaks for the accuracy of the final results.

Consider the e_1 transformation, which is the simplest one. If three partial sums S_{n-1}, S_n, and S_{n+2} of a series are known, then

Figure 28. Eulerized series for inward spherical solidification.

$$e_1(S_n) = \frac{S_{n+1}\,S_{n-1} - S_n^2}{S_{n+1} + S_{n-1} - 2S_n} \tag{385}$$

The success of the e_1 transformation in improving the convergence results because it yields the exact sum if applied to a geometric series. Therefore, it works best on series with nearly geometric coefficients.

To illustrate the application of Shanks transformation, we consider the three-term strained coordinate solution for τ, eq 250. Rewrite eq 250 as

$$\tau = \tau_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2 \tag{386}$$

where

$$\tau_0 = \frac{1}{4}\left(1 - \psi^2\right) + \frac{1}{2}\psi^2\ln\psi$$

$$\tau_1 = \frac{\left(1 - \psi^2\right) + \left(1 + \psi^2\right)\ln\psi}{2\ln\psi}$$

$$\tau_2 = \frac{15\left(1 - \psi^2\right)^2 + 21\left(1 - \psi^4\right)\ln\psi + 12\left(1 + \psi^4\right)\ln^2\psi + 3\left(1 - \psi^4\right)\ln^3\psi}{96\psi^2\ln^4\psi}$$
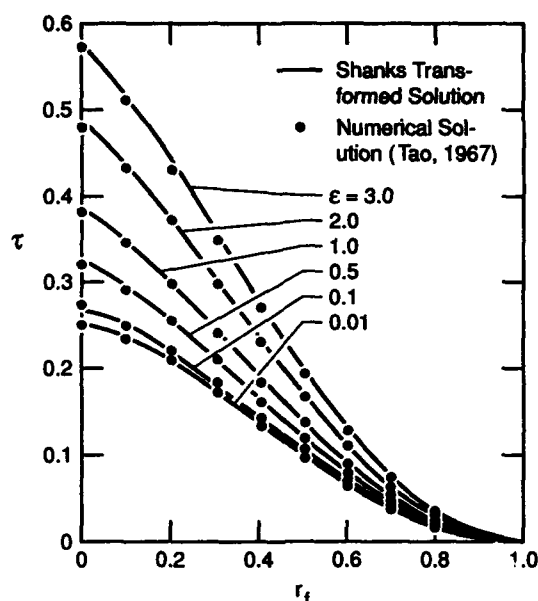
Figure 29. Shanks-transformed solution for inward cylindrical solidification.

If we note that $S_{n-1} = \tau_0$, $S_n = \tau_0 + \varepsilon\tau_1$ and $S_{n+1} = \tau_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2$, the application of Shanks transformation (eq 385) to eq 386 gives

$$e_1 = \tau^*(\text{Shanks}) = \frac{\tau_0\tau_1 - \varepsilon(\tau_0\tau_2 - \tau_1^2)}{\tau_1 - \varepsilon\tau_2} \tag{387}$$

Figure 29 compares the Shanks transformed solution given by eq 387 with the numerical results of Tao (1967). The agreement is exceptionally good even at $\varepsilon = 3$. This is a big improvement over the predictions of eq 386 which gives valid results only up to $\varepsilon = 0.8$ (Asfar et al. 1979).

10. CONCLUDING REMARKS

This review has demonstrated the usefulness of perturbation techniques to analyze heat transfer problems involving freezing and melting. The perturbation approach has proved effective and convenient in one-dimensional situations, and thus the discussion was mostly confined to these situations. However, the method has also enjoyed limited success with two-dimensional cases; such studies have been briefly mentioned here but further details can be found in the appropriate references. Despite their limited success in more complex problems, perturbation methods often prove invaluable in illuminating the physics of the problem.

This monograph has been written to serve two purposes. The first purpose is to assist the unfamiliar reader in understanding the perturbation techniques and seeing, with the help of detailed mathematics, how these techniques are applied to specific problems. The second purpose is to bring together in a single publication the essence of a large body of literature on the subject so that it can serve as a reference for future studies.

LITERATURE CITED

Afsar, O.R., A. Aziz and M.A. Soliman (1979) A uniformly valid solution for inward cylindrical solidification. *Mechanical Research Communications.*, **6**: 325.

Andrews, J.G. and D.R. Atthey (1975) *Moving Boundary Problems in Heat Flow and Diffusion.* Oxford: Clarendon Press.

Antaki, P.J. (1990) Asymptotic analysis of laser-induced sublimation for small Stefan number. *5th American Institute of Aeronautic and Aerospace/American Society of Mechanical Engineers Thermophysics and Heat Transfer Conference,* Seattle, Washington.

Aziz, A. (1978) New asymptotic solution for the variable property Stefan problem. *Chemical Engineering Journal,* **16**: 65.

Aziz, A. and J.Y. Benzies (1976) Application of perturbation techniques to heat transfer problems with variable thermal properties. *International Journal of Heat and Mass Transfer,* **19**: 271.

Aziz, A. and V.J. Lunardini (1991) Application of local similarity method to nonsimilar conduction controlled freezing problems. *International Communications in Heat and Mass Transfer,* **18**: 813.

Aziz, A. and V.J. Lunardini (1992) An assessment of prediction methods for the thickness of the active layer in permafrost regions. *Proceedings of the 1992 OMAE Conference,* Calgary, **4**: 131.

Aziz, A. and T.Y. Na (1984) *Perturbation Methods in Heat Transfer.* New York: Hemisphere.

Beckett, P.M. (1981) On the use of series solution applied to solidification problems. *Mechanical Research Communications,* **8**: 169.

Carslaw, H.S. and J.C. Jaeger (1959) *Conduction of Heat in Solids.* Oxford: Clarendon Press.

Cervantes, J., C. Trevino and F. Mendez (1990) An asymptotic analysis for transient freezing and laminar flow in a circular tube. *International Heat Transfer Conference,* paper No. 12–PC–19.

Charach, Ch. and P. Zoglin (1985) Solidification in a finite, initially overheated slab. *International Journal of Heat and Mass Transfer,* **28**: 2261.

Chung, B.T.F. and L.T. Yeh (1975) Solidification and melting of materials subject to convection and radiation. *Journal of Spacecrafts and Rockets,* **12**: 329.

Chung, B.T.F. and L.T. Yeh (1976) Freezing and melting of materials with variable properties and arbitrary heat fluxes. *American Institute of Aeronautic and Aerospace Journal,* **4**: 388.

Churchill, S.W. and L.B. Evans (1971) Calculation of freezing in a semi-infinite region. *Journal of Heat Transfer,* **93**: 234.

Goodman, T.R. (1958) The heat balance integral and its application to problems involving a change of phase. *Transactions of American Society of Mechanical Engineers,* **80**: 335.

Gutman, L.N. (1986) On the problem of heat transfer in phase-change materials for small Stefan numbers. *International Journal of Heat and Mass Transfer,* **29**: 921.

Gutman, L.N. (1987) On the problem of heat transfer in a phase-change slab initially not at the critical temperature. *Journal of Heat Transfer,* **109**: 5.

Howarth, J.A. (1985) A note on two-dimensional linearized perturbations of the Neumann problem. *International Journal of Heat and Mass Transfer,* **28**: 301.

Howarth, J.A. (1987) Solidification of a sphere with constant heat flux at the boundary. *Mechanical Research Communications,* **14**: 135.

Howarth, J.A. (1990) The Stefan problem for a slightly wary wall. *Mechanical Research Communications,.* **17**: 25.

Huang, C.L. and Y.P. Shih (1975) Perturbation solution for planar solidification of a saturated liquid with convection at the wall. *International Journal of Heat and Mass Transfer,* **18**: 1481.

Jiji, L.M. and S. Weinbaum (1978) Perturbation solution for melting or freezing in annular regions initially not at the fusion temperature. *International Journal of Heat and Mass Transfer,* **21**: 581.

Kane, D.L., L.D. Hinzman and J.P. Zarling (1991) Thermal response of the active layer to climatic warming in a permafrost environment. *Cold Regions Science and Technology,* **19**: 111.

Krieth, F. and F.E. Romie (1955) A study of the thermal diffusion equation with boundary conditions corresponding to solidification or melting of materials initially at the fusion temperature. *Proceedings of the Physical Society,* Section B, **68**: 277.

Kuo, Y.H. (1953) On the flow of an incompressible viscous fluid past a flat plate at moderate Reynolds number. *Journal of Mathematics and Physics,* **32**: 83.

Kuo, Y.H. (1956) Viscous flow along a flat plate moving at high supersonic speeds. *Journal of Aeronautical Science,* **23**: 125.

Lamé, G. and B.P.E. Clapeyron (1831) Memoire sur la solidification par refroidissement d'un globe liquide. *Annals of Chemistry and Physics,* **47**: 250–256.

Landau, H.G. (1950) Heat conduction in a melting solid. *Quarterly of Applied Mathematics,* **8**: 81.

Lighthill, M.J. (1949) A technique for rendering approximate solutions to physical problems uniformly valid. *Philosophy Magazine,* **40**: 1179.

Linstedt, A. (1882) Ueber die Integration einer fur die Strorungstheorie wichtigen Differentialgleichung. *Astron. Nach.,* **103**: 211.

Lock, G.S.H. (1969) On the use of asymptotic solutions to plane ice-water problems. *Journal of Glaciology,* **8**: 285.

Lock, G.S.H. (1971) On the perturbation solution of the ice-water layer problem. *International Journal of Heat and Mass Transfer,* **14**: 642.

Lock, G.S.H., J.R. Gunderson, D. Quon and J.K. Donnelly (1969) A study of one-dimensional ice formation with special reference to periodic growth and decay. *International Journal of Heat and Mass Transfer,* **12**: 1343.

Lock, G.S.H. and R.H. Nyren (1971) Analysis of fully-developed ice formation in a convectively-cooled circular tube. *International Journal of Heat and Mass Transfer,* **14**: 825.

Lozano, C.J. and R. Reemsten (1981) On Stefan problem with emerging free boundary. *Numerical Heat Transfer,* **4**: 239.

Lunardini, V.J. (1981) *Heat Transfer in Cold Climates.* New York: Van Nostrand Reinhold.

Lunardini, V.J. (1991) *Heat Transfer with Freezing and Thawing.* Amsterdam, The Netherlands: Elsevier Science Publishers.

Milanez, L.F. and J.L. Boldrini (1988) Perturbation solution for spherical solidification by convective cooling. *Journal of Thermophysics,* **2**: 88.

Milanez, L.F. and K.S.R. Ismail (1984) Inward solidification of spheres: numerical and experimental analysis. American Society of Mechanical Engineers, Paper No. 84–WA/HT–9.

Nayfeh, A.H. (1981) *Introduction to Perturbation Techniques.* New York: John Wiley.

Neumann, F. (ca. 1860) Lectures given in the 1860's, cf. Riemann-Weber, *Die partiellen Differentialgleichungen der mathematischen Physik. 5th edition,* 1912, vol. 2, pp. 117–121.

Ozisik, M.N. (1980) *Heat Conduction.* New York: John Wiley.

Parang, M., D.M. Crocker and B.D. Haynes (1990) Perturbation solution for spherical and cylindrical solidification by combined convective and radiative cooling. *International Journal of Heat and Fluid Flow,* **11**: 142.

Pedroso, R.I. and G.A. Domoto (1973a) Exact solution by perturbation method for planar solidification of a saturated liquid with convection at the wall. *International Journal of Heat and Mass Transfer,* **16**: 1816.

Pedroso, R.I. and G.A. Domoto (1973b). Technical note on planar solidification with fixed wall temperature and variable thermal properties. *Journal of Heat Transfer,* **99**: 553.

Pedroso, R.I. and G.A. Domoto (1973c) Perturbation solutions for spherical solidification of saturated liquids. *Journal of Heat Transfer,* **95**: 42.

Pedroso, R.I. and G.A. Domoto (1973d) Inward spherical solidification-solution by the method of strained coordinates. *International Journal of Heat and Mass Transfer,* **16**: 1037.

Poincaré, H. (1982) *New Methods of Celestial Mechanics.* Vol. I–III (English transl.), NASA TTF–450, 1967.

Pritulo, M.F. (1962) On the determination of uniformly accurate solutions of the differential

equations by the method of perturbation of coordinates. *Journal of Applied Mathematics and Mechanics*, **26**: 661.

Prud'homme, M., T.H. Nguyen and D.L. Nguyen (1989) A heat transfer analysis for solidification of slabs, cylinders, and spheres. *Journal of Heat Transfer*, **111**: 699.

Rathjen, K.A. and L.M. Jiji (1970) Transient heat transfer in fins undergoing phase transformation. *Proceedings of the Fourth International Heat Transfer Conference*, Section Cu 2.6, pp. 1–11.

Riley, D.S., F.T. Smith and G. Poots (1974) The inward solidification of spheres and circular cylinders. *International Journal of Heat and Mass Transfer*, **17**: 1507.

Schulze, H.J., P.M. Beckett, J.A. Howarth and G. Poots (1983) Analytical and numerical solutions to two-dimensional moving interface problems with applications to the solidification of killed steel ingots. *Proceedings of the Royal Society A, 385: 313.*

Seeniraj, R.V. and T.K. Bose (1982) One-dimensional phase change problems with radiation-convection. *Journal of Heat Transfer*, **104**: 1811.

Shanks, D. (1955) Nonlinear transformations of divergent and slowly convergent sequences. *Journal of Mathematics and Physics*, **34**: 1.

Siegel, R. and J.M. Savino (1966) An analysis of the transient solidification of a flowing warm liquid on a convectively cooled wall. *Proceedings of the Third International Heat Transfer Conference*, **4**: 141.

Soward, A.M. (1980) A unified approach to Stefan's problem for spheres and cylinders. *Proceedings of the Royal Society A*, **373**: 131.

Sparrow, E.M., J.W. Ramsey and R.G. Kemink (1979) Freezing controlled by natural convection. *Journal of Heat Transfer*, **101**: 578.

Stefan, J. (1891) Uber die Theorie des Eisbildung, insbesondere uber die Eisbildung im Polarmeere. *Annals of Physics and Chemistry*, Neue Folge, Bd. 42, Ht 2, 269–286.

Stephan, K. and B. Holzknecht (1974) Heat conduction in solidification of geometrically simple bodies. *Warme Stoffubertrag*, 7: 200.

Stewartson, K. and R.T. Waechter (1976) On Stefan's problem for spheres. *Proceedings of the Royal Society A*, **348**: 415.

Tao, L.C. (1967) Generalized numerical solution of freezing a saturated liquid in cylinders and spheres. *American Institute of Chemical Engineers Journal*, **13**: 165.

Tsien, H.S. (1956) The Poincaré-Lighthill-Kuo Method. *Advances in Applied Mechanics*, **4**: 281.

Van Dyke, M. (1975) *Perturbation Methods in Fluid Mechanics*. Annotated Edition. Stanford, California: Parabolic Press.

Weinbaum, S. and L.M. Jiji (1977) Singular perturbation theory for melting and freezing in finite domains initially not at the fusion temperature. *Journal of Applied Mechanics*, **34**: 25.

Westphal, K.O. (1967) Series solution of freezing problem with the fixed surface radiating into a medium of arbitrary varying temperature. *International Journal of Heat and Mass Transfer*, **10**: 195.

Yan, M.M. and P.N.S. Huang (1979) Perturbation solutions to phase change problem subjected to convection and radiation. *Journal of Heat Transfer*, **101**: 96.

Yeh, L.T. and B.T.F. Chung (1977) Phase change in a radiating and convecting medium with variable thermal properties. *Journal of Spacecraft and Rockets*, **5**: 178.

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13. ABSTRACT (Maximum 200 words)

Heat transfer with change of phase (freezing or melting) is important in numerous scientific and engineering applications. Since the pioneering works of Lamé and Clapeyron, Neumann and Stefan, a number of analytical and numerical techniques have been developed to deal with freezing and melting problems. One such analytical tool is the method of perturbation expansions, which is the main focus of this work. The report begins with a review of the perturbation theory and outlines the regular perturbation method, the method of strained coordinates, the method of matched asymptotic expansions, and the recently developed method of extended perturbation series. Next, the applications of these techniques to phase change problems in Cartesian, cylindrical, and spherical systems are discussed in detail. Although the bulk of the discussion is confined to one-dimensional situations, the report also includes two- and three-dimensional cases where admittedly the success of these techniques has so far been limited. The presentation is sufficiently detailed that even the reader who is unfamiliar with the perturbation theory can understand the material. However, at the same time, the discussion covers the latest literature on the subject and therefore should serve as a state-of-the-art review.

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