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Non-Uniqueness in Dynamic Rate-Independent Non-Associated Plasticity

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13. ABSTRACT *(Maximum 200 words)*

Ground shock calculations often use rate-independent plasticity models with non-associated flow. By means of a "constructive" mathematical proof, it is demonstrated here that this approach leads to multiple (and therefore spurious) solutions for dynamic initial/boundary value problems; in other words, uniqueness is lost. Although a simple class of multiple solutions is presented, the results are shown to be generally valid, implying that normality of flow is a necessary (as well as a sufficient) condition for uniqueness in dynamic applications of rate-independent plasticity.

This finding implies an inherent lack of robustness in numerical analyses utilizing models based on non-associated flow. Because the use of such models is widespread in ground shock (and structural) calculations, the fact that uniqueness breaks down is significant; it casts serious doubt on any (and every) calculation based on these constitutive representations. It is strongly recommended that all ground shock calculators promptly abandon their reliance on such models. To describe the situation most bluntly, these models do not provide a rational basis for the computations needed for prediction, design and/or analysis applications. In all cases for which an associated flow rule is deemed inadequate to fit observed material behavior, an approach other than rate-independent plasticity must be sought in order to represent such behavior in a reliable, self-consistent and rational manner.

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SUMMARY

Many ground shock calculations performed for the Defense Nuclear Agency utilize rate-independent plasticity models with non-associated (i.e., non-normal) flow rules to represent the behavior of geological materials. In such models the yield surface is used to limit the shear and/or tensile stresses, while the flow rule is used to independently define the direction of plastic straining (thereby limiting the dilatancy). The behavior of constitutive formulations of this kind in dynamic applications is analyzed in this report. By means of a "constructive" proof, it is demonstrated mathematically that these models lead to multiple, and therefore spurious, solutions in wave propagation situations; in other words, uniqueness of solution breaks down for dynamic initial/boundary value problems. Although a simple class of multiple solutions is presented to complete the proof for a very simple type of problem, the conclusion is quite general; it is relevant to all of the practically used ground shock (and some of the structural) plasticity models. The result implies that normality of flow is a necessary (as well as a sufficient) condition for uniqueness in dynamic applications of any rate-independent plasticity model.

Obviously, these findings also imply an inherent lack of robustness in numerical analyses based on non-associated plasticity. Because the use of these models is widespread in ground shock (and structural) calculations, the fact that they can produce multiple (and therefore spurious) solutions is significant; it casts serious doubt on the validity of the specific results of any (and every) calculation based on this kind of constitutive representation. Therefore it is strongly recommended that ground shock calculators promptly abandon their reliance on such models in all work. To describe the situation most bluntly, these models do not provide a rational basis for the computations needed for prediction, design and/or analysis applications. In all cases for which an associated flow rule is deemed inadequate to fit observed material behavior, an approach other than rate-independent plasticity must be sought in order to represent such behavior in a reliable, self-consistent and rational manner.

In view of the fact that non-associated plasticity models have already been extensively used in analyses for DNA, it is recommended that some effort be made to quantitatively assess the consequences of their use. In particular, the extent to which the model assumptions and parameters can affect the numerical solutions to various types of problems should be investigated.

CONVERSION TABLE

CONVERSION FACTORS FOR U.S. CUSTOMARY TO
METRIC (SI) UNITS OF MEASUREMENT

MULTIPLY →	***BY →***	***TO GET***
TO GET ←	***BY ←***	***DIVIDE***
bar	1.000 000 X E + 2	kilopascal (kPa)
degree Farenheit	$(t_{oF} - 32) / 1.8$	degrees Celsius
foot	3.048 000 X E - 1	meter (m)
foot-pound-force	1.355 818	joule (J)
inch	2.540 000 X E - 2	meter (m)
kip (1000 lbf)	4.448 222 X E + 3	newton (N)
kip/inch ^{2} (ksi)	6.894 757 X E + 3	kilopascal (kPa)
kips per foot	14.5932	kilonewtons per meter
kips per sq. ft. (ksf)	4.788 X E + 3	pascal (Pa)
ksi	6.894 757 X E + 6	pascal (Pa)
pound	0.4536	kilogram (kg)
pound-force (lbs avoirdupois)	4.448 222	newton (N)
pound-force inch	1.129 848 X E - 1	newton-meter (N · m)
pound-force/inch	1.751 268 X E + 2	newton/meter (N/m)
pound-force/foot ^{2}	4.788 026 X E - 2	kilopascal (kPa)
pound-force/inch ^{2} (psi)	6.894 757	kilopascal (kPa)
pressure (psi)	6.894 757 X E + 3	pascal (Pa)
square inch	6.452 X E - 4	meter ^{2} (m ^{2})
square foot	9.290 X E - 2	meter ^{2} (m ^{2})
slug	1.459 390 X E + 1	kilogram (kg)

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SECTION 1
INTRODUCTION

To represent geological materials in ground shock calculations, analysts often use rate-independent plasticity models which are non-associated, i.e., which utilize plastic strain increments not directed along the normal to the yield surface. Although such models are often fit to quasistatic material tests, they are used in dynamic ground shock analyses in a rate-independent formulation, with only the values of the material constants being adjusted to account for dynamic effects. In such models the yield surface is used to limit the shear and tensile stresses, while the flow rule is used to independently define the direction of plastic strain (and thereby limit the dilatancy exhibited by the model).

The wave propagation behavior of these models is analyzed in this report. Although a considerable amount of test data is consistent with non-associated plastic flow (Lade 1988)[1], it is shown below that the assumption that this applies for all strain rates is not appropriate in dynamic analysis. Viscosity, size effects, etc., in real materials may affect the flow direction for strain rates of importance; in fact such effects may preclude <u>any</u> plasticity-based formulation from working well. The present analysis proves that the rate-independent, non-associated plasticity used in practice is not rational (i.e., it leads to improperly posed problems possessing multiple - and therefore spurious - solutions) whenever inertia effects are included (as is always the case for wave propagation applications).

This report presents the general equations of rate-independent plasticity in Section 2. This is followed by a derivation of the plastic tangent stiffness tensor in Section 3, an analysis of certain infinitesimal load-unload cycles in Section 4, and a derivation of the general equations governing elastic-plastic wave propagation in Section 5. Finally, the results of Sections 2-5 are utilized in Section 6 to construct a simple initial/boundary value problem for which the existence of multiple solutions is easily demonstrated. It has long been known that associated flow is a sufficient condition for uniqueness in plasticity. This report proves that it is also a necessary condition for uniqueness for the rate-independent models used in practical analyses of ground shock.

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1 P.V. Lade, "Effects of Voids and Volume Changes on the Behavior of Frictional Materials", Intl. J. Num. Anal. Meth. Geomech., Vol. 12, No. 4, p.351-370, 1988.

SECTION 2
RATE-INDEPENDENT NON-ASSOCIATED PLASTICITY

Non-uniqueness will be proven by demonstrating that multiple solutions can occur during an infinitesimally short time interval *dt*. During such an interval the changes in the displacement field itself are infinitesimal. Therefore we may linearize the geometric treatment without loss of generality, provided that we adopt a reference configuration corresponding to the (possibly deformed) current geometry at the instant of interest. If we adopt such an "updated Lagrangian" approach, then we may utilize the linearized "small displacement" kinematic strain decomposition during the interval *dt*,

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{E} + \dot{\varepsilon}_{ij}^{P} \tag{2.1}$$

where the components of the strain rate tensor, denoted by $\dot{\varepsilon}_{ij}$, are measured with respect to a reference configuration which represents the geometry at the instant of interest. The superscripts denote the elastic and plastic strain rates, and the elastic strain is related to the stress through the elastic stiffness. (In this report we adopt the engineering mechanics convention that stresses and strains are positive in tension).

Aside from the kinematic decomposition of the deformation into elastic and plastic strains, all rate-independent plasticity models are characterized by means of four main features. These are the elastic behavior (to be discussed later), the yield condition, hardening (if any), and the flow rule. The yield condition can be written most generally as $f(\sigma_{ij}, \kappa_k)=0$, where σ_{ij} are the components of the stress tensor and the κ_k, $k=1...n$, denote the n hardening parameters in the model (which are absent in the special case of ideal plasticity). The yield condition may be considered a hypersurface, called the yield surface, in a six-dimensional stress space. (Each point in such a space corresponds to a stress state, and the yield surface contains those points corresponding to stresses at which plastic deformation can occur). Without loss of generality we can associate values $f<0$ with elastic states "within" the yield surface, while the region $f>0$ "outside" the yield surface is excluded by the theory. This exclusion implies that $f=0$ for as long as plastic yielding occurs, so that differentiation of the yield condition with time gives $\dot{f} = 0$, the "plastic compatibility condition"

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \kappa_k} \dot{\kappa}_k = 0 \tag{2.2}$$

which must be satisfied during plastic response. (In this report we adopt the notation $\dot{} \equiv \frac{d}{dt}$ and utilize the Einstein summation convention concerning subscripted indices, except where explicitly indicated otherwise).

Because hardening (if any) proceeds according to specified "hardening rules" as plastic straining occurs,

$$\dot{\kappa}_k = Y^k_{ij}\,\dot{\varepsilon}^P_{ij} \tag{2.3}$$

where the Y^k_{ij} are coefficients used to represent the hardening behavior. Equation (2.2) therefore becomes

$$n^Y_{ij}\,\dot{\sigma}_{ij} = H_{ij}\,\dot{\varepsilon}^P_{ij} \tag{2.4}$$

in which

$$n^Y_{ij} \equiv \frac{\partial f}{\partial \sigma_{ij}} \;\; ; \;\; \text{and} \;\; H_{ij} \equiv -\,Y^k_{ij}\,\frac{\partial f}{\partial \kappa_k} \tag{2.5}$$

It should be noted that the tensor n^Y_{ij} corresponds to the outward-drawn "normal" to the yield surface in stress space and possesses the symmetry $n^Y_{ij} = n^Y_{ji}$ (because of the symmetry of σ_{ij}).

To complete the model, the flow rule must be specified. This is done in terms of a "direction" of plastic strain n^P_{ij} so that the flow rule can be written generally as

$$\dot{\varepsilon}^P_{ij} = \lambda\, n^P_{ij} \tag{2.6}$$

in which, without loss of generality, n^P_{ij} is chosen so that $\lambda>0$ during plastic deformation ($\lambda=0$ during elastic deformation), and the magnitudes of n^P_{ij} and λ are scaled so that

$$n^P_{ij} n^P_{ij} = n^Y_{ij} n^Y_{ij} \tag{2.7}$$

If the "normality condition" $n_{ij}^{P} = n_{ij}^{Y}$ is satisfied, the flow rule is said to be normal, or associated; otherwise $n_{ij}^{P} \neq n_{ij}^{Y}$ and the flow rule is said to be non-associated. It should be noted that $n_{ij}^{P} = n_{ji}^{P}$ (because of the symmetry of ε_{ij}).

Summarizing, rate-independent plasticity models postulate that material deformation consists of elastic and plastic components, and all such models exhibit four features:

a) an elastic stiffness tensor that defines the functional relationship between the stress and the elastic strain,

b) a yield function that defines the plastic states of the material, and which is defined in terms of the stress as well as hardening parameters (except in the case of ideal plasticity),

c) hardening rules which specify the way the hardening parameters evolve,

d) a flow rule that specifies the "direction" of plastic straining.

Particular choices for the arbitrary functions in a-d define the individual models which are used in practical applications, and which make up the broad class of models examined in this report.

SECTION 3
DERIVATION OF THE TANGENT STIFFNESS TENSOR

We begin our analysis of the wave propagation characteristics of plasticity models by deriving the general form of the tangent (elastic-plastic) stiffness tensor. The incremental relationship between the stress and strain rates for elastic response is

$$\dot{\sigma}_{ij} = C_{ijkl}\,\dot{\varepsilon}^{E}_{kl} \tag{3.1}$$

where C_{ijkl} are the components of the (fourth rank) elastic stiffness tensor. It is well known that C_{ijkl} possesses the symmetries $C_{ijkl} = C_{ijlk} = C_{jikl} = C_{klij}$. In the plastic case substituting eq. (2.1) gives

$$\dot{\sigma}_{mn} = C_{mnpq}\,(\dot{\varepsilon}_{pq} - \dot{\varepsilon}^{P}_{pq}) \tag{3.2}$$

Multiplying this equation by n^{Y}_{mn} and substituting eqs. (2.4) and (2.6) leads to

$$\lambda H_{mn} n^{P}_{mn} = n^{Y}_{mn} C_{mnpq}(\dot{\varepsilon}_{pq} - \lambda n^{P}_{pq}) = \lambda H_{pq} n^{P}_{pq} \tag{3.3}$$

where some dummy indices have been interchanged. Equation (3.3) can be solved for λ,

$$\lambda = Q n^{Y}_{mn} C_{mnpq}\,\dot{\varepsilon}_{pq} \tag{3.4}$$

in which

$$Q = \frac{1}{\left(n^{Y}_{mn} C_{mnpq} n^{P}_{pq} + H_{pq} n^{P}_{pq}\right)} \tag{3.5}$$

In standard dynamic computational procedures the constitutive behavior is utilized to calculate the stresses from strains (which, in turn, are computed from the field equations). We therefore consider $\dot{\varepsilon}_{ij}$ to be the input here and $\dot{\sigma}_{ij}$ to be the desired output quantity. According to eq. (3.1) the quantity $C_{mnpq}\,\dot{\varepsilon}_{pq}$ represents the rate of change of stress in the case of elastic response (i.e., $\dot{\varepsilon}^{P}_{pq} = 0$), so the first of eqs. (2.5) implies that the scalar $n^{Y}_{mn}\,C_{mnpq}\,\dot{\varepsilon}_{pq}$ represents $\dot{f}^{E}$, the hypothetical rate of change of the value of the yield function in that case. Because the definition of f implies that a negative value of $\dot{f}^{E}$ would indeed be consistent with elastic behavior, plasticity can occur only for non-negative values of $\dot{f}^{E}$ (otherwise the stress response would not be unique). Furthermore, if $\dot{f}^{E}=0$, eq. (3.4) indicates that $\lambda = 0$, so that $\dot{\varepsilon}^{P}_{ij}=0$ from eq. (2.6). Therefore, $\dot{f}^{E}=0$ also implies that the response is elastic (this situation is called neutral loading). For these reasons, plastic yielding ($\dot{\varepsilon}^{P}_{ij} \neq 0$) will occur if and only if

$$n^{Y}_{mn}\, C_{mnpq}\, \dot{\varepsilon}_{pq} > 0 \tag{3.6}$$

Because $\lambda>0$ in this case, eq. (3.4) implies that $Q>0$. For consistency, therefore, n^{P}_{pq} must be defined so that the denominator of eq. (3.5) is always positive. Substituting eqs. (2.1), (2.6) and (3.4) into (3.1) gives the incremental stress-strain relation

$$\dot{\sigma}_{ij} = C_{ijkl}\, \dot{\varepsilon}_{kl} - QC_{ijkl}\, n^{P}_{kl}\, n^{Y}_{mn} C_{mnpq}\, \dot{\varepsilon}_{pq} \tag{3.7}$$

which (after interchanging the dummy indices k,l and p,q in the last term) gives

$$\dot{\sigma}_{ij} = C_{ijkl}\, \dot{\varepsilon}_{kl} - QC_{ijpq}\, n^{P}_{pq}\, n^{Y}_{mn} C_{mnkl}\, \dot{\varepsilon}_{kl} \tag{3.8}$$

Invoking the symmetry $C_{mnkl} = C_{klmn}$ leads to

$$\dot{\sigma}_{ij} = \left(C_{ijkl} - QC_{ijpq}\, n^{P}_{pq}\, C_{klmn}\, n^{Y}_{mn}\right)\dot{\varepsilon}_{kl} \tag{3.9}$$

The quantity in the parenthesis of eq. (3.9) is the desired tangent stiffness modulus, L_{ijkl}. It may be written more conveniently as

$$L_{ijkl} = C_{ijkl} - Q m^{P}_{ij} m^{Y}_{kl} \tag{3.10}$$

where

$$m^{P}_{ij} \equiv C_{ijpq}\, n^{P}_{pq} \quad ; \quad \text{and} \quad m^{Y}_{ij} \equiv C_{ijpq}\, n^{Y}_{pq} \tag{3.11}$$

Note that for associated flow $m^{P}_{ij} = m^{Y}_{ij}$ so that, by virtue of eq. (3.10), the tangent stiffness tensor possesses the "stress-strain" symmetry $L_{ijkl} = L_{klij}$. For general non-associated flow, however, eq. (3.10) implies that the tangent stiffness tensor does not possess this particular symmetry, although, of course, $L_{ijkl} = L_{jikl} = L_{ijlk}$ in all cases.

SECTION 4
WORK DONE IN AN INFINITESIMAL LOAD-UNLOAD CYCLE

Consider the response of a rate-independent plastic material subjected to an infinitesimal load-unload strain cycle. (In order to do this, we refine an analysis previously described in (Sandler and Rubin 1987)[2], which in turn was partially based on arguments first presented in (Il'iushin 1961)[3]). Refer to Figure 1, which shows a two-dimensional schematic representation of an infinitesimally small region of the six-dimensional stress space. Point O corresponds to any initial stress state σ_{ij}^O on the yield surface, represented locally by the line OY, while the inclined line OS represents (locally) the surface orthogonal to the "flow direction" n_{ij}^P (this surface is called the plastic potential surface). Let $\Delta\varepsilon_{kl}$ be an infinitesimal strain increment from the state O that causes the response to be plastic. The resulting stress, shown in Figure 1 at P, is given by

$$\sigma_{ij}^P = \sigma_{ij}^O + L_{ijkl}\Delta\varepsilon_{kl} \tag{4.1}$$

(The figure indicates some hardening of the yield surface during the loading increment, but this detail is not significant; for ideal plasticity P lies on OY). The unloading segment PQ, which corresponds to the strain increment $-\Delta\varepsilon_{kl}$, is elastic and results in the final stress

$$\sigma_{ij}^Q = \sigma_{ij}^P - C_{ijkl}\Delta\varepsilon_{kl} \tag{4.2}$$

The work (energy) required to deform a unit volume of the material through the load-unload strain cycle C ($\Delta\varepsilon_{ij}$ followed by $-\Delta\varepsilon_{ij}$) is given by

$$\Delta W \equiv \int_C \sigma_{ij} d\varepsilon_{ij} = \int_O^P \sigma_{ij} d\varepsilon_{ij} + \int_P^Q \sigma_{ij} d\varepsilon_{ij} \tag{4.3}$$

For the infinitesimal cycle $\pm\Delta\varepsilon_{ij}$ the Mean Value Theorem can be applied to obtain

2 I. Sandler and D. Rubin, "The Consequences of Non-Associated Plasticity in Dynamic Problems", in Constitutive Laws for Engineering Materials - Theory and Applications, Ed. by C.S. Desai, et al, Elsevier, p. 345-352, 1987.

3 A.A. Il'iushin, "On the Postulate of Plasticity", PMM, Vol. 25, No. 3, p. 503-507, 1961.

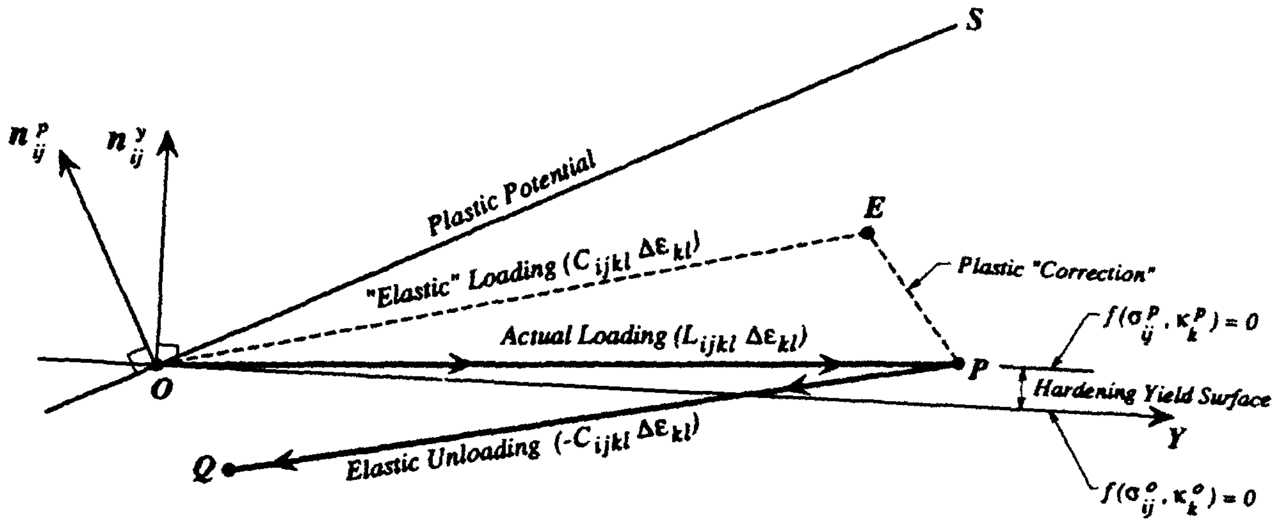

Figure 4.1. Depiction in stress space of the response of a rate-independent non-associated plasticity model to an infinitesmal strain cycle $\pm\Delta\varepsilon_{kl}$.

$$\Delta W = \frac{1}{2}\left(\sigma_{ij}^{O} + \sigma_{ij}^{P}\right)\Delta\varepsilon_{ij} + \frac{1}{2}\left(\sigma_{ij}^{P} + \sigma_{ij}^{Q}\right)(-\Delta\varepsilon_{ij}) = \frac{1}{2}\left(\sigma_{ij}^{O} - \sigma_{ij}^{Q}\right)\Delta\varepsilon_{ij} \tag{4.4}$$

Referring back to eqs. (3.10) and (4.1,4.2) this becomes

$$\Delta W = \frac{1}{2} Q m_{ij}^{P} m_{kl}^{Y} \Delta\varepsilon_{kl}\Delta\varepsilon_{ij} \tag{4.5}$$

Substitution of eqs. (3.11) finally gives

$$\Delta W = \frac{1}{2} Q\left(C_{ijpq} n_{pq}^{P} \Delta\varepsilon_{ij}\right)\left(C_{klrs} n_{rs}^{Y} \Delta\varepsilon_{kl}\right) \tag{4.6}$$

Note that for associated flow $n_{ij}^{P} = n_{ij}^{Y}$, so that ΔW can never be negative because the two expressions enclosed in parenthesis in eq. (4.6) are equal and Q is positive.

In the non-associated case, it is easy to find strain increments $\Delta\varepsilon_{kl}$ which satisfy the pair of inequalities

$$n_{ij}^{Y} C_{ijkl} \Delta\varepsilon_{kl} > 0 > n_{ij}^{P} C_{ijkl} \Delta\varepsilon_{kl} \tag{4.7}$$

For example, consider the strain increment defined by the equation

$$C_{ijkl} \Delta\varepsilon_{kl} = n_{ij}^{Y} - n_{ij}^{P} \tag{4.8}$$

In this case, using eq. (2.7), we can calculate the quantities

$$n_{ij}^{Y} C_{ijkl} \Delta\varepsilon_{kl} = n_{ij}^{Y} n_{ij}^{Y} - n_{ij}^{Y} n_{ij}^{P} = \frac{1}{2}\left(n_{ij}^{Y} n_{ij}^{Y} + n_{ij}^{P} n_{ij}^{P} - 2 n_{ij}^{Y} n_{ij}^{P}\right) = \frac{|n_{ij}^{Y} - n_{ij}^{P}|^{2}}{2} > 0 \tag{4.9}$$

$$n_{ij}^{P} C_{ijkl} \Delta\varepsilon_{kl} = n_{ij}^{P} n_{ij}^{Y} - n_{ij}^{P} n_{ij}^{P} = n_{ij}^{Y} n_{ij}^{P} - n_{ij}^{Y} n_{ij}^{Y} = -n_{ij}^{Y} C_{ijkl} \Delta\varepsilon_{kl} < 0 \tag{4.10}$$

so that eq. (4.7) will indeed be satisfied. A derivation of the complete (six-dimensional) set of $\Delta\varepsilon_{kl}$ for which eq. (4.7) holds is given in the Appendix.

Consider the geometric interpretation of eq. (4.7). Its first inequality merely implies, from eq. (3.6), that the strain increment $\Delta\varepsilon_{kl}$ causes plastic loading, with the quantity $C_{ijkl}\Delta\varepsilon_{kl}$ representing the hypothetical stress increment for elastic instead of plastic response. This hypothetical "elastic stress increment" is shown in Figure 1 as the dashed line OE. According to the second inequality in (4.7), the elastic stress increment must form a negative inner product with n_{ij}^{P}. Therefore (4.7) merely requires that the line OE lie within the "wedge" YOS. We have just shown that paths such as OE exist for non-associated flow, but (4.7) can never be satisfied in the associated case, because its left and right sides, which have opposite signs, then become equal. (In other words the "wedge" YOS in Figure 1 collapses into the single line OY when the flow rule is associated).

Let us now return to eq. (4.6). Inequalities (4.7) imply that the factors enclosed in parentheses in eq. (4.6) are of opposite sign, so that the work ΔW must be negative (because $Q > 0$). Therefore, instead of requiring that work be dissipated during all strain cycles involving plastic response, the non-associated material model actually supplies energy whenever the cycle $\pm\Delta\varepsilon_{ij}$ satisfies the inequalities (4.7). In the next two sections we will show how the possibility of such "energy generation" causes uniqueness to break down and introduces spurious signals into dynamic initial/boundary value problems.

SECTION 5
ELASTIC-PLASTIC WAVE PROPAGATION

In order to understand the behavior of non-associated plasticity models in dynamic problems, we will first analyze the propagation of simple plane waves. Consider plastic loading waves that propagate with wave speed c into material at rest in a homogeneous (uniform) stress/strain state O at yield, and produce motion behind the wave front in a single spatial direction (which will generally be different from the direction of propagation of the wave) for all of the particles. The situation is depicted in Figure 2, in which the unit vectors α_i and β_i represent, respectively, the (constant) direction of wave propagation (normal to the wave front) and the direction of particle motion (displacement, velocity and acceleration). As shown in the figure, the "local" coordinate system, dx_i, has its origin at an arbitrary point on the wave front at some arbitrarily chosen instant. Let $ds \equiv cdt - \alpha_j\, dx_j$ denote the infinitesimal distance of point dx_j behind the wave front at a later time dt. The particle displacement field $g(ds)\beta_i$ due to the wave produces the displacement field

$$u_i\,(dx_j\,,dt) = u_i^O(dx_j) + g\,(ds)\,\beta_i = u_i^O(0) + \frac{\partial u_i^O}{\partial x_j} dx_j + g\,(ds)\,\beta_i \tag{5.1}$$

where $u_i^O(dx_j)$ is the initial displacement field in strain state ε_{ij}^O, and g is any continuous, non-decreasing function of ds behind the wave front and zero ahead of it. Therefore $g > 0$ and $g' > 0$. The remaining relevant field quantities are derivable from u_i by means of the equations

$$v_i \equiv \frac{\partial u_i}{\partial t} = \dot{g}\,\beta_i = cg'\,\beta_i \tag{5.2}$$

$$a_i \equiv \frac{\partial v_i}{\partial t} = \ddot{g}\,\beta_i = c^2 g''\,\beta_i \tag{5.3}$$

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij}^O + \frac{1}{2}(-\alpha_j\,\beta_i - \alpha_i\,\beta_j\,)g' \tag{5.4}$$

$$\sigma_{ij} = \sigma_{ij}^O + L_{ijkl}\,(\varepsilon_{kl} - \varepsilon_{kl}^O\,) = \sigma_{ij}^O + \frac{1}{2}L_{ijkl}\,(-\alpha_l\,\beta_k - \alpha_k\,\beta_l\,)g' = \sigma_{ij}^O - L_{ijkl}\,\alpha_k\,\beta_l\,g' \tag{5.5}$$

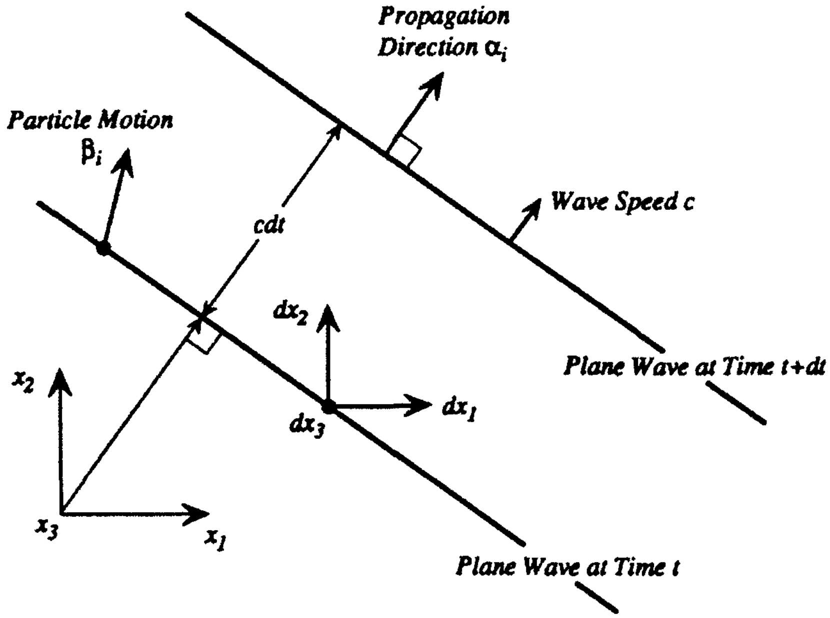

Figure 5.1. Plane wave propagating in α_i direction with particle motion in β_i direction.

where v_i and a_i are the velocity and acceleration, respectively, σ_{ij}^O is the initial (uniform) stress, and $' \equiv \frac{d}{ds}$. (To obtain the above equations the chain rule,

$$\frac{\partial g}{\partial x_j} = \frac{dg}{ds}\frac{\partial s}{\partial x_j} = -\alpha_j\, g' \ ; \ \text{ and } \ \frac{\partial g}{\partial t} = \frac{dg}{ds}\frac{\partial s}{\partial t} = c\, g' \tag{5.6}$$

as well as the "strain" symmetry $L_{ijkl} = L_{ijlk}$ were invoked). The equation of motion is

$$\rho a_i = \frac{\partial \sigma_{ij}}{\partial x_j} \tag{5.7}$$

in which ρ is the density of the material. Substituting eqs. (5.3,5.5,5.6) into eq. (5.7) gives

$$\rho c^2 g'' \beta_i = L_{ijkl}\, \alpha_k\, \beta_l\, \alpha_j\, g'' \tag{5.8}$$

behind the wave front. By interchanging dummy subscripts in the last equation we may write

$$g''\left(\alpha_k\, L_{iklj}\, \alpha_l\, \beta_j - \rho c^2 \beta_i\right) = g''\left(\alpha_k\, L_{iklj}\, \alpha_l - \rho c^2 \delta_{ij}\right)\beta_j = 0 \tag{5.9}$$

in which δ_{ij} is the Kronecker delta (unity if i=j, zero if $i \neq j$). For a (non-trivial) wave solution the "loading rate" g'' will not be zero, so that wave propagation requires that

$$\left(A_{ij} - M\, \delta_{ij}\right)\beta_j = 0 \tag{5.10}$$

in which

$$A_{ij} \equiv \alpha_k\, L_{iklj}\, \alpha_l \ ; \ \text{ and } \ M \equiv \rho c^2 \tag{5.11}$$

Equations (5.10,5.11), which must be satisfied for a physical wave to exist, form an eigenvalue problem. The three eigenvalues M of the 3×3 matrix A_{ij} represent the values of the quantity ρc^2 for which wave propagation can occur (i.e., for which non-zero eigenvectors β_i are possible). These values determine the wavespeeds c which are characteristic of the material (at its current state O) for the direction of propagation α_i, while the three eigenvectors β_i correspond to the directions of the particle motion in the different types of wave which may occur. We will return to this point later.

Equations (5.1) through (5.11) apply as well to elastic wave propagation if the tangent stiffness L_{ijkl} is replaced by the elastic stiffness C_{ijkl}. Define

$$A^E_{ij} \equiv \alpha_k\, C_{iklj}\, \alpha_l = \alpha_k\, C_{kijl}\, \alpha_l = \alpha_k\, C_{jlki}\, \alpha_l = \alpha_l\, C_{jkli}\, \alpha_k = \alpha_k\, C_{jkli}\, \alpha_l = A^E_{ji} \tag{5.12}$$

so that the eigenvalue problem corresponding to elastic waves is

$$A^E_{ij}\, \beta_j = M^E \beta_i \tag{5.13}$$

where M^E is an eigenvalue of A^E_{ij}. Because C_{ijkl} is positive definite, the matrix A^E_{ij} is also positive definite and symmetric in view of eq. (5.12); therefore it has three real and positive eigenvalues, denoted by M^E_k and a corresponding set of orthonormal eigenvectors $\underline{z}_k$. If we define z_{ki} as the i th component of $\underline{z}_k$, we can write

$$A^E_{ij}\, z_{kj} = M^E_k\, z_{ki} \quad \text{(no sum on } k\text{)} \;\; ; \quad \text{where } \; M^E_1 \geq M^E_2 \geq M^E_3 > 0 \; \text{ and } \; z_{ki} z_{li} = \delta_{kl} \tag{5.14}$$

Therefore three different elastic waves are possible (for $k = 1,2$ or 3) in which $\beta_i = z_{ki}$. Also, note that $-\underline{z}_k$ can be chosen instead of $\underline{z}_k$ as an eigenvector of A^E_{ij}; we shall choose each $\underline{z}_k$ in such a way that $z_{ki}\alpha_i \geq 0$.

As an example, let us consider linear isotropic elasticity. In this simple case the elastic stiffness is expressible as

$$C_{iklj} = (K - \frac{2G}{3})\delta_{ik}\delta_{lj} + G(\delta_{il}\delta_{kj} + \delta_{ij}\delta_{kl}) \tag{5.15}$$

where $K > 0$ and $G > 0$ are the bulk and shear moduli of the material, so that

$$A^E_{ij} = \alpha_k\, C_{iklj}\, \alpha_l = (K - \frac{2G}{3})\, \alpha_i\, \alpha_j + G\, \alpha_j\, \alpha_i + G\, \delta_{ij}\, \alpha_k\, \alpha_k$$

$$= (K + \frac{G}{3})\, \alpha_i\, \alpha_j + G\, \delta_{ij} \tag{5.16}$$

because $\alpha_k\, \alpha_k = |\alpha_k|^2 = 1$. Multiplying eq. (5.16) by z_{kj} and subtracting $M^E_k\, z_{ki}$ from both sides gives

$$A^E_{ij}\, z_{kj} - M^E_k\, z_{ki} = (K + \frac{G}{3})\, \alpha_i\, (\alpha_j\, z_{kj}\,) + (G - M^E_k) z_{ki} = 0 \qquad \text{(no sum on } k\text{)} \tag{5.17}$$

which, after multiplying by α_i, leads to

$$A^E_{ij}\, \alpha_i\, z_{kj} - M^E_k\, \alpha_i\, z_{ki} = (K + \frac{4G}{3} - M^E_k)(\alpha_i\, z_{ki}\,) = 0 \quad \text{(no sum on } k\text{)} \tag{5.18}$$

so that either $M^E_k = K + \frac{4G}{3}$ or $\alpha_i\, z_{ki} = 0$ must be satisfied. For the first possibility, which - for reasons that will become apparent shortly - we will label k=1, substitution for M^E_1 into eq. (5.17) implies that $\alpha_i\, (\alpha_j\, z_{1j}\,) = z_{1i}$ so that $z_{1i}\, \alpha_i\, (\alpha_j\, z_{1j}\,) = (\alpha_j\, z_{1j}\,)^2$ must be equal to $z_{1i}\, z_{1i} = \delta_{11} = 1$. Because we have chosen z_{1i} so that $z_{1i}\alpha_i \geq 0$, $z_{1i}\alpha_i = +1$, implying that $z_{1i} = \alpha_i$; such a wave is said to be a longitudinal, primary or P wave. For the second possibility, $\alpha_i\, z_{ki} = 0$ implies that z_{ki} is perpendicular to α_i so that such a wave is a transverse, shear, secondary or S wave; in this case eq. (5.17) implies $M^E = G$. Because two orthogonal eigenvectors may be chosen in the plane perpendicular to α_i, $M^E = G$ represents a double root and each line in the plane represents one of the possible directions of polarization of the S wave. Therefore, z_{2i} can lie in any direction perpendicular to α_i with z_{3i} perpendicular to both α_i and to z_{2i}. Summarizing the isotropic linearly elastic case, the wave speeds must satisfy the inequalities $M^E_1 = K + \frac{4G}{3} > M^E_2 = M^E_3 = G > 0$, and the direction of particle motion z_{1i} must coincide with α_i, while z_{2i} and z_{3i} can be chosen as any two mutually perpendicular directions in the plane perpendicular to α_i. We also note that the strict inequality $M^E_1 > M^E_2$ appears to hold for all elastic solid behavior (anisotropic as well as isotropic); we will utilize this fact later.

We now return to the analysis of plastic loading waves, as described by eqs. (5.10,5.11). Applying eqs. (3.10) and (5.12) leads to

$$A_{ij} = \alpha_k\, C_{iklj}\, \alpha_l - Q\, \alpha_k\, m^P_{ik}\, m^Y_{lj}\, \alpha_l = A^E_{ij} - Q\, \alpha_k\, m^P_{ik}\, m^Y_{lj}\, \alpha_l \tag{5.19}$$

so that

$$A_{ij}\, \beta_j = A^E_{ij}\, \beta_j - Q\, m^P_{ik}\, \alpha_k\, (m^Y_{lj}\, \alpha_l\, \beta_j) = M\, \beta_i \tag{5.20}$$

Define a new quantity γ_i as the solution of the system of linear equations

$$z_{ji}\,\gamma_j = \beta_i \tag{5.21}$$

so that, in view of the last of eqs. (5.14), $\gamma_p = \delta_{pj}\,\gamma_j = z_{pi}\,z_{ji}\,\gamma_j = z_{pi}\,\beta_i$. Multiplying eq. (5.20) by z_{qi} and substituting eq. (5.21) leads to

$$z_{qi}\,A^E_{ij}\,z_{pj}\,\gamma_p - Q\,z_{qi}\,m^P_{ik}\,\alpha_k\,(m^Y_{lj}\,\alpha_l\,z_{pj}\,\gamma_p\,) = M\,z_{qi}\,z_{pi}\,\gamma_p = M\,\delta_{qp}\,\gamma_p \tag{5.22}$$

From the first of eqs. (5.14)

$$z_{qi}\,A^E_{ij}\,z_{pj} = z_{qi}\,M^E_p\,z_{pi} = M^E_p\,\delta_{pq} \qquad \text{(no sum on } p\text{)} \tag{5.23}$$

so that, eq. (5.22) implies

$$[\,M - M^E_q\,]\,\gamma_q = Q\,z_{qi}\,m^P_{ik}\,\alpha_k\,(-m^Y_{lj}\,\alpha_l\,z_{rj}\,\gamma_r\,) \qquad \text{(no sum on } q\text{)} \tag{5.24}$$

or

$$\gamma_q = \frac{Q\,z_{qi}\,m^P_{ik}\,\alpha_k}{M - M^E_q}\,a \qquad \text{(no sum on } q\text{)} \tag{5.25}$$

where a represents the term in parenthesis in eq. (5.24). Now

$$a \equiv -m^Y_{kl}\,\alpha_k\,z_{pl}\,\gamma_p = -C_{klij}\,n^Y_{ij}\,\alpha_k\,\beta_l = -n^Y_{ij}\,\frac{1}{2}\left(C_{ijkl} + C_{ijlk}\right)\alpha_k\,\beta_l$$

$$= n^Y_{ij}\,C_{ijkl}\,\frac{1}{2}\,(-\alpha_k\,\beta_l - \alpha_l\,\beta_k\,) = n^Y_{ij}\,C_{ijkl}\,(\varepsilon_{kl} - \varepsilon^O_{kl})/g' > 0 \tag{5.26}$$

where the last inequality is inferred from eq. (3.6) together with $g' > 0$ for the plastic loading wave. Furthermore, from eqs. (5.25,5.26),

$$-a = m^Y_{ij}\,\alpha_i\,z_{pj}\,\gamma_p = \sum_{p=1}^{3}(\alpha_i\,m^Y_{ij}\,z_{pj}\,)(\,z_{pl}\,m^P_{lk}\,\alpha_k\,)\,\frac{Qa}{M - M^E_p} \tag{5.27}$$

Because $a > 0$ for plastic loading, eq. (5.27) gives

$$\frac{p_1 y_1}{M - M_1^E} + \frac{p_2 y_2}{M - M_2^E} + \frac{p_3 y_3}{M - M_3^E} + \frac{1}{Q} = 0 \tag{5.28}$$

in which

$$p_i \equiv - \alpha_k m_{kl}^P z_{il} \quad \text{and} \quad y_i \equiv - \alpha_k m_{kl}^Y z_{il} \tag{5.29}$$

Equation (5.28) actually represents the secular or characteristic equation for the eigenvalues M_i in the plastic loading wave; it is equivalent to a cubic equation for M. The quantities p_i and q_i depend upon the direction of propagation and on the *elastic* eigenvectors corresponding to that direction.

Let us replace the unknown plastic wave speed M in eq. (5.28) by $\tau = M - M_1^E$ and let $M_1^E - M_2^E = b$ and $M_1^E - M_3^E = c$, where b and c are positive. After clearing of fractions eq. (5.28) becomes a cubic in τ,

$$\tau^3 + [Q(p_i\, y_i)+b+c]\tau^2 + [Qp_1\, y_1\, (b+c) + Qp_2\, y_2\, c + Qp_3\, y_3\, b]\tau + Qp_1\, y_1\, bc = 0 \tag{5.30}$$

The sign of the constant term in a real cubic equation is always opposite to that of at least one of the roots (which, of course, must be real). Therefore, if $p_1 y_1$ is negative (which cannot occur in the associated case, because then $p_1 = y_1$), at least one value of τ must be positive. This means that eq. (5.28) possesses a real root $M > M_1^E$, so that the corresponding plastic wave travels faster than any of the elastic waves. It was shown in (Sandler and Rubin 1987) how such a fast plastic wave speed leads to non-uniqueness; that argument will be reformulated in the next section, but first we need to determine the conditions under which $p_1 y_1$ is negative. In order to do this, note that

$$y_1 = - \alpha_k m_{kl}^Y z_{1l} = n_{ij}^Y C_{ijkl}(-\alpha_k z_{1l}) \tag{5.31}$$

and

$$p_1 = - \alpha_k m_{kl}^P z_{1l} = n_{ij}^P C_{ijkl}(-\alpha_k z_{1l}) \tag{5.32}$$

Comparing eqs. (5.31,5.32) with eq. (4.7), it is clear that the condition $p_1 y_1 < 0$ is equivalent to requiring that the "stress increment" associated with the "strain path" $-\alpha_k z_{1l}$ be within the wedge in Figure 1. For this reason, any value of α_i for which p_1 and y_1 have opposite signs can be used to construct a "strain path" in the wedge, and represents a direction of propagation for which plastic waves travel faster than elastic waves.

In order to prove that such propagation directions α_i always exist for the non-associated flow models used in practical applications, we must introduce a new concept and definition. Note that in all such practical plasticity models the yield surface limits the shear stresses at many (if not all) stress states, while the flow rule permits the relatively large shear strains associated with yield or failure. Let us describe the tensor m_{ij}^Y (or m_{ij}^P) as "shear-like" if at least one of its eigenvalues is positive and at least one is negative. In other words, one of the principal directions of a shear-like tensor represents tension (or extension) while another represents compression (or contraction). With this definition we can say that all practical plasticity models have many stress states for which m_{ij}^Y and/or m_{ij}^P is shear-like.

Let m_{kl} be any shear-like tensor and consider the value of the product $\alpha_k\, m_{kl}\, z_{1l}$ when α_k lies in one of the principal directions of m_{kl}. Then $\alpha_k\, m_{kl} = \lambda_m\, \alpha_l$, where λ_m is the eigenvalue of m_{kl} corresponding to the chosen principal direction. The product

$$\alpha_k\, m_{kl}\, z_{1l} = \lambda_m\, \alpha_l\, z_{1l} \tag{5.33}$$

has the same sign as the eigenvalue λ_m, because we have chosen the sign of $\underline{z}_1$ in such a way that $\alpha_l\, z_{1l} > 0$. (We ignore the possibility $\alpha_l\, z_{1l} = 0$ because it arises only for degenerate elastic behavior; note that $\alpha_l\, z_{1l} = +1$ for isotropic elasticity). Because m_{kl} is shear-like, the three λ_m have both positive and negative eigenvalues among them, so that the product $\alpha_k\, m_{kl}\, z_{1l}$ (which is a continuous function of α_k) must pass through zero as α_k spans all possible spatial directions. Therefore, for all practical rate-independent plasticity models, there exist states O and directions α_k for which y_1 (and/or p_1) passes through zero. (For example, in the isotropic elastic case $z_{1i} = \alpha_i$, so that y_1 and p_1 are the simple quadratic forms $-\alpha_i m_{ij}^Y \alpha_j$ and $-\alpha_i m_{ij}^P \alpha_j$, respectively; since m_{ij}^Y and m_{ij}^P are shear-like, these forms must pass through zero as α_i spans the directions between those corresponding to the maximum and minimum principal values of m_{ij}^Y and m_{ij}^P). It is clear that in the non-associated case, $m_{ij}^P \neq m_{ij}^Y$, the directions corresponding to $y_1 = 0$ will be different from those for $p_1 = 0$. (This can be rigorously proven by means of a straightforward, but lengthy, argument).

As a consequence, the product $p_1 y_1$ must take on negative values (because its factors pass through zero one at a time). Therefore in all practical non-associated plasticity models there will exist states O and directions α_i for which plastic waves can propagate with a sp--d consistent with $M > M_1^E$.

SECTION 6
FAST PLASTIC WAVES AND NON-UNIQUENESS

We now proceed in a manner similar to (Sandler and Rubin 1987) in order to show that the existence of plastic waves with $M > M_1^E$ implies non-uniqueness. We will do this by constructing multiple (actually, infinitely many) solutions to a single initial/boundary value problem. Consider a half space of material initially at rest (at yield state O) but subjected to a prescribed boundary traction which varies linearly with time in such a way that it can produce a response involving only an elastic (i.e., unloading) primary wave. We shall show that this problem has alternative solutions (involving plastic response) if the normal α_i to the boundary plane lies in a direction for which $p_1 \, y_1 < 0$ in state O.

The general nature of these "spurious" (but nevertheless mathematically valid) alternative solutions is shown in Figure 3. In this figure the abscissa is $\alpha_i \, x_i$, which represents the position of any point measured from the boundary plane of the half-space, $\alpha_i \, x_i = 0$. The ordinate is time, t, measured from some arbitrarily chosen instant t=0. The initial conditions at $t = 0$ are $\sigma_{ij} = \sigma_{ij}^O$ and $v_i = 0$ with the boundary traction $\alpha_j \, \sigma_{ij}$ at the equilibrium value $\alpha_j \, \sigma_{ij}^O$. The half-space is subjected to the traction boundary condition (which holds along the ordinate in Figure 3), $\alpha_j \, \sigma_{ij} = \alpha_j \, \sigma_{ij}^O - B p_1 \, z_{1i} \, t$ where B is an arbitrarily prescribed positive constant. As shown in the figure, the alternative solutions that we are about to construct for this problem consist of a quiescent Zone 1, followed by the front OF (propagating with wave speed c) of a unidirectional plastic loading wave in Zone 2, followed in turn by a transition OV (which travels with speed V) to elastic behavior in Zones 3,4,5 and 6. The lines OP, OS and OT separating the elastic zones are characteristics whose slopes are the inverses of the three elastic wave speeds. For simplicity, we will construct only continuous, piecewise-linear alternative solutions, i.e., those for which the velocity vector and stress tensor are linear functions of x_i and of t in each of the zones shown in Figure 3.

The unidirectional plastic loading wave in Zone 2 has the motion direction β_i defined by eqs. (5.21,5.25). For a piecewise linear solution the loading rate g'' in Zone 2 is a positive constant, so that $g' > 0$ varies linearly with distance behind the wave front. From eqs. (5.2,5.5), v_i, and σ_{ij} vary linearly with distance behind the wave front,

$$v_i = c \, \beta_i \, g' = c \, z_{mi} \, \gamma_m \, g'' s \qquad (6.1)$$

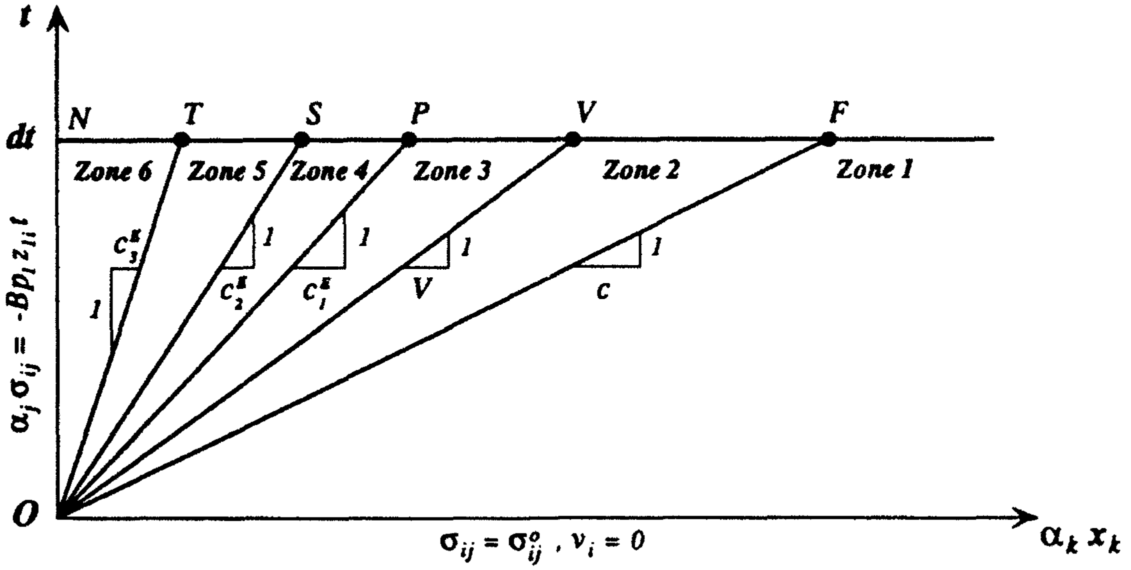

Figure 6.1. Characteristic space-time construction showing spurious waves (Resulting From Homogenous Initial / Boundary Conditions) :

Zone 1 is quiescent

Zone 2 involves plastic loading

Zones 3, 4, 5, and 6 involve elastic unloading

c = *Plastic Loading Wave Speed*

V = *Speed of Elastic-Plastic Interface*

c_1^E = *Primary Elastic Wave Speed*

c_2^E = *Secondary Elastic Wave Speed*

c_3^E = *Tertiary Elastic Wave Speed*

$$\sigma_{ij} = \sigma_{ij}^{O} - L_{ijkl}\, \alpha_k\, z_{ml}\, \gamma_m\, g'' s \tag{6.2}$$

where $s \equiv ct - \alpha_p\, x_p$ is the distance behind the wave front. Along the plastic-elastic interface, line OV in Figure 3, the velocity v_i^V and the stress σ_{ij}^V may be obtained by setting $t = \alpha_p\, x_p/V$ to get

$$v_i^V = c(\frac{c}{V} - 1) g'' \gamma_m \alpha_p\, x_p\, z_{mi} \tag{6.3}$$

$$\sigma_{ij}^V = \sigma_{ij}^O - (\frac{c}{V} - 1) g'' \gamma_m\, L_{ijkl}\, \alpha_k\, \alpha_p\, x_p\, z_{ml} \tag{6.4}$$

The governing equations for the elastic zones consist of the equation of motion,

$$\frac{\partial v_i}{\partial t} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} \tag{6.5}$$

and the constitutive relation,

$$\frac{\partial \sigma_{ij}}{\partial t} = C_{ijkl} \frac{1}{2} \left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) = C_{ijkl} \frac{\partial v_l}{\partial x_k} \tag{6.6}$$

Let R_m and $S_m$be six arbitrary real constants, and let $T_{ij} = T_{ji}$ be another six real constants which satisfy the three equations

$$T_{ij}\, \alpha_j = 0 \tag{6.7}$$

It is easy to show (by direct substitution) that the functions

$$v_i = (R_m\, t + S_m\, \alpha_p\, x_p\,) z_{mi} \tag{6.8}$$

and

$$\sigma_{ij} = \sigma_i^O + (T_{ij} + R_m\, \rho z_{mi}\, \alpha_j\,)\, \alpha_p\, x_p + S_m\, C_{ijkl}\, \alpha_k\, z_{ml}\, t \tag{6.9}$$

identically satisfy the governing eqs. (6.5,6.6), if one invokes the identity

$$\frac{\partial}{\partial x_j} = \frac{\partial(\alpha_p\, x_p\,)}{\partial x_j} \frac{\partial}{\partial(\gamma_p\, x_p)} = \alpha_j \frac{\partial}{\partial(\alpha_p\, x_p)} \tag{6.10}$$

and uses eq. (6.7). Given suitable auxiliary conditions (such as initial or boundary conditions for v_i and σ_{ij}), the relationships (6.7,6.8,6.9) form a system of twelve equations which determine the twelve constants R_m, $S_m$andT_{ij}. We can eliminate the T_{ij} by multiplying eq. (6.9) by α_j and using eqs. (6.7) and (5.12,5.14) to get

$$\alpha_j \left(\sigma_{ij} - \sigma_{ij}^O \right) = \sum_{m=1}^{3} (R_m \, \rho \alpha_p \, x_p + S_m \, M_m^E t \,) z_{mi} \tag{6.11}$$

Now eqs. (6.8,6.11) form a system of six equations for the R_m and S_m, with the T_{ij} being obtainable directly from eq. (6.9) after R_m and S_m are determined. The new system can be further simplified by multiplying each equation by z_{ni} (uncoupling the terms corresponding to different values of m) to get

$$tR_n + \alpha_p \, x_p \, S_n = v_i \, z_{ni} \tag{6.12}$$

$$\alpha_p \, x_p \, R_n + (c_n^E)^2 \, tS_n = \alpha_j \, (\sigma_{ij} - \sigma_{ij}^O) z_{ni}/\rho \qquad \text{(no sum on } n) \tag{6.13}$$

in which $c_n^E \equiv \sqrt{M_n^E/\rho}$ is the n^{th} elastic wave speed.

Along line OV in Figure 3, where $\alpha_p \, x_p = Vt$, the values of v_i and σ_{ij} in eqs. (6.12,6.13) must equal v_i^V and σ_{ij}^V as given by eqs. (6.3,6.4). Therefore, by utilizing eqs. (5.11) we get (after some algebra),

$$R_n = -\frac{cV + (c_n^E)^2}{V^2 - (c_n^E)^2} c(c-V) g'' \gamma_n \qquad \text{(no sum on } n) \tag{6.14}$$

$$S_n = \frac{c^2 - V^2}{V^2 - (c_n^E)^2} \, cg'' \gamma_n \qquad \text{(no sum on } n) \tag{6.15}$$

in Zone 3.

In the remainder of the elastic region, R_n and S_n must be constant except along the characteristics $(\alpha_p \, x_p)^2 = (c_n^E \, t)^2$, where eqs. (6.12,13) become singular. Along these lines the theory of hyperbolic partial differential equations permits the "constants" to have different values in the two regions on opposite sides of a characteristic as long as the system (6.12,6.13) remains satisfied. In particular, along $\alpha_p x_p = c_n^E t$ the system becomes

$$R_n + c_n^E S_n = v_i z_{ni}/t \qquad \text{(no sum on } n\text{)} \tag{6.16}$$

$$R_n + c_n^E S_n = \alpha_j (\sigma_{ij} - \sigma_{ij}^O) z_{ni}/(\rho c_n^E t) \qquad \text{(no sum on } n\text{)} \tag{6.17}$$

Because the left sides of eqs. (6.16,6.17) involve only the combination $(R_n + c_n^E S_n)$, the individual values of the constants may be discontinuous across the characteristic line if the combination is preserved. Let $\overline{R}_n$ and $\overline{S}_n$ denote the values of the constants behind the n^{th} characteristic. Then

$$\overline{R}_n + c_n^E \overline{S}_n = R_n + c_n^E S_n = -\frac{c-c_n^E}{V+c_n^E} c(c-V) g'' \gamma_n \qquad \text{(no sum on } n\text{)} \tag{6.18}$$

As each of the characteristics OP, OS and OT in Figure 3 is crossed in passing from Zone 3 to Zone 6, another pair R_n, S_n jumps to $\overline{R}_n$, $\overline{S}_n$. For Zone 6 eq. (6.9) implies

$$\alpha_j (\sigma_{ij} - \sigma_{ij}^O) = (\overline{R}_m \rho \alpha_p x_p + \overline{S}_m M_m^E t) z_{mi} \qquad \text{for} \quad 0 \le \alpha_p x_p < c_3^E t \tag{6.19}$$

The boundary condition for the problem at hand is $\alpha_j (\sigma_{ij} - \sigma_{ij}^O) = -Bp_1 z_{1i} t$ on $\alpha_p x_p = 0$ for all $t \ge 0$ (i.e., along line ON in Figure 3). Because the z_{mi} are linearly independent vectors, eq. (6.19) can satisfy this boundary condition if, and only if, $\overline{S}_2$, and $\overline{S}_3$ are zero and

$$\overline{S}_1 M_1^E = -Bp_1 \tag{6.20}$$

The elastic region is therefore characterized by the following sets of constants:

R_1, S_1, R_2, S_2, R_3 and S_3	in Zone 3
$\overline{R}_1, \overline{S}_1, R_2, S_2, R_3$ and S_3	in Zone 4
$\overline{R}_1, \overline{S}_1, \overline{R}_2, 0, R_3$ and S_3	in Zone 5
$\overline{R}_1, \overline{S}_1, \overline{R}_2, 0, \overline{R}_3$ and 0	in Zone 6

At this point we have completed the construction of a set of alternative solutions to the simple initial/boundary value problem posed at the beginning of this section. However, we have not yet proven that the stress given by eq. (6.9) is actually consistent with the assumed elastic behavior in Zones 3, 4, 5 and 6. In order to do this, note that eq. (6.9) gives the rate of change of stress in elastic regions as

$$\dot{\sigma}_{ij} = E_m\, C_{ijkl}\, \alpha_k\, z_{ml} \tag{6.21}$$

in which

$$E_m = \begin{cases} \dfrac{c^2 - V^2}{V^2 - (c_m^E)^2}\, c g''\, \gamma_m & \text{when} \quad \dfrac{\alpha_p x_p}{V} < t < \dfrac{\alpha_p x_p}{c_m^E} \\ \left.\begin{array}{ll} -Bp_1/M_1^E & \text{for} \quad m = 1 \\ 0 & \text{for} \quad m = 2 \text{ or } 3 \end{array}\right\} & \text{when} \quad t > \dfrac{\alpha_p x_p}{c_m^E} \end{cases} \tag{6.22}$$

(E_m is undefined at $t = \dfrac{\alpha_p x_p}{c_m^E}$).

The loading rate relative to the yield surface, $\dot{f}^{\mathrm{E}}$, is

$$n_{ij}^Y\, \dot{\sigma}_{ij} = E_m\, n_{ij}^Y\, C_{ijkl}\, \alpha_k\, z_{ml} = E_m\, m_{ij}^Y\, \alpha_k\, z_{ml} = -E_m\, y_m \tag{6.23}$$

In Zone 3 this becomes,

$$n_{ij}^Y\, \dot{\sigma}_{ij} = Qacg'' \sum_{m=1}^{3} \left[\frac{c^2 - V^2}{V^2 - (c_m^E)^2} \right] \frac{p_m\, y_m}{M - M_m^E} \tag{6.24}$$

after using eqs. (5.25,5.29). The sign of the $m = 1$ term on the right side of eq. (6.24) is negative because Q, a, c and g'' are all positive, $c > V > c_1^E > 0$ (so that the expression in brackets is positive), and $p_1\, y_1$ is negative. Furthermore, the bracketed expression can be made arbitrarily large in magnitude for $m = 1$ by choosing the value of V to be only slightly larger than c_1^E. Because c_2^E and c_3^E are less than c_1^E, the $m = 1$ term will then dominate eq. (6.24) and determine its sign. Therefore values of V can be chosen for which the loading rate is negative in Zone 3 (and therefore consistent with the presumed elastic behavior of Figure 3).

In Zones 4, 5 and 6 the loading rate is, according to eqs. (6.23,6.22),

$$n_{ij}^{Y}\,\dot{\sigma}_{ij} = Bp_1\,y_1/M_1^E + Ng'' \tag{6.25}$$

in which

$$N = \begin{cases} Qac\displaystyle\sum_{m=2}^{3}\frac{c^2 - V^2}{V^2 - (c_m^E)^2}\,\frac{p_m\,y_m}{M - M_m^E} & \text{in Zone 4} \\[2ex] Qac\dfrac{c^2 - V^2}{V^2 - (c_3^E)^2}\,\dfrac{p_3\,y_3}{M - M_3^E} & \text{in Zone 5} \\[2ex] 0 & \text{in Zone 6} \end{cases} \tag{6.26}$$

Now, the value of g'' can be chosen small enough so that the first term on the right side of eq. (6.25) dominates the second. Then, because B and M_1^E are positive and $p_1\,y_1$ is negative, the loading rates in Zones 4, 5 and 6 are negative (and the response is elastic in those zones as required in Figure 3).

Summarizing, we have constructed (and confirmed the validity of) multiple solutions to a simple initial/boundary value problem. These solutions represent alternatives to the simple solution (involving only primary elastic unloading waves and corresponding to $g'' = 0$). The existence of such multiple solutions (involving appropriate values of V and $g'' > 0$) proves that non-associated flow leads to non-uniqueness. Because the plasticity formulation analyzed here is completely general, it extends over the entire class of rate-independent plasticity models used for practical ground shock calculations.

SECTION 7
CONCLUSION

Many ground shock calculations performed for the Defense Nuclear Agency utilize rate-independent plasticity models with non-associated (i.e., non-normal) flow rules to represent the behavior of geological materials. In such models the yield surface is used to limit the shear and/or tensile stresses, while the flow rule is used to independently define the direction of plastic straining (thereby limiting the dilatancy). The behavior of constitutive formulations of this kind in dynamic applications has been analyzed in this report. By means of a "constructive" proof, it has been demonstrated mathematically that such models lead to multiple, and therefore spurious, solutions in wave propagation situations; in other words, uniqueness of solution breaks down for dynamic initial/boundary value problems.

Although a simple class of multiple solutions was presented to complete the proof for a very simple type of problem, the conclusion is quite general; the proof makes no specific modeling assumptions, except rate-independence, shear-like plasticity and non-singular elastic behavior (and is therefore relevant to all of the ground shock, and some of the structural, plasticity models used in practice). The loss of uniqueness is found to be closely related to the energy generating properties of the models according to Il'iushin's postulate (even though the second law of thermodynamics may be satisfied). The result implies that normality of flow is a necessary (as well as a sufficient) condition for uniqueness in dynamic applications of any rate-independent plasticity model.

Obviously, these findings imply an inherent lack of robustness in numerical analyses based on non-associated plasticity. Because the use of these models is widespread in ground shock (and structural) calculations, the fact that they can produce multiple (and therefore spurious) solutions is significant; it casts serious doubt on the validity of the specific results of any (and every) calculation based on this kind of constitutive representation. Therefore it is strongly recommended that DNA promptly abandon its reliance on such models in all work performed for the Agency. To describe the situation most bluntly, these models do not provide a rational basis for the computations needed for prediction, design and/or analysis applications. In all cases for which an associated flow rule is deemed inadequate to fit observed material behavior, an approach other than rate-independent plasticity must be sought in order to represent such behavior in a reliable, self-consistent and rational manner.

In view of the fact that non-associated plasticity models have already been extensively used in analyses for DNA, it is recommended that some effort be made to quantitatively assess the consequences of their use. In particular, the extent to which the model assumptions and parameters can affect the numerical solutions to various types of problems should be investigated.

APPENDIX
DERIVATION OF THE STRAIN INCREMENTS SATISFYING INEQUALITY (4.7)

In this Appendix we present the entire six-dimensional set of strain increments $\Delta\varepsilon_{kl}$ which satisfy eq. (4.7) whenever the flow rule is non-associated. Consider the ratio of each of the components n_{ij}^{Y} to the corresponding component n_{ij}^{P}. Either these ratios are all the same (i.e. independent of i and j) or two (or more) of them differ. If the former possibility holds, we can let r be the common ratio, so that $n_{ij}^{Y} = r\, n_{ij}^{P}$ for all i,j. But in that case $n_{ij}^{Y} n_{ij}^{Y} = r^2\, n_{ij}^{P} n_{ij}^{P}$, so that $r = \pm 1$. When $r = +1$ the flow rule is associated, so this situation is not relevant here. For $r = -1$, $n_{ij}^{Y} = -n_{ij}^{P}$, and eq. (4.7) holds for <u>any</u> $\Delta\varepsilon_{kl}$ satisfying eq. (3.6). The only remaining possibility in the case of non-associated flow is that two (or more) of the six ratios n_{ij}^{Y}/n_{ij}^{P} are different. Then we can select at least two pairs of subscripts m,n and p,q in such a way that

$$\frac{n_{mn}^{Y}}{n_{mn}^{P}} \neq \frac{n_{pq}^{Y}}{n_{pq}^{P}} \quad \text{(no "sum" on } m,n \text{ or on } p,q\text{)} \tag{A.1}$$

Let h_1, h_2, h_3 and h_4 be four arbitrary real numbers. Form a symmetric tensor h_{ij} which has zero components in the m,n and p,q positions and has h_1, h_2, h_3 and h_4 as the other four independent components (in any order). Also define $h^{Y} \equiv n_{ij}^{Y} h_{ij}$, $h^{P} \equiv n_{ij}^{P} h_{ij}$ and D as

$$D \equiv n_{mn}^{Y} n_{pq}^{P} - n_{pq}^{Y} n_{mn}^{P} \neq 0 \tag{A.2}$$

Then two additional arbitrary real numbers, μ and ν, together with h_1, h_2, h_3 and h_4 define a six-parameter family of "elastic stress increments"

$$C_{ijkl}\Delta\varepsilon_{kl} \equiv \mu[(1-\nu)n_{ij}^{Y} - (1+\nu)n_{ij}^{P}] + h_{ij} + (h^{Y} n_{mn}^{P} - h^{P} n_{mn}^{Y})U_{ijpq} - (h^{Y} n_{pq}^{P} - h^{P} n_{pq}^{Y})U_{ijmn} \tag{A.3}$$

in which $U_{ijmn} \equiv \frac{1}{2D}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$. By multiplying eq. (A.3) by n_{ij}^{Y} and n_{ij}^{P}, respectively, it is straightforward to verify that eq. (4.7) is satisfied for all $\mu > 0$ and $|\nu| < \dfrac{1 - n_{ij}^{Y} n_{ij}^{P}}{1 + n_{ij}^{Y} n_{ij}^{P}}$.

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