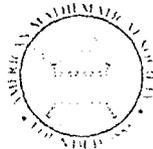


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## Inverse Scattering and Applications

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## Inverse Scattering and Applications

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on Inverse Scattering on the Line  
held June 7–13, 1990  
at the University of Massachusetts, Amherst  
with support from the National Science Foundation,  
the National Security Agency,  
and the Office of Naval Research

D. H. Sattinger  
C. A. Tracy  
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## Preface

This conference covered a variety of topics in inverse problems: inverse scattering problems on the line; inverse problems in higher dimensions; inverse conductivity problems; and numerical methods. In addition, problems from statistical physics were covered, including monodromy problems, quantum inverse scattering, and the Bethe ansatz. One of the aims of the conference was to bring together researchers in a variety of areas of inverse problems. All of these areas have seen intensive activity in recent years.

### Inverse conductivity problems

This class of problems was discussed by David Isaacson and Margaret Cheney of Rensselaer Polytechnic Institute and by Gunther Uhlmann of the University of Washington. Uhlmann discussed his work with John Sylvester on the problem of determining anisotropic conductivities in a region from measurements made on the boundary. These measurements may include the Dirichlet-Neumann map or knowledge of the geodesics. Margaret Cheney discussed various algorithms for reconstructing the conductivities from the data: these included iterative methods, and Calderon's methods. David Isaacson discussed experimental work being carried out at Rensselaer Polytechnic Institute and ended his talk with an intriguing videotape of actual inverse imaging experiments on a human subject (himself).

Adrian Nachman, of the University of Rochester, gave an overview of inverse scattering and conductivity problems. Joyce McLaughlin, of Rensselaer Polytechnic Institute, presented recent results on inverse spectral problems for second order differential operators.

### Numerical methods

Vladimir Rokhlin of Yale University described a numerical algorithm for inverse scattering based on a Riccati equation for the impedance function combined with certain trace formulae for the unknown functions. Numerical experiments performed in one dimension have shown themselves to be stable, rigorous, and extremely efficient. He hopes to be able to extend the methods to two and three dimensional problems.

### Soliton problems

One dimensional inverse scattering methods are a fundamental tool in the theory of completely integrable systems. Percy Deift of the Courant Institute opened the

conference with a beautiful summary of the theory of inverse scattering for  $n$ th order ordinary differential operators. Thanks to recent work by Xin Zhou and Deift, this theory is now complete. Thomas Kappeler of Brown University discussed action angle variables for the periodic KdV equation. Richard Beals of Yale University spoke on his recent work with D. Sattinger on action angle variables for integrable systems based on first order  $n \times n$  isospectral operators. The construction of action angle variables for these infinite dimensional completely integrable systems is based on the scattering transform.

Scattering theory was also used by Bjorn Birnir of University of California, Santa Barbara and S. Kichenessamy of the Courant Institute in their (independent) work showing that only the Sine-Gordon equation can support breather solutions.

M. Wickerhauser of the University of Georgia reported on joint work with R. Coifman of Yale University on some of the special problems of the scattering transform for the Benjamin-Ono equation. Their work gives estimates for some previously formal work associated with the Benjamin-Ono hierarchy.

S. Venakides of Duke University reported on joint work with P. Deift of the Courant Institute and R. Oba of Tulane University on the Toda Shock problem. Long time asymptotic analysis of the explicit solution is carried out by the inverse scattering method. Residual oscillations are derived and analyzed when the initial velocity exceeds a critical value. The results are in agreement with earlier numerical experiments by Straub and Holian, and Flaschka and McLaughlin.

David McLaughlin of Princeton University discussed chaos and heteroclinic orbits of perturbed integrable systems.

### **Three dimensional problems**

A. Ramm of Kansas State University and T. Aktosun of the University of Texas at Dallas presented their work on three dimensional problems. Ramm talked about the C Property and Aktosun talked on the Wiener-Hopf factorization of the scattering operator in three dimensions, based on ideas of R. Newton.

### **Statistical physics**

A number of problems in statistical physics lead to problems involving inverse monodromy or inverse scattering, and several of the talks addressed these areas. V. Korepin, of the University of New York at Stonybrook, discussed correlation functions for the quantized version of the nonlinear Schrödinger equation. In many cases, the correlation functions satisfy nonlinear differential equations of Painlevé type. The Painlevé equations, in turn, are associated in a direct way with certain monodromy problems; in fact, the monodromy problems play a role analogous to the isospectral operators in the theory of completely integrable systems. Inverse monodromy problems thus play an important role. John Palmer of the University of Arizona talked about the Cauchy Riemann operators associated with such inverse monodromy problems and their infinite dimensional determinants as tau functions

for the problem. The tau functions are in fact the partition function of statistical mechanics. Hank Thacker of the University of Virginia talked about related topics including spin chains and vertex models. Craig Tracy spoke on monodromy problems in higher dimensions, specifically some isomonodromy problems for the Laplacian on the Poincaré disk. The two point correlation function can be expressed in terms of Painlevé VI.

During the course of the conference, Persi Diaconis, who was attending the other conference at Amherst, overheard mention of the "Bethe ansatz" during an informal discussion at coffee break. It developed that there was a connection between the order/disorder transitions in "card shuffling" problems that Diaconis has been working on, and the Bethe ansatz method used in connection with the statistical problems being discussed by Korepin and Thacker. Diaconis agreed to give a special lecture, at 8:30 a.m. Sunday morning, on his work on order/disorder transitions. Several discussions resulted, and a round table session took place on Monday evening to understand the relationships.

D. H. Sattinger

WIENER-HOPF FACTORIZATION  
IN MULTIDIMENSIONAL INVERSE SCHRÖDINGER SCATTERING<sup>1</sup>

Tuncay Aktosun<sup>2</sup> and Cornelis van der Mee<sup>3</sup>

ABSTRACT. We consider a Riemann-Hilbert problem arising in the study of the inverse scattering for the multidimensional Schrödinger equation with a potential having no spherical symmetry. It is shown that under certain conditions on the potential, the corresponding scattering operator admits a Wiener-Hopf factorization. The solution of the Riemann-Hilbert problem can be obtained using a similar factorization for the unitarily dilated scattering operator. We also study the connection between the Wiener-Hopf factorization and the Newton-Marchenko integral operator.

1. RIEMANN-HILBERT PROBLEM IN QUANTUM SCATTERING. Consider the  $n$ -dimensional Schrödinger equation ( $n \geq 2$ )

$$(1.1) \quad \Delta\psi + k^2\psi = V(x)\psi$$

where  $x \in \mathbf{R}^n$ ,  $\Delta$  is the Laplacian,  $k^2$  is energy, and  $V(x)$  is the potential. In nonrelativistic quantum mechanics the behavior of a particle in the force field of  $V(x)$  is governed by (1.1).

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Mathematics Subject Classification (1991). 35Q15, 81U05, 35R30, 47A40.

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<sup>3</sup>The author is supported by the National Science Foundation under grant DMS 8823102.

We assume that  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  in some sense which will be made precise in the next paragraph, but we do not assume any spherical symmetry for  $V(x)$ . As  $|x| \rightarrow \infty$ , the wavefunction  $\psi$  behaves as

$$\psi(k, x, \theta) = e^{ik\theta \cdot x} + ie^{-\frac{\pi}{4}i(n-1)} e^{ik|x|} |x|^{\frac{1-n}{2}} A(k, \frac{x}{|x|}, \theta) + o(|x|^{-\frac{1-n}{2}})$$

where  $\theta \in S^{n-1}$  is a unit vector in  $\mathbf{R}^n$  and  $A(k, \theta, \theta')$  is the scattering amplitude. The scattering operator  $S(k)$  is defined as

$$S(k, \theta, \theta') = \delta(\theta - \theta') + i \left(\frac{k}{2\pi}\right)^{\frac{n-1}{2}} A(k, \theta, \theta').$$

where  $\delta$  is the Dirac delta distribution. In operator notation the above equation becomes

$$S(k) = \mathbf{I} + i \left(\frac{k}{2\pi}\right)^{\frac{n-1}{2}} A(k).$$

All our results presented in this paper hold for real and locally square-integrable potentials  $V(x) \in L^2_{\text{loc}}(\mathbf{R}^n)$  belonging to the class  $\mathbf{B}_\alpha$  with  $0 \leq \alpha < 2$ . Here  $\mathbf{B}_\alpha$ ,  $\alpha \in [0, 2)$ , denotes the class of potentials such that for some  $s > \frac{3}{2} - \frac{1}{n}$ ,  $(1 + |x|^2)^s V(x)$  is a bounded linear operator from  $H^\alpha(\mathbf{R}^n)$  into  $L^2(\mathbf{R}^n)$ , where  $H^\alpha(\mathbf{R}^n)$  denotes the Sobolev space of order  $\alpha$ . For the reader whose interest is restricted to the case  $n = 3$ , the following conditions on the potential will be sufficient:

1. There exist positive constants  $a$  and  $b$  such that for all  $y \in \mathbf{R}^3$  we have

$$\int_{\mathbf{R}^3} dx |V(x)| \left( \frac{|x| + |y| + a}{|x - y|} \right)^2 \leq b.$$

2. There exist constants  $c > 0$  and  $s > 1/2$  such that  $|V(x)| \leq c(1 + |x|^2)^{-s}$  for all  $x \in \mathbf{R}^3$ .
3. There exist constants  $\gamma > 0$  and  $\beta \in (0, 1]$  such that  $\int_{\mathbf{R}^3} dx |x|^\beta |V(x)| < \gamma$ .
4.  $k = 0$  is not an exceptional point. This condition is satisfied if there are neither bound states nor half-bound states at zero energy.

The inverse quantum scattering problem consists of recovering the potential  $V(x)$  for all  $x$  when  $S(k)$  is known for all  $k$ . Information about molecular, atomic, and subatomic particles is usually obtained from scattering experiments. An important problem in physics

is to understand the forces between these particles. Solving the inverse scattering problem is equivalent to the determination of the force from the scattering data. For a review of the methods and open problems for 3-D inverse scattering prior to 1989 we refer the reader to [Ne89] and [CS89]. None of the methods developed to solve the multidimensional inverse problem have led to a complete and satisfactory solution yet, but there has been a lot of progress made in this research area especially during the last ten years. The methods to solve the multidimensional inverse scattering problem include the Newton-Marchenko method [Ne80, Ne81, Ne82], the generalized Gel'fand-Levitan method [Ne74, Ne80, Ne81, Ne82], the  $\bar{\delta}$  method [NA84, BC85, BC86, NH87], the generalized Jost-Kohn method [Pr69, Pr76, Pr80, Pr82], a method that uses the Green's function of Faddeev [Fa65, Fa74, Ne85], and the generalized Muskhelishvili-Vekua method [AV91b]. The principal idea behind the methods of Newton-Marchenko, generalized Gel'fand-Levitan, and generalized Muskhelishvili-Vekua is to formulate the inverse scattering problem as a Riemann-Hilbert problem and to transform this latter problem into an integral equation that uses the scattering data in its kernel and its inhomogeneous term. Then, the potential is recovered from the solution of the integral equation. Here we will solve the same Riemann-Hilbert problem by using a Wiener-Hopf factorization for operator functions utilizing some results of Gohberg and Leiterer [GL73].

In the Schrödinger equation  $k$  appears as  $k^2$ , and as a result  $\psi(-k, x, \theta)$  is also a solution whenever  $\psi(k, x, \theta)$  is a solution. These two solutions are related to each other by the functional equation [Ne80]

$$\psi(k, x, \theta) = \int_{S^{n-1}} d\theta' S(k, -\theta, \theta') \psi(-k, x, \theta')$$

or equivalently

$$(1.2) \quad f_+(k, x, \theta) = \int_{S^{n-1}} d\theta' G(k, x, \theta, \theta') f_-(k, x, \theta'), \quad k \in \mathbf{R}$$

where

$$f_{\pm}(k, x, \theta) = e^{\mp ik\theta \cdot x} \psi(\pm k, x, \pm\theta)$$

and

$$(1.3) \quad G(k, x, \theta, \theta') = e^{-ik(\theta - \theta') \cdot x} S(k, -\theta, -\theta').$$

For potentials specified in the beginning of this section, in the absence of bound states,  $f_{\pm}$  has an analytic extension in  $k \in \mathbf{C}^{\pm}$ . If there are bound states, these can be removed by the reduction technique [Ne89] before the analysis is carried out. Let us suppress the  $x$ -dependence and write (1.2) in vector form as

$$f_+(k) = G(k)f_-(k), \quad k \in \mathbf{R}.$$

or equivalently as

$$(1.4) \quad X_+(k) = G(k)X_-(k) + [G(k) - \mathbf{I}]\hat{1}, \quad k \in \mathbf{R}.$$

where

$$X_{\pm}(k) = f_{\pm}(k) - \hat{1}.$$

For potentials considered in this paper  $X_{\pm} \in L^2(S^{n-1})$ , the Hilbert space of square integrable functions on  $S^{n-1}$ , and the strong limit of  $f_{\pm}$  is  $\hat{1}$  as  $k \rightarrow \infty$  in  $\mathbf{C}^{\pm}$ . Note that in our notation  $\mathbf{I}$  denotes the identity operator on  $L^2(S^{n-1})$  and  $\hat{1}$  denotes the vector in  $L^2(S^{n-1})$  such that  $\hat{1}(\theta) = 1$  for  $\theta \in S^{n-1}$ . Hence, (1.4) constitutes a Riemann-Hilbert problem: Given  $G(k)$ , determine  $X_{\pm}(k)$ . Note also that from (1.3) it is seen that  $G(k)$  is the unitarily dilated scattering operator.

2. SOLUTION OF THE RIEMANN-HILBERT PROBLEM. We have the following result concerning the Wiener-Hopf factorization of the operator  $G(k)$  that appears in the Riemann-Hilbert problem (1.4). In order to keep the discussion short, we assume that there are no bound states. If there are bound states, these can be removed by a reduction technique [Ne80, Ne89] before the factorization is accomplished. For details we refer the reader to [AV90].

**THEOREM 1.** *For potentials as specified in Section 1,  $G(k)$  defined in (1.3) has a (left) Wiener-Hopf factorization; i.e., there exist operators  $G_+(k)$ ,  $G_-(k)$ , and  $D(k)$  such that  $G(k) = G_+(k)D(k)G_-(k)$  where*

1.  $G_+(k)$  is continuous in  $\mathbf{C}^+$  in the operator norm of  $\mathcal{L}(L^2(S^{n-1}))$  and is boundedly invertible there. Here  $\mathcal{L}(L^2(S^{n-1}))$  denotes the Banach space of linear operators acting on

$L^2(S^{n-1})$ . Similarly,  $G_-(k)$  is continuous in  $\mathbf{C}^-$  in the operator norm of  $\mathcal{L}(L^2(S^{n-1}))$  and is boundedly invertible there.

2.  $G_+(k)$  is analytic in  $\mathbf{C}^+$  and  $G_-(k)$  is analytic in  $\mathbf{C}^-$

3.  $G_+(\pm\infty) = G_-(\pm\infty) = \mathbf{I}$ .

4.  $D(k) = P_0 + \sum_{j=1}^m \left(\frac{k-i}{k+i}\right)^{\rho_j} P_j$ , where  $P_1, \dots, P_m$  are mutually disjoint, rank-one projections, and  $P_0 = \mathbf{I} - \sum_{j=1}^m P_j$ . The (left) partial indices  $\rho_1, \dots, \rho_m$  are nonzero integers. In case there are no partial indices: i.e., when  $D(k) = \mathbf{I}$ , the resulting Wiener-Hopf factorization becomes canonical.

Note that, as seen from (1.3),  $G(k)$  is a unitary transform of the scattering operator  $S(k)$ . In particular, when  $x = 0$ ,  $G(k)$  reduces to  $S(k)$ . The proof of Theorem 1 uses some results of Gohberg and Leiterer regarding factorization of operator functions on contours in the complex plane [GL73]. When  $S(k)$  is boundedly invertible, is a compact perturbation of the identity, and  $\tilde{S}(\xi) = S(i\frac{1+\xi}{1-\xi})$  is uniformly Hölder continuous on the unit circle  $\mathbf{T}$  in the complex plane, its unitary transform  $G(k)$  also satisfies these three conditions and admits a Wiener-Hopf factorization. The Hölder-continuity of  $\tilde{S}(\xi)$  and  $\tilde{G}(\xi) = G(i\frac{1+\xi}{1-\xi})$  can be established using either an additive representation of the scattering amplitude or a multiplicative representation. We refer the reader to [AV90] for the proof that uses an additive representation of the scattering amplitude and to [AV91a] for the proof that uses a multiplicative representation of the scattering amplitude. The conditions on the potential in 3-D specified in Section 1 were used in the additive representation, and the conditions specified in that section in  $n$ -D were used in the multiplicative representation. We also refer the reader to [Ne90] for various results related to the Wiener-Hopf factorization of the scattering operator: in this reference Professor Newton introduced a related factorization called the Jost function factorization and studied the relationship between these two factorizations: in this reference Theorem 5.1 gives a characterization of the scattering operator for the existence of a potential.

The solution of the Riemann-Hilbert problem (1.4) is obtained in terms of the Wiener-Hopf factors of  $G(k)$  [AV90] and is given as

$$(2.1) \quad X_+(k) = [G_+(k) - \mathbf{I}] \hat{\mathbf{I}} + G_+(k) \sum_{\rho_j > 0} \frac{\phi_j(k)}{(k+i)^{\rho_j}} \pi_j$$

$$(2.2) \quad X_-(k) = [G_-(k)^{-1} - \mathbf{I}]\hat{\mathbf{1}} + G_-(k)^{-1} \sum_{\rho_j > 0} \frac{\phi_j(k) \pi_j + [(k+i)^{\rho_j} - (k-i)^{\rho_j}] P_j \hat{\mathbf{1}}}{(k-i)^{\rho_j}},$$

provided  $P_j \hat{\mathbf{1}} = 0$  whenever  $\rho_j < 0$ . Here  $\pi_j$  is a fixed nonzero vector in the range of  $P_j$ , and  $\phi_j(k)$  is an arbitrary polynomial of degree less than  $\rho_j$  associated with each  $\rho_j > 0$ . We can state our result as follows.

**THEOREM 2.** *For potentials as specified in Section 1, the Riemann-Hilbert problem (1.4) has a solution if and only if  $P_j \hat{\mathbf{1}} = 0$  whenever  $\rho_j < 0$ . When this happens, the solution is given by (2.1) and (2.2). The solution, if it exists, is unique when the operator  $G(k)$  has no positive partial indices.*

A simple condition that assures the unique solvability of the Riemann-Hilbert problem (1.4) is given by  $\sup_{k \in \mathbf{R}} \|S(k) - \mathbf{I}\| < 1$ , where the norm is the operator norm in  $\mathcal{L}(L^2(S^{n-1}))$ . If this holds, neither the scattering operator  $S(k)$  nor its unitary transform  $G(k)$  has any partial indices. As a result, in this case, (1.4) is uniquely solvable.

**3. PARTIAL INDICES.** In this section we relate the partial indices of the unitarily dilated scattering operator given in (1.3) to the Newton-Marchenko integral operator [Ne89]. We also discuss the relationship between solutions of the Riemann-Hilbert problem and the Newton-Marchenko integral equation. The proofs of the results stated in this section will be published elsewhere.

We let  $Q$  be the operator on  $L^2(S^{n-1})$  such that  $(Qf)(\theta) = f(-\theta)$ . As in [Ne89] we define

$$(3.1) \quad \mathbf{G}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik\alpha} [G(k) - \mathbf{I}]Q$$

and we also define the operators  $\mathcal{G}$ ,  $\mathcal{G}^*$ , and  $\mathcal{H}^*$  on  $L^2(\mathbf{R}^+)$

$$(3.2) \quad (\mathcal{G}\eta)(\alpha) = \int_0^{\infty} d\beta \mathbf{G}(\alpha + \beta) \eta(\beta), \quad \alpha > 0$$

$$(3.3) \quad (\mathcal{G}^*\eta)(\alpha) = \int_0^{\infty} d\beta \mathbf{G}(-\alpha - \beta) \eta(\beta), \quad \alpha > 0$$

$$(\mathcal{H}^*\eta)(\alpha) = \int_0^{\infty} d\beta \mathbf{G}(-\alpha + \beta) \eta(\beta), \quad \alpha > 0.$$

The Fourier transform maps  $L^2(\mathbf{R}^+)$  onto the Hardy space of analytic operator functions  $X_+(k)$  on  $\mathbf{C}^+$  such that

$$\sup_{\epsilon > 0} \int_{-\infty}^{\infty} dk \|X_+(k + i\epsilon)\|_{L^2(S^{n-1})}^2 < +\infty.$$

We will denote this Hardy space by  $\mathbf{H}_2^+$ .

Defining

$$\begin{aligned} \eta(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik\alpha} X_+(k) \\ f(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik\alpha} [G(k) - \mathbf{I}]\hat{\mathbf{i}}. \end{aligned}$$

from (1.4) we obtain

$$(3.4) \quad \eta(\alpha) = \int_{-\infty}^{\infty} d\beta \mathbf{G}(\alpha + \beta) \eta(\beta) + Q\eta(-\alpha) + f(\alpha), \quad \alpha \in \mathbf{R}.$$

Since  $X_+ \in \mathbf{H}_2^+$ , we have  $\eta(\alpha) = 0$  for  $\alpha < 0$ . Hence, we see that (3.4) is equivalent to

$$(3.5) \quad \begin{cases} \eta(\alpha) = \int_0^{\infty} d\beta \mathbf{G}(\alpha + \beta) \eta(\beta) + f(\alpha), & \alpha > 0 \\ 0 = \int_0^{\infty} d\beta \mathbf{G}(-\alpha + \beta) \eta(\beta) + Q\eta(\alpha) + f(-\alpha), & \alpha > 0. \end{cases}$$

We can write (3.5) in the form

$$(3.6) \quad \begin{cases} \eta = \mathcal{G}\eta + f \\ (Q + \mathcal{H}^*)\eta = -f^*, \end{cases}$$

where  $f^*(\alpha) = f(-\alpha)$ . Since (1.4) and (3.6) are equivalent, it follows that every solution  $X_+ \in \mathbf{H}_2^+$  of the Riemann-Hilbert problem (1.4) leads to a solution  $\eta \in L^2(\mathbf{R}^+)$  of (3.6), and conversely. The first equation in (3.6) is the Newton-Marchenko integral equation and  $\mathcal{G}$  is the Newton-Marchenko integral operator.

Since  $\tilde{G}(\xi)$  is Hölder continuous on  $\mathbf{T}$ ,  $\tilde{G}(\xi) - \mathbf{I}$  is a compact operator, and  $\tilde{G}(\xi)$  is boundedly invertible for all  $\xi \in \mathbf{T}$ , it follows that  $G(k)$  has a (left) Wiener-Hopf factorization [GL73, AV90, AV91a]. In that case, we can solve the Riemann-Hilbert problem (1.4) in terms of the Wiener-Hopf factors of  $G(k)$  and obtain the following [AV90, AV91a].

**PROPOSITION 3.** *There are finitely many, namely  $\sum_{\rho_j > 0} \rho_j$ , linearly independent solutions of the homogeneous problem (1.4) where  $F(k) \equiv 0$ . The inhomogeneous terms  $F(k)$*

for which at least one solution of the Riemann-Hilbert problem (1.4) exists, form a closed subspace of  $L^2(\mathbf{R})$  of co-dimension equal to  $-\sum_{\rho_j < 0} \rho_j$ .

Due to the fact that (1.4) and (3.6) are equivalent problems, the above results imply that for all  $f, f^* \in L^2(\mathbf{R}^+)$ , we have the following.

**COROLLARY 4.** *There are  $\sum_{\rho_j > 0} \rho_j$  linearly independent solutions  $\eta$  of the homogeneous problem  $(Q + \mathcal{H}^*)\eta = 0$ . The right-hand sides  $-f^*$  for which at least one solution  $\eta$  of the equation  $(Q + \mathcal{H}^*)\eta = -f^*$  exists, form a closed subspace of  $L^2(\mathbf{R}^+)$  of finite co-dimension equal to  $-\sum_{\rho_j < 0} \rho_j$ .*

The partial indices of the operator  $G(k)$  given in (1.3) is related to the Newton-Marchenko operator  $\mathcal{G}$  as in the following theorem. Note that  $\mathcal{G}$  and  $\mathcal{G}^*$  are defined in (3.2) and (3.3).

**THEOREM 5.** *The partial indices of  $G(k)$  satisfy*

$$\begin{aligned} \sum_{\rho_j > 0} \rho_j &= \dim \text{Ker} (\mathbf{I} - \mathcal{G}) + \dim \text{Ker} (\mathbf{I} + \mathcal{G}), \\ - \sum_{\rho_j < 0} \rho_j &= \dim \text{Ker} (\mathbf{I} - \mathcal{G}^*) + \dim \text{Ker} (\mathbf{I} + \mathcal{G}^*). \end{aligned}$$

Hence,  $G(k)$  has a canonical factorization if and only if 1 and  $-1$  are not eigenvalues of  $\mathcal{G}$  and  $\mathcal{G}^*$ .

Combining the result of Theorem 5 given above and the results in Lemma 4.3 and Theorem 4.7 in [Ne90], we have the following result. In the absence of bound states, for potentials whose scattering operators belong to the admissible class defined in [Ne90], there are no partial indices. Also using Theorem 5 above and Corollary 4.5 in [Ne90], we see that not only the sum index of  $G(k)$  is independent of  $x$  [AV90, AV91a], but also the sum of the negative partial indices of  $G(k)$  is independent of  $x$  and the sum of the positive partial indices of  $G(k)$  is independent of  $x$ . Since  $\sup_{k \in \mathbf{R}} \|G(k) - \mathbf{I}\| = \sup_{k \in \mathbf{R}} \|S(k) - \mathbf{I}\|$ , noting that  $G(k) = S(k)$  for  $x = 0$ , it also follows that  $\mathcal{G}$  and  $\mathcal{G}^*$  do not have eigenvalues  $\pm 1$  if  $\sup_{k \in \mathbf{R}} \|S(k) - \mathbf{I}\| < 1$ . Thus, the Newton-Marchenko integral equation is uniquely solvable if  $\sup_{k \in \mathbf{R}} \|S(k) - \mathbf{I}\| < 1$ . Here the norms are the operator norm on  $L^2(S^{n-1})$ .

4. **CONCLUSION.** If the potential in (1.1) causes bound states, the analysis given in Sections 1, 2, and 3 remains valid, provided we replace  $G(k)$  by the reduced operator

$G^{\text{red}}(k)$  obtained after removing the bound states by the reduction technique of Newton [Ne89, AV90]. Theorem 5 given in Section 3 remains valid for  $G(k)$  even in the presence of bound states.

Combining the result of Theorem 5 given above and the result in Lemma 4.3 in [Ne90], we have the following result. When there are bound states, for potentials whose scattering operators belong to the admissible class defined in [Ne90], the number of bound states  $\mathcal{N}$  for the Schrödinger equation (1.1) is related to the sum of the negative partial indices of  $G(k)$  as

$$\mathcal{N} = -\frac{1}{2} \sum_{\rho_j < 0} \rho_j.$$

For the same class of potentials, there are still no positive partial indices of  $G(k)$ .

Using Theorem 5 above and Corollary 4.5 in [Ne90], it follows, even if there are bound states, that both the sum of the negative partial indices of  $G(k)$  and that of the positive partial indices of  $G(k)$  are independent of  $x$ .

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## Complete integrability of "Completely Integrable" systems

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There is a well-known hierarchy of commuting flows associated with an  $n \times n$  spectral problem in one variable. These flows are Hamiltonian with respect to natural symplectic and Poisson structures on the manifold of potentials. It is common to speak of complete integrability, in analogy with classical mechanics, although in the infinite dimensional case there is no question of the number of independent commuting flows being half the number of degrees of freedom.

The scattering transform linearizes these flows and decouples the spectral modes, but for  $n > 2$  it does not trivially decouple the symplectic and Poisson structures. We find action-angle variables which do decouple these structures and thus show that the system on the scattering side is a direct integral of finite dimensional completely integrable systems, for general  $n$ .

An important tool is the analysis of a natural 2-form and symplectic foliation of  $SL(n)$ . Reductions of the system require further analysis, e.g. in  $SU(n)$ . As an application we obtain the complete integrability of the three wave interaction equation and note that even on the scattering side one must adjoin a nonlinear flow at each mode in order to have a complete system of commuting Hamiltonian flows.

The detailed version of this paper will be published elsewhere.

This is an expository account of results concerning the classical complete integrability of nonlinear evolution equations which are solvable by the inverse scattering or inverse spectral method. The method allows the question of complete integrability to be posed and (sometimes) answered in a finite-dimensional setting. In the process one encounters an interesting 2-form and symplectic foliation on classical Lie groups.

Section 1 gives the classical background and a discussion of the KdV equation. The  $n \times n$  spectral problem and the associated Hamiltonian flows are described in section 2 and the main theorems are stated. The proofs depend on an analysis of a 2-form and symplectic foliation on the groups  $SL(n)$  and  $SU(n)$ , described in section 3. Details will appear in [BS1].

### §1. Complete integrability in dimensions $2N, 2$ , and $\infty$ .

Classically a Hamiltonian flow with  $N$  degrees of freedom is described in suitable

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coordinates  $(x_1, \dots, x_N, y_1, \dots, y_N)$  by Hamilton's equations

$$(1.1) \quad \dot{x}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j}.$$

The structure of these equations is linked to the geometry of the symplectic form

$$\Omega = \Sigma dx_j \wedge dy_j$$

and the dual Poisson bracket for functions

$$\{f, g\} = \Sigma \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right).$$

In fact, an arbitrary function changes along the trajectories (1.1) according to its Poisson bracket with the Hamiltonian function  $H$ :

$$\dot{f} = \{f, H\}.$$

The flow (1.1) is said to be *completely integrable* if there are  $N$  independent integrals of the motion which are in involution, i.e. functions  $I_1, \dots, I_N$  such that

$$(1.2) \quad \{I_j, H\} = 0 = \{I_j, I_k\}, \quad dI_1 \wedge \dots \wedge dI_N \neq 0.$$

It was shown by Liouville that (1.2) is equivalent to the existence of action-angle variables for (1.1): coordinate functions  $p_1, \dots, p_N, q_1, \dots, q_N$  such that

$$\{p_j, H\} = 0, \quad \Omega = \Sigma dp_j \wedge dq_j.$$

The last condition can be written in terms of the Poisson bracket as

$$(1.3) \quad \{p_j, q_k\} = \delta_{jk}; \quad \{p_j, p_k\} = \{q_j, q_k\} = 0.$$

In fact, one may take  $p_j = I_j$ , and Liouville's method finds the  $q_j$  by quadratures [W].

Any Hamiltonian flow with 1 degree of freedom is completely integrable: take  $I_1 = H$  (if  $dH \neq 0$ ). An example is the *mathematical pendulum*

$$(1.4) \quad \ddot{x} = -c \sin x, \quad c = \text{constant.}$$

with  $y = \dot{x}$  and  $H(x, y) = \frac{1}{2}y^2 - c \cos x$ . Action-angle variables for (1.4) are Jacobi elliptic functions. Equation (1.4) appears in a natural but surprising way for the scattering data for the 3-wave interaction equation: see end of §2.

Somewhat more generally, a system of  $k$  commuting flows with Hamiltonians  $H_1, \dots, H_k$  in involution is completely integrable if there are  $N$  independent functions  $I_j$  which are invariant under each flow and in involution with each other.

The question of complete integrability in a nontrivial nonlinear infinite-dimensional context first arose in connection with the KdV equation

$$(1.5) \quad u_t - 6uu_x + u_{xxx} = 0.$$

Kruskal and co-workers discovered that (1.5) has a countable family of polynomial conservation laws

$$(1.6) \quad \dot{I}_k = 0, \quad I_k = \int P_k(u, \frac{\partial u}{\partial x}, \dots, (\frac{\partial}{\partial x})^k u) dx.$$

It was shown by Gardner [Ga] that (1.5) is a Hamiltonian flow with respect to the Poisson bracket

$$(1.7) \quad \{F, G\} = \int \frac{\delta F}{\delta u} \frac{d}{dx} \left( \frac{\delta G}{\delta u} \right) dx;$$

moreover the constants of motion  $I_k$  in (1.6) are in involution. Of course one can no longer simply count degrees of freedom to see whether there are enough  $I_k$ 's for complete integrability. However, Gardner, Greene, Kruskal, and Miura [GGKM] had shown that (1.5) can be integrated exactly by using the scattering theory of the linear spectral problem

$$(1.8) \quad \psi_{xx}(x, t, \xi) + [\xi^2 - u(x, t)]\psi(x, t, \xi) = 0.$$

In fact, ignore the  $t$ -dependence for a moment and suppose  $u$  vanishes at  $\infty$  in  $x$ . Then for real  $\xi$ , (1.8) has solutions  $\psi_{\pm} \sim \exp(\pm i x \xi)$  as  $x \rightarrow \pm \infty$ , and these solutions are related by

$$(1.9) \quad \psi_+(x, \xi) = a(\xi)\psi_-(x, -\xi) + b(\xi)\psi_-(x, \xi), \quad |a|^2 - |b|^2 = 1.$$

In the absence of point spectrum, the potential  $u$  can be recovered from the reflection coefficient  $b/a$  [F]. If the potential  $u(\cdot, t)$  evolves according to (1.6), then  $a, b$  evolve according to

$$(1.10) \quad \frac{\partial}{\partial t} a(\xi, t) = 0, \quad \frac{\partial}{\partial t} b(\xi, t) = 8i\xi^3 b(\xi, t).$$

This suggests that the modulus and argument of  $b$  might serve as action-angle variables. Indeed the complete integrability of KdV was established by Zakharov and Faddeev [ZF] by showing that the functions

$$(1.11) \quad p(\xi) = \frac{\xi}{\pi} \log |a(\xi)|^2 = \frac{\xi}{\pi} \log [1 + |b(\xi)|^2], \quad q(\xi) = \arg b(\xi)$$

satisfy the continuous version of (1.3):

$$(1.12) \quad \{p(\xi), q(\eta)\} = \delta(\xi - \eta), \quad \{p(\xi), p(\eta)\} = \{q(\xi), q(\eta)\} = 0.$$

The equations (1.10) show that the scattering data  $a, b$  not only linearize the KdV flow, but also decouple the modes  $\xi$ . Similarly (1.12) shows that the scattering data decouple the Hamiltonian structure (1.7) as a direct integral of 2-dimensional structures. Combining the two, one has decomposed KdV as a direct integral of Hamiltonian flows with 1 degree of freedom.

There are now a large number of physically interesting nonlinear evolution equations which are known to be similar to KdV in the following ways (see, e.g. [BuC], [NMPZ]):

- (a) they are Hamiltonian with respect to a Poisson structure similar to (1.7);
- (b) there is a countable family of conserved quantities like (1.6) which are in involution;
- (c) there is an associated linear spectral problem whose scattering theory (if known) gives a procedure for solving the nonlinear initial value problem exactly.

Such equations and systems are commonly called "integrable" or, because of (a), (b), "completely integrable". It is (c), however, which allows one to pose the question of complete integrability in a very precise sense: *does the scattering data provide action-angle variables?* This is essentially a finite-dimensional question, at least after the fact, because for each appropriate value of the spectral parameter  $\xi$ , the scattering data lives in a finite-dimensional space. In most of the cases where the answer was known, the dimension of this space is two, so in some sense the problem is reduced to the trivial case of classical complete integrability. Examples are KdV, the cubic nonlinear Schrodinger equation [ZM2], the sine-Gordon equation [FT1], or any equation whose spectral problem is a  $2 \times 2$  system of AKNS-ZS type: see [FT2]. The complete integrability of the 3-wave interaction equation, linked to a  $3 \times 3$  system, was investigated by Manakov [M].

In the remainder of this paper, we discuss  $n \times n$  systems and the associated flows. Proofs will appear elsewhere [BS1]. In a separate paper [BS2] we prove complete integrability of the Gelfand-Dikii flows [GD]. These flows are associated to eigenvalue problems for higher order ordinary differential operators and include the Boussinesq equation. In this paper, as in the discussion of the KdV flow above, we consider only the case of purely continuous spectral data. Discrete data poses a different type of question, which is studied in [BK].

## §2. The $n \times n$ spectral problem; flows; scattering data

The isospectral problem is

$$(2.1) \quad \psi_x(x, z) = zJ\psi(x, z) + q(x)\psi(x, z), \quad z \in \mathbf{C}, \quad \psi(x, z) \in GL(n, \mathbf{C}).$$

The potential  $q$  is off-diagonal with Schwartz-class entries and  $J$  is diagonal with distinct eigenvalues. General references are [BY], [BC1], [BC2], [C], [Ge], [Ne], [Sa], [Sh].

The question of complete integrability can be reduced to the case  $J + J^* = 0$ , which we now assume. We also reorder rows and columns so that

$$J = \text{diag}(i\lambda_1, \dots, i\lambda_n), \quad \lambda_1 > \lambda_2 > \dots > \lambda_n.$$

There is a Hamiltonian structure on the manifold of potentials  $q$  given by the symplectic form and Poisson bracket

$$(2.2) \quad \Omega = \int \text{tr} \{ \delta q(x) \wedge [adJ]^{-1} \delta q(x) \} dx,$$

$$(2.3) \quad \{F, G\} = \int \text{tr} \left[ J \cdot \frac{\delta F}{\delta q} \right] \frac{\delta G}{\delta q} dx.$$

For each traceless constant diagonal matrix  $\mu$  there is a hierarchy of flows

$$(2.4) \quad \dot{q} = [J, F_{k+1, \mu}]$$

where the  $F_{k, \mu}$  are polynomials in  $q$  and its derivatives  $[Sa]$  which are defined recursively by

$$F_{0, \mu} = \mu; \quad [J, F_{k+1, \mu}] = \frac{dF_{k, \mu}}{dx} + [q, F_{k, \mu}], \quad \lim_{x \rightarrow -\infty} F_{k+1, \mu}(x) = 0.$$

The flows (2.4) are Hamiltonian and in involution.

For real  $\xi$  there are normalized solutions  $\psi_{\pm}(x, \xi)$  of (2.1) linked by the *scattering matrix*  $s(\xi) \in sl(n, \mathbf{C})$ :

$$(2.5) \quad \lim_{x \rightarrow \pm\infty} \psi_{\pm}(x, \xi) \exp[-x\xi J] = 1, \quad \psi_{-}(x, \xi) = \psi_{+}(x, \xi) s(\xi), \quad \xi \in \mathbf{R}.$$

If  $\int \|q(x)\| dx < 1$  then the *scattering map*  $q \rightarrow s$  is injective. Under the flow (2.4), the scattering matrix evolves linearly:

$$(2.6) \quad \dot{s}(\xi, t) = [\xi^k \mu, s(\xi, t)].$$

A complication is that  $s$  is subject to nonlocal constraints: the upper (resp. lower) principal minors of  $s$  are boundary values of functions which are holomorphic in the lower (resp. upper) half plane. This reflects the fact that the potential  $q$  maps the line to a space of dimension  $n^2 - n$ , while the scattering matrix  $s$  maps to a space of dimension  $n^2 - 1$ . Minimal scattering data can be obtained from  $s$  by factoring

$$(2.7) \quad \begin{aligned} s_{\pm} &= sv_{\pm}, \quad s_{+} \text{ and } v_{-} \text{ are upper triangular,} \\ s_{-} \text{ and } v_{+} &\text{ are lower triangular, } (v_{\pm})_{jj} = 1. \end{aligned}$$

The *scattering data*  $(v_{+}, v_{-})$  is a map to a space of dimension  $n^2 - n$ . The scattering matrix  $s$  can be reconstructed from  $(v_{+}, v_{-})$  by a process which includes solving  $n - 1$  scalar Riemann-Hilbert factorization problems.

In terms of scattering data, the symplectic form and Poisson bracket are

$$(2.8) \quad \Omega = \frac{1}{4\pi i} \int_{\mathbf{R}} \text{tr}[v_+^{-1}(\delta v_+) \wedge s_+^{-1} \delta s_+ - v_-^{-1}(\delta v_-) \wedge s_-^{-1} \delta s]$$

$$(2.9) \quad \{s_{jk}(\xi), s_{\ell m}(\eta)\} = \pi i s_{jm}(\xi) s_{\ell k}(\eta) [\text{sgn}(\ell - j) - \text{sgn}(m - k)] \delta(\xi - \eta) \\ + s_{jk}(\xi) s_{\ell m}(\eta) [\delta_{j\ell} - \delta_{km}] \text{p.v.} \frac{1}{\xi - \eta}$$

where  $\text{sgn}(0) = 0$  and p.v. denotes the principal value; [BC2], [BS1]. See [M] for the Poisson bracket in the case  $n = 3$  and [Sk], [KD] for  $R$ -matrix formulations.

**THEOREM A.** *There are functions  $a_\nu, b_\nu, l \leq \nu \leq \frac{1}{2}(n^2 - n)$  defined on a dense open subset of  $SL(n, \mathbf{C})$  such that the composed functions  $p_\nu = a_\nu \circ s, q_\nu = b_\nu \circ s$  are action-angle variables for the flows (2.4): the Hamiltonian for (2.4) is a linear combination of the functionals  $\int \xi^k p_\nu(\xi) d\xi$  and*

$$(2.10) \quad \Omega = \sum_{\nu} \int_{\mathbf{R}} d p_{\nu} \wedge d q_{\nu}$$

$$(2.11) \quad \{p_{\mu}(\xi), q_{\nu}(\eta)\} = \delta_{\mu\nu} \delta(\xi - \eta); \quad \{p_{\mu}(\xi), p_{\nu}(\eta)\} = \{q_{\mu}(\xi), q_{\nu}(\eta)\} = 0.$$

In the case  $n = 2$ , there are just two functions which can be taken to be

$$(2.12) \quad p(\xi) = \log[s_{11}(\xi)s_{22}(\xi)], \quad q(\xi) = i \log[s_{12}(\xi)/s_{21}(\xi)];$$

in the general case the  $p_{\nu}$  and  $iq_{\nu}$  can be taken to be logarithms of products and ratios of suitable minors of  $s$ .

This gives complete integrability of (2.4) in the complex sense. The most important examples for (2.4) involve *reduction*, i.e. restriction to a submanifold of the manifold of potentials  $q$ , for which the form is real. An important case is the reduction

$$(2.13) \quad q(x) + q(x)^* = 0$$

which leads to real  $\Omega$  and to  $s(\xi) \in SU(n)$ . In this case, one would like real action-angle variables. The functions given above for  $n = 2$  are real on  $SU(2)$ , but in general, real action-angle variables necessarily involve more complicated functions of the matrix entries.

**THEOREM B.** *With  $n = 3$ , the functions  $a_\nu$  and  $b_\nu$  in Theorem A can be chosen so as to be real on  $SU(n)$ .*

This proves complete integrability for the 3-wave interaction. A different approach, using nonlocal functions of  $s(\xi)$ , was taken by Manakov [M].

One should note the following: the flows (2.4) do not constitute a full set of commuting Hamiltonian flows in any sense, when  $n \geq 3$ . The minimal scattering data has pointwise dimension  $\frac{1}{2}(n^2 - n)$ , while the space of flows (2.6) has pointwise dimension  $n - 1$  (complex dimensions in the general case, real dimensions in the case of the reduction (2.13)). The flows associated to the additional constants of the motion are not linear in the scattering data (as elements of the linear space  $M_n(\mathbf{C})$ ). Thus for  $n \geq 3$  the existence of the infinite family of commuting flows (3.4) is not in itself very convincing evidence of complete integrability.

A final remark: under the natural third flow occurring in Theorem B to supplement the two linear flows (2.6) at a given point  $\xi$ , the modulus  $|s_{22}(\xi)|$  is fixed, while  $\arg s_{22}(\xi)$  obeys the pendulum equation (1.4)!

**§3. A 2-form and symplectic foliation on  $SL(n)$  and  $SU(n)$ .**

Given  $s$  in  $SL(n) = SL(n, \mathbf{C})$ , or  $SL(n, \mathbf{R})$ , denote the upper and lower principal minors by

$$(3.1) \quad d_\ell^+(s) = \det(s_{jk})_{j,k \leq \ell}; \quad d_\ell^-(s) = \det(s_{jk})_{j,k \geq \ell}.$$

If no  $d_\ell^\pm(s)$  vanishes,  $s$  has two unique factorizations (2.7). In terms of these factorizations, we may define a 2-form on (a dense open subset of)  $SL(n)$  by adapting (2.8):

$$(3.2) \quad \Omega_n = \text{tr}[v_+^{-1}(dv_+) \wedge s_+^{-1}ds_+ - v_-^{-1}(dv_-) \wedge s_-^{-1}ds_-].$$

As a form on  $SL(n)$ , it is not obvious that  $\Omega_n$  is closed; it is certainly not symplectic, since the rank in a neighborhood of the identity is  $n^2 - n$ .

**THEOREM C.** *There are functions  $p_\nu, q_\nu$ ,  $1 \leq \nu \leq \frac{1}{2}(n^2 - n)$ , such that*

$$(3.3) \quad \Omega_n = \sum dp_\nu \wedge dq_\nu.$$

*In particular,  $\Omega_n$  is closed.*

Although the proof of this theorem is purely algebraic, it is motivated by observations from scattering theory. First, the result is not difficult when  $n = 2$ :  $p_\nu$  and  $q_\nu$  can be chosen as in (2.12). Second, the general result should follow from the result for  $n = 2$ , because the matrix  $J$  of section 2 can be taken as a limit of "generic" complex  $J$  for which the scattering data are  $2 \times 2$  matrices living on  $\frac{1}{2}(n^2 - n)$  lines in the complex plane [BC1]. These ideas lead to a multiplicative decomposition of  $v = v_-^{-1}v_+$  and a corresponding additive decomposition of  $\Omega_n$ .

The functions (3.1) lead to a natural foliation of  $SL(n)$  by the functions

$$(3.3) \quad \varphi_j(s) = d_j^+(s)/d_{j+1}^-(s), \quad 1 \leq j \leq n.$$

**PROPOSITION.** *The foliation of  $SL(n)$  by the functions (3.3) is symplectic for  $\Omega_n$ , i.e. the pullback of  $\Omega_n$  to each leaf is a symplectic form on the leaf.*

The symplectic foliation gives a (degenerate) Poisson structure  $(\ , \ )$  on  $SL(n)$ : on each leaf  $L$  the symplectic structure determines a bracket  $(\ , \ )_L$  and the global bracket  $(\ , \ )$  is characterized by

$$(f, g)|_L = (f|_L, g|_L).$$

It follows from (3.2) that

$$(3.4) \quad (p_\mu, q_\nu) = \delta_{\mu\nu}, \quad (p_\mu, p_\nu) = (q_\mu, q_\nu) = 0.$$

As one might begin to expect, the functions denoted  $p_\nu, q_\nu$  here are the functions denoted  $a_\nu, b_\nu$  in Theorem A. This is further confirmed by the fact that the Poisson bracket  $(\ , \ )$  is the local part of the Poisson bracket as computed in (2.9):

**THEOREM D.** *The Poisson bracket of matrix elements is given by*

$$(3.5) \quad (s_{jk} \ s_{\ell m}) = \frac{1}{4} s_{jm} s_{\ell k} [\operatorname{sgn}(\ell - j) - \operatorname{sgn}(m - k)], \quad \operatorname{sgn}(0) = 0.$$

The formula (3.5) was first calculated for  $n = 2$  and  $n = 3$  by Lu [L], who conjectured the general formula and pointed out its relation to the classical limit of a quantum group structure defined in [D]. The proof in the general case proceeds through a reduction to the cases  $n \leq 4$ .

To complete the proof of Theorem A, one needs to show that the possible nonlocal terms in (2.11) vanish.

The 2-form  $\Omega_n$  makes sense as a complex 2-form on  $SU(n)$ , but it can be shown that

$$(3.6) \quad i\Omega_n \text{ is a real 2-form on } SU(n).$$

Moreover, the foliation functions (3.3) have modulus 1 on  $SU(n)$ , so the foliation is determined by their arguments and thus the Poisson bracket  $-i(f, g)$  is real on  $SU(n)$  when the functions  $f$  and  $g$  are real.

**THEOREM E.** *For  $SU(3)$  the functions  $p_\nu$  and  $q_\nu$  in Theorem C can be chosen so that  $p_\nu$  and  $iq_\nu$  are real.*

In fact one can choose  $p_j = \log|s_{jj}|$  and the  $q_j$  may be found by Liouville's method: they are elliptic functions of the matrix entries; see [BS1].

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## On the Determinant Theme for Tau Functions, Grassmannians, and Inverse Scattering

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**ABSTRACT.** One investigates relations between tau functions, dressing kernels, wave functions, and spectral asymptotics for KdV, KP, and AKNS situations in a determinant context where emphasis is on the continuous spectrum.

### 1. Background (cf. [2-5, 8, 13, 14, 19, 22-26])

Consider first the KdV situation for (\*)  $L\psi = (D^2 + q)\psi = -k^2\psi$  and  $\psi_t = B\psi = -4\psi_{xxx} - 6q\psi_x - 3q_x\psi$  with  $q_t + 6qq_x + q_{xxx} = 0$  ( $q$  real). One defines Jost solutions for (\*) with  $f \sim \exp(\pm ikx)$  as  $x \rightarrow \pm\infty$  and writing  $T = s_{11}$ ,  $R = s_{21}$ ,  $R_L = s_{12}$ , and  $rf(k) = f^-(k) = f(-k)$ , we have  $Tf_- = Rf_+ + f_+$  with  $Tf_+ = Rf_- + f_-$ . Assume there are no bound states and say  $q \in S$  (Schwartz space). The classical picture involves  $F(z, t) = (1/2\pi) \int_{-\infty}^{\infty} R(k, 0) \exp(ikz + 8ik^3t) dk$  with  $K$  the solution of the Marčenko (M) equation

$$K(x, y, t) + F(x + y, t) + \int_x^{\infty} K(x, s, t)F(s + y, t) ds = 0$$

for  $y > x$ . Then  $q(x, t) = 2D_x K(x, x, t)$  satisfies KdV. Now introduce hierarchy variables  $x = (x_1, x_3, \dots)$ ,  $x = x_1$  sometimes,  $x_3 \sim t_3 = 4t$ , and unless otherwise specified for hierarchy variables  $x, y$  we stipulate  $x_{2n+1} = y_{2n+1}$  for  $n \geq 1$ . Set  $\tilde{\xi}(x, k) = \tilde{\xi} = i \sum_0^{\infty} x_{2n+1} k^{2n+1}$  and  $\psi_0(x, k) = \exp(\tilde{\xi}(x, k))$  with

$$F(x, y) = \int_{\Gamma} \psi_0(x, k) \psi_0(y, k) d\Lambda = \langle \psi_0(x, k), \psi_0(y, k) \rangle_{\Lambda},$$

where  $\Gamma, \Lambda$  can be in general any "suitable" curve and measure ( $\Gamma = (-\infty, \infty)$  and  $d\Lambda = R_0 dk/2\pi$  classically). Let  $\Phi_n$  be the  $n \times n$

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matrix with entries  $F(s_i, s_j)$  with  $\Omega_n$  the  $(n+1) \times (n+1)$  matrix having first row  $F(x, y), F(x, s_1), \dots, F(x, s_n)$ , first column  $F(x, y), F(s_1, y), \dots, F(s_n, y)$ , and  $\Phi_n$  for the remainder. Then following [23–26] one solves the Fredholm integral equation

$$(1 + F_x)K = -F \quad (F_x \sim \int_x^\infty \equiv \int_{x_1}^\infty)$$

in the form  $K_+(x, y) = K(x, y) = D(x, y)/\tau(x)$ , where

$$(1) \quad \begin{aligned} \tau(x) &= 1 + \sum_1^\infty (1/n!) \int_x^\infty \cdots \int_x^\infty \det \Phi_n \Pi_1^n ds_i; \\ D(x, y) &= -F(x, y) - \sum_1^\infty (1/n!) \int_x^\infty \cdots \int_x^\infty \det \Omega_n \Pi_1^n ds_i. \end{aligned}$$

This leads to  $q(x) = 2D_x K(x, x) = 2D_x^2 \log \tau(x)$  ( $x = (x_1, x_3, \dots)$ ,  $x_3 \sim 4t$ , etc.). One defines wave functions now as

$$\psi_+ = \psi_0 + \int_x^\infty K_+(x, s) \psi_0(s, k) ds,$$

and similarly there is a  $K_-$  based on  $\int_{-\infty}^x$  with

$$T\psi_- = \exp(-\tilde{\xi}) + \int_{-\infty}^x K_-(x, s) \exp(-\tilde{\xi}(s, k)) ds$$

(for KdV  $K_- = (1 + K_+^T)^{-1}$ ). One can write (cf. [2–8])

$$(2) \quad \begin{aligned} K_+(x, y) &= -\langle \psi_+(x, k), \psi_0(y, k) \rangle_\Lambda \quad (y > x); \\ K_-(x, y) &= \langle \psi_+(x, k), \psi_0(y, k) \rangle_\Lambda \quad (y < x) \end{aligned}$$

and the vertex operator equation (VOE) is  $\tilde{X}_-(k)_\tau = \exp(\tilde{\xi}) \tilde{G}_-(k) \tau = \psi_+ \tau$ , where

$$\begin{aligned} \tilde{X}_\pm(k) \tau &= \exp(\mp \tilde{\xi}) \tilde{G}_\pm(k) \tau = \exp(\mp \tilde{\xi}) \tau_\pm \\ &= \exp(\mp \tilde{\xi}) \tau(x_1 \pm 1/ik, x_3 \pm 1/3ik^3, \dots). \end{aligned}$$

The relation  $(1 + K_-^T)(1 + K_+) = 1$  leads directly (via Fourier transform) to the classical completeness relation (C)

$$(1/2\pi) \int_{-\infty}^\infty T\psi_-(y, k) \psi_+(x, k) dk = \delta(x - y).$$

This is based entirely on the structure  $\psi_+ = (1 + K_+) \exp(\tilde{\xi})$  and  $T\psi_- = (1 + K_-) \exp(-\tilde{\xi})$  with  $(1 + K_-^T) = (1 + K_+)^{-1}$  (for KP a similar argument applies for completeness using Laplace transforms—see below and cf. [3–6, 22]).

We indicate also some minimal background for KP (cf. [5, 6, 10, 15, 16, 19–25, 27]). Thus writing  $\partial_n = \partial/\partial x_n$ ,  $\partial = \partial/\partial x_1$  ( $x = (x_1, x_2, \dots)$ ), one requires a Lax operator  $L = \partial + u_1(x)\partial^{-1} + \dots$ , a gauge operator

$P = 1 + w_1(x)\partial^{-1} + \dots$  such that  $LP = P\partial$ , and hierarchy equations  $\partial_n L = [B_n, L]$ , where  $B_n = (L^n)_+$ . Set now  $\xi = \sum x_n k^n$  with  $w = P \exp(\xi) = (1 + \sum_1^\infty w_j k^{-j}) \exp(\xi) = \hat{w} \exp(\xi)$  and  $w^* = (P^*)^{-1} e^{-\xi}$ . The dressing picture involves upper and lower Volterra operators  $K_\pm$  such that, for  $n \geq 2$ ,  $(\partial_n - B_n)(1 + K_\pm) = (1 + K_\pm)(\partial_n - \partial^n)$ . Then  $P \sim 1 + K_+$ ,  $(P^*)^{-1} \sim (1 + K_+^T)^{-1}$  and formal residue calculations lead (when they make sense) to the Hirota bilinear formula (H)  $\int_C w(x, k) w^*(y, k) dk = 0$ , where  $C$  is a circle at " $\infty$ ." This is proved first for  $x_n = y_n$  ( $n \geq 2$ ) and then extended to arbitrary  $x, y$ . A corresponding completeness relation ( $\tilde{C}$ )  $(1/2\pi i) \int_{-i\infty}^{i\infty} w(x, k) w^*(y, k) dk = \delta(x - y)$  for  $x_n = y_n$  ( $n \geq 2$ ) is proved as above for (C) (without extension to  $x, y$  arbitrary). Hence heuristically we record (cf. [3, 5, 6, 22])

**THEOREM 1.1.** *The conceptual background for (H) and ( $\tilde{C}$ ) is equivalent, namely,  $w = P \exp(\xi)$  and  $w^* = (P^*)^{-1} \exp(-\xi)$ , with  $P \sim 1 + K_+$ , etc.*

**REMARK 1.2.** The problem is, of course, that for half plane analytic wave functions  $w, w^*$  (or  $\psi_+, T\psi_-$  in KdV), the residue calculations generally make no sense. However, the Hirota formula (H) is derived in many geometric and algebraic contexts where residue calculations do make sense (cf. [3, 10, 15, 16, 19-22, 25]) and its geometrical content must have a version in the case of no discrete spectrum with tau functions constructed as above, for example based on continuous spectrum. One wants to preserve the algebra of the hierarchy framework in the scattering situation and this is discussed in §2.

## 2. The Hirota bilinear identity in the Grassmannian picture for KdV

We go to the Grassmann picture of scattering developed in [12, 18] and refer to [3, 4] for details (cf. also [11, 15, 19, 27]). Let  $H^+ = FL^2[0, \infty)$  and  $H^- = FL^2(-\infty, 0]$  (Fourier transform) be the standard Hardy spaces with  $p: L^2 \rightarrow H^+$  the orthogonal projection. The Grassmannian is  $GR = \{H \subset L^2; (1-p): H \rightarrow H^- \text{ is 1-1, onto, with continuous inverse}\}$ . One works with  $R = s_{21} \in S$  here (KdV situation with no bound states) and it is shown in [18] that  $H \in GR$  corresponds uniquely to  $s_{21} = R$  via  $H = L^2 \cap \{f; f^- + Rf \in H^+\}$ . The approach of [12, 18] is part of a program on the geometry of KdV and one works there from the viewpoint of algebraic curves and divisors. The addition theory in [12], for example, is equivalent to the hierarchy (or a substitute for the hierarchy) and involves updating  $R$  via  $\tilde{R} = R \exp(2ikx)$  (or eventually  $\tilde{R} = R \exp(2\tilde{\xi})$ ) and  $\tilde{R}^+ = \tilde{R}(\omega - k)/(\omega + k)$  ( $\text{Im } \omega > 0$ ). We define the Baker-Akhiezer (BA) function for  $\tilde{R}$  as  $\tilde{e}_+ = \psi_+ \exp(-\tilde{\xi})$  and set  $\tilde{e}_- = \psi_- \exp(\tilde{\xi})$  so  $T\tilde{e}_- = \tilde{e}_+ + \tilde{R}\tilde{e}_+$ . Let  $e_n = (-i/\sqrt{\pi}(k+i))((k-i)/(k+i))^n$

be a basis for the Hardy spaces and write

$$(3) \quad \begin{aligned} F_x \varphi(s) &= \int_{x_1}^{\infty} F(s, \xi, x_3, \dots) \varphi(x) d\xi; \\ F^x \varphi(s) &= \int_0^{\infty} F(s + x_1, \sigma + x_1, x_3, \dots) \varphi(\sigma) d\sigma. \end{aligned}$$

Then  $\text{Tr} F_x = \text{Tr} F^x$  so the theta functions constructed in [12, 18] involve  $\Theta(\tilde{R}) = \det(1 + pr\tilde{R}|_{H^+}) \sim \det(1 + F^x) = \det(1 + F_x) = \tau(x)$  for  $F = \langle \psi_0(x, k), \psi_0(y, k) \rangle_{\Lambda}$  as above ( $x_{2n+1} = y_{2n+1}$ ,  $n \geq 1$ ). Using techniques of [25], one knows that vertex operator action  $\tilde{G}_-(\omega)\tau = \tau_-$  ( $\tau$  based on  $\tilde{R}$ ) gives a tau function based on  $\tilde{R}^+$ , and one obtains an alternative proof of a result in [18], namely  $(\tilde{R}^- = \tilde{R}(\omega + k)/(\omega - k))$

**THEOREM 2.1.** *We have  $\tilde{e}_+(\omega) = \Theta(\tilde{R}^+)/\Theta(\tilde{R}) \sim \psi_+ = \tilde{X}_- \tau/\tau$  and  $T\tilde{e}_- = \Theta(\tilde{R}^-)/\Theta(\tilde{R}) \sim T\psi_- = \tilde{X}_+ \tau/\tau$ .*

In the context of divisor theory, of course the proofs in [12, 18] are to be preferred, but this version exhibits the equivalent vertex operator geometry which plays a role in the Hirota formula to follow. What we do is take the proof of the Hirota bilinear identity in [15], based on loop groups over  $S^1$ , and transport it to the geometry of the Hardy spaces via  $\lambda^n \rightarrow e_n$ . We then introduce a formal residue calculation at “ $\infty$ ” which embodies this geometry, so the Hirota formula has the same appearance as before. Thus the residue calculation is artificial but the geometrical facts expressed through it are genuine. We let  $\Lambda e_n = e_{n+1}$ , and vertex operator action can be expressed for KdV via  $Q_\omega = (1 - \Lambda/\omega)(1 + \Lambda/\omega)^{-1} (\sim \tilde{G}_-(\omega))$ . Now  $H^0 = H^+ = W = (1 + rR)H^+ = wH^+ = \binom{w_+}{w_-} H^+$  with  $w_+ = 1 + prR$  and  $w_- = (1 - p)rR$ ; in this notation  $\widehat{W} = (1 + r\tilde{R})H^+$  leads to an important map  $\tilde{w}_- = \hat{w}_- \hat{w}_+^{-1} = (1 - p)r\tilde{R}(1 + pr\tilde{R})^{-1}$  with  $\tilde{w}_-^* \sim (1 + pr\tilde{R})^{-1} pr\tilde{R}$  acting in  $H^- (\widehat{W} = \binom{\tilde{w}_+}{\tilde{w}_-} H^+)$ . Setting  $\tilde{w}_-(e_m) = \sum_{s=-\infty}^{-1} w_{sm} e_s$  with  $\tilde{w}_-^*(e_p) = \sum_0^{\infty} w_{pm} e_m$ , the recipe in [15, 27] calls for a BA function  $\hat{\psi}_H(\hat{\gamma}, \omega)$  of  $H$  expressed in the form ( $\gamma \sim \exp(\xi(x, \Lambda))$  action,  $\hat{\gamma} = \gamma^{*-1}$ )

$$(4) \quad \begin{aligned} \hat{\psi}_H(\hat{\gamma}, \omega) &= -\overline{(1 - \tilde{w}_-^*(e_{-1}))}|_{e_n \rightarrow \omega^n} = 1 - \sum_0^{\infty} w_{-1,m} \omega^{-m-1} \\ &= \det(1 + \mu^{-1} \nu \tilde{w}_-); \quad Q_\omega \sim \begin{pmatrix} \mu & \nu \\ 0 & \pi \end{pmatrix}. \end{aligned}$$

Now express the  $(e_n, e_m) = \delta_{nm}$  geometry in  $H^\pm$  via  $e_n \rightarrow \omega^n$  (with a  $-1/2\pi i$  adjustment) so that  $(e_n, e_m) = (1/2\pi i) \int_C \omega^{n-m-1} d\omega$  with  $C$  a circle at “ $\infty$ .” Then for  $\hat{\psi}_W = (1 + \tilde{w}_-(e_0))|_{e_n \rightarrow \omega^n}$ , we get by construction  $(1/2\pi i) \int_C \widehat{\psi}_H(\hat{\gamma}, \omega) \hat{\psi}_W(\gamma, \omega) d\omega = 0$ . The Hirota formula results by stip-

ulating  $\psi_W = \hat{\psi}_W e^{\xi(x, \omega)} \in W = H^\perp$  and  $\psi_H \in H$  for different  $\gamma$  actions so we can state

**THEOREM 2.2.** *The Hirota bilinear identity can be written formally as  $\int_C \psi_W(\gamma, \omega) \overline{\psi_H(\gamma', \omega)} d\omega = 0$ , expressing a genuine perpendicularity of Grassmann objects, and*

$$(5) \quad \overline{\psi_H(\gamma, \omega)} = \overline{\det(1 + \mu^{-1} \nu \hat{w}_-)} = \hat{e}_+(\omega) = \Theta(\tilde{R}^+) / \tilde{\Theta}(R).$$

### 3. AKNS and KdV: Connections of tau functions, spectral data, and dressing kernels

Given the importance of tau functions and their ubiquitous appearance, we note that for KdV

$$(6) \quad a(k) = \lim_{x \rightarrow -\infty} \tau_-(x, k) / \tau(x) = \lim_{x \rightarrow -\infty} (1 + e^{-ikx} \hat{K}_+(x, k)).$$

In particular, for suitable general  $F_\Lambda = \langle \psi_0(x, k), \psi_0(y, k) \rangle_\Lambda$ , one constructs  $K_\Lambda (= K_+^\Lambda)$ ,  $\tau_\Lambda$ ,  $q_\Lambda$ , etc. in §1 and (6) serves as one criterion for "spectrality." One can construct various potentials  $q_\Lambda$  by the determinant method from  $F_\Lambda$ , analogous to Newton-Sabatier techniques in inverse scattering, and the study of such situations and their spectral properties (if any) is of interest (cf. [2, 4, 7]). For AKNS in the form  $Q = \sum_0^\infty Q_j \zeta^{-j} = \begin{pmatrix} h & e \\ f & -h \end{pmatrix}$ ,  $Q^n = \sum_0^n Q_j \zeta^{n-j}$ ,  $Q_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3$ ,  $Q_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ ,  $\partial_n Q = [Q^n, Q]$ , etc. (cf. [1, 4, 14, 22]) one writes, e.g.,  $Q = F Q_0 F^{-1}$ ,  $\partial_n F = Q^n F$ , etc. One can use the tau functions in [22] for example, if suitable regions  $\text{Im } \zeta > 0$  or  $\text{Im } \zeta < 0$  are isolated (following [13] for NLS) and then asymptotically

$$(7) \quad F \sim (1/\tau) \begin{pmatrix} X_- \tau & -(i/2\zeta) X_+ \sigma \\ (i/2\zeta) X_- \rho & X_+ \tau \end{pmatrix},$$

where  $X_\pm \tau = \exp(\pm i \sum \zeta^k t_k) \tau_\pm$ ,  $\tau_\pm = \tau(t_k \pm i/2k\zeta^k)$ ,  $\sigma = \tau e_1$ ,  $\rho = \tau f_1$ ,  $e = \sum_1^\infty e_j \zeta^{-j}$ ,  $f = \sum_1^\infty f_j \zeta^{-j}$ ,  $h = \sum_0^\infty h_j \zeta^{-j}$ ,  $e_1 = q$ ,  $f_1 = r$ ,  $h_1 = 0$ ,  $h_0 = -i$ , etc. Determinant constructions are developed in [17] and a dressing framework in [13], which we follow here. Thus, for AKNS column wave functions  $\varphi, \hat{\varphi}, \psi, \hat{\psi}$  (cf. [8]) we write  $S_+ = (\varphi \psi)$ ,  $S_- = (\hat{\psi} - \hat{\varphi})$ ,  $S_- = S_+ S$ ,  $S = (1/a) \begin{pmatrix} 1 & -b \\ -b & 1 \end{pmatrix} (a\hat{a} + b\hat{b} = 1)$ ,  $a = W(\varphi, \psi)$ , etc.,  $E = \exp(-i\zeta x \sigma_3)$ ,  $G_- = S_- E^{-1}$ ,  $G_+ = a E S_+^{-1}$ , and  $F_0 = \exp(-i \sum_1^\infty \zeta^k t_k \sigma_3)$ . For basic  $F$  in  $\text{Im } \zeta < 0$  or  $\text{Im } \zeta > 0$ , one takes the functions  $F_- = G_- F_0$  and  $F_+ = G_+^{-1} F_0$  (corresponding to different tau functions) and then, analogous to (6), one finds

$$(8) \quad F_+ : \overline{(\tau_-/\tau)} = 1/a = \overline{(\tau_+/\tau)}; \quad F_- : \overline{(\tau_-/\tau)} = \hat{a} = \overline{(\tau_+/\tau)},$$

where  $\overline{^\pm f} = \lim_{x \rightarrow \pm\infty} f(x)$ . There are many results in [4] about determinant constructions, spectral forms of kernels, completeness, Marčenko equations, dressing kernels, and structures of kernels for AKNS, and many formulas in

[2] about connections of tau functions with spectral data, dressing kernels, vertex operator equations, and asymptotics in general.

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## An Overview of Inversion Algorithms for Impedance Imaging

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### Introduction

Impedance imaging systems [BB, NGI] apply currents to the surface of a body, measure the resulting voltages, and use these data to reconstruct an approximation to the conductivity in the interior.

A simple mathematical formulation is the following. The potential  $u$  satisfies

$$(1) \quad \nabla \cdot \sigma \nabla u = 0$$

in the interior of a body  $\Omega$  of conductivity  $\sigma$ . A boundary condition that corresponds to applying currents is

$$(2) \quad \sigma \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = j,$$

where  $\nu$  is the outward unit normal and  $j$  is the applied current density. The applied current density must satisfy the conservation of charge condition

$$(3) \quad \int_{\partial \Omega} j = 0.$$

The resulting voltages are given by

$$(4) \quad u|_{\partial \Omega} = v.$$

The voltages are not uniquely determined by (1) and (2) until we add a condition specifying a ground. A convenient condition is

$$(5) \quad \int_{\partial \Omega} v = 0.$$

The inverse conductivity problem is, from the knowledge of the map  $R: j \rightarrow v$ , to find the conductivity  $\sigma$ .

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This is actually a simplification of the true problem. This model leaves out the effects of the electrodes [CING]. In addition, for experiments done at nonzero frequencies, the model should include the electric permittivity as well as the conductivity [IC1]. In this case, the conductivity can be replaced by  $\sigma + i\omega\epsilon$ , where  $\omega$  is the frequency and  $\epsilon$  is the electric permittivity. The problem of reconstructing the conductivity then becomes the problem of reconstructing the electrical impedance.

### Relation to scattering

The impedance imaging problem should be interesting to those who like scattering theory, because impedance imaging is closely related to inverse scattering. This can be seen by making the substitution  $u = \sigma^{-1/2}\psi$  in (1). This transforms (1) into the Schrödinger equation

$$(-\Delta + q)\psi = 0, \quad \text{where} \quad q = \frac{\Delta\sigma^{1/2}}{\sigma^{1/2}}.$$

Nachman [N] has shown that knowledge of the Neumann-to-Dirichlet map for the Schrödinger equation is equivalent to knowledge of boundary measurements of the Green's function that satisfies

$$(-\Delta + q)G = \delta.$$

This models a wave propagation problem in which one sets off point sources everywhere on the boundary and listens to the responses on the boundary. By taking the boundary to infinity, one can then obtain the scattering amplitude or far-field pattern [DB].

Actually, scattering problems are idealizations of boundary value problems. No physical measurement can really be made at infinity. There is therefore no loss in generality in considering only boundary value problems.

Another connection between the impedance imaging problem and scattering is that the Sylvester-Uhlmann uniqueness proof [SU] (that the boundary map uniquely determines the conductivity) used scattering theory.

Essentially, the inverse conductivity problem can be thought of as equivalent to an inverse scattering problem at fixed frequency.

### Applications

Impedance imaging is also interesting because of its applications. Its use for medical imaging is being explored at Rensselaer [GIN] and many other institutions. The reason it may be useful for medical applications is that different tissues in the body have different electrical properties [SK]. Thus images of the electrical properties should enable one to visualize the different organs of the body.

Impedance imaging has also been used to detect flaws, such as cracks, in metals. This work is being done at General Electric R&D [ESIC].

Impedance imaging may also be useful in visualizing multiphase fluid flow [XPB]. A second group at Rensselaer has begun work on this application.

### Iterative methods

There are many iterative methods being used, but they all use roughly the same algorithm. This algorithm is

- (a) guess  $\sigma$
- (b) solve the forward problem
- (c) compare the solution to the measurements
- (d) update  $\sigma$
- (e) go to step (b).

The main difference among the various algorithms is the functional used in step c). One commonly used functional, the one used in the images produced so far at Rensselaer [CINGS], is the output least squares functional; another is the functional [KM], which has the property that Ohm's law is satisfied at the minimum. There are many additional constraints that can be added in, but we will not discuss these here.

### Linearized methods

**A. The Applied Potential Tomography (APT) system.** The APT system of Barber and Brown [BB] is the only commercial impedance imaging system. Their system applies currents only on two neighboring electrodes and measures the voltages on all the other electrodes. It uses a linearized reconstruction method based on backprojecting along the equipotential lines for a homogeneous body. The work of Beylkin [B] and Vogelius and Santosa [SV] shows that this method gives an approximate solution to the linearized inverse problem.

**B. The Calderón method.** In [C], Calderón gave a formula that can be used to solve the linearized inverse problem exactly. To obtain his formula, he used (1) and the equation for the homogeneous problem

$$(6) \quad \nabla^2 v = 0.$$

He then applied Green's theorem to the identity

$$(7) \quad \int_{\Omega} (v \nabla \cdot \sigma \nabla u - u \nabla^2 v) = 0.$$

This results in

$$(8) \quad \int_{\Omega} (\sigma - 1) \nabla u \cdot \nabla v = \int_{\partial \Omega} \left( v \sigma \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right).$$

The right side of (8) is data. Calderón then used two particular solutions of (1) and (6) involving the complex vector

$$(9) \quad \zeta = \frac{\xi + i\xi^{\perp}}{2}, \quad \xi \in \mathbb{R}^n, \quad \xi \cdot \xi^{\perp} = 0.$$

It has the property that  $\zeta \cdot \zeta = 0$ . The special solution of (6) is

$$(10) \quad v = e^{-i\bar{\zeta} \cdot x} - 1, \quad x \in \Omega.$$

The special solution of (1) is obtained by specifying

$$(11) \quad \sigma \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = \frac{\partial}{\partial \nu} (e^{-i\zeta \cdot x}).$$

If the conductivity is close to one, then  $u$  is close to

$$(12) \quad u \sim e^{-i\zeta \cdot x} - 1.$$

If we use this expression for  $u$  in (8), then we obtain

$$(13) \quad \frac{-|\xi|^2}{2} \int_{\Omega} (\sigma - 1) e^{-i\xi \cdot x} dx \approx \int_{\partial \Omega} \left( v \sigma \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right).$$

Thus the linearized inverse problem can be solved by inverting the Fourier transform.

**C. Exact solutions.** Calderón's method has been used to obtain exact solutions to the linearized inverse problem in special cases.

In the "two-ring" case, when the body is a homogeneous circular disk containing a concentric disk of constant conductivity, the linearized reconstruction [II] contains a whole sequence of rings. The outermost ring has the correct radii and correct conductivity. The inner rings have radii converging exponentially to zero. The conductivities of these inner rings are not correct. Nevertheless, it is clear that the linearized reconstruction contains useful information.

The "three-ring" case is similar [CII]. Each discontinuity gives rise to a sequence of rings whose radii converge exponentially to zero.

These examples show that the solution of the linearized problem contains more discontinuities than the solution of the nonlinear problem. Nevertheless, useful information can be extracted from the linearized solutions.

**D. Modified Calderón method.** These examples suggest that Calderón's method might be useful in practice. The problem is that it requires boundary data that are highly oscillatory. This is bad for two reasons. First, highly oscillatory data contain mainly information about the boundary of the body [I]. Any signal due to conductivity changes deep inside the body tends to be buried in the noise. Second, highly oscillatory data cannot be applied with only a limited number of electrodes. This suggests that we should change basis from the highly oscillatory exponentials to a basis of smooth functions such as the trigonometric functions.

We thus expand the boundary data (in the two-dimensional, circular case) as

$$(14) \quad \frac{\partial}{\partial \nu} e^{-i\zeta \cdot x} = \frac{\partial v}{\partial \nu} = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \quad \theta \in [0, 2\pi].$$

We use this expansion in (13), obtaining [IC2]

$$(15) \quad \sigma - 1 \approx \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} \delta V_{n+1, m+1} \left( \frac{-\bar{\mu} \cdot \nabla}{2} \right)^n \left( \frac{-\mu \cdot \nabla}{2} \right)^m \delta,$$

where  $\delta$  is the Dirac delta function,  $\mu = (1, i)$ ,  $\bar{\mu} = (1, -i)$ , and

$$(16) \quad \delta V_{n, m} = \int_{\partial\Omega} e^{-im\theta} (R_0 - R) e^{in\theta} d\theta;$$

here  $R_0$  is the boundary map in the homogeneous case (6). The quantities (16) are now the data; they are good from the signal-to-noise point of view. The formula (15) gives an exact solution to the linearized inverse problem. The obvious difficulty is that each term on the right side is singular.

These singularities can be understood with the following example. Suppose one computes the Fourier transform of a Gaussian by expanding the Gaussian in a Taylor series and Fourier transforming each term. The result is an infinite sum again containing derivatives of delta functions. In fact, the sum can be written as the heat operator applied to a delta function. This shows how it is possible to add up an infinite number of singular terms all supported in a neighborhood of the origin and obtain a result that is smooth and nonzero everywhere.

However, (15) presents computational difficulties. Clearly some sort of regularization is needed to make the formula useful in practice.

**E. The moment method.** The linearized inverse problem can also be formulated in terms of a moment problem. This formulation has been explored in [CW] and [BAG].

To obtain a moment problem, we merely use different special solutions in (8). In particular, we use

$$(17) \quad u \approx \frac{1}{n} r^{|n|} e^{in\theta}, \quad v = \frac{1}{m} r^{|m|} e^{im\theta}.$$

We also expand the conductivity in its Fourier series as

$$(18) \quad (\sigma - 1)(r, \theta) = \sum_{k=-\infty}^{\infty} \eta_k(r) e^{ik\theta}.$$

Then (8) reduces to the moment problem

$$(19) \quad \int_0^1 \eta_{m-n}(r) r^{n+m-1} dr = (\text{data})_{n, m}.$$

Such a moment problem can be solved by taking linear combinations of (19) to build up orthogonal polynomials out of the powers of  $r$  appearing on the left side; the orthogonality can then be used to solve for  $\eta$ . For example, one can build up the Legendre polynomials

$$(20) \quad P_l(r) = \sum_{i=1}^l a_i r^i.$$

However, this step is unstable, because the coefficients  $a_i$  rapidly become very large.

**F. Comparison of linear methods.** We can summarize the above methods in the following table. Here the second column refers to the measurement precision needed to achieve a given resolution.

method	required precision	algorithm
Barber-Brown	higher	stable
Calderón	higher	stable
modified Calderón	lower	must be regularized
moment method	lower	unstable step

Thus we see that stable algorithms seem to require high-precision data. If one measures with current patterns for which less precision is required and changes basis to obtain the data needed for a stable algorithm, then this change of basis is itself unstable. We expect to see a similar phenomenon in the full nonlinear case.

### Nonlinear methods

**A. The Sylvester-Uhlmann-Nachman method.** This method is based on the discovery [SU] that, at least in dimensions higher than two, there are solutions of (1) that approach  $\sqrt{\sigma} \exp(i\zeta \cdot x)$  for large  $|\zeta|$ . The boundary values of these solutions can be obtained from the map  $R$  by solving an integral equation [N]. Using these special solutions in (8) allows the inverse problem to be solved by an inverse Fourier transform or by a Cauchy-type integral formula [N].

It is not known whether this method can be made into an algorithm that will work in practice. However, it is one of the most promising approaches to the full nonlinear problem. Another approach worth exploring is based on invariant imbedding.

**B. Invariant imbedding.** The idea of invariant imbedding is to imbed the conductivity in a one-parameter family of conductivities. We will consider the particular family in which the conductivity is truncated outside the disk of radius  $a$ . Outside the disk of radius  $a$ , the conductivity  $\sigma_a$  is defined to be a constant. One can then obtain an equation for  $\partial R / \partial a$ , the change in the data with respect to the parameter  $a$ . This equation, which is called the invariant imbedding equation, involves only knowledge of the conductivity on the circle of radius  $a$ .

This suggests that the inverse problem might be solved by the following layer-stripping algorithm. Let us first consider an outline of the algorithm,

and then go into more detail below. We will denote by  $R(a, b)$  the boundary map on the circle of radius  $b$  corresponding to  $\sigma_a$ .

- (1) Measure  $R(r_0, r_0)$ .
- (2) Find  $\sigma$  on the boundary  $r = r_0$ .
- (3) Use a finite difference approximation to the invariant imbedding equation to obtain the boundary map corresponding to the conductivity  $\sigma_{r_0-\Delta}$ :

$$R(r_0 - \Delta, r_0) \approx R(r_0, r_0) - \Delta \left. \frac{\partial R(a, r_0)}{\partial a} \right|_{a=r_0}.$$

- (4) Propagate the measurements to the inner surface to obtain  $R(r_0 - \Delta, r_0 - \Delta)$ .
- (5) Replace  $r_0$  by  $r_0 - \Delta$  and repeat, starting with step (2).

Thus this algorithm proceeds by first finding the conductivity on the outer boundary (which is only a mildly ill-posed step), and then using this information to synthesize the data we would have measured if we could have made measurements on a surface inside the body. By repeating this process over and over, we successively strip away layers of the body, finding the conductivity as we go in.

This approach has some apparent advantages. First, part of the ill-posedness is made explicit, in step (4). This problem has already been studied [CF]. Second, a naive version of this algorithm uses only  $L \times L$  matrices, as opposed to the  $L^2 \times L^2$  matrices needed by a naive least-squares algorithm. (Here  $L$  is the number of electrodes.) Third, the layer-stripping algorithm requires fewer operations than a naive least-squares one. However, because of the instability of the algorithm some modifications are needed to make it work [CISI, SCII].

### The invariant imbedding equation (step (3))

The idea behind the invariant imbedding equation is the following. First, we note that the operator  $R$  is the restriction to the boundary of the Green's function  $G_a$  defined by

$$(21a) \quad \nabla \cdot \sigma_a(x) \nabla G_a(x, y) = -\delta(x - y) \quad \text{in } \Omega,$$

$$(21b) \quad \sigma_a \left. \frac{\partial G_a}{\partial \nu} \right|_{\partial \Omega} = \text{const},$$

$$(21c) \quad \int_{\partial \Omega} G_a = 0.$$

If we differentiate (21) with respect to  $a$ , we obtain

$$(22a) \quad \nabla \cdot \sigma_a \nabla \frac{\partial G}{\partial a} = -\nabla \cdot \frac{\partial \sigma}{\partial a} \nabla G \quad \text{in } \Omega,$$

$$(22b) \quad \sigma_a \frac{\partial}{\partial \nu} \frac{\partial G}{\partial a} \Big|_{\partial \Omega} = \frac{-\partial \sigma}{\partial a} \frac{\partial G}{\partial \nu},$$

$$(22c) \quad \int_{\partial \Omega} \frac{\partial G}{\partial a} = 0.$$

If we then multiply (21a) by  $\partial G/\partial a$ , multiply (22a) by  $G$ , subtract the resulting equations, and use Green's theorem, we obtain

$$(23) \quad -\int_{\partial \Omega} \left( G \sigma \frac{\partial}{\partial \nu} \frac{\partial G}{\partial a} - \frac{\partial G}{\partial a} \sigma \frac{\partial G}{\partial \nu} \right) = \frac{-\partial G}{\partial a} + \int_{\partial \Omega} G \frac{\partial \sigma}{\partial a} \frac{\partial G}{\partial \nu} - \int_{\Omega} \nabla G \cdot \frac{\partial \sigma}{\partial a} \nabla G.$$

We now note that (21b) and (22c) imply that the second term on the left side vanishes. By (22b), the first term on the left cancels with the second term on the right. Finally, we use the fact that  $\partial \sigma/\partial a = (\sigma - 1)\delta$  to obtain

$$(24) \quad \frac{\partial G}{\partial a}(x, y) = -\int_{\partial \Omega_a} (\sigma(z) - 1) \nabla G(x, z) \cdot \nabla G(y, z) dS_z,$$

where  $\partial \Omega_a$  denotes the circle of radius  $a$ . The gradients can be decomposed into radial and angular components. The radial derivatives are known from (21b), and the angular derivatives can be obtained from the data. We note that the right side of (24) depends entirely on quantities on the circle  $r = a$ .

Actually, the above derivation is too simplistic and (24) is not correct. This is because almost everything in sight is singular. If we follow the above plan, being careful to interpret everything in the distribution sense, then we obtain

$$(25) \quad \frac{\partial}{\partial a} \langle j_1, R_a j_2 \rangle = \frac{1}{a} \int_0^{2\pi} [1 - \sigma(a, \theta)] \frac{\partial u_1}{\partial \theta}(a, \theta) \frac{\partial u_2}{\partial \theta}(a, \theta) d\theta \\ - a \int_0^{2\pi} \left( 1 - \frac{1}{\sigma(a, \theta)} \right) j_1(\theta) j_2(\theta) d\theta,$$

where  $u_i$  satisfies

$$(26) \quad \nabla \cdot \sigma_a \nabla u_i = 0 \quad \text{in } \Omega, \\ \sigma_a \frac{\partial u_i}{\partial \nu} \Big|_{\partial \Omega} = j_i.$$

Equation (25) is the invariant imbedding equation.

The quantity  $\partial u/\partial \theta$  appearing on the right side of (25) can be calculated from the Fourier series of  $u$  on the boundary. In particular, if

$$u = \sum_{n=-\infty}^{\infty} v_n e^{in\theta},$$

then

$$\frac{\partial u}{\partial \theta} = \sum_{n=-\infty}^{\infty} i n v_n e^{in\theta}.$$

We note that the factor of  $n$  appearing on the right side is one way in which the ill-posedness of the problem manifests itself.

### How to get the conductivity on the boundary (step (2))

It is easy to see how to get the conductivity on the boundary if we write (21a) as

$$(27) \quad \sigma \nabla^2 G + \nabla \sigma \cdot \nabla G = -\delta.$$

The leading order singularity of  $G$  satisfies

$$(28) \quad \sigma \nabla^2 G_0 = -\delta,$$

which has solutions of the form

$$(29) \quad G_0(x, y) = c \frac{\log|x-y|}{\sigma(x)} + (\text{smoother stuff}).$$

Thus we can write  $G$  as

$$(30) \quad G(x, y) = \frac{c \log|x-y|}{\sigma(x)} + (\text{smoother stuff}).$$

This enables us to obtain the conductivity as

$$(31) \quad \sigma(x) = \lim_{y \rightarrow x} \frac{c \log|x-y|}{G(x, y)}.$$

More detailed, rigorous proofs that  $\sigma$  on the boundary can be recovered from boundary measurements may be found in [KV, SU].

### How to propagate the measurements to the inner surface (step (4))

The measurements can be propagated through an annulus of constant conductivity by noting that the current density and voltage on the inner circle are linear functions of the current density and voltage on the outer circle. Explicitly, we write  $\tilde{v}$  and  $\tilde{j}$  for the voltage and current density on the inner circle of radius  $a$ , and  $v$  and  $j$  for the corresponding quantities on the outer circle of radius  $r_0$ . The linear relation between these quantities can be written as

$$(32) \quad \tilde{v} = Av + Bj, \quad \tilde{j} = Cv + Dj.$$

The operators  $A$ ,  $B$ ,  $C$ , and  $D$  can be found by separation of variables. In particular, the Fourier coefficients  $\tilde{v}_n$  of  $\tilde{v}$  can be written as

$$(33) \quad \tilde{v}_n = \frac{1}{2} \left[ \left( \frac{a}{r_0} \right)^n + \left( \frac{a}{r_0} \right)^{-n} \right] v_n + \frac{r_0}{2n} \left[ \left( \frac{a}{r_0} \right)^n + \left( \frac{a}{r_0} \right)^{-n} \right] j_n,$$

where  $v_n$  and  $j_n$  are the Fourier coefficients of  $v$  and  $j$ , respectively. The formula for  $\tilde{j}_n$  is similar. Note that the terms  $(a/r_0)^{-n}$  on the right side

of (33) are very large if  $a$  is small or  $n$  is large. This is a symptom of the ill-posedness of this step.

By the definition of the boundary map  $R = R(r_0, r_0)$ , we can write

$$(34) \quad \hat{v} = (AR + B)j, \quad \hat{j} = (CR + D)j.$$

If we denote by  $R(a, a)$  the boundary map on the circle of radius  $a$ , then the relation  $\hat{v} = R(a, a)\hat{j}$  implies

$$(35) \quad R(a, a) = (AR + B)(CR + D)^{-1}.$$

### The Riccati equation

Generally the method of invariant imbedding involves a Riccati equation. In this case, we can obtain a Riccati equation by combining steps (3) and (4). However, it is not clear whether it is a good idea to do this, because the Riccati equation recombines a number of unstable steps that may be easier to analyze separately.

Algorithms based on these invariant imbedding ideas have produced reconstructions from synthetic data [CISI, SCII]. It remains to be seen whether such algorithms can be used with experimental data.

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ON THE CONSTRUCTION OF INTEGRABLE  
XXZ HEISENBERG MODELS WITH ARBITRARY SPIN

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ABSTRACT. The use of finite-dimensional representations of the quantum group  $[SU(2)]_q$  to the construction of spin  $S$  integrable systems with XXZ anisotropy in the quantum inverse scattering framework is discussed. To obtain a physical spin chain operator the representation of the quantum group has to be self-adjoint. This requirement gives rise to a commensuration condition between the value of  $S$  and the anisotropy of the system.

Recently, surprising connections have been unveiled between certain quantum deformations of Lie-algebras [1-3] and the theory of integrable quantum systems, especially within the framework of the quantum inverse scattering method (QISM) [4,5]. While quantum group symmetries manifest themselves in various ways [6-8] in the context of integrable systems they seem to be particularly useful in the construction of higher spin integrable spin chains [1,9-11].

In this note this application of quantum groups is studied in some detail. It is well known that a solution of the Yang-Baxter relation

$$(1) \quad R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)$$

defines an integrable two-dimensional vertex model with row-to-row transfer matrix

$$(2) \quad \mathcal{T}_N(u) = R_{1,a}(u) \otimes R_{2,a}(u) \otimes \dots \otimes R_{N,a}(u)$$

or, equivalently, a (1+1)-dimensional quantum spin chain with Hamiltonian

$$(3) \quad \mathcal{H} = \sum_{i=1}^N \mathcal{H}_{i,i+1}, \quad \mathcal{H}_{i,i+1} = -i\mathcal{P}_{i,i+1} \frac{\partial}{\partial u} R_{i,i+1}(u) \Big|_{u=u_0}$$

where  $\mathcal{P}_{i,i+1}$  is the permutation operator and  $u_0$  is the special value of the *spectral parameter*  $u$  where the transfer matrix reduces to a shift-operator.

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In (1) the so-called  $R$ -matrix  $R_{ij} \otimes \mathbf{1}_k$  is a linear operator acting on the space  $V_i \otimes V_j \otimes V_k$ . In the following I shall use  $V_i \sim \mathbb{C}^{q_i}$  for some integer  $q_i$  so that  $V_i$  can be thought of as the Hilbert-space of a spin  $S = (q_i - 1)/2$ .

Clearly, the simplest nontrivial case is  $q_i \equiv 2$  for  $i = 1, 2, 3$ . Several solutions are known for this case. Here I shall concentrate on the  $R$ -matrix corresponding to the 6-vertex model (or, equivalently, the spin- $\frac{1}{2}$  XXZ Heisenberg chain with anisotropy  $\Delta = \cos \gamma$ ) which can be written as

$$(4) \quad R_{6v}(u) = \begin{pmatrix} \frac{1}{2}(e^{u+i\gamma/2}\mathbf{S}_0 - e^{-u-i\gamma/2}\mathbf{S}_1) & i \sin \gamma \mathbf{S}^- \\ i \sin \gamma \mathbf{S}^+ & \frac{1}{2}(e^{u+i\gamma/2}\mathbf{S}_1 - e^{-u-i\gamma/2}\mathbf{S}_0) \end{pmatrix}$$

with  $\mathbf{S}_{0,1} = \exp(\pm i\gamma \mathbf{S}^z)$ ,  $\mathbf{S}^\pm$  and  $\mathbf{S}^z$  are spin- $\frac{1}{2}$  representations of  $SU(2)$ .

Before attempting to find a solution of the Yang-Baxter equation (1) with  $q_i > 2$  for all  $i$ , let us consider the case  $q_1 = q_2 = 2$ ,  $q_3 = N > 2$

$$(5) \quad R_{12}^{2,2}(u-v)R_{13}^{2,N}(u)R_{23}^{2,N}(v) = R_{23}^{2,N}(v)R_{13}^{2,N}(u)R_{12}^{2,2}(u-v).$$

(Superscripts are used to indicate the dimension of the spaces in which  $R$  acts nontrivially.) It will turn out that this detour allows for a simple way to generalize the 6-vertex model to higher-dimensional ones. For  $R_{12}^{2,2}$  in (5) I use the expression (4).  $R^{2,N}$  are  $2 \times 2$ -matrices in  $V_{1,2}$ , respectively, with operator-valued elements acting in  $V_3$ . To find a solution of the Yang-Baxter equation I need an ansatz for  $R^{2,N}(u)$ . The easiest ansatz one can think of is to take the expression (4) and replace the  $SU(2)$  spin- $\frac{1}{2}$  operators  $\mathbf{S}$  by unknown operators  $L$  acting on  $\mathbb{C}^N$ . Inserting this ansatz into the Yang-Baxter equation (5) one finds that they are indeed satisfied provided that the  $L$ -operators satisfy the following relations:

$$(6) \quad [L_0, L_1] = 0, \quad [L^+, L^-] = \frac{1}{2i \sin \gamma} (L_0^2 - L_1^2),$$

$$L^\pm L_0 = e^{\mp i\gamma} L_0 L^\pm, \quad L^\pm L_1 = e^{\pm i\gamma} L_1 L^\pm.$$

The Casimir operators of the algebra defined by (6) are easily obtained by taking the *quantum determinant* [5] of the  $R$ -matrix

$$(7) \quad d_q(u) = (R^{2,N}(u - i\frac{\gamma}{2}))_{1,1}(R^{2,N}(u + i\frac{\gamma}{2}))_{2,2} - (R^{2,N}(u - i\frac{\gamma}{2}))_{1,2}(R^{2,N}(u + i\frac{\gamma}{2}))_{2,1}.$$

(By construction this quantity commutes with all the elements of  $R^{2,N}$ .) Expanding  $d_q(u)$  in linearly independent hyperbolic functions of  $u$  one finds the two central elements of the quantum group  $[SU(2)]_q$  to be  $\mathcal{L}_1 = L_0 L_1$  and

$$(8) \quad \mathcal{L}_2 = \frac{1}{2 \sin^2 \gamma} \left( \sin^2 \gamma (L^+ L^- + L^- L^+) - \frac{1}{2} \cos \gamma ((L_0)^2 + (L_1)^2) \right).$$

Above I have constructed a solution of the Yang-Baxter equation (5) provided that there exists an  $N$ -dimensional representation of the operators  $L_{0,1}$ ,  $L^\pm$  satisfying the relations (6).

To construct such representations let us work in a basis where both  $L_0$  and  $L_1$  are diagonal, i. e.  $L_{0,1}|l_0, l_1\rangle = l_{0,1}|l_0, l_1\rangle$ . Using the commutation relations (6) one obtains  $L^\pm|l_0, l_1\rangle \sim |e^{\pm i\gamma}l_0, e^{\mp i\gamma}l_1\rangle$ . For any irreducible  $N$ -dimensional representation of the quantum group (6) the set of operators  $\{(L^\pm)^k, 0 \leq k < M\}$  has to be linear independent for  $M \leq N$  but linear dependent for  $M > N$ . This requires  $L^\pm$  to be either idempotent or nilpotent.

A necessary condition for  $L^\pm$  to be idempotent is that  $e^{i\gamma}$  be an  $N$ -th root of unity. A special case of this representation has been studied recently by Bazhanov and Stroganov [11]: They have shown that the corresponding solution of the Yang-Baxter equation generates the integrable  $N$ -state chiral Potts model [12,13].

Here I shall concentrate on the only other possible choice for a finite dimensional representation of the quantum group operators, namely nilpotent  $L^\pm$ . For the representation to be irreducible this requires  $(L^\pm)^M \neq 0$  for  $M = 1, \dots, N-1$  and  $(L^\pm)^N = 0$  where  $N = 2S+1$  is the dimension of the representation. In analogy to the construction of a spin- $S$  representation of  $SU(2)$  one introduces an operator  $L^z = \text{diag}(S, S-1, \dots, -S)$ . Then the representation of the quantum group is given by

$$(9) \quad \begin{aligned} L_{0,1}|m\rangle &= e^{\pm i\gamma L^z}|m\rangle = e^{\pm i\gamma m}|m\rangle, & m = -S, \dots, S \\ L^-|m\rangle &= d_m|m-1\rangle, & m = -S+1, \dots, S \\ L^+ &= (L^-)^\top. \end{aligned}$$

The commutation relations (6) are satisfied if one chooses the matrix elements of  $L^\pm$  to be [1]

$$(10) \quad d_m^2 = \frac{\sin \gamma(S+m) \sin \gamma(S-m+1)}{\sin^2 \gamma}.$$

Note that this representation is reducible if  $\gamma$  takes one of the values  $\gamma = (k\pi/n)$  with integer  $1 \leq k < n$  and  $n = 2, 3, \dots, 2S$ . Furthermore, it is self-adjoint only if

$$(11) \quad \begin{aligned} d_m^2 &> 0 & \forall m = -S+1, \dots, S \\ &\text{or} \\ d_m^2 &< 0 & \forall m = -S+1, \dots, S. \end{aligned}$$

(These inequalities are equivalent to the requirement that certain bound states in the spectrum of the spin chain Hamiltonian are allowed [10].)

With this representation of the quantum group  $[SU(2)]_q$  I have found a solution to the Yang-Baxter equation (5):

$$(12) \quad R_{hw}^{2,2S+1}(u) = \begin{pmatrix} \sinh(u + i\gamma(\frac{1}{2} + L^z)) & (i \sin \gamma)L^- \\ (i \sin \gamma)L^+ & \sinh(u + i\gamma(\frac{1}{2} - L^z)) \end{pmatrix}.$$

From this expression a solution of the Yang-Baxter equation with  $V_i \sim \mathbb{C}^{2S+1}$  is easily obtained [1,10]. It defines a vertex model with current-conservation on the vertices and currents  $-S, \dots, +S$  on the bonds [14]. In the Hamiltonian limit it is a spin- $S$  generalization

of the XXZ Heisenberg chain [1,9,10,15]. The commensuration conditions (11) ensure the full set of symmetries in the vertex model and the self-adjointness of the Hamiltonian [10]. The spectrum of the Hamiltonian for values of  $\gamma$ ,  $S$  satisfying (11) has been obtained using the algebraic Bethe Ansatz [9]. From this solution it is known that this system has massless excitations only. The critical properties depend on the details of the relation between  $\gamma$  and  $S$ . For given  $S$ , each of the allowed intervals defined by (11) corresponds to a different universality class. In general, the continuum limit can be described in terms of products of Gaussian and parafermionic fields [10,16].

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## A GEOMETRIC CONSTRUCTION OF SOLUTIONS OF MATRIX HIERARCHIES

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**ABSTRACT.** We associate to each maximal commutative algebra in  $gl_m(\mathbb{C})$  a hierarchy of differential-difference equations and show how we can construct solutions of the hierarchy starting with a Grassmann manifold.

### 1. The equations.

Let  $\mathfrak{h}$  be a maximal commutative subalgebra of  $gl_m(\mathbb{C})$  and let  $\{E_\alpha, 1 \leq \alpha \leq r\}$  be a basis of  $\mathfrak{h}$ . We consider loops

$$\sum_{i \in \mathbb{Z}} h_i \lambda^i, h_i \in \mathfrak{h},$$

that converge on a neighbourhood of the unit circle. The group  $L(\exp(\mathfrak{h}))$  consists of all the loops in  $\mathfrak{h}$  that are invertible on some neighbourhood of  $|\lambda| = 1$ . In  $L(\exp(\mathfrak{h}))$  we consider the subgroups

$$\Gamma_+ = \left\{ \exp \left( \sum_{i \geq 1} \sum_{1 \leq \alpha \leq r} t_{i\alpha} E_\alpha \lambda^i \right) \in L(\exp(\mathfrak{h})) \right\} \text{ and}$$

$$\Gamma_- = \left\{ \exp \left( \sum_{i \leq 0} h_i \lambda^i \right) \in L(\exp(\mathfrak{h})), \text{ with } h_0 \in \exp(\mathfrak{h}) \right\}.$$

Then there exists a subgroup  $\Delta$  of  $L(\exp(\mathfrak{h}))$ , whose elements are all of the form

$$\delta = \sum_{i=-N}^N h_i \lambda^i, h_i \in \mathfrak{h},$$

and that satisfies  $\Gamma_+ \cap \Delta = \Gamma_- \cap \Delta = \{\text{Id}\}$  and  $L(\exp(\mathfrak{h})) = \Gamma_- \cdot \Delta \cdot \Gamma_+$ .

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Let  $R$  be the ring of meromorphic functions on  $\Gamma_+$ . We denote the partial derivative w.r.t.  $t_{i\alpha}$  by  $\partial_{i\alpha}$ . Since the identity belongs to  $\mathfrak{h}$ , we have  $\text{Id} = \sum_{\alpha} c_{\alpha} E_{\alpha}$  and we put  $\partial = \sum_{\alpha} c_{\alpha} \partial_{i\alpha}$ . We write  $Ps(\partial)$  for the ring of pseudo differential operators in  $\partial$  with coefficients from  $gl_m(R)$ . For each  $P = \sum p_i \partial^i$  in  $Ps(\partial)$  we denote by  $P_+$  the differential operator part  $\sum_{i \geq 0} p_i \partial^i$  of  $P$ . Our interest is to find operators  $L$  and  $V_{\alpha}$  in  $Ps(\partial)$  of the form

$$L = \partial + \sum_{j>0} l_j \partial^{-j} \text{ and } V_{\alpha} = E_{\alpha} + \sum_{j>0} v_{j\alpha} \partial^{-j} \quad (1.1)$$

that satisfy the following Lax-equations

$$[L, V_{\alpha}] = [V_{\alpha}, V_{\beta}] = 0 \quad (1.2)$$

$$\partial_{i\alpha}(L) = [(L^i V_{\alpha})_+, L] = [B_{i\alpha}, L] \quad (1.3)$$

$$\partial_{i\alpha}(V_{\beta}) = [B_{i\alpha}, V_{\beta}]. \quad (1.4)$$

These equations give rise to non-linear differential equations for the coefficients  $l_j$  and  $v_{j\alpha}$ . If  $\mathfrak{h}$  consists of the diagonal matrices and  $E_{\alpha}$  is the diagonal matrix with 1 as  $(\alpha, \alpha)$ -entry and elsewhere zeros, then this system of equations was introduced by the Sato school, see [3], and is called the multicomponent KP-hierarchy. We will call (1.2), (1.3) and (1.4) the equations of the  $\mathfrak{h}$ -hierarchy. Also this hierarchy possesses a linearization. For that we introduce a  $Ps(\partial)$ -module  $M$  consisting of all formal products

$$\left\{ \sum_{j=-\infty}^N f_j \lambda^j \right\} \exp\left( \sum_{i \geq 1} \sum_{1 \leq \alpha \leq r} t_{i\alpha} E_{\alpha} \lambda^i \right) = \left\{ \sum_{j=-\infty}^N f_j \lambda^j \right\} g(\lambda),$$

where  $f_j \in gl_m(R)$ . An element  $g \in gl_m(R)$  acts on such expressions by simply multiplying the left factor on the left with  $g$ . The operator  $\partial$  acts on  $M$  according to

$$\partial \left\{ \sum f_j \lambda^j \right\} g(\lambda) = \left\{ \sum \partial(f_j) \lambda^j + \sum f_j \lambda^{j+1} \right\} g(\lambda).$$

These operations determine a  $Ps(\partial)$ -module structure on  $M$ .

Consider operators  $L$  and  $V_{\alpha}$  of the form 1.1. Let  $\delta = \sum \delta_i \lambda^i$  be an element of  $\Delta$ . We call  $\psi \in M$  a wavefunction of type  $\delta$  for the operators  $(L, V_{\alpha})$ , if it satisfies the equations

$$L(\psi) = \lambda \psi, V_{\alpha} \psi = \psi E_{\alpha}, \partial_{i\alpha}(\psi) = B_{i\alpha} \psi$$

and if it has the form

$$\psi = \left\{ \left( \text{Id} + \sum_{j=1}^{\infty} \alpha_j \lambda^{-j} \right) \left( \sum \delta_j \lambda^j \right) \right\} g(\lambda) = \hat{\psi} \delta g(\lambda).$$

By construction the wavefunction  $\psi$  determines the operators  $L$  and  $V_{\alpha}$  completely and each set of operators  $(L, V_{\alpha})$  that possesses a wavefunction of type  $\delta$  satisfies the equations (1.2), (1.3) and (1.4).

## 2. The Grassmann setting.

Before we give the construction of solutions to the  $\mathfrak{h}$ -hierarchy, we have to adjust some ingredients from the Grassmann picture in [2] to our situation. Let  $\mathcal{H}$  be the Hilbertspace  $L^2(S^1, \mathbb{C}^m)$ . Its elements are of the form  $\sum_{i \in \mathbb{Z}} \alpha_i \lambda^i$ ,  $\alpha_i \in \mathbb{C}^m$ . The space  $\mathcal{H}$  decomposes as  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ , where  $\mathcal{H}_+ = \{\sum_{i \geq 0} \alpha_i \lambda^i \in \mathcal{H}\}$ . We denote the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{H}_+$  by  $p_+$ . As in [2], we associate to this decomposition a Grassmann manifold that is a homogeneous space for the restricted linear group  $GL_{\text{res}}(\mathcal{H})$ . The connected component containing  $\mathcal{H}_+$  is denoted by  $Gr$  and the connected component  $GL_{\text{res}}^0(\mathcal{H})$  of  $GL_{\text{res}}(\mathcal{H})$  acts transitively on it. Since  $L(\exp(\mathfrak{h}))$  is commutative we can define an action of  $L(\exp(\mathfrak{h}))$  on  $\mathcal{H}$  by

$$\left(\sum \alpha_i \lambda^i\right) \mapsto \left(\sum \alpha_i \lambda^i\right)(\Sigma h_j \lambda^j).$$

All these automorphisms belong to  $GL_{\text{res}}(\mathcal{H})$  and thus one has embedded  $L(\exp(\mathfrak{h}))$  in  $GL_{\text{res}}(\mathcal{H})$ . Let  $\Delta^\circ$  be the group  $\Delta \cap GL_{\text{res}}^0(\mathcal{H})$ .

Every  $W \in Gr$  can be obtained as the image of an embedding  $w$  of  $\mathcal{H}_+$  into  $\mathcal{H}$  such that  $w_+ = p_+ \circ w$  has the form "identity + trace-class" and  $w_- = (\text{Id} - p_+) \circ w$  is Hilbert-Schmidt. The space of all such embeddings we denote by  $P$ . To lift the action of  $GL_{\text{res}}^0(\mathcal{H})$  on  $Gr$  to one on  $P$  we need the extension  $G$  of  $GL_{\text{res}}^0(\mathcal{H})$  defined by

$$G = \{(g, q) \in GL_{\text{res}}^0(\mathcal{H}) \times \text{Aut}(\mathcal{H}_+), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, aq^{-1} - \text{Id} \text{ is trace-class}\}$$

The group  $G$  acts on  $P$  by  $w \mapsto gwq^{-1}$ . Elements of  $GL_{\text{res}}^0(\mathcal{H})$  of the form  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$  or  $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$  can directly be lifted to  $G$  by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, a\right) \text{ and } \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, a\right).$$

This will be assumed from now on. For each embedding  $w$  as above we define a function  $\tau_w : G \rightarrow \mathbb{C}$  by

$$\tau_w\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}, q^{-1}\right) = \det(aw_+q^{-1} + bw_-q^{-1}).$$

## 3. The construction of solutions.

Let  $\delta \in \Delta^\circ$ . With  $\delta$  we associate the open set  $Gr(\delta)$  of  $Gr$  consisting of all  $W$  in  $Gr$  for which there exists a  $\gamma$  in  $\Gamma_+$  such that the orthogonal projection of  $W\gamma^{-1}\delta^{-1}$  onto  $\mathcal{H}_+$  is a bijection. For  $W \in Gr(\delta)$  we consider

$$\Gamma(\delta, W) = \{\gamma \in \Gamma_+, p_+ : W\gamma^{-1}\delta^{-1} \rightarrow \mathcal{H} \text{ is bijective}\}.$$

Let  $\{e_i, 1 \leq i \leq m\}$  be the standardbasis of  $\mathbb{C}^m$ . For each  $W$  in  $Gr(\delta)$  and  $\gamma \in \Gamma(\delta, W)$  we define  $\hat{\psi}_{W,i}(\gamma, \lambda)$  as the unique element in  $W\delta^{-1}\gamma^{-1}$  that is mapped by  $p_+$  onto  $e_i$ . Let  $\psi_{W,i}(\gamma, \lambda)$  be the element in  $W$  given by  $\gamma\delta \cdot (\hat{\psi}_{W,i}(\gamma, \lambda))$ . Form matrices  $\hat{\psi}_W^\delta$  and  $\psi_W^\delta$  by taking the  $\{\hat{\psi}_{W,i}^\delta\}$  resp.  $\{\psi_{W,i}^\delta\}$  as rows. These matrices have the form

$$\hat{\psi}_W^\delta(\gamma, \lambda) = \text{Id} + \sum_{m=1}^{\infty} \alpha_m(\gamma) \lambda^{-m}$$

and

$$\psi_W^\delta(\gamma, \lambda) = \{\text{Id} + \sum_{m=1}^{\infty} \alpha_m(\gamma) \lambda^{-m}\} \delta g(\lambda).$$

Hence  $\psi_W^\delta$  has the proper form for a wavefunction of type  $\delta$ .

For  $\zeta \in \mathbb{C}$ ,  $|\zeta| > 1$ , and  $1 \leq i \leq m$ , we consider the element  $q_\zeta^{(i)}$  of  $Gl_{\text{res}}^{(0)}(\mathcal{H})$  given by

$$(h_1(\lambda), \dots, h_m(\lambda)) \mapsto (\dots, h_{i-1}(\lambda), (1 - \frac{\lambda}{\zeta})h_i(\lambda), h_{i+1}(\lambda), \dots).$$

For  $i \neq j$ , we define  $\Lambda_{i/j}$  in  $Gl_{\text{res}}^0(\mathcal{H})$  by

$$(h_1(\lambda), \dots, h_m(\lambda)) \mapsto (\dots, \lambda h_i(\lambda), \dots, \lambda^{-1} h_j(\lambda), \dots).$$

Now we can formulate our first result.

**3.1. Theorem.** Let  $w \in P$  be such that  $w(\mathcal{H}) = W$ .

(a) Then we have for all  $i$ ,  $1 \leq i \leq m$ , that the  $(i, i)$ -entry of  $\hat{\psi}_W^\delta$  is given by

$$(\hat{\psi}_W^\delta(\gamma, \zeta))_{ii} = \frac{\tau_{\delta^{-1}w}(\gamma q_\zeta^{(i)})}{\tau_{\delta^{-1}w}(\gamma)}.$$

(b) For  $i \neq j$ , there is a lifting  $\hat{\Lambda}_{i/j}$  of  $\Lambda_{i/j}$  to  $G$  such that

$$(\hat{\psi}_W^\delta(\gamma, \zeta))_{ji} = \frac{\tau_{\delta^{-1}w}(\gamma \hat{\Lambda}_{i/j}^{-1} q_\zeta^{(i)})}{\tau_{\delta^{-1}w}(\gamma)}.$$

The formulae in theorem 3.1 are a generalization and a geometric version of formulae for the multicomponent  $KP$ -hierarchy, see [3]. This theorem implies the coefficients  $\alpha_m$  in the powerseries of  $\hat{\psi}_W^\delta$  are meromorphic on  $\Gamma_+$ . We have come now to the final result.

**3.2. Theorem.**

(a) Let  $\delta \in \Delta^\circ$ . If  $W \in Gr(\delta)$ , then  $\psi_W^\delta$  is a wavefunction of type  $\delta$  for a set of pseudodifferential operators  $(L^\delta, V_\alpha^\delta)$  of the form 1.1. We write  $B_{i\alpha}^\delta$  for  $((L^\delta)^i V_\alpha^\delta)_+$ .

(b) If  $W \in Gr(\delta_1)$  and  $W \in Gr(\delta_1 \delta_2)$  then there is  $aU_{\delta_1}^{\delta_2} = (U_{\delta_1}^{\delta_2})_+$  in  $Ps(\partial)$  such that

$$\psi_W^{\delta_1 \delta_2} = (U_{\delta_1}^{\delta_2}) \psi_W^{\delta_1}.$$

This relation leads to the following differential-difference equations of Toda-type

$$B_{i\alpha}^{\delta_1 \delta_2} U_{\delta_1}^{\delta_2} = \partial_{i\alpha} U_{\delta_1}^{\delta_2} + U_{\delta_1}^{\delta_2} B_{i\alpha}^{\delta_1}.$$

(c) For each  $\rho$  in  $\Gamma_-$ , the spaces that  $W$  and  $\rho \cdot W$  give the same solutions of the  $\mathfrak{h}$ -hierarchy.

### 3.3. Remarks

(i) The second part of this theorem gives a geometric interpretation of the differential-difference equations in [1]

(ii) Among the solutions constructed above one can find solutions of well-known equations like the  $N$ -wave equation, the AKNS hierarchy and the Davey-Stewartson II.

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**Lax Pairs, Recursion Operators  
and  
the Perturbation of Nonlinear Evolution Equations**

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**Introduction**

As is well known the methods of inverse scattering can be applied to study the solutions of many integrable nonlinear evolution equations [1]. However, many of these equations are only approximations to the physical systems of interest. For example, in the study of ion-acoustic solitons in plasmas[20], one first assumes that the physical system can be described by the two fluid model, consisting of four coupled, nonlinear partial differential equations. As this system is difficult to solve, one makes further approximations, which magically leads to one nonlinear evolution equation, such as the Korteweg-deVries (KdV) equation.

Experimentalists then try to observe the special solutions of these nonlinear evolution equations; however, much of the physics has been neglected along the way in uncovering the equations governing the evolution of the dynamical variables. Should we expect these special solutions to exist in reality? One way to investigate the answers to this question is to try to put back some of the terms, which have been neglected in the analysis, or to add some physical disturbances to the original model, and see how they may modify our integrable systems.

In this talk a general procedure is given for setting up a direct perturbation method for studying such perturbations of nonlinear evolution equations, which are solvable by the Inverse Scattering Transform. The linearized operator can be paired with the so-called recursion operator to form an additional Lax pair for the nonlinear evolution equation. The eigenfunctions of this spectral problem, which are typically related to products of the eigenfunctions of the original Lax pair, provide the natural basis in which to expand the first order correction.

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### The General Method

We consider the perturbed evolution equation

$$u_t + \mathcal{N}[u] = \epsilon \mathcal{P}[u]. \quad (1)$$

The steps of the proposed method are as follows:

#### GENERAL METHOD

- I. Linearize evolution equation
- II. Find the Perturbation Basis
- III. Invert the Linear Operator
- IV. Obtain Secularity Conditions

In order to implement step I, we assume that the solution of the perturbed equation is close to that of the unperturbed equation. This is accomplished by assuming an expansion for  $u$ ,

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots, \quad (2)$$

and by allowing various parameters, such as the amplitude and the velocity, to vary on a slow time scale,  $\tau = \epsilon t$ .

Inserting these expansions into equation (1) and equating the coefficients of the powers of the small parameter  $\epsilon$ , we arrive at the hierarchy of equations

$$u_{0t} + \mathcal{N}[u_0] = 0 \quad (3)$$

$$u_{1t} + \mathcal{N}'[u_0]u_1 = \mathcal{P}[u_0] - u_{0\tau} \quad (4)$$

$$\dots \mathcal{L}u_n = F_n, \quad n \geq 1. \quad (5)$$

The operator  $\mathcal{N}'[u_0]$  denotes the Fréchet derivative of  $\mathcal{N}$  at  $u_0$ . The first equation confirms that we are close to a solution of the unperturbed equation. The remaining equations involve the linearized version of the unperturbed equation and complicated driving terms,  $F_n$ , resulting from the linearization process.

We now seek to invert the linear operator  $\mathcal{L}$  in order to obtain the corrections  $u_n$ . In the present method this is accomplished by expanding each  $u_n$  in the natural set of basis states, consisting of the eigenfunctions of the linear operator. Namely, we assume that we can find a complete set of eigenfunctions  $\Omega(x, t; \lambda)$ , such that

$$\mathcal{L}\Omega(x, t; \lambda) = \omega(\lambda)\Omega(x, t; \lambda), \quad (6)$$

and that they satisfy orthogonality relations with the adjoint basis of the form:

$$\begin{aligned} \langle \Omega(x, t; \lambda) | \Omega^A(x, t; \lambda') \rangle &= \int_{-\infty}^{\infty} \Omega(x, t; \lambda) \Omega^A(x, t; \lambda') dx \\ &= N(\lambda) \delta(\lambda - \lambda'). \end{aligned} \quad (7)$$

Having completed step II we now carry out step III by expanding  $u_n$  as

$$u_n(x, t) = \int_C f_n(\lambda, t) \Omega(x, t; \lambda) d\lambda, \quad (8)$$

where  $C$  is an appropriate contour. Operating on (8) with  $\mathcal{L}$ , we obtain from equation (5):

$$F_n = \int [f_{n_t}(\lambda, t) + \omega(\lambda) f_n(\lambda, t)] \Omega(x, t; \lambda) d\lambda. \quad (9)$$

Multiplying by the adjoint and integrating over  $x$  gives

$$[f_{n_t}(\lambda, t) + \omega(\lambda) f_n(\lambda, t)] = \frac{\langle F_n | \Omega^A(x, t; \lambda) \rangle}{N(\lambda)}. \quad (10)$$

Solving for  $f_n(\lambda, t)$ , given  $f_n(\lambda, 0)$ , we can write out the solution for  $u_t$ .

It may happen that our solution is not bounded in time. However, from the introduction of the slow time scale, we use the unknown time dependence of our free parameters to eliminate the secularities in time. In the study of soliton perturbations, these secularity conditions determine the soliton shape and velocity.

### The Role of Recursion Operators

The key to carrying out such an analysis is the ability to find the perturbation basis and the adjoint states, as well as to prove that the basis is complete in some suitable space and satisfies the orthogonality relations in equation (7). These questions can be answered by turning to the theory of recursion operators, or strong symmetries, which we now sketch [2, 15, 17, 19].

We consider the (unperturbed) evolution equation

$$u_t + \mathcal{N}[u] = 0. \quad (11)$$

The associated linearized equation can be written as

$$v_t = -\mathcal{N}'[u]v, \quad (12)$$

where previously we had defined  $\mathcal{N}'[u]$  in (4).

Now,  $u_t = \mathcal{M}[u]$  is a symmetry of equation (11) if

$$\mathcal{N}'[u]\mathcal{M}[u] - \mathcal{M}'[u]\mathcal{N}[u] = 0 \quad (13)$$

for all solutions  $u$  of (11). In particular,  $u_t = Du$  is a trivial symmetry, where  $D \equiv \partial_x$ .

Other symmetries of (11) can be generated from the trivial symmetry through the use of a recursion operator,  $R[u]$ , of the form [19]

$$R[u] = a_p[u]D^p + a_{p-1}[u]D^{p-1} + \dots + a_0[u] + a_{-1}[u]D^{-1} + \dots + a_{-m}[u]D^{-m}. \quad (14)$$

Namely, if

$$R[u]_t = [R[u], N'[u]], \quad (15)$$

for all solutions  $u$  of equation (11), then for every  $n > 0$ ,  $u_t = (R[u])^n Du$  is a symmetry of equation (11).

Note that equation (15) is in the form of a Lax equation. Namely, this equation results as a consistency condition for the Lax pair

$$R[u]\phi = \sigma(\lambda)\phi \quad \phi_t = -N'[u]\phi. \quad (16)$$

In other words, the recursion operator gives the spectral problem and the time evolution is governed by the linearized operator. Thus, the eigenfunctions of the linearized operator  $\mathcal{L}$  are the same as those for the recursion operator.

Furthermore, if we know the ordinary Lax pair for the given nonlinear evolution equation, we can write the eigenfunctions of the linearized operator in terms of the eigenfunctions of the standard associated spectral problem. Assuming that the standard Lax pair is given by

$$L\psi = 0 \quad \psi_t = M\psi, \quad (17)$$

we seek a transformation  $\phi = T(\psi, \lambda)$  between the time evolution equations for  $\psi$  and  $\phi$ . Now, inserting  $\psi = T^{-1}(\phi, \lambda)$  into the spectral problem in (17) we obtain the recursion operator in (16).

As an example, we consider the KdV equation. The well known Lax pair for the KdV equation is given by [1]

$$\begin{aligned} D^2\psi + (u + \lambda^2)\psi &= 0 \\ \psi_t &= -D^3\psi - 3(u - \lambda^2)D\psi. \end{aligned} \quad (18)$$

The linearized equation is found as

$$\phi_t = -D^3\phi - 6uD\phi - 6u_x\phi. \quad (19)$$

It can be shown that  $\phi = (D^{-1}\psi)^{1/2}$  will transform the time evolution equation in (18) to the linearized equation (19). Transforming the spectral part of the Lax pair and using (16) yields the desired recursion operator:

$$R[u] = D^2 + 4u + 2u_x D^{-1} \quad \sigma(\lambda) = -4\lambda^2. \quad (20)$$

Therefore, for the perturbation theory for the KdV equation, the perturbation basis consists of the states

$$\Omega(x, t; \lambda) = \psi(x, t; \lambda) = D(\phi^2(x, t; \lambda)). \quad (21)$$

Actually, after selecting the correct contour in (8), one finds that the residues lead us to include several bound state eigenfunctions,  $\{\Omega_k(x, t), \Lambda_k(x, t)\}$ , whose form can be found in [5]. Sachs has shown that this basis is complete over the space of continuous,  $L^1$  functions, providing that  $u_0$  is bounded in the Sobolev  $L^2_1$  norm [18]. The proof relies on the use of the recursion operator, though it is not referred to as such. The orthogonality relations can also be obtained using the recursion operator, though Newell has shown how to compute these from the spectral problem [16].

### Application: KdV Equation

As an example of how this method works, we consider solving the perturbed KdV problem

$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= \epsilon F[u] \\ u(x, 0) &= 2\eta^2 \operatorname{sech}^2 \eta x. \end{aligned} \quad (22)$$

We assume an asymptotic expansion of the form

$$\begin{aligned} u(x, t) &= u_0(x, t) + \epsilon u_1(x, t) + \dots \\ u_0(x, t) &= 2\eta^2 \operatorname{sech}^2 \eta \left(x - \frac{1}{\epsilon} x_0 - x_1\right), \end{aligned} \quad (23)$$

introduce the time scales  $T = t, \tau = \epsilon t$ , and allow the soliton parameters  $\eta, x_0$ , and  $x_1$  to depend on the slow time scale  $\tau$ .

Inserting the above expansions, we find that the lowest order equation is satisfied if  $x_{0\tau} = 4\eta^2$ . The first order equation is

$$\mathcal{L}u_1 = -4\eta\eta_\tau v - 2\eta\eta_\tau \phi v_\phi + 2\eta^3 x_{1\tau} v_\phi + F[u_0] \equiv \mathcal{F}, \quad (24)$$

where  $\mathcal{L}$  is the linearized KdV operator:

$$\mathcal{L} \equiv \partial_T - 4\eta^3 \partial_\phi + 6\eta \partial_\phi u_0 + \eta^3 \partial_\phi^3, \quad (25)$$

and

$$v = \operatorname{sech} \phi, \quad \phi = \eta \left(x - \frac{1}{\epsilon} x_0 - x_1\right).$$

Our aim now is to invert  $\mathcal{L}u_1 = \mathcal{F}$ . We expand  $u_1$  in the basis found in the last section:

$$u_1 = \int_{-\infty}^{\infty} d\lambda f(\lambda, t) \Omega(x, t; \lambda) + f_1(t) \Omega_1(x, t) + g_1(t) \Lambda_1(x, t). \quad (26)$$

Operating on  $u_1$  with  $\mathcal{L}$  and solving for the expansion coefficients, we find

$$f(\lambda, t) = f(\lambda, 0)e^{8i\lambda^3 t} + \int_0^t dt' \frac{\langle \mathcal{F} | \Omega^A \rangle}{2\pi i \lambda a^2(\lambda)} e^{8i\lambda^3(t-t')} \quad (27)$$

$$g_1(t) = g_1(0)e^{8\eta^3 t} - 2i\eta \int_0^t dt' \langle \mathcal{F} | \Omega_1^A \rangle e^{8\eta^3(t-t')} \quad (28)$$

$$f_1(t) = f_1(0)e^{8\eta^3 t} - 4S\eta^2 t g_1(0)e^{8\eta^3 t} - 2i\eta \int_0^t dt' \langle \mathcal{F} | \Lambda_1^A \rangle e^{8\eta^3(t-t')} \\ - 96\eta^3 \int_0^t dt' \int_0^{t'} dt'' \langle \mathcal{F} | \Omega_1^A \rangle e^{8\eta^3(t-t'')}. \quad (29)$$

Using the explicit forms for the eigenfunctions [5], we can write out the solution of our perturbed equation to first order in  $\epsilon$ . However, if we write out the last two terms of  $u_1$ , we have for  $B \equiv f_1 \Omega_1 + g_1 \Lambda_1$ ,

$$B = \tilde{g}_1 [\text{sech}^2 \phi + \frac{1}{2} \phi (\text{sech}^2 \phi)_\phi] + \tilde{h}_1 (\text{sech}^2 \phi)_\phi, \quad (30)$$

where

$$\tilde{g}_1 \equiv \int_0^t dt' \langle \mathcal{F} | \text{sech}^2 \phi \rangle \\ \tilde{h}_1 \equiv -\frac{1}{2} \int_0^t dt' \langle \mathcal{F} | [\phi + 8\eta^3(t-t')] \text{sech}^2 \phi + \tanh \phi \rangle. \quad (31)$$

When the inner products are time independent, which is the case for many perturbations, these coefficients will grow in time. This leads to an unbounded solution. We can eliminate such secular behavior by requiring that

$$\langle \mathcal{F} | \text{sech}^2 \phi \rangle = 0 \quad \langle \mathcal{F} | \phi \text{sech}^2 \phi + \tanh \phi \rangle = 0. \quad (32)$$

Inserting the driving terms from equation (24) into (32), we obtain the time dependence of the soliton parameters:

$$\eta_\tau = \frac{1}{4\eta} \int_{-\infty}^{\infty} F[u_0] \text{sech}^2 \phi d\phi \\ x_{1\tau} = \frac{1}{4\eta^3} \int_{-\infty}^{\infty} F[u_0] [\phi \text{sech}^2 \phi + \tanh \phi] d\phi. \quad (33)$$

The correction in (26) now becomes

$$u_1 = \int_{-\infty}^{\infty} d\lambda \int_0^t dt' \int_{-\infty}^{\infty} dx' \frac{\mathcal{F}(x', t') \Omega^A(x', t'; \lambda)}{2\pi i \lambda a^2(\lambda)} e^{8i\lambda^3(t-t')} \Omega(x, t; \lambda). \quad (34)$$

From an asymptotic analysis of this integral, we find that there is a decaying oscillatory tail plus shelf behind the soliton [3,5,11,16]. The size of this shelf, which is due to a singularity

at  $\lambda = 0$ , can be estimated as [5.11]

$$u_1 \sim \frac{1}{4\eta^3} \int_{-\infty}^{\infty} F[u_0] \tanh^2 \phi \, d\phi. \quad (35)$$

Shelves generally occur for dissipative perturbations, in which case one has to obtain information from the second order in the perturbation theory. Doing this one finds that a correction to the soliton shift has to be made. Namely, we obtain [3.5.6.11]

$$x_{1\tau} = \frac{1}{4\eta^3} \int_{-\infty}^{\infty} F[u_0] [\phi \operatorname{sech}^2 \phi + \tanh \phi + \tanh^2 \phi] \, d\phi. \quad (36)$$

### Conclusion

We have seen a sketch of the proposed method applied to the perturbed KdV equation. Results have been found for a variety of specific perturbations, leading to previous results, as well as producing some new ones [3.11,14]. Examples of such studies include the damped [5] and stochastic KdV [4] equations. We have also used these results to study the effects of truncation errors in discretizations of the KdV equation [9].

Other equations which have been investigated are the sine-Gordon [5], the nonlinear Schrödinger (NLS) [5], the derivative NLS, the coupled NLS, the Toda lattice [8], and evolution equations supporting loop solitons. An indication of how this method can be carried out for the Kadomtsev-Petviashvili equation was given in [7].

The method discussed here is a natural approach to perturbation studies of soliton dynamics, which is similar to methods seen in other works [10,13,16]. One of the uses of such an approach is to obtain more details about the first order correction term. This is important when the growth of the oscillations can have adverse effects on the system under study, such as the study of dynamics under random perturbations, or the effects of discretization error in numerical simulations of these equations. There are many other equations, possessing Lax Pairs and recursion operators, which still need to be studied for their behavior under perturbations. In such cases this method would be appropriate.

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**Time and Temperature Dependent Correlation Function  
of Impenetrable Bose Gas Field Correlator  
in the Impenetrable Bose Gas**

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**Abstract**

Asymptotics of the temperature time-dependent two-point field correlator of two fields for one-dimensional bosons with infinite point-like repulsion is calculated.

## **1 Introduction**

Asymptotics of temperature time-dependent correlation functions of impenetrable Bose gas are given in this paper. The Hamiltonian of this quantum integrable model [1] is

$$H = \int dz (\partial_z \psi^+ \partial_z \psi + c \psi^+ \psi^+ \psi \psi - \hbar \psi^+ \psi) \quad (1)$$

with coupling constant  $c = +\infty$ . Fields  $\psi, \psi^+$  are canonical Bose fields

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in one space dimension, i.e.

$$[\psi(z_1, t), \psi^+(z_2, t)] = \delta(z_1 - z_2). \quad (2)$$

Thermodynamics of the model was constructed in [2]. The chemical potential  $h$  defines the gas density  $D$ . At temperature  $T > 0$ ,  $D \rightarrow 0$  as  $h \rightarrow -\infty$  and  $D \rightarrow \infty$  as  $h \rightarrow +\infty$ . In the state of the thermodynamic equilibrium the distribution of gas particles (possessing momenta  $k$  and energies  $\epsilon(k) = k^2 - h$ ) is given by the Fermi weight  $\rho(k)$

$$\begin{aligned} \rho(k) &= \frac{1}{2\pi} (1 + \exp\{\frac{k^2 - h}{T}\})^{-1}, \\ D &= \int_{-\infty}^{\infty} \rho(k) dk. \end{aligned} \quad (3)$$

Professor C.N. Yang drew our attention to the problem of the calculation of time and space dependent correlation functions at finite temperature.

The two point field correlator considered is defined as usual by

$$\langle \psi(z_2, t_2) \psi^+(z_1, t_1) \rangle_T = \frac{\text{tr}(\exp\{-H/T\} \psi(z_2, t_2) \psi^+(z_1, t_1))}{\text{tr}(\exp\{-H/T\})}. \quad (4)$$

The equal-time ( $t_1 = t_2$ ) temperature correlator was completely described in papers [3-5] where the integrable partial differential equations for it were derived and asymptotics in various regimes was obtained (for zero-temperature equal-time correlator it was done earlier in paper [6]). It should be noted that the starting point for consideration in papers [3-6] was the representation of the equal-time correlator as the first Fredholm minor of a linear integral operator [7].

The analogous representation for time-dependent correlator (4) was given in [8], which was used in [9] to derive integrable differential equations

that completely describing the time-dependent correlator. These results are briefly discussed in the next Section.

## 2 Correlator in terms of Fredholm determinant

Correlator (4) depends essentially only on the rescaled variables

$$x \equiv \frac{1}{2}(z_1 - z_2)\sqrt{T}; t \equiv \frac{1}{2}(t_2 - t_1)T; \beta \equiv h/T \quad (5)$$

which can be easily established using the representation of paper [8]. So one has

$$\langle \psi(z_2, t_2)\psi^+(z_1, t_1) \rangle_T = \sqrt{T} g(x, t, \beta). \quad (6)$$

For  $g$  one has [8,9]

$$g(x, t, \beta) = -\frac{1}{2\pi} \exp\{2it\beta\} b_{++} \det(I + V_T) \quad (7)$$

where the Fredholm determinant of the linear integral operator  $V_T$  acting on the whole real axis enters the right hand side. The kernel of this operator is

$$V_T(\lambda, \mu) = (\lambda - \mu)^{-1} [e_+(\lambda)e_-(\mu) - e_+(\mu)e_-(\lambda)], \quad (8)$$

with

$$\begin{aligned} e_-(\lambda) &= \frac{1}{\pi} \sqrt{\theta(\lambda)} \exp\{it\lambda^2 + ix\lambda\}, \\ e_+(\lambda) &= e_-(\lambda) E(\lambda), \\ E(\lambda) &= \int_{-\infty}^{\infty} \frac{d\nu}{\nu - \lambda} \exp\{-2it\nu^2 - 2ix\nu\}. \end{aligned} \quad (9)$$

The rescaled Fermi weight  $\theta(\lambda)$  is

$$\theta(\lambda) = (1 + \exp\{\lambda^2 - \beta\})^{-1}. \quad (10)$$

Function  $b_{++}$  in (7) is defined as

$$\begin{aligned} b_{++} &= B_{++} - G, \\ G &= \int_{-\infty}^{\infty} d\mu \exp\{-2it\mu^2 - 2ix\mu\}. \end{aligned} \quad (11)$$

“Potential”  $B_{++}$  is one of the potentials  $B_{ik}, C_{ik}(i, k = +, -)$ :

$$\begin{aligned} B_{ik} &= \int_{-\infty}^{\infty} e_i(\mu) f_k(\mu) d\mu; \\ C_{ik} &= \int_{-\infty}^{\infty} \mu e_i(\mu) f_k(\mu) d\mu, \\ B_{+-} &= B_{-+}, \end{aligned} \quad (12)$$

where functions  $f_{\pm}$  are the solutions of equations

$$f_{\pm}(\lambda) + \int_{-\infty}^{\infty} V_T(\lambda, \mu) f_{\pm}(\mu) d\mu = e_{\pm}(\lambda) \quad (13)$$

### 3 Differential equations for correlator

Let us define a vector function  $\vec{f}$ :

$$\vec{f}(\lambda) = \begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix}. \quad (14)$$

Differentiating (13) with respect to  $x, t$  and applying operator  $(2\lambda\partial_{\beta} + \partial_{\lambda})$  (this is done similarly to the equal-time case [3-5]) one obtains a Lax representation

$$L(\lambda)\vec{f} = 0, \quad M(\lambda)\vec{f} = 0, \quad N(\lambda)\vec{f} = 0 \quad (15)$$

with

$$\begin{aligned} L(\lambda) &= \partial_x + i\lambda\sigma_3 - 2iQ, \\ M(\lambda) &= -i\partial_t + \lambda^2\sigma_3 - 2\lambda Q - V, \\ N(\lambda) &= 2\lambda\partial_{\beta} + \partial_{\lambda} + 2i\lambda t\sigma_3 + ix\sigma_3 - 4itQ - 2\partial_{\beta}U, \end{aligned} \quad (16)$$

with

$$\begin{aligned}
 Q &= \begin{pmatrix} 0 & b_{++} \\ B_{--} & 0 \end{pmatrix}, \\
 V &= \begin{pmatrix} 2b_{++}B_{--} & i\partial_x b_{++} \\ -i\partial_x B_{--} & -2B_{--}b_{++} \end{pmatrix}, \\
 U &= \begin{pmatrix} -B_{+-} & b_{++} \\ -B_{--} & B_{+-} \end{pmatrix}.
 \end{aligned}$$

All the operators  $L(\lambda), M(\lambda), N(\lambda)$  should commute at arbitrary value of spectral parameter  $\lambda$ , which lead to nonlinear differential equations for potentials  $B_{ik}$ . Introducing notations

$$\begin{aligned}
 g_- &\equiv \exp\{-2it\beta\} B_{--}; \quad g_+ \equiv \exp\{2it\beta\} b_{++}, \\
 n &\equiv g_+g_-; \quad p \equiv g_- \partial_x g_+ - g_+ \partial_x g_-
 \end{aligned} \tag{17}$$

one obtains first of all the separated nonlinear Schrödinger equations

$$\begin{aligned}
 -i\partial_t g_+ &= 2\beta g_+ + \frac{1}{2} \partial_x^2 g_+ + 4g_+^2 g_- \\
 +i\partial_t g_- &= +2\beta g_- + \frac{1}{2} \partial_x^2 g_- + 4g_-^2 g_+ \\
 -2i\partial_t n &= \partial_x p
 \end{aligned} \tag{18}$$

Equations containing the  $\beta$  derivative are

$$\frac{\partial_\beta \partial_x g_+}{g_+} = \frac{\partial_\beta \partial_x g_-}{g_-} \equiv \phi(x, t, \beta) \tag{19}$$

(new function  $\phi$  is defined here) and

$$\begin{aligned}
 -i\partial_t \phi + 4\partial_\beta p &= 0 \\
 \partial_x \phi + 8\partial_\beta n + 2 &= 0
 \end{aligned} \tag{20}$$

This is a complete set of equations for potentials  $b_{++}(B_{++})$  and  $B_{--}$ . Initial data at  $t = 0$  for these equations can be extracted from equal-time correlator  $g(x, t = 0, \beta) = g(x, \beta)$  completely described in [4,5]. Other potentials  $B, C$

(12) are defined in terms of these two. Namely,

$$\begin{aligned}\partial_x B_{+-} &= 2in; \partial_t B_{+-} = -p; \\ \partial_\beta B_{+-} &= -\frac{ix}{2} - \frac{i\phi}{4}\end{aligned}\quad (21)$$

and

$$\begin{aligned}C_{++} &= \frac{i}{2}\partial_x B_{++} + B_{++}B_{+-} - 2GB_{+-} \\ C_{--} &= -\frac{i}{2}\partial_x B_{--} - B_{+-}B_{--} \\ \partial_x(C_{+-} + C_{-+}) &= (B_{++} - 2G)\partial_x B_{--} - B_{--}\partial_x B_{++}\end{aligned}\quad (22)$$

Correlator (6),(7) is essentially a products of potential  $g_+$ (17) and  $\exp\{\sigma\}$ ,

$$\sigma = \ln \det(I + V_T)$$

It is easy to get an equation involving only  $g_+$  expressing from the first of equations (18)

$$g_- = \frac{1}{8}(-2i\partial_t g_+ - 4\beta g_+ + \partial_x^2 g_+)g_+^{-2}\quad (23)$$

and substituting this expression in remaining equations (18)-(20). At  $t = 0$  function  $\sigma$  is known [4,5]. The derivatives of  $\sigma$  can be expressed in terms of potentials B,C as follows

$$\begin{aligned}\partial_x \sigma &= -2iB_{+-}, \\ \partial_t \sigma &= -2iGB_{--} - 2i(C_{+-} + C_{-+}) \\ \partial_\beta \sigma &= -2it\partial_\beta(C_{+-} + C_{-+}) - 2ix\partial_\beta B_{+-} - 2itB_{--}\partial_\beta B_{++} + \\ &+ 2it(B_{++} - 2G)\partial_\beta B_{--} + 2(\partial_\beta B_{++})(\partial_\beta B_{--}) - 2(\partial_\beta B_{+-})^2\end{aligned}\quad (24)$$

(the calculation of  $\partial_\beta \sigma$  is not very straightforward; it is done analogously to

the corresponding calculation in the equal-time case [4]). This completely defines the correlation function.

## 4 Matrix Riemann Problem

The most interesting from the point of view of physics is the asymptotic behavior of correlation functions at large time and distance. As in the equal-time case [3,5], it can be calculated using the matrix Riemann problem of the inverse scattering method for classical integrable equations (18)-(20) describing the correlator. Consider the matrix Riemann problem for  $2 \times 2$  matrix-valued function  $\chi(\lambda)$  ( $\det \chi(\lambda) \neq 0$ ) which is holomorphic for  $Im \lambda > 0$  and  $Im \lambda < 0$ , and equal to the unit matrix  $I$  at  $\lambda = \infty$ ,  $\chi(\infty) = I$ . The boundary values on the real axis are related as

$$\chi^-(\lambda) = \chi^+(\lambda)G(\lambda) \quad (25)$$

The conjugating matrix is

$$G(\lambda) = I + 2\pi i \begin{pmatrix} -e_+(\lambda)e_-(\lambda) & e_+^2(\lambda) \\ -e_-^2(\lambda) & e_+(\lambda)e_-(\lambda) \end{pmatrix} \quad (26)$$

with functions  $e_{\pm}(\lambda)$  just the same as given in (9). In complete analogy with the equal-time case [5] one can show that the standard singular integral equations for this Riemann problem are equivalent to integral equations (13), the solutions  $f_{\pm}(\lambda)$  being simply expressed in terms of  $\chi(\lambda)$ , namely,

$$\begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix} = \chi^+(\lambda) \begin{pmatrix} e_+(\lambda) \\ e_-(\lambda) \end{pmatrix} \quad (27)$$

Potentials B,C (12) are then readily extracted from the  $(1/\lambda)$ -expansion at  $\lambda \rightarrow \infty$ :

$$\chi(\lambda) = I + (\Psi_1/\lambda) + (\Psi_2/\lambda^2) + \dots,$$

$$\Psi_1 = \begin{pmatrix} -B_{-+}, B_{++} \\ -B_{--}, B_{+-} \end{pmatrix}; \quad \psi = \begin{pmatrix} -C_{-+}, C_{++} \\ -C_{--}, C_{+-} \end{pmatrix} \quad (28)$$

The calculation of asymptotics of the potentials at  $x \rightarrow \infty, t \rightarrow \infty$  is similar to the simpler case  $x \rightarrow \infty, t = 0$  [5]. Nevertheless there are essential difficulties due to the appearance of stationary phase point in the singular integral equations. Using then partial differential equations (18)-(24), one restores the asymptotics of correlator (4)-(7). The derivation and complete answers will be given in more detailed paper. Below the first results are formulated. It should be emphasized, however, that our approach gives a systematic method to calculate the asymptotic expansion (see[5]).

## 5 Asymptotics

Consider first the case of negative chemical potential ( $h < 0$ ). In this region we obtained the following expression for the main term of the asymptotics of correlator (4)-(7):

$$\begin{aligned} & \langle \psi(z_2, t_2) \psi^+(z_1, t_1) \rangle_T = \\ & = a \sqrt{T} (4t)^{\frac{(\nu-1)^2}{2}} \exp\left\{2it\beta + \frac{ix^2}{2t}\right\} \times \\ & \times \exp\left\{\frac{-1}{\pi} \int_{-\infty}^{\infty} |x + 2\mu t| \ln \left(\frac{e^{\mu^2 - \beta} + 1}{e^{\mu^2 - \beta} - 1}\right) d\mu\right\} \\ & (x \equiv \frac{1}{2}(z_1 - z_2)\sqrt{T}; t = \frac{1}{2}(t_2 - t_1)T; \\ & \beta = (h/T) < 0; x \rightarrow +\infty, t \rightarrow +\infty) \end{aligned} \quad (29)$$

Here

$$\nu = \frac{1}{\pi} \ln \left| \frac{\exp\left\{\left(\frac{x}{2t}\right)^2 - \beta\right\} + 1}{\exp\left\{\left(\frac{x}{2t}\right)^2 - \beta\right\} - 1} \right|$$

and coefficient  $a$  depends on  $\beta$  and the ratio  $(x/t)$  only. We would like to

emphasize that G.G. Varzugin took part in derivation of (29). The expression for the coefficient  $a$  in (29) is complicated. To write it down let us introduce the following notations

$$\begin{aligned}\lambda_0 &= -\frac{x}{2t} & \nu &= \frac{1}{\pi} \ln \left( \frac{e^{\lambda_0^2 - \beta} + 1}{e^{\lambda_0^2 - \beta} - 1} \right) \\ \psi_0 &= -\frac{3\pi}{4} + \arg \Gamma(i\nu) + \frac{1}{\pi} \int_{-\infty}^{\infty} \sin(\lambda_0 - \mu) \ln |\mu_0 - \lambda| \frac{\partial}{\partial \mu} \ln \left( \frac{e^{\mu^2 - \beta} - 1}{e^{\mu^2 - \beta} + 1} \right) d\mu\end{aligned}\quad (30)$$

Now coefficient  $a$  in (29) can be represented as

$$\begin{aligned}a &= -\frac{1}{4} \sqrt{\nu} \exp \left\{ \frac{1}{2} (\lambda_0^2 - \beta) + i\psi_0 + \frac{1}{2} \nu^2 - \int_{-\infty}^{\beta} \left( i\frac{\nu}{2} + \nu \frac{\partial \psi_0}{\partial \beta} \right) d\beta + \right. \\ &\quad \left. + \frac{1}{2\pi^2} \int_{-\infty}^{\beta} \left[ \frac{\partial}{\partial \beta} \int_{-\infty}^{\infty} d\mu \sin(\mu - \lambda_0) \ln \left( \frac{e^{\mu^2 - \beta} - 1}{e^{\mu^2 - \beta} + 1} \right) \right]^2 d\beta \right\}\end{aligned}\quad (31)$$

It is interesting to compare (29) with the result for the equal-time correlator obtained earlier [3,5]:

$$\begin{aligned}&\langle \psi(z_2, 0) \psi^+(z_1, 0) \rangle_T = \\ &= \sqrt{T} a_0(\beta) \exp \{ -x(2\sqrt{|\beta|} + c(\beta)) \} \\ &\quad \left( x \equiv \frac{1}{2} (z_1 - z_2) \sqrt{T} \rightarrow \infty, \beta = h/T < 0 \right)\end{aligned}\quad (32)$$

with

$$c(\beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\mu \ln \left| \frac{\exp\{\mu^2 - \beta\} + 1}{\exp\{\mu^2 - \beta\} - 1} \right| \quad (33)$$

and

$$\begin{aligned}a_0(\beta) &= \frac{\rho_0}{|\beta|^{1/2}} \exp \left\{ -\frac{|\beta|^{1/2}}{\pi} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \beta} \ln \left( \frac{\exp\{\mu^2 - \beta\} + 1}{\exp\{\mu^2 - \beta\} - 1} \right) \right\} \times \\ &\quad \times \exp \left\{ \frac{1}{2} \int_{-\infty}^{\beta} d\tau \left( \frac{dc(\tau)}{d\tau} \right)^2 \right\}\end{aligned}$$

( $\rho_0$  is a constant).

For positive chemical potential ( $h > 0$ ) one obtains

$$\begin{aligned}
 & \langle \psi(z_2, t_2) \psi^+(z_1, t_1) \rangle_T = \\
 & = a \sqrt{T} (4t)^{\frac{1}{2} \nu^2} \exp \left\{ -\frac{1}{\pi} \int_{-\infty}^{\infty} |x + 2\mu t| \ln \left| \frac{e^{\mu^2 - \beta} + 1}{e^{\mu^2 - \beta} - 1} \right| d\mu + o\left(\frac{\ln^4 t}{t}\right) \right\} \\
 & x = \frac{1}{2}(z_1 - z_2) \sqrt{T}, t = \frac{1}{2}(t_2 - t_1) T; \beta = \frac{h}{T} > 0 \\
 & \beta = \frac{h}{T} > 0; x \rightarrow +\infty; t \rightarrow +\infty
 \end{aligned} \tag{34}$$

This is in agreement with the previously obtained answer for the equal-time correlator [3,5]:

$$\begin{aligned}
 & \langle \psi(z_2, 0) \psi^+(z_1, 0) \rangle_T = \\
 & = \rho_{\infty} \sqrt{\frac{T}{\pi}} \exp \left\{ -\frac{1}{2} \int_{\beta}^{\infty} d\tau \left( \frac{dc(\tau)}{d\tau} \right)^2 \right\} \exp \{-x c(\beta)\} \\
 & \left( x \equiv \frac{1}{2}(z_1 - z_2) \sqrt{T} \rightarrow +\infty, \beta \equiv \frac{h}{t} > 0 \right)
 \end{aligned} \tag{35}$$

Here function  $c(\beta)$  is given in (33) and  $\rho_{\infty}$  is a constant of paper [10] ( $\rho_{\infty} = \pi e^{-1/2} 2^{-1/3} A^{-6}$ ,  $A$  being the Glaisher constant).

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## Breathers and the sine-Gordon Equation.

Satyanad Kichenassamy

### 1 Introduction.

The purpose of this paper is to report on a recent non-existence result for breather solutions of nonlinear Klein-Gordon equations [1,2], and to mention some technical improvements thereof.

One of the by-products of the method of Inverse Scattering to solve nonlinear evolution equations is the construction of the "breather" solutions of the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0. \quad (1)$$

The simplest of them is a periodic solution, with period  $2\pi/\sqrt{1-\varepsilon^2}$  ( $0 < \varepsilon < 1$ ) which tends to zero as  $|x| \rightarrow \infty$ ; it is given by

$$u_{SG} = 4 \arctan \left( \frac{\varepsilon \cos(t\sqrt{1-\varepsilon^2})}{\sqrt{1-\varepsilon^2} \cosh \varepsilon x} \right). \quad (2)$$

This solution, which can be made arbitrarily small in, say, the  $L^2$  norm, by taking the parameter  $\varepsilon$  sufficiently small, has remarkable properties, that are shared neither by solutions of the (linear) Klein-Gordon equation, nor by those of other soliton equations which, like the Korteweg-de Vries equation, are related to a self-adjoint eigenvalue problem. We will not detail these properties here.

Solutions with behavior analogous to that of (2) have not been found in other equations of the form:

$$u_{tt} - u_{xx} + u + g(u) = 0. \quad (3)$$

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with  $g(u) = g_2 u^2 + g_3 u^3 + \dots$ , and this leads to the possibility that breathers might be specific to completely integrable systems.

The two questions before us are therefore:

1. What should be called a breather solution of a (non-integrable) wave equation?
2. Do such solutions exist for (3)?

We here summarize, with a few additions, the answers to these questions given in [1], briefly comment on their proofs (in §2), and discuss some improvements in §3. (A discussion of the literature on this problem can be found in §7 of [1].)

1. A breather is a time periodic solution of (3) which decays to zero as  $x$  goes to infinity. (This definition is discussed and extended in [2].)
2. There are formal breathers:

$$u = \varepsilon u_1(\varepsilon x, t\sqrt{1-\varepsilon^2}) + \varepsilon^2 u_2(\varepsilon x, t\sqrt{1-\varepsilon^2}) + \dots \quad (4)$$

for (3), characterized by periodicity, decay and parity conditions, if and only if

$$\frac{5}{6}g_2^2 - \frac{3}{4}g_3 > 0. \quad (5)$$

3. If series (4) defines a solution of (3) analytic in  $\varepsilon$ ,  $\varepsilon x$  and  $t\sqrt{1-\varepsilon^2}$  for  $|\varepsilon| < \frac{1}{\sqrt{2}} + \delta$  for any  $\delta > 0$  and, say,  $x, t$  close to the real axis, and if this function has an expansion in powers of  $\varepsilon$ ,  $e^{-\varepsilon x}$  and  $\cos(t\sqrt{1-\varepsilon^2})$  convergent for large  $x$ , then  $u + g(u) = \alpha^{-1} \sin(\alpha u)$ . (In this case, the series does have the asserted analyticity property.)
4. If  $g$  is *odd*, and

$$|g_m| \leq \alpha^{m-1}/m!$$

for some  $\alpha > 0$ , then eq. (3) does have a time periodic solution on some interval  $(A, +\infty)$  given by a power series in  $e^{-\varepsilon x}$ , convergent for large  $x$  and small  $\varepsilon$ . This solution is the only one which is odd in  $\cos(t\sqrt{1-\varepsilon^2})$  and tends to zero exponentially as  $x$  tends to  $+\infty$ .

*Remarks:* 1. We have thus achieved a characterization of the sine-Gordon equation among nonlinear Klein-Gordon equations. This is a criterion for complete integrability of (3) via the Zakharov-Shabat eigenvalue problem.

2. The integrability here corresponds to a *smoothness* property of special solutions.

3. The convergence result in the odd case justifies the perturbative approach taken here: There is a natural candidate for a breather for every small  $\varepsilon$ , but it is never analytic except in the integrable case.

4. The construction of the first formal solution of course gives approximate breathers with any desired precision.

## 2 Sketch of proofs.

The proofs are based on the existence, for very general functions  $g$ , of two formal periodic solutions of (3). We first introduce a parameter  $\varepsilon$  and two scaled variables  $\xi$  and  $\tau$  by

$$\xi = \varepsilon x, \tau = t\sqrt{1 - \varepsilon^2},$$

and let  $u = \varepsilon v$ . Equation (3) then becomes:

$$v_{\tau\tau} + v - \varepsilon^2(u_{\xi\xi} + u_{\tau\tau}) + g(\varepsilon v)/\varepsilon = 0,$$

and we seek  $v$  periodic in  $\tau$  with period  $2\pi$ , tending to zero as  $\xi$  goes to  $\pm\infty$ . We also expect  $u$  to depend smoothly on  $\varepsilon$ , as in (2). The corresponding  $u$  will then be called a *breather* solution of (3).

The *first formal solution* is in increasing powers of  $\varepsilon$ :

$$u = \sum_{k \geq 1} \varepsilon^k u_k(\xi, \tau). \quad (6)$$

We assume  $u_k$  has the same properties as  $u$ . If the  $u_k$  are even in  $\xi$  and  $\tau$ , they can be uniquely determined by from  $u_1$ . On the other hand, if  $u \neq 0$ , then we must have:

$$u_1 = \pm \sqrt{\frac{2}{\lambda}} (\cosh \xi)^{-1} \cos \tau$$

where  $\lambda = 5g_2^2/6 - 3g_3/4$  must be positive; otherwise formal breathers do not exist.

The *second formal solution* is in increasing powers of  $e^{-\xi}$ :

$$u = \sum_{k \geq 1} e^{-k\xi} v_k(\varepsilon, \tau). \quad (7)$$

We again take  $v_k$  even in  $\tau$ . This series is determined by its first term,  $v_1$ , which must have the form  $a(\varepsilon) \cos \tau$ . No condition on  $g$  similar to (5) is required here.

The existence of the first formal solution proves statement 2. in the Introduction.

To prove 3., one shows that if  $u$  had the given analyticity property, then it would have an expansion of type (6) which could be rearranged in the form (7) with coefficients holomorphic in  $\varepsilon$  for  $|\varepsilon| < \frac{1}{\sqrt{2}} + \delta$ . One then proves, using the analyticity again, that  $a(\varepsilon)$  is always nonzero, so that after a translation in  $\xi$ , one may assume  $a(\varepsilon) \equiv 1$ . One next proves by induction that the  $v_k$  possess poles which approach  $\varepsilon = 0$  as  $k$  increases, and that all the poles of modulus  $\leq 1/\sqrt{2}$  disappear if and only if  $g(u) = \alpha^{-1} \sin(\alpha u)$ ,  $\alpha^{-1} \sinh(\alpha u)$ , or  $u$ , only the first of these giving rise to breathers as one verifies directly.

To prove 4., one uses a majorant method based on comparison with the solution of the sinh-Gordon equation corresponding to (2). One obtains the uniqueness statement by a variant of the stable manifold theorem.

### 3 Remarks.

1. Because of translation invariance, it is apparent that some condition, similar to the parity ones used here, is needed in order to fix the solution under consideration. On the other hand, we next very briefly outline a technique which enables one to relate the general formal solutions to the even ones.

The idea is this: we know [1] that series (4) is determined by an induction whereby at the  $k$ th step, the harmonics of  $u_k$  different from the first and the first harmonic of  $u_{k-2}$  are found. They were uniquely determined because of the parity assumptions. Now if we drop these assumptions at some given step  $k$ , we find that  $u_k$  and  $u_{k-2}$  differ from their values in the "even" case by terms of the form  $-\tau_m \sin \tau$  and  $\xi_{m-2} S_r$  (with the notation  $u_1 = S(x) \cos \tau$ ;  $u_1$  can clearly be taken of this form, with  $S$  even, after an initial translation.) The observation is then that replacing  $\xi$  and  $\tau$  by  $\xi - \xi_{m-2} \varepsilon^{m-2}$  and  $\tau - \tau_m \varepsilon^m$  produces a new formal solution which coincides with the "even" one upto order  $k$  inclusive, apart possibly from the first harmonic of  $u_{k-1}$ . One may carry on the process by induction.

2. The presence of poles in the expansion (6) can be interpreted in terms of "resonances" as follows: Considering (3) with periodic conditions in time defines a dynamical system in a space of periodic pairs  $(u(x, t); u_x(x, t))$  with  $x$  playing the role of a "time" variable, for which orbits homoclinic to the zero solution correspond to breathers. Now the poles we obtain here are exactly those for which the eigenvalues 1 and  $\varepsilon$  of this system resonate with one of the others. (This, incidentally, can happen in the odd case only if  $\varepsilon$  is pure imaginary.) The extension of our definition of breathers to other "nearly integrable equations," and more examples of this resonance phenomenon will be given in a paper in preparation.

3. One may ask whether our emphasis on rapidly decreasing solutions is justified. In this direction, one can prove (see [3] for the precise statement and proof) that solutions which decay faster than  $1/r^2$  must have  $\iint (1 + r^2)^s |u(x, t)|^2 dx dt < \infty$  for every  $s > 0$ . This follows from a general decay result for linear wave equations. This dichotomy is of course not surprising when compared to similar results for ordinary differential equations.

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## Localized solitons for the Ishimori equation\*

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### Abstract

The coherent structures for the Ishimori-II equation are studied. General form of the exact solutions of the Ishimori-II equation with nontrivial boundaries is found. It is shown that there exist the four (essentially the three) different types (ss, sb, bs, bb) of localized solitons for this equation with the time-independent boundaries.

One of the main features of nonlinear partial differential equations solvable by the inverse spectral transform (IST) method is the existence of the localized soliton solutions (see e.g. [1-3]). In the  $2 + 1$ -dimensions such exponentially localized solitons have been constructed recently for the Davey-Stewartson (DS)-I equation in the paper [4]. Scattering of these localized solitons, their properties and general initial-boundary value problem for the DS-I equation have been studied by different methods in [4-8].

The present paper is devoted to the study of the localized solitons for

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the Ishimori equation [9]

$$\begin{aligned} \vec{S}_t(x, y, t) + \vec{S} \times (\vec{S}_{xx} + \alpha^2 \vec{S}_{yy}) + \varphi_x \vec{S}_y + \varphi_y \vec{S}_x &= 0, \\ \varphi_{xx} - \alpha^2 \varphi_{yy} + 2\alpha^2 \vec{S}(\vec{S}_x \times \vec{S}_y) &= 0 \end{aligned} \quad (1)$$

where  $\vec{S} = (S_1, S_2, S_3)$  is the three-dimensional unit vector

$$\vec{S}^2 = 1, \varphi(x, y, t)$$

is a scalar field and  $\alpha^2 = \pm 1$ . Equation (1) is the 2 + 1-dimensional integrable generalisation of the Heisenberg ferro-magnet model equation  $\vec{S}_t + \vec{S} \times \vec{S}_{xx} = 0$  (isotropic Landau-Lifshitz equation). The Ishimori equation (1) is of the great interest since it is the first example of the integrable nonlinear spin-one field model on the plane. An important feature of equation (1) is the existence of the classes of the topologically nontrivial and nonequivalent solutions which are classified by the topological charge  $Q = \frac{1}{4\pi} \iint_{-\infty}^{+\infty} dx dy \vec{S}(\vec{S}_x \times \vec{S}_y)$  [9].

The applicability of the IST method to the Ishimori equation (1) is based on its equivalence to the commutativity condition  $[L_1, L_2] = 0$  of the operators [9]

$$\begin{aligned} L_1 &= \alpha \partial_y + P \partial_x, \\ L_2 &= i \partial_t + 2P \partial_x^2 + (P_x + \alpha P_y P - i \alpha^3 P \varphi_x + i \varphi_y) \partial_x \end{aligned} \quad (2)$$

where  $P \equiv \vec{S}(x, y, t) \vec{\sigma}$ ,  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  are Pauli matrices and  $\partial_x \equiv \frac{\partial}{\partial x}$ ,  $\partial_y \equiv \frac{\partial}{\partial y}$ ,  $\partial_t \equiv \frac{\partial}{\partial t}$ . The standard initial value problem for the Ishimori-I ( $\alpha = i$ ) and Ishimori-II ( $\alpha = 1$ ) equations with the vanishing boundary values ( $\vec{S} \xrightarrow{x, y \rightarrow \infty} (0, 0, -1)$ ,  $\varphi \xrightarrow{x, y \rightarrow \infty} 0$ ) has been solved in the papers [10-12] with the use of the  $\bar{\partial}$ -method ( $\alpha = i$  [10,11]) and nonlocal Riemann-Hilbert problem ( $\alpha = 1$  [12]) method.

Here we present the results concerning the construction of the exponentially localized solutions of the Ishimori-II equation with the nontrivial boundaries [13]. In the characteristic coordinates  $\xi = \frac{1}{2}(y+x)$ ,  $\eta = \frac{1}{2}(y-x)$  equation (1) at  $\alpha = 1$  is equivalent to the equation

$$\begin{aligned} \vec{S}_t + \frac{1}{2} \vec{S} \times (\vec{S}_{\xi\xi} + \vec{S}_{\eta\eta}) + \frac{1}{2} \left( \int_{-\infty}^{\eta} d\eta' \vec{S} (\vec{S}_{\xi} \times \vec{S}_{\eta'}) + 2U_2(\xi, t) \right) \vec{S}_{\xi} \\ - \frac{1}{2} \left( \int_{-\infty}^{\xi} d\xi' \vec{S} (\vec{S}_{\xi'} \times \vec{S}_{\eta}) + 2U_1(\eta, t) \right) \vec{S}_{\eta} = 0 \end{aligned} \quad (3)$$

due to the equalities

$$\begin{aligned} \varphi_{\xi} &= \int_{-\infty}^{\eta} d\eta' \vec{S} (\vec{S}_{\xi} \times \vec{S}_{\eta'}) + 2U_2(\xi, t). \\ \varphi_{\eta} &= \int_{-\infty}^{\xi} d\xi' \vec{S} (\vec{S}_{\xi'} \times \vec{S}_{\eta}) + 2U_1(\eta, t) \end{aligned} \quad (4)$$

where  $U_1(\eta, t)$  and  $U_2(\xi, t)$  are arbitrary scalar functions. So, we are interesting in the exact solutions of the Ishimori-II equation with the boundary values

$$\begin{aligned} \vec{S}(x, y, t) &\xrightarrow{x^2 \oplus y^2 \rightarrow \infty} (0, 0, -1), \\ \varphi(x, y, t) &\xrightarrow{x^2 \oplus y^2 \rightarrow \infty} 2\partial_{\xi}^{-1} U_2(\xi', t) + 2\partial_{\eta}^{-1} U_1(\eta', t) \end{aligned} \quad (5)$$

with given functions  $U_1(\eta, t)$  and  $U_2(\xi, t)$ .

We will use the method proposed by Fokas and Santini for the DS-I equation [6-8]. Within the framework of this approach one needs: 1) to derive the equations which solve the inverse problem for the linear equation  $L_1\psi = 0$ , associated with the given integrable equation, 2) to

solve the inverse problem equations for the degenerated inverse problem data. 3) to construct the modified second auxiliary equation  $L_2\psi = 0$  and to find the corresponding time evolution of the inverse problem data. 4) to construct the exact factorized solutions of the linear evolution equations for the inverse problem data. 5) using these solutions and reconstruction formulae for the degenerated data. to calculate the exact solutions of the soliton equation with the nontrivial boundaries.

Following this scheme, one gets for the Ishimori-II equation (3) the following results [13].

1. The inverse problem equations for the linear auxiliary equation  $L_1\psi = 0$  where the operator  $L_1$  is given by (2) are generated by the corresponding nonlocal Riemann-Hilbert problem [12,13]. Namely transiting to the functions  $\chi$  defined by

$$\psi = \chi(\xi, \eta, \lambda) \begin{pmatrix} e^{\frac{i\xi}{\lambda}}, & 0 \\ 0, & e^{-\frac{i\eta}{\lambda}} \end{pmatrix} \quad (6)$$

where  $\lambda$  is a complex parameter, one can construct the solutions  $\chi^+$  and  $\chi^-$  which are bounded and analytic in the upper ( $Ym\lambda > 0$ ) and lower ( $Ym\lambda < 0$ ) half-planes, respectively. and such that  $\chi^+ - \chi^-$  at the real axis ( $Ym\lambda = 0$ ) is expressed nonlocally via  $\chi^-$ . The solution of this nonlocal Riemann-Hilbert problem is given by the linear integral equation

$$\chi^-(\xi, \eta, \lambda) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \int \frac{dp dq}{p^2 q^2} \frac{\chi^-(\xi, \eta, p) \Sigma_p f(p, q) \Sigma_q^{-1}}{q - \lambda + i0} \quad (7)$$

where

$$\begin{aligned} \Sigma_p(\xi, \eta) &= \begin{pmatrix} e^{\frac{i\xi}{p}} & 0 \\ 0 & e^{\frac{-i\eta}{p}} \end{pmatrix}, \\ f(p, q) &= \begin{pmatrix} 0, & T^+(p, q) \\ -T^-(p, q), & -\int_{-\infty}^{+\infty} \frac{dk}{k^2} T^-(p, k) T^+(k, q) \end{pmatrix} \end{aligned} \quad (8)$$

and

$$\begin{aligned} T^+(p, q) &= \frac{1}{4\pi} \iint_{-\infty}^{+\infty} d\xi d\eta e^{\frac{-i\xi}{p} - \frac{i\eta}{q}} ((P(\xi, \eta) + \sigma_3)(\partial_\xi - \partial_\eta + \frac{i}{q})\chi^+(\xi, \eta, q))_{12} \\ T^-(p, q) &= \frac{1}{4\pi} \iint_{-\infty}^{+\infty} d\xi d\eta e^{\frac{i\xi}{p} + \frac{i\eta}{q}} ((P(\xi, \eta) + \sigma_3)(\partial_\xi - \partial_\eta + \frac{i}{q})\chi^-(\xi, \eta, q))_{21}. \end{aligned} \quad (9)$$

The formulae which allow us to reconstruct  $\vec{S}$  and  $\varphi$  via  $\chi$  are of the form

$$\begin{aligned} \vec{S}(\xi, \eta, t) &= -tr(\vec{\sigma} g \sigma_3 g^{-1}), \\ \varphi(\xi, \eta, t) &= 2i \log \det g + 2\partial_\xi^{-1} U_2(\xi', t) + 2\partial_\eta^{-1} U_1(\eta', t). \end{aligned} \quad (10)$$

where  $g(\xi, \eta, t) = \chi(\xi, \eta, t, \lambda = 0)$ .

The formulae (7) and (10) form the complete set of the equations which solve the inverse problem for the Ishimori-II equation. The functions  $T^+(p, q)$  and  $T^-(p, q)$  are the inverse problem data [12,13].

2. For the degenerated inverse problem data. i.e.

$$T^\pm(p, q) = \sum_{k=1}^{N_\pm} T_k^\pm(p) \tilde{T}_k^\pm(q) \quad (11)$$

where  $T_k^\pm(p), \tilde{T}_k^\pm(q)$  are arbitrary functions, the inverse problem equation (7) is reduced to the system of algebraic equations. As a result, the

solution  $\chi^-(\lambda)$  is found in the closed explicit form [13]:

$$\begin{aligned} \begin{pmatrix} \chi_{11}^-(\lambda) \\ \chi_{21}^-(\lambda) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sqrt{2\pi} \sum_{j=1}^{N_-} F_j (\tilde{T}_j^-(\lambda) e^{-\frac{i\xi}{\lambda}})^-, \\ \begin{pmatrix} \chi_{12}^-(\lambda) \\ \chi_{22}^-(\lambda) \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sqrt{2\pi} \sum_{j=1}^{N_+} G_j (\tilde{T}_j^+(\lambda) e^{\frac{i\xi}{\lambda}})^- \end{aligned} \quad (12)$$

where

$$(f(\lambda))^{\pm} \doteq \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\lambda' \frac{f(\lambda')}{\lambda' - (\lambda \pm i0)}$$

and

$$\begin{aligned} F_j &= \sum_{k=1}^{N_-} (1 - A)_{jk}^{-1} \left( \sum_{\ell=1}^{N_+} \alpha_{k\ell}^-(\eta) \tau_{\ell}^+(\xi) \right), \\ G_j &= \sum_{k=1}^{N_+} (1 - B)_{jk}^{-1} \begin{pmatrix} \tau_k^+(\xi) \\ \sum_{\ell=1}^k \alpha_{k\ell}^+(\xi) & \tau_{\ell}^-(\eta) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{N_+} \alpha_{ik}^-(\eta) \alpha_{kj}^+(\xi), \quad B_{\ell m} = \sum_{k=1}^{N_-} \alpha_{\ell k}^+(\xi) \alpha_{km}^-(\eta), \\ \tau_k^{\pm}(\rho) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^2} T_k^{\pm}(\lambda) e^{\pm \frac{i\rho}{\lambda}}, \\ \alpha_{jk}^{\pm}(\rho) &= \mp \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^2} T_j^{\pm}(\lambda) e^{\pm \frac{i\rho}{\lambda}} (\tilde{T}_k^{\mp}(\lambda) e^{\mp \frac{i\rho}{\lambda}})^{\pm}. \end{aligned}$$

The formulae (10), (12) give the exact solutions of the Ishimori-II equation which depend on the  $2(N_+ + N_-)$  arbitrary functions of one variable.

3. Nontrivial boundaries (5) demand the modification of the second operator  $L_2 \rightarrow L_2 + \Delta$  which is associated with the Ishimori equation. Since the compatibility condition for the system

$$L_1 \psi = 0, \quad (L_2 + \Delta) \psi = 0$$

again should be equivalent to the Ishimori equation, then  $L_1 \Delta \psi = 0$ . So, the functions  $\psi$  and  $\Delta \psi$  obey the same differential equation. But the corresponding integral equations are different. Comparing them, one finds [13]

$$\Delta = -\frac{i}{2\lambda^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} \int_{-\infty}^{+\infty} \frac{d\lambda'}{\lambda'^2} \gamma(\frac{1}{\lambda} - \frac{1}{\lambda'}), & 0 \\ 0, & \int_{-\infty}^{+\infty} \frac{d\lambda'}{\lambda'^2} \hat{\gamma}(\frac{1}{\lambda} - \frac{1}{\lambda'}) \end{pmatrix} \quad (13)$$

where

$$\gamma(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\xi e^{i\xi\lambda} U_2(\xi, t), \quad \hat{\gamma}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\eta U_1(\eta, t) e^{-i\eta\lambda}.$$

Now, the equation  $(L_2 + \Delta)\psi = 0$  where  $L_2$  is given by (2) and  $\Delta$  by (13) implies that the full Fourier transforms of the inverse problem data

$$\begin{aligned} \hat{T}^+(\xi, \eta, t) &= i \iint_{-\infty}^{+\infty} \frac{dp}{p^2} \frac{dq}{q^2} q T^+(p, q) e^{\frac{i\xi}{p} + \frac{i\eta}{q}}, \\ \hat{T}^-(\xi, \eta, t) &= -i \iint_{-\infty}^{+\infty} \frac{dp}{p^2} \frac{dq}{q^2} q T^-(p, q) e^{\frac{-i\eta}{p} - \frac{i\xi}{q}} \end{aligned} \quad (14)$$

obey the linear equations [13]

$$\hat{T}_t^\pm(\xi, \eta, t) \pm \frac{i}{2} (\hat{T}_{\xi\xi}^\pm + \hat{T}_{\eta\eta}^\pm) + U_2(\xi, t) \hat{T}_\xi^\pm - U_1(\eta, t) \hat{T}_\eta^\pm = 0. \quad (15)$$

These linear evolution equations for the inverse spectral data play a fundamental role in the whole of our construction. Emphasize that equations (15) coincide with the linearized equation (3) for  $S_\pm \doteq S_1 \pm iS_2$ . In the weak field limit one has  $\hat{T}^\pm(\xi, \eta, t) = \pi S_\pm(\xi, \eta, t)$ .

Note also that for the real  $\vec{S}$  one has

$$qT^-(p, q) = -p(T^+(q, p))^*$$

or

$$\hat{T}^-(\xi, \eta, t) = (\hat{T}^+(\xi, \eta, t))^* \quad (16)$$

So, for real  $\vec{S}$  it is sufficient to consider one equation (15), for instance, the equation for  $\hat{T}^+$ . In what follows we will discuss the case of the real-valued  $\vec{S}$ .

4. Equations (15) admit the separation of variables. So, the equation for  $\hat{T}^+$  has the solutions of the form

$$\hat{T}^+(\xi, \eta, t) = 2\pi \sum_{i,j} \rho_{ij} X_i(\xi, t) Y_j(\eta, t) \quad (17)$$

where  $\rho_{ij}$  are arbitrary constants and  $X_i$  and  $Y_j$  obey the equations

$$\begin{aligned} iX_{it} + \frac{1}{\eta} X_{i\xi\xi} + iU_2(\xi, t) X_{i\xi} &= 0, \\ iY_{jt} - \frac{1}{2} Y_{j\eta\eta} - iU_1(\eta, t) Y_{j\eta} &= 0. \end{aligned} \quad (18)$$

The inverse problem data  $\hat{T}^+$  of the form (17) are degenerated one and the corresponding functions  $T_k^+(p)$ ,  $\bar{T}_k^+(q)$  from (11) are

$$\begin{aligned} T_k^+(p, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi X_k^*(\xi, t) e^{\frac{-i\xi}{p}}, \\ T_k^-(q, t) &= \frac{i}{\sqrt{2\pi}} \sum_j \rho_{kj} \int_{-\infty}^{+\infty} d\eta \frac{1}{q} Y_j^*(\eta, t) e^{\frac{-i\eta}{q}}. \end{aligned} \quad (19)$$

For such a data, using the formula (12), one gets [13]

$$g(\xi, \eta, t) = \begin{pmatrix} 1 - \langle X, (1 - \rho a \rho^+ b)^{-1} \rho a \rho^+ X^* \rangle, \langle Y^*, \rho^+ (1 - b \rho a \rho^+)^{-1} X^* \rangle \\ -\langle X, (1 - \rho a \rho^+ b)^{-1} \rho Y \rangle, 1 + \langle Y^*, \rho^+ (1 - b \rho a \rho^+)^{-1} b \rho Y \rangle \end{pmatrix} \quad (20)$$

where  $\langle X, Y \rangle = \sum_i X_i Y_i$ ,  $(\rho)_{ij} = \rho_{ij}$  and

$$a_{ik}(\eta) = \int_{-\infty}^{\eta} d\eta' Y_k^*(\eta', t) \partial_{\eta'} Y_i(\eta', t),$$

$$b_{ik}(\xi) = - \int_{-\infty}^{\xi} d\xi' X_k(\xi', t) \partial_{\xi'} X_i^*(\xi', t).$$

Then the formula (10) gives us the solution  $\vec{S}$  and  $\varphi$  of the Ishimori-II equation. So each set of exact solutions of equations (18) gives rise to the exact solution of the Ishimori-II equation (3) with given boundaries  $U_1(\eta, t)$  and  $U_2(\xi, t)$  via the formulae (12) and (20). These formulae play a central role in the theory of the coherent structures for the Ishimori-II equation [13].

5. The problem now is to solve the linear equations (18). Here we will consider the case of time-independent boundaries  $U_1(\eta)$  and  $U_2(\xi)$ . In this case equations (18) admit the further separation of variables, namely

$$X_i(\xi, t) = e^{\frac{-i}{2} \lambda_i^2 t} X_i(\xi),$$

$$Y_j(\eta, t) = e^{\frac{i}{2} \lambda_j^2 t} Y_j(\eta) \quad (21)$$

where  $X_i(\xi)$  and  $Y_j(\eta)$  obey the ordinary differential equations

$$X_{i\xi\xi} + 2iU_2(\xi)X_{i\xi} + \lambda_i^2 X_i = 0,$$

$$Y_{j\eta\eta} - 2iU_1(\eta)Y_{j\eta} + \lambda_j^2 Y_j = 0. \quad (22)$$

The solutions  $X$  and  $Y$  of equations (22) can be expressed via the solutions of the  $2 \times 2$  matrix spectral problem. Namely, if the functions  $V_1(z)$  and  $V_2(z)$  are the solutions of the specialized Zacharov-Shabat spectral problem

$$\begin{pmatrix} -\partial_z & U \\ U & \partial_z \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = i\lambda \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad (23)$$

then the functions

$$\begin{aligned} X(\xi) &= -(V_2(\xi) + iV_1(\xi))e^{-i \int_{-\infty}^{\xi} d\xi' (U_2(\xi'))} \\ Y(\eta) &= (V_1(\eta) + iV_2(\eta))e^{i \int_{-\infty}^{\eta} d\eta' U_1(\eta')} \end{aligned} \quad (24)$$

obey equations (22) [13]. For the calculation of the function  $X(\xi)$  one must use  $U_2(\xi)$  as  $U$  in (23) while for calculation  $Y(\eta)$  one must use  $U_1(\eta)$  as  $U$  in (23).

Spectral problem (23) has been studied in detail by Wadati in [14, 15] in connection with the IST integration of the modified Korteweg-de Vries equation. The discrete spectrum of the problem (23) consists from the points located symmetrically with respect to the imaginary axis  $Re\lambda = 0$ . The points  $\lambda_n = i\beta_n$  ( $Ym\beta_n = 0$ ) correspond to the soliton potential  $U$  and the pairs of points  $\lambda_{\pm} = \pm\alpha + i\beta$  ( $\alpha, \beta > 0$ ) correspond to the breathers. The general  $N_1$  solitons  $N_2$  breathers potential  $U$  is of the form [14, 15]

$$U(z) = 2 \frac{d}{dz} Ym \ell n \det(1 + iM) \quad (25)$$

and the corresponding eigenfunctions are

$$\begin{pmatrix} V_{1n}(z) \\ V_{2n}(z) \end{pmatrix} = \sum_{m=1}^{N_1+2N_2} (1 + M^2)^{-1}_{nm} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda_m z} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sum_k M_{mk} e^{i\lambda_k z} \quad (26)$$

where  $(N_1 + 2N_2) \times (N_1 + 2N_2)$  matrix  $M$  looks like

$$M_{nm} = \frac{C_m e^{i(\lambda_n + \lambda_m)z}}{\lambda_n + \lambda_m} \quad (27)$$

where  $c_m$  are arbitrary constants ( $n, m = 1, \dots, N_1 + 2N_2$ ).

So, for the boundary functions  $U_1(\eta)$  and  $U_2(\xi)$  chosen in the form (25) we have the exact solutions of equations (22) given by the formulae (24), (26).

Thus, the formulae (10), (20), (21), (24), (26) give us the class of exact solutions of the Ishimori-II equation with the boundary functions  $U_2(\eta)$  and  $U_2(\xi)$  of the form (25). As far as the functions  $V_{1n}(z), V_{2n}(z)$  given by (26), the field variables  $\vec{S}(\xi, \eta, t)$  and  $\varphi(\xi, \eta, t)$  are exponentially localized in all directions on the plane  $(x, y)$ . These solutions are the localized breather type solutions of the Ishimori-II equation [13].

The solitons ( $s$ ) and breathers ( $b$ ) are quite different transparent potentials. As a result, we have four different types of the exact solutions of the Ishimori-II equation:

$$\vec{S}_{(N,M)}^{ss}(\xi, \eta, t), \quad \vec{S}_{(N,M)}^{sb}(\xi, \eta, t), \quad \vec{S}_{(N,M)}^{bs}(\xi, \eta, t), \quad \vec{S}_{(N,M)}^{bb}(\xi, \eta, t)$$

which correspond to the choices of  $X$  and  $Y$  as the pure soliton or breather eigenfunctions. The solutions  $\vec{S}^{sb}$  and  $\vec{S}^{bs}$  are, obviously, related by the interchange  $\xi \leftrightarrow \eta, N \leftrightarrow M$ . So, we have the three essentially different types of exact solutions

$$\vec{S}_{(N,M)}^{ss}, \quad \vec{S}_{(N,M)}^{sb}, \quad \vec{S}_{(N,M)}^{bb} \quad (28)$$

where the integers  $N$  and  $M$  correspond to the  $N$ -soliton (breather) boundary  $U_2(\xi)$  and  $M$ -soliton (breather) boundary  $U_1(\eta)$ . All these

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where the integers  $N$  and  $M$  correspond to the  $N$ -soliton (breather) boundary  $U_2(\xi)$  and  $M$ -soliton (breather) boundary  $U_1(\eta)$ . All these

solutions are calculated explicitly. Their properties and the simplest examples are considered in the detailed paper [13]. The case of the time-dependent boundaries will be discussed elsewhere.

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## Tau Functions<sup>1,2</sup>

John Palmer

### Abstract

The  $\tau$ -functions introduced by M. Sato, T. Miwa and M. Jimbo in work on Holonomic Quantum Fields and monodromy preserving deformation theory are discussed. In some cases they can be shown to be determinants of differential operators in a determinant bundle formalism that has some technical and conceptual advantages.

In a series of five long papers titled "Holonomic Quantum Fields I-V," published in the years 1978-1980, the mathematicians M. Sato, T. Miwa, and M. Jimbo (SMJ henceforth) revealed an unexpected connection between certain two dimensional quantum field theory models and the theory of monodromy preserving deformations of linear differential equations [27]. Very briefly the connection is this. The central objects of the field theory, called correlation functions or vacuum expectations could be expressed in terms of the solutions to certain nonlinear deformation equations associated with monodromy preserving deformations of linear differential equations. This work grew out of a penetrating analysis of an earlier result (1976) of Wu, McCoy, Tracy, and Barouch [33] on the scaling behavior of the correlation functions for the two dimensional Ising model on a lattice. The WMTB result is that a certain scaling limit of the two point correlation of the Ising model (a model of ferromagnetism in two dimensions) can be expressed in terms of Painlevé functions. The Painlevé functions satisfy nonlinear differential equations in the plane with a characteristic property first analysed in detail by Painlevé around the turn of the century [13]. Roughly speaking the property that singles out the class of Painlevé equations from the general class of equations with rational nonlinearities is that the solutions are to have at worst pole type singularities away from the manifest singularities of the equation itself.

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The notion of monodromy preserving deformations for linear differential equations in the complex plane was first considered in papers of R. Fuchs [9] and L. Schlesinger [29] and R. Garnier [10]. Painlevé functions arise in the integration of the deformation equations formulated by Fuchs (in particular Painlevé functions of the 6<sup>th</sup> kind) and in [10],[11] many special cases of the deformation equations were integrated in terms of Painlevé functions. Part of the *genius* of the SMJ analysis of the WMTB result is their realization that monodromy preserving deformations play a central role in accounting for the appearance of Painlevé functions in the scaling behavior of the Ising model. To see this required generalizing the notion of monodromy preserving deformations for linear differential equations in the complex plane to a theory of monodromy preserving deformations for the (elliptic) Dirac equation in the plane. The correlation function for the Ising model generalizes in their theory to the (Euclidean) correlation function of a Holonomic Quantum Field, a function they refer to as a  $\tau$ -function. It is impossible to do justice to the wide spectrum of ideas in the original five papers of SMJ much less the subsequent development of a theory in the irregular singular case (which was started by H. Flaschka and A. Newell in [8] and further developed by M. Jimbo, T. Miwa, and K. Ueno in [14]) in a short article such as this. Instead I will concentrate on a problem that has intrigued me for some years now in the case where the resolution is simplest to explain.

Consider a linear differential equation on  $\mathbf{P}^1$  with rational coefficients:

$$(1) \quad \frac{dY}{dz} = A(z)Y$$

where  $A(z)$  is an  $n \times n$  matrix valued function with rational entries. In the case that  $A(z)$  has only simple poles in the finite plane one can do a partial fraction expansion to obtain:

$$A(z) = \sum_{\nu}^p \frac{A_{\nu}}{z - a_{\nu}}$$

where each  $A_{\nu}$  is an  $n \times n$  matrix. Infinity will be a regular point for the differential equation provided that  $\sum_{\nu} A_{\nu} = 0$ . For simplicity we will suppose this is the case. The local existence theory in the complex plane tells one that in a neighborhood of any regular point for the differential equation (1) one can find a fundamental set of solutions  $\{y_1(z), \dots, y_n(z)\}$ . If one analytically continues such a solution  $y_j$  around a singular point  $a_{\nu}$  it does not in general return to the solution  $y_j$  but to a linear combination  $\sum_k M_{kj}^{\nu} y_k$ . The matrices  $M^{\nu}$  are called *monodromy matrices*. We can normalize the choice of such matrices by considering a matrix valued fundamental solution:

$$Y(z) := [y_1(z), \dots, y_n(z)]$$

normalized so that  $Y(\infty) = \text{identity}$ . Under a simple circuit of the singular point

$a_\nu$  such a matrix fundamental solution will transform:

$$Y \rightarrow YM^\nu$$

In 1912 Schlesinger posed the question: How must the coefficient matrices  $A_\nu$  depend on the poles  $\{a_1, \dots, a_n\}$  so that the monodromy matrices  $M^\nu$  do not depend on the location of the poles  $a_1, \dots, a_n$ . He discovered that these coefficients must satisfy a nonlinear system of equations:

$$(2) \quad d_a A_\nu = - \sum_{\mu \neq \nu} [A_\nu, A_\mu] \frac{da_\nu - da_\mu}{a_\nu - a_\mu}$$

where  $[A, B] = AB - BA$  is the usual commutator. These equations are known as the *Schlesinger equations* or sometimes the deformation equations. As I mentioned earlier special cases of these equations were integrated in terms of Painlevé transcendents by Garnier [10],[11].

It will be useful to recall that work on monodromy preserving deformations evolved from the consideration of a problem first posed by Riemann and later included in Hilbert's famous list of problems at the 1900 International Congress (it is number 21). The "classical Riemann-Hilbert problem" is the problem of finding a linear differential equation with simple poles that has arbitrarily prescribed monodromy matrices. Riemann was lead to consider this problem after he achieved considerable success in analyzing the global structure of the space of solutions to the hypergeometric equation by concentrating on the monodromy properties of solutions. Actually what I have referred to (for later convenience) as the "classical Riemann-Hilbert problem" does not always have a solution and indeed it is not the form of the problem posed by Hilbert. An isolated singularity for a linear differential equation in the complex plane with otherwise locally holomorphic coefficients is said to be a *regular singular point* if there is a matrix fundamental solution whose entries are polynomially bounded near the singular point. The problem of finding a linear differential equation with regular singularities and prescribed monodromy does always have a solution. Solutions to this problem were offered by Hilbert, Plemelj, and Birkhoff [2] at the beginning of this century. In the 1950's Röhrl [23] solved a generalization of this problem for Riemann surfaces and more recently multidimensional generalizations have been considered by Deligne [7]. These modern developments are the subject of the theory of "D-modules" [15].

Returning to the one dimensional situation a simple pole is always a regular singular point (the method of Frobenius produces solutions) but the converse is not true. The distinction I've introduced will be useful in describing one version of the significance of the  $\tau$ -function introduced by SMJ.

We turn now to the consideration of the  $\tau$ -function introduced by SMJ. They show that for solutions  $A_\nu(a)$  to the Schlesinger equations (2) above the right hand

side of the following equation is exact:

$$(3) \quad d_a \log(\tau(a)) = \frac{1}{2} \sum_{\nu \neq \mu} \text{Tr}(A_\nu A_\mu) \frac{da_\nu - da_\mu}{a_\nu - a_\mu}$$

I would like to make some observations concerning this formula. The first is that this equation gives the fundamental connection between the quantum field theory whose correlation function is  $\tau(a)$  and the deformation theory for the matrices  $A_\nu(a)$ . The second is that the right hand side has manifest singularities on the sets

$$\Delta_{\mu\nu} = \{a \in \mathbf{C}^p \mid a_\mu = a_\nu\}$$

for  $\mu \neq \nu$ . The third observation is that the solutions to the Schlesinger equations  $A_\nu(a)$  may have pole type singularities themselves. The final observation is that equation (3) is the closest thing to a definition of the  $\tau$ -function that appears in the paper "Holonomic Quantum Fields II" (the paper concerned with the Riemann-Hilbert version of their theory). The reason for this is that there are some serious difficulties in making rigorous mathematical sense of massless holonomic quantum fields. Massless holonomic fields have been constructed in a special case by A. Carey, S.N.M. Ruijsenaars and J. Wright [4] but their results do not, so far as I am aware, allow one to come to grips with (3). One reason for this is that the  $\tau(a)$  which appears in (3) is not quite the "physical" correlation function but the analytic continuation to "imaginary times." This analytic continuation is the source of the multivaluedness in the "function" defined by (3), and it is hard to deal with mathematically.

These difficulties make the following results of B. Malgrange [16] especially interesting. First Malgrange proves the Painlevé property for solutions to the Schlesinger equations (not everyone was happy with Schlesinger's original proof). Secondly he shows that the  $\tau$ -function defined by (3) (or more precisely a close relative of (3)) is a well defined holomorphic function on the simply connected covering of  $\mathbf{C}^p - \cup \Delta_{\mu\nu}$ . Finally he shows that  $\tau(a) = 0$  precisely at those values of  $a$  where the solutions to the Schlesinger equations have a pole and this set in turn is the same as the set of points where the "classical Riemann-Hilbert problem" fails to have a solution (it is precisely to describe this result that we have singled out the "classical Riemann-Hilbert problem" even though neither Riemann nor Hilbert may ever have posed the problem in exactly this fashion). These results of Malgrange are the only results I am aware of that establish a conceptual significance for the  $\tau$ -function in the associated deformation theory.

In the rest of this article I would like to describe yet another way to think about what a  $\tau$ -function "really" is. Before I do this I will explain what motivated

this work. Monodromy preserving deformations of linear differential equations in the complex plane is not the only setting in which SMJ introduced  $\tau$ -functions. Indeed, as has already been mentioned, the original example of scaled correlations for the Ising model was tied to a deformation theory for the Dirac equation. One would like to generalize the results of Malgrange to this setting, but for a number of reasons this is not straightforward. In particular Malgrange makes heavy use of the relationship between the solution to the Riemann-Hilbert problem and the problem of trivializing holomorphic vector bundles (a connection first exploited by Röhrl [23]). There is no strict analogue of this connection in the Dirac case. In *Determinants of Cauchy-Riemann operators as  $\tau$ -functions* [19] I propose to reinterpret the Malgrange analysis in the following manner. I introduce a differential operator  $\bar{\partial}_{a,L}$  whose domain incorporates functions with prescribed branching and monodromy,  $M^\nu = \exp(2\pi i L_\nu)$  at the point  $a_\nu$ . The  $\tau$ -function is then:

$$(4) \quad \tau(a) = \det(\bar{\partial}_{a,L})$$

The operator  $\bar{\partial}_{a,L}$  is a close relative of the familiar Cauchy-Riemann operator  $\bar{\partial}_z := \frac{1}{2}(\partial_x + i\partial_y)$  on  $\mathbf{P}^1$ . To get a feeling for the significance of the operator  $\bar{\partial}_{a,L}$  it is useful to recall a standard reformulation of the classical Riemann-Hilbert problem. One seeks a multivalued holomorphic invertible matrix valued function  $Y(z)$  with branch type singularities at each of the points  $a_\nu$  such that in a neighborhood of each of these points we have:

$$Y(z) = \Phi_\nu(z)(z - a_\nu)^{L_\nu}$$

for some invertible matrix valued function  $\Phi_\nu(z)$  which is holomorphic in some neighborhood of the branch point  $a_\nu$ . If one has such a function  $Y(z)$ , then it is easy to see that the function  $Y'(z)Y(z)^{-1} = A(z)$  is single valued on  $\mathbf{P}^1$  with at worst simple poles at the points  $a_\nu$ . Thus  $A(z)$  must be rational and one obtains the differential equation (1) as an automatic consequence of solving the problem in this form.

The columns of  $Y(z)^{-1}$  have the right local behavior to be in the domain of  $\bar{\partial}_{a,L}$  but they are not global sections of the appropriate bundle and so are not globally in the domain of  $\bar{\partial}_{a,L}$  (if they were our operator  $\bar{\partial}_{a,L}$  would have at least an  $n$  dimensional null space instead of being generically invertible). This will be explained later. For the present it is useful to think of  $Y(z)$  as a *gauge transformation* (when it exists) transforming  $\bar{\partial}_z$  into  $\bar{\partial}_{a,L}$ . That is:

$$\bar{\partial}_{a,L}f(z) = Y(z)^{-1}\bar{\partial}_zY(z)f(z)$$

It is important to understand that while this equation is correct when  $Y(z)$  exists it is possible to define  $\bar{\partial}_{a,L}$  even when the classical Riemann-Hilbert problem does not have a solution. One observes then that the classical Riemann-Hilbert problem has a solution precisely when the operator  $\bar{\partial}_{a,L}$  is gauge equivalent (by a singular gauge transformation to be sure) to the standard Cauchy-Riemann operator  $\partial_z$ . Note that in the case that the classical Riemann-Hilbert problem has a solution, the  $\tau$ -function is the determinant of a similarity transform of a fixed differential operator. The usual finite dimensional determinant is a similarity invariant and the fact that the  $\tau$ -function depends on  $a$  in this case happens to be related to what is called a "gauge anomaly" in the physics literature [1], [24], [3].

Something which doubtless requires further comment in understanding (4) is the notion of a determinant for a differential operator. The sense which we give to (4) requires the notion of determinant bundles over families of Fredholm maps — a notion that was introduced by D. Quillen in [24]. Before turning in more detail to the consideration of (4) let me stop to summarize the different point of view that is implicit in (4). First, the operator  $\bar{\partial}_{a,L}$  is the fundamental object of the theory. This operator is simple to define — it is not necessary, for example, to solve the Riemann-Hilbert problem to define  $\bar{\partial}_{a,L}$ . The significance of the vanishing of the  $\tau$ -function is easy to understand. The  $\tau$ -function vanishes precisely where the operator  $\bar{\partial}_{a,L}$  fails to be invertible. How does the deformation theory arise in this picture? It turns out that to make sense of the determinant in a natural way requires certain asymptotics for the Green function of  $\bar{\partial}_{a,L}$  in a neighborhood of the branch points  $\{a_1, \dots, a_n\}$ . The Schlesinger deformation theory may be reinterpreted to give an independent characterization of the relevant asymptotics, leading in particular to (3). Most importantly this view of the  $\tau$ -function does generalize to the Dirac case. One can introduce a Dirac operator with a domain that contains functions with appropriate branching and monodromy. The SMJ deformation theory [27 III] independently characterizes the asymptotics of the Green function for this operator and the  $\tau$ -function may be understood in a natural fashion as the determinant of this differential operator.

Since this article is to appear in the proceedings of a conference on inverse scattering, it is appropriate for me to mention a connection with spectrum preserving deformation theory that is possibly illuminating. Following work of M. Sato [25] and M. Sato and Y. Sato [26] which interpreted certain KdV flows as induced linear flows on infinite dimensional Grassmannians and introduced  $\tau$ -functions as generating functions for Plücker coordinates, Daté, Jimbo, Kashiwara, and Miwa [5] found a group representational significance for the  $\tau$ -functions for KdV in the context

of their results connecting certain features of KdV theory with the representation theory of Kac-Moody algebras. A geometric synthesis of these results can be found in the paper *Loop groups and equations of KdV type* by G. Segal and G. Wilson where a rigorous account of the theory of  $\tau$ -functions for KdV is also presented [28]. In this theory the  $\tau$ -function has many guises--it is the object of Hirota's bilinear operator analysis, it is the parametrization of the orbit of a family of solutions to the KdV hierarchy, it is the determinant of the projection onto the reference subspace in the Grassmannian, it is a generating function for Plücker coordinates, and etc. One guise not mentioned there but which can be inferred from the Krichever construction and a determinant bundle map which I will explain shortly is that the  $\tau$ -function is also the determinant of a Cauchy-Riemann operator acting on the sections of a line bundle over a Riemann surface. More precisely there is a family of line bundles parametrized by the flow variables in the KdV hierarchy. This version of the  $\tau$ -function is mentioned by E. Witten [32] in a paper which also contains the suggestion that the Baker function might be regarded as the asymptotics of the Green function for the associated Cauchy-Riemann operator. Witten's discussion is not mathematically precise, but I believe the constructions in [19] can be used to give a mathematically rigorous account of his ideas. There are a number of other models in spectrum preserving deformation theory where  $\tau$ -functions appear, including the Toda lattice and the Landau-Lifschitz equations [6]. I believe that the introduction of the right differential operator (the putative operator whose Green function has asymptotics governed by the Baker functions of the spectrum preserving deformation theory) would go a long way towards making manifest the similarities in the theory of monodromy and spectrum preserving deformation theory. Both theories might be realized as auxiliary results in the analysis of Green's functions for certain families of differential operators. A related project is to find an appropriate determinant bundle trivialization to define the  $\tau$ -functions for the analogue of the operator  $\bar{\partial}_{a,L}$  from [19] in the irregular singular generalizations of [14]. This has been done and it might have implications for quantum 2D gravity where Painlevé functions associated with monodromy preserving deformations of irregular singular points arise [18].

Let me return now to the ideas which are involved in making sense of (4). Suppose that  $H_1$  and  $H_2$  are two Hilbert spaces. Let  $Fred(H_1, H_2)$  denote the space of Fredholm maps from  $H_1$  to  $H_2$ . These are the linear maps with finite dimensional kernel in  $H_1$  whose range has finite codimension in  $H_2$ . The space  $Fred(H_1, H_2)$  is not connected. The index distinguishes the components of  $Fred(H_1, H_2)$ . If  $T$  is a

Fredholm map then the index of  $T$  is defined by:

$$\text{Ind}(T) := \dim(\ker T) - \dim(\text{coker} T)$$

D. Quillen has defined a holomorphic line bundle over  $\text{Fred}(H_1, H_2)$ . It is somewhat simpler and it will suffice for our purposes to restrict our attention to the determinant bundle over the Fredholm maps with index 0,  $\text{Fred}_0(H_1, H_2)$  (this is roughly like restricting oneself to square matrices in the finite dimensional case). If  $T$  is a Fredholm map with index 0 then there exist invertible maps  $q : H_1 \rightarrow H_2$  with  $q^{-1}T$  a trace class perturbation of the identity on  $H_1$ . We will refer to such a  $q$  as an *admissible parametrix* for  $T$ . We take the fiber in the determinant bundle over  $T$  to be equivalence classes of pairs  $(q, \alpha)$  where  $q$  is an admissible parametrix for  $T$  and  $\alpha$  is a complex number. Two such pairs  $(q_k, \alpha_k)$  for  $k = 1, 2$  are equivalent if:

$$\alpha_1 = \alpha_2 \det(q_1^{-1} q_2)$$

See Malgrange [17]. The multiplicative property of determinants makes it possible to check that this is indeed an equivalence relation. The map which sends  $T$  into the "relative determinant"  $(q, \det(q^{-1}T))$  (where  $q$  is any admissible parametrix for  $T$ ) is not a function on the set  $\text{Fred}_0$ , but it is not hard to check that it is a section of the determinant bundle. This section is called the *canonical section*  $\sigma$ . If  $\mathcal{F}$  is a family of operators in  $\text{Fred}_0$  then making a choice of a relative determinant for each element of the family  $\mathcal{F}$  is morally equivalent to finding a trivialization (that is a nonvanishing section  $\delta$ ) of the line bundle:

$$\det \rightarrow \mathcal{F}$$

The determinant for the family  $\mathcal{F}$  determined by the trivialization  $\delta$  is:

$$\det(T) = \frac{\sigma(T)}{\delta(T)}$$

Why is the notion of a determinant bundle superior to the notion of regularizing determinants using admissible parametrices? If one has a family  $\mathcal{F}$  of Fredholm operators depending on parameters it may be difficult to construct admissible parametrices for this family that depend smoothly on the parameters. Indeed, it can happen that there is a global obstruction to the construction of such families of parametrices. The information needed to patch together local families of parametrices so that a global determinant is well defined is cruder than this and is codified in the determinant bundle. It can, of course, happen that the line bundle  $\det \rightarrow \mathcal{F}$  is non-trivial — indeed one of the real advantages of the determinant bundle idea is

that nebulous idea of defining a relative determinant for a family of maps becomes the geometric problem of trivializing a line bundle. The topological (and holomorphic) obstructions in this problem are understood in the theory of Chern classes [12]. The study of gauge anomalies in quantum field theory, for example, is much illuminated by the calculation of Chern classes for a determinant bundle over the space of gauge potentials modulo gauge equivalence [1].

There are also advantages even in the "favorable case" that the bundle  $\det \rightarrow \mathcal{F}$  does have a trivialization. For example, if  $\mathcal{F}$  is a manifold the bundle  $\det \rightarrow \mathcal{F}$  may have a flat connection. A local trivialization is then obtained by integrating such a flat connection (and if  $\mathcal{F}$  is simply connected one obtains a global trivialization in this manner). Such a trivialization may be geometrically natural but quite awkward to frame in the setting of local parametrices. This is precisely the situation that arises for the operator  $\bar{\partial}_{u,L}$ : the  $\tau$ -function arises from the integration of a flat connection on the determinant bundle but no attempt is made to construct a smooth system of local parametrices giving this result.

I will conclude this discussion of (4) by saying a few words about Cauchy-Riemann operators and the main technical result used in [19] to localize the calculation of the  $\tau$ -function and obtain (3). Suppose that  $E \rightarrow X$  is a  $C^\infty$  vector bundle over a Riemann surface  $X$ . A Cauchy-Riemann operator on  $E$  is a first order linear differential operator which takes a section  $f$  of  $E$  to a section  $Df$  of  $E \otimes \Omega^{0,1}$ , the bundle  $E$  tensored with the bundle of  $(0,1)$  forms, and which has the following form relative to a local parameter  $z$  on  $X$  and a local frame for  $E$ :

$$Df(z) = d\bar{z}(\bar{\partial}_z + A(z))f(z)$$

where  $A(z)$  is a  $\text{rank}(E) \times \text{rank}(E)$   $C^\infty$  matrix valued function. The different possible Cauchy-Riemann operators on  $E$  parametrize the different possible holomorphic structures for the bundles  $E$ . A local section  $s$  of  $E$  will be holomorphic with respect to the complex structure defined by the Cauchy-Riemann operator  $D$  if and only if  $Ds = 0$ . If  $X$  is compact then it is well known that one can introduce a Sobolev space  $H^1(E)$  so that  $D$  becomes a Fredholm map from  $H^1(E)$  to  $L^2(E \otimes \Omega^{0,1})$  (note that for  $X = \mathbf{C}$  and  $D = d\bar{z}\bar{\partial}_z$  the map  $D$  is not Fredholm from  $H^1$  to  $L^2$  so the restriction to compact  $X$  is important here). The index of a Cauchy-Riemann operator  $D$  on a bundle  $E$  over a compact Riemann surface  $X$  turns out to be given by topological data for the bundle  $E$  and the space  $X$ . The formula is the index theory version of the Riemann-Roch theorem:

$$\text{Ind}(D) = (1 - g)r + d$$

where  $g = \text{genus}(X)$ ,  $r = \text{rank}(E)$  and  $d = \text{degree}(E)$  which is also the first Chern class of  $E$ . Working on  $X = \mathbf{P}^1$  we have  $g = 0$  and so to get  $\text{Ind}(D) = 0$  it is necessary to have  $r + d = 0$ . We achieve this in [19] by working on  $r$  copies of the bundle over  $\mathbf{P}^1$  with  $d = -1$  (this bundle is a "square root" of the tangent bundle and is sometimes referred to as the spin bundle). By working in this setting we succeed in defining a family of Fredholm maps  $\bar{\partial}_{a,L}$  with index 0.

Finally I will describe the main technical result used to localize the description of the determinant bundle over the family of operators  $\bar{\partial}_{a,L}$ . Away from branch cuts emanating from the points  $\{a_1, \dots, a_n\}$  the differential operator  $\bar{\partial}_{a,L}$  acts just like the "standard" Cauchy-Riemann operator on the spin bundle (the action does not depend on  $a$ ). This makes it natural to localize  $\bar{\partial}_{a,L}$  in the following manner. Let  $\gamma$  be a smooth simple closed curve which surrounds the branch cuts for  $\bar{\partial}_{a,L}$ . Let  $W(a)$  denote the subspace of  $H^{\frac{1}{2}}(\gamma)$  which consists of boundary values on  $\gamma$  of functions  $f$  in the domain of  $\bar{\partial}_{a,L}$  with  $\bar{\partial}_{a,L}f(z) = 0$  for  $z$  in the interior of  $\gamma$ . We now localize  $\bar{\partial}_{a,L}$  in the *exterior* of the curve  $\gamma$  by letting it act on sections defined in the exterior of  $\gamma$  which have boundary values in  $W(a)$ . This is natural from the Fredholm point of view since the kernel and cokernel of the localized operator can be identified with the kernel and cokernel of the full operator  $\bar{\partial}_{a,L}$ . As remarked above the "differential operator" part of the localization is independent of  $a$ , the subspace  $W(a)$  now contains all the information about the variation of the family of operators with  $a$ . Now let  $\mathcal{F}$  denote a family of Cauchy-Riemann operators on the spin bundle which are fixed in the exterior of  $\gamma$ . For  $D \in \mathcal{F}$  let  $W_\gamma(D)$  denote the subspace of  $H^{\frac{1}{2}}(\gamma)$  obtained as boundary values of sections mapped to 0 by  $D$  in the interior of  $\gamma$ . Then the image of  $W_\gamma$  is contained in a certain Grassmannian  $Gr$  of subspaces of  $H^{\frac{1}{2}}(\gamma)$  closely related to the Grassmannians discussed in [22]. G. Segal and G. Wilson have defined a *det* line bundle over this Grassmannian (the line over a subspace  $W$  is the analogue in infinite dimensions of the highest exterior power of  $W$ ). The main technical result of [19] is then that the map  $W_\gamma : \mathcal{F} \rightarrow Gr$  lifts to a map  $\tilde{W}_\gamma$  from the determinant bundle over  $\mathcal{F}$  to the *det*<sup>\*</sup> bundle over  $Gr$  (*det*<sup>\*</sup> is the dual of the determinant bundle). Roughly speaking, what makes this lift natural is that there is a way to construct an admissible parametrix for  $D \in \mathcal{F}$  given an *admissible frame* for the subspace  $W_\gamma(D)$ . An admissible frame for a subspace  $W_\gamma(D)$  in the Grassmannian is a map that inverts the projection of this subspace onto the reference subspace of the Grassmannian up to a trace class perturbation. I refer the reader to [28] or [22] for a construction of the *det* bundle over  $Gr$  which uses the data of admissible frames to define the fiber in much the same way that we defined the fiber of the determinant bundle over  $\text{Fred}_0$  using

admissible parametrices.

The problem we consider then is the trivialization of the  $\det^*$  bundle over the family of subspaces  $W(a)$ . We reinterpret the results of Malgrange to get a flat connection on this bundle and the  $\tau$ -function is obtained by integrating this connection to get a trivialization which is then compared with the canonical section.

The map  $\hat{W}_\gamma$  is a mathematically precise version of an idea that can be found in Witten [32]. It has interesting connections with loop groups and conformal field theory (see [3] for one simple example) and it is also useful in making precise the ideas in the first part of Witten [32]. It would be interesting in this regard to see Witten's version of Baker functions and the connection with Ward identities in conformal field theory worked out in the mathematically well defined setting of  $\det$  and  $\det^*$  bundles.

As I mentioned earlier one may interpret the  $\tau$ -function for the Dirac case in the plane in much the same manner [20]. I am currently working with C. Tracy and R. Narayanan to define a  $\tau$ -function for the Dirac operator in the hyperbolic plane. C. Tracy [30] and [21] has already worked out the deformation theory that should be relevant for this problem and he will describe this work elsewhere in these proceedings. The survey article [31] is another good place to read in more detail about some of the historical developments and more recent efforts to generalize the SMJ deformation theory.

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## Inverse problems in anisotropic media

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### §0. Introduction

In this paper we consider two inverse problems for anisotropic conductivities. In Section 1 we consider the problem of determining the anisotropic conductivity of a body  $\Omega$  from electrical measurements at  $\partial\Omega$ . In Section 2 we consider the kinematic problem in seismology; i.e. can one recover a metric on  $\Omega$  from the lengths of geodesics joining points at the boundary (the so called travel times)? Both problems have the common feature that there is an obstruction to uniqueness. A diffeomorphism of  $\bar{\Omega}$  that fixes the boundary gives rise to a new metric (conductivity) with the same measurements. We show in both problems that the harmonic map equation is useful in breaking the diffeomorphism invariance for the linearized problem at the euclidean conductivity (euclidean metric for the second problem).

### §1. The inverse conductivity problem

Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  with smooth boundary. The conductivity of  $\Omega$  is represented by a symmetric positive definite matrix  $\gamma = (\gamma^{ij})_{i,j=1,\dots,n}$  in  $\bar{\Omega}$  which we assume to be smooth.

If we put a potential  $f$  on  $\partial\Omega$  (assume  $f \in H^{\frac{1}{2}}(\partial\Omega)$ ), the induced potential  $u$  in  $\Omega$  satisfies the Dirichlet problem

$$(1.1) \quad L_\gamma u = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (\gamma^{ij} \frac{\partial u}{\partial x_j}) = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = f.$$

The *Dirichlet to Neumann map* is the map

$$(1.2) \quad f \xrightarrow{\Lambda} \Lambda_\gamma(f) = \sum_{i,j=1}^n \nu^i \gamma^{ij} \frac{\partial u}{\partial x_i} |_{\partial\Omega} dS$$

where  $\nu^i$  denotes the  $i$ th component of the unit euclidean conormal and  $dS$  denotes eu-

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clidean surface measure on  $\partial\Omega$ . One reason for interpreting the Neumann data as an  $(n-1)$  form is that physically it is not the value of the current at one point but rather the current flux across a portion of the boundary that can be measured. The map  $\Lambda$  is also called the *voltage to current map*.

The amount of energy necessary to maintain a potential  $f$  on the boundary is given by the quadratic form

$$(1.3) \quad Q_\gamma(f) = \sum_{i,j=1}^n \int_\Omega \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$$

with  $u$  solution of (1.1).

If we denote by  $Q_\gamma(\cdot, \cdot)$  the bilinear form associated to (1.3), then  $\Lambda_\gamma$  is the unique self adjoint map associated to  $Q_\gamma$ , i.e.

$$(1.4) \quad Q_\gamma(f, g) = \int_{\partial\Omega} f \Lambda_\gamma(g).$$

We are going to study the injectivity of  $\Lambda$ , or equivalently, injectivity of  $\mathcal{Q}$ , where  $\mathcal{Q}$  is defined by

$$(1.5) \quad f \stackrel{\mathcal{Q}}{=} Q_\gamma(f).$$

Significant progress has been made on this question for isotropic conductivities, that is, in the case in which the relationship between voltage and current is independent of direction. In euclidean coordinates, an isotropic conductivity,  $\gamma$  has the form  $\gamma^{ij}(x) = \alpha(x)\delta_{ij}$  where  $\delta_{ij}$  denotes the Kronecker delta and  $\alpha$  is a smooth positive function in  $\bar{\Omega}$ . In this case injectivity of  $\Lambda$  is known for piecewise real analytic conductivities in dimension  $n \geq 2$  ([K-V I]) and  $C^2$ -conductivities in dimension  $n \geq 3$  ([S-U I]), locally near constants in 2 dimensions ([S-U II]) and globally for "most conductivities" in two dimensions ([Su-U]).

However, injectivity of  $\Lambda$  is not valid in general. If  $\psi : \bar{\Omega} \rightarrow \bar{\Omega}$  is a diffeomorphism which is the identity on boundary of  $\Omega$  and

$$(1.6) \quad \tilde{\gamma} = \frac{((D\psi)^t \circ \gamma \circ D\psi)}{|\det D\psi|} \circ \psi^{-1}$$

then (see [K-V II])

$$(1.7) \quad \Lambda_{\tilde{\gamma}} = \Lambda_\gamma.$$

This can be understood invariantly by computing the action of a diffeomorphism of the boundary on  $\Lambda_\gamma$  (see [S]). It is natural to conjecture that (1.6) is the only obstruction to uniqueness. More precisely

**Conjecture 1.** Let  $\gamma, \tilde{\gamma}$  be (anisotropic) conductivities such that  $\Lambda_\gamma = \Lambda_{\tilde{\gamma}}$ , then there exists a diffeomorphism  $\psi : \bar{\Omega} \rightarrow \bar{\Omega}$ , such that  $\psi|_{\partial\Omega} = \text{Identity}$  and (1.6) holds.

The conjecture has been proved in some cases. Sylvester ([S]) proved it in two dimensions in the case that  $\gamma$  and  $\tilde{\gamma}$  are  $C^3$  close to a constant conductivity. Lee and Uhlmann

([L-U]) proved it in the real analytic category in dimensions  $n \geq 3$  under some additional assumptions on  $\Omega$ . It is worth noting that in dimensions  $n \geq 3$ , the inverse conductivity problem is equivalent to the geometric problem of determining the Riemannian metric of a domain from the Dirichlet to Neumann map. More precisely if  $g = (g_{ij}(x))$  denotes a Riemannian metric in  $\bar{\Omega}$  and  $\Delta_g$  denotes the Laplace-Beltrami operator associated to  $g$ , which is given in local coordinates by

$$(1.8) \quad \Delta_g u = \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}} \left( \frac{\partial}{\partial x_i} (g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_j} u) \right)$$

where  $g^{ij} = (g_{ij})^{-1}$  and  $\det g(x) = \det(g_{ij}(x))$ . If  $u$  solves the Dirichlet problem

$$(1.9) \quad \begin{aligned} \Delta_g u &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f \end{aligned}$$

then the Neumann data is given by

$$(1.10) \quad \Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^n \nu^i g^{ij} \frac{\partial u}{\partial x_i} |_{\partial\Omega} dS$$

where both  $dS$  and  $\nu$  are associated to the euclidean metric. The Dirichlet to Neumann map is given by

$$(1.11) \quad f \rightarrow \Lambda_g(f).$$

In dimension  $n \geq 3$ ,  $\Lambda_\gamma = \Lambda_g$  if

$$g_{ij} = (\det \gamma^{k\ell})^{\frac{1}{n-2}} (\gamma^{ij})^{-1} \text{ or } \gamma^{ij} = (\det g_{k\ell})^{-\frac{1}{2}} (g_{ij})^{-1}.$$

In terms of  $\Lambda_g$  conjecture 1 is equivalent to

**Conjecture 2.** Let  $n \geq 3$  and  $g, h$  be  $C^\infty$  Riemannian metrics in  $\bar{\Omega}$  so that  $\Lambda_g = \Lambda_h$ . Then there exists a diffeomorphism  $\psi : (\bar{\Omega}, g) \rightarrow (\bar{\Omega}, h)$ ,  $\psi|_{\partial\Omega} = \text{Identity}$  such that  $\psi^*h = g$  where  $\psi^*h$  denotes the pull back of the metric  $h$  under the diffeomorphism  $\psi$ .

These two conjectures remain open in general . There are two main difficulties. The first one is the construction of special solutions analogous to the exponential growing solutions constructed by the authors in the isotropic case ([S-U I, II]). This was done by Sylvester in [S] in two dimensions; the use of isothermal coordinates allowed the construction in this case.

The second difficulty is how to break the diffeomorphism invariance . Jack Lee proposed the use of harmonic maps for this purpose. We show that this is successful in proving the linearization of conjectures 1 or 2 at the constant conductivity (see Section 1) as well as for the linearization at the euclidean metric of the inverse kinematic problem (see Section 2). In a paper in preparation, the authors, G. Mendoza and J. Lee are considering the non-linear case for the last problem ([L-M-S-U]).

We digress now to discuss the harmonic map equation. For a general reference see [H]. We shall only consider the case where the domain and range of the map is  $\bar{\Omega}$ , with  $\Omega$  a smooth bounded domain in  $\mathbf{R}^n$ .

Let  $f : (\bar{\Omega}, g) \rightarrow (\bar{\Omega}, h)$  be a smooth map where  $g$  and  $h$  are Riemannian metrics in  $\bar{\Omega}$ . The *energy* associated to the map  $f$  is given in local coordinates by

$$(1.12) \quad E(f) = \sum_{\alpha, \beta, i, j=1}^n \int_{\Omega} g^{ij}(x) h_{\alpha\beta}(f(x)) \frac{\partial f^{\alpha}}{\partial x_i} \frac{\partial f^{\beta}}{\partial x_j} \sqrt{\det g} dx.$$

The Euler-Lagrange equation associated to the quadratic form (1.12) is given by the non-linear elliptic system

$$(1.13) \quad \frac{-2}{\sqrt{\det g}} \sum_{\alpha, i, j=1}^n \frac{\partial}{\partial x_j} ((\sqrt{\det g}) g^{ij} h_{\alpha\beta} \frac{\partial f^{\alpha}}{\partial x_i}) + \sum_{\alpha, \gamma, i, j=1}^n g^{ij} \frac{\partial h_{\alpha\gamma}}{\partial f_{\beta}} \frac{\partial f^{\alpha}}{\partial x_i} \frac{\partial f^{\gamma}}{\partial x_j} = 0 \quad \forall \beta$$

**Definition 1.14** A  $C^{\infty}$  map  $f : (\bar{\Omega}, g) \rightarrow (\bar{\Omega}, h)$  is called *harmonic* if it is a critical point of (1.12) (i.e. it is a solution of (1.13)).

Note that if  $h$  is the Euclidean metric, then (1.13) simply states that the components  $f$  are harmonic functions with respect to the metric  $g$ .

We are going to reduce conjecture 2 to the proof of a uniqueness theorem by means of the following Proposition, which follows readily from the definition of a harmonic map.

**(1.15) Proposition.** *Let  $(\bar{\Omega}, g)$  and  $(\bar{\Omega}, h)$  be two smooth bounded domains with Riemannian metrics  $g$  and  $h$ . Suppose there is a harmonic map*

$$(1.16) \quad \psi : (\bar{\Omega}, g) \rightarrow (\bar{\Omega}, h) \text{ such that } \psi|_{\partial\Omega} = \text{Identity and } \psi \text{ a diffeomorphism.}$$

Then

$$\text{Identity} : (\bar{\Omega}, g) \rightarrow (\bar{\Omega}, \psi^* h)$$

is harmonic.

We shall show that conjecture 2 is reduced to prove

**Conjecture 3.** Suppose  $g$  and  $h$  are Riemannian metrics on  $\bar{\Omega}$  and that  $\text{Identity} : (\bar{\Omega}, g) \rightarrow (\bar{\Omega}, h)$  is harmonic and  $\Lambda_g = \Lambda_h$ . Then  $g = h$ .

**(1.17) Proposition.** *Conjecture 3  $\implies$  Conjecture 2 if there exists a harmonic map  $\psi$  satisfying (1.16).*

**Proof.** Assume  $\Lambda_g = \Lambda_h$  and the existence of a harmonic map  $\psi$  satisfying (1.16). Then we have  $\Lambda_g = \Lambda_h = \Lambda_{\psi^* h}$ . Now using Proposition 1.15 and Conjecture 3 we conclude that  $g = \psi^* h$ .

The solvability of the harmonic Dirichlet problem (1.16) is known if  $h$  has nonpositive sectional curvature ([H]) or if  $g$  and  $h$  are sufficiently close in the  $C^3$  topology to the euclidean metric ([L-M-S-U]).

Thus, we have reduced the proof of Conjecture 1 or 2 to the uniqueness statement in Conjecture 3, under the additional assumption of the existence of an harmonic map which is the identity on the boundary.

In the rest of this section we prove that the linearization at the identity of conjecture 3 holds. In analogy with (1.3) the quadratic form associated to  $\Lambda_g$  is given by

$$(1.18) \quad Q_g(f, g) = \sum_{i,j=1}^n \int_{\Omega} g^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \sqrt{\det g} dx$$

with  $u, v$  solution of  $\Delta_g u = \Delta_g v = 0$  in  $\Omega$ ;  $u|_{\partial\Omega} = f, v|_{\partial\Omega} = g$ . We consider the linearization of  $Q$  at the euclidean metric in the direction of the quadratic form  $m \in C_0^\infty(\Omega)$

$$(1.19) \quad dQ_m(f, g) = \lim_{\epsilon \rightarrow 0} \frac{Q_{e+\epsilon m}(f, g) - Q_e(f, g)}{\epsilon}.$$

A computation yields:

$$(1.20) \quad dQ_m(f, g) = \sum_{i,j=1}^n \int_{\Omega} (m_{ij} - \frac{1}{2} tr m) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

where  $\Delta u = \Delta v = 0$  in  $\Omega$ ;  $u|_{\partial\Omega} = f, v|_{\partial\Omega} = g$  and  $tr m = (\sum_{i=1}^n m_{ii})\delta_{ij}$ . We will also denote  $tr m$  the function  $tr m = \sum_{i=1}^n m_{ii}$ .

As Calderón ([C]) did for the isotropic case, we take

$$(1.21) \quad u = e^{x \cdot \xi}, v = e^{-x \cdot \bar{\xi}}$$

where  $\xi \in \mathbf{C}^n, \xi = \frac{1}{2}(\eta + ik)$  with  $\eta, k \in \mathbf{R}^n$  and  $\langle \eta, k \rangle = 0, |\eta| = |k|$ . Substituting (1.21) in (1.20) we obtain

$$(1.22) \quad \sum_{i,j=1}^n \int_{\Omega} (m_{ij} - \frac{1}{2} tr m) e^{i(x,k)} (\eta_i \eta_j + k_i k_j) = 0.$$

We rewrite (1.22) in the form

$$(1.23) \quad k^t (\hat{m} - \frac{1}{2} tr \hat{m}) k + \eta^t (\hat{m} - \frac{1}{2} tr \hat{m}) \eta = 0$$

where  $t$  denotes transpose and  $\hat{\cdot}$  the Fourier transform.

Now the fact that the identity is a harmonic map implies the following system of  $n$  first order linear partial differential equations for  $m = g - h$  ( $g$  is the euclidean metric in this computation):

$$(1.24) \quad -2 \sum_{j=1}^n \frac{\partial}{\partial x_j} (m_{j\beta}) + \frac{\partial}{\partial x_\beta} tr m = 0 \text{ in } \Omega, \quad \beta = 1, \dots, n.$$

Taking the Fourier transform of (1.24) we obtain

$$(1.25) \quad -2 \sum_{j=1}^n k_j \hat{m}_{j\beta}(k) + k_\beta tr \hat{m}(k) = 0. \quad \beta = 1, \dots, n.$$

Let us take  $k = (1, 0, \dots, 0)$ ,  $\eta \in k^\perp$  with  $|\eta| = |k| = 1$ .

Using (1.25) we get

$$(1.26) \quad \begin{aligned} \hat{m}_{1\beta}(k) &= 0, \quad \beta = 2, \dots, n \\ \hat{m}_{11}(k) &= \frac{1}{2} \operatorname{tr} \hat{m}(k). \end{aligned}$$

Using (1.23) we obtain

$$(1.27) \quad \hat{m}_{\beta\gamma} - \frac{1}{2} \operatorname{tr} \hat{m}(k) = -(\hat{m}_{11} - \frac{1}{2} \operatorname{tr} \hat{m})(k), \quad \beta = 2, \dots, n, \quad \gamma = 2, \dots, n.$$

Combining (1.26) and (1.27) we conclude  $\operatorname{tr} \hat{m}(k) = 0$ .

Using (1.26) and (1.27) again we see that  $\hat{m}_{ij}(k) = 0$   $i, j = 1, \dots, n$ .

Rotating coordinates shows that  $\hat{m}(k) = 0 \forall k$  and therefore  $m = 0$ .

## §2. The inverse kinematic problem in seismology

In this section we show that knowledge of the Dirichlet to Neumann map for the wave equation associated to the Laplace Beltrami operator to a metric  $g$  on a domain, determines the length of the geodesics joining points of the boundary. The inverse kinematic problem is to recover the metric knowing these lengths, which are sometimes called travel times. Finally we show, using again the harmonic map equation, that the linearized inverse kinematic problem is injective at the euclidean metric.

Let  $\Omega$  be a smooth bounded domain in  $\mathbf{R}^n$  and  $g$  a smooth Riemannian metric on  $\bar{\Omega}$ . We consider the initial boundary value problem

$$(2.1) \quad \begin{aligned} \left( \frac{\partial^2}{\partial t^2} - \Delta_g \right) u &= 0 \text{ in } \Omega \times (0, T), \quad T > 0 \\ u|_{t=0} &= \frac{\partial u}{\partial t}|_{t=0} = 0 \text{ in } \Omega \\ u|_{\Omega \times (0, T)} &= f. \end{aligned}$$

We define the (hyperbolic) Dirichlet to Neumann map by

$$(2.2) \quad \Lambda_g^h(f) = \sum_{i,j=1}^n (\nu^i g^{ij} \frac{\partial u}{\partial x_j})|_{\partial\Omega} dS$$

where  $u$  is a solution of (2.1).

As in the elliptic case, it is easy to see that the map

$$(2.3) \quad g \xrightarrow{\Lambda_g^h} \Lambda_g^h$$

is not injective since  $\Lambda_{\psi^*g}^h = \Lambda_g^h$  for any diffeomorphism  $\psi : \bar{\Omega} \rightarrow \bar{\Omega}$  such that  $\psi|_{\partial\Omega} = \text{Identity}$  (it is injective in the isotropic case [R-S]). We first show that knowledge of  $\Lambda_g^h$  determines the Taylor series of  $g$  at  $\partial\Omega$  in boundary normal coordinates.

For each  $q \in \partial\Omega$ , let  $\gamma_q$  be the unit-speed geodesic starting at  $q$  and normal to  $\partial\Omega$ . If  $\{x^1, \dots, x^{n-1}\}$  are local coordinates for  $\partial\Omega$  near  $q$  we can extend them smoothly to

functions on a neighborhood of  $q \in \partial\Omega$  in  $\bar{\Omega}$  by letting them be constant along each normal geodesic  $\gamma_q$ . If we denote by  $x^n$  the length parameter along each  $\gamma_q$ , it follows easily that  $\{x^1, \dots, x^n\}$  form coordinates in  $\bar{\Omega}$  in a neighborhood of  $q$ , which we call *boundary normal coordinates*. In these coordinates  $x^n > 0$  in  $\Omega$  and  $\partial\Omega$  is locally defined by  $x^n = 0$ . In boundary normal coordinates the metric  $g$  takes the form

$$(2.4) \quad g = \sum_{\alpha, \beta=1}^{n-1} g_{\alpha\beta}(x) dx^\alpha dx^\beta + (dx^n)^2$$

**(2.5) Proposition.** *Let  $g_0, g_1$  be smooth Riemannian metrics on  $\bar{\Omega}$ . Assume  $\Lambda_{g_0}^h = \Lambda_{g_1}^h$ . Then there exists a diffeomorphism  $\psi : \bar{\Omega} \rightarrow \bar{\Omega}$ ,  $\psi|_{\partial\Omega} = \text{Identity}$  such that  $\psi^*g_1 = g_0$  to infinite order on  $\partial\Omega$ .*

**Proof.** Let  $N_k$  denote boundary normal coordinates for the metric  $g_k$ . We will make a change of variables which is the identity on the boundary so that both metrics look simultaneously like (2.4) near  $\partial\Omega$ . If we let  $h_1 = (N_1^{-1}N_0)^*g_1$ , then we have that both  $h_1$  and  $g_0$  look in the new coordinates as in (2.4) with a fixed coordinate system  $(x_1, \dots, x_{n-1}, x_n)$ . Notice that  $(N_1^{-1}N_0)_*$  maps the boundary normal corresponding to the metric  $g_0$  into the one corresponding to the metric  $g_1$ , fixing coordinates on the boundary. Notice that  $\Lambda_{h_1} = \Lambda_{g_1} = \Lambda_{g_0}$ . Hence, by replacing  $g_1$  with  $h_1$  if necessary, we may work in coordinates where both  $g_0$  and  $g_1$  have the form (2.4).

Let us take a point  $(x_0, t_0) \in \partial\Omega \times (0, T)$  and let  $\xi \in \mathbf{R}^{n-1}$ ,  $\tau \in \mathbf{R}$  be fixed vectors such that  $\tau^2 - \sum_{i,j=1}^{n-1} g_k^{ij}(x_0, t_0)\xi_i\xi_j > 0$ . Then there are solutions of

$$(2.6) \quad \left( \frac{\partial^2}{\partial t^2} - \Delta_{g_k} \right) u_k = 0$$

$$u_k|_{t=0} = \frac{\partial u_k}{\partial t}|_{t=0} \text{ in } \Omega, \quad k = 0, 1$$

which near  $(x_0, t_0)$  has the asymptotic form

$$(2.7) \quad u_k \sim e^{i\lambda\varphi^k} \sum_{j=0}^{\infty} A_j^k \lambda^{-j}, \quad k = 0, 1$$

where both  $\varphi$  and  $A$  are functions of  $(x, t, \xi, \tau)$  satisfying equations (2.8) and (2.9) below. The meaning of  $\sim$  is the standard one, namely

$$|\partial_x^\alpha \partial_t^\beta (u_k - e^{i\lambda\varphi^k} \sum_{j=0}^N A_j^k \lambda^{-j})| = O(\lambda^{-(N+1)+|\alpha|+|\beta|})$$

uniformly near  $(x_0, t_0)$  for  $(\xi, \tau)$  in a compact set. Solutions of (2.6) as in (2.7) can be constructed by propagating the boundary data near  $(x_0, t_0)$  backwards in time and then solving the forward wave equation  $(\frac{\partial^2}{\partial t^2} - \Delta_{g_k})u_k = 0$  with the initial data obtained at  $t = 0$  (see Chazarain([Ch]) for more details).

By comparing powers of  $\lambda$  we obtain the transport equations. For  $k = 0, 1$  and for  $(x, t)$  near  $(x_0, t_0)$ , the following hold

$$(2.8) \quad \left( \frac{\partial \varphi^k}{\partial t} \right)^2 - \sum_{i,j=1}^n g_k^{ij}(x) \frac{\partial \varphi^k}{\partial x_i} \frac{\partial \varphi^k}{\partial x_j} = 0$$

$$\varphi^k|_{\partial\Omega \times (0, T)} = x' \cdot \xi + t\tau, k = 0, 1$$

where  $x' = (x_1, \dots, x_{n-1})$ , and  $g_k^{\alpha\beta}(x) = (g_k^{\alpha\beta}(x))^{-1}$ . In addition,

$$(2.9) \quad L^k A_0^k = 0$$

$$A_0^k|_{\partial\Omega \times (0, T)} = 1$$

$$\sqrt{-1} L^k A_{j+1}^k = (\partial_t^2 - \Delta_{g_k}) A_j^k, \quad \forall j \geq 0$$

$$A_{j+1}^k|_{\partial\Omega \times (0, T)} = 0, \quad \forall j \geq 0$$

where

$$(2.10) \quad L^k = 2 \frac{\partial \varphi^k}{\partial t} \frac{\partial}{\partial t} - 2 \frac{\partial \varphi^k}{\partial x_n} \frac{\partial}{\partial x_n} - 2 \sum_{\alpha, \beta=1}^{n-1} g_k^{\alpha\beta} \frac{\partial}{\partial x_\alpha} \varphi^k \frac{\partial}{\partial x_\beta} + \left( \frac{\partial^2 \varphi^k}{\partial t^2} - \Delta_{g_k} \varphi^k \right)$$

To show that we can solve (2.8) we notice that in our coordinates both  $g_k^{nn}(x_0, t_0) = 1$ . Therefore we can rewrite (2.8) as

$$(2.11) \quad \left( \frac{\partial \varphi^k}{\partial x_n} \right)^2 = \left( \frac{\partial \varphi^k}{\partial t} \right)^2 - \sum_{\alpha, \beta=1}^{n-1} g_k^{\alpha\beta}(x) \frac{\partial \varphi^k}{\partial x_\alpha} \frac{\partial \varphi^k}{\partial x_\beta}$$

$$\varphi^k|_{x_n=0} = x' \cdot \xi + t\tau.$$

This is a non-characteristic first order non-linear PDE which can be solved (locally) by Hamilton-Jacobi theory.

The transport equation (2.9) is noncharacteristic since the coefficient of  $\frac{\partial}{\partial x_n}$  in  $L^k$  is  $-2 \frac{\partial \varphi^k}{\partial x_n}$  which is non-zero in a neighborhood of  $(x_0, t_0)$ .

Now the fact that  $\Lambda_{g_0} = \Lambda_{g_1}$  implies that

$$(2.12) \quad \frac{\partial}{\partial \nu_{g_0}} u_0|_{\partial\Omega \times (0, T)} = \frac{\partial}{\partial \nu_{g_1}} u_1|_{\partial\Omega \times (0, T)}.$$

Computing the powers of  $\lambda$  in the asymptotic expansion of both sides of (2.12) we get

$$(2.13) \quad \frac{\partial \varphi^0}{\partial \nu_{g_0}}(x_0, t_0, \xi, \tau) = \frac{\partial \varphi^1}{\partial \nu_{g_1}}(x_0, t_0, \xi, \tau).$$

and

$$(2.14) \quad \sqrt{-1} \frac{\partial}{\partial \nu_{g_0}} \varphi^0 A_{j+1}^0 + \frac{\partial}{\partial \nu_{g_0}} A_j^0 = \sqrt{-1} \frac{\partial}{\partial \nu_{g_1}} \varphi^1 A_{j+1}^1 + \frac{\partial}{\partial \nu_{g_1}} A_j^1 \quad \forall j \geq 0.$$

$$\frac{\partial}{\partial \nu_{g_0}} A_0^0 = \frac{\partial}{\partial \nu_{g_1}} A_0^1$$

at  $(x_0, t_0, \xi, \tau)$ .

A computation yields

$$\frac{\partial \varphi^k}{\partial \nu_{g_k}} = \frac{1}{\sqrt{\det g_k}} \frac{\partial}{\partial x_n} \varphi^k, \quad k = 0, 1.$$

Thus, (2.13) implies that

$$\frac{1}{\sqrt{\det g_0}} \frac{\partial}{\partial x_n} \varphi^0 = \frac{1}{\sqrt{\det g_1}} \frac{\partial}{\partial x_n} \varphi^1 \text{ at } (x_0, t_0, \xi, \tau).$$

The eikonal equation (2.11) allows us to conclude that at the point  $(x_0, t_0, \xi, \tau)$

$$\frac{1}{\det g_0} (\tau^2 - \sum_{\alpha, \beta=1}^{n-1} g_0^{\alpha\beta}(x_0, t_0) \xi_\alpha \xi_\beta) = \frac{1}{\det g_1} (\tau^2 - \sum_{\alpha, \beta=1}^{n-1} g_1^{\alpha\beta}(x_0, t_0) \xi_\alpha \xi_\beta).$$

Taking  $\tau > 0$ ,  $\xi = 0$ , we find that  $\det g_0 = \det g_1$ . Next, we obtain

$$\sum_{i, j=1}^{n-1} g_0^{ij}(x_0, t_0) \xi_i \xi_j = \sum_{i, j=1}^{n-1} g_1^{ij}(x_0, t_0) \xi_i \xi_j$$

proving that the metrics coincide at the boundary. To prove that the normal derivatives coincide, we use (2.14). We have that

$$\frac{\partial}{\partial x_n} A_0^0 = \frac{\partial}{\partial x_n} A_0^1 \text{ at } (x_0, t_0, \xi, \tau)$$

Now from the transport equation (2.9) we see that

$$\frac{\partial}{\partial x_n} A_0^k = \frac{\partial^2}{\partial x_n^2} \varphi^k + E^k(x) \frac{\partial}{\partial x_n} \varphi^k + R^k \text{ at } (x_0, t_0, \xi, \tau)$$

where  $R^k$  involves only time and tangential derivatives of  $\varphi^k$ . We have that  $R^0 = R^1$  at  $(x_0, t_0, \xi, \tau)$  and we also have that  $E^0 = E^1$  at  $(x_0, t_0, \xi, \tau)$ . Therefore we conclude that

$$\frac{\partial^2}{\partial x_n^2} \varphi^0 = \frac{\partial^2}{\partial x_n^2} \varphi^1 \text{ at } (x_0, t_0, \xi, \tau).$$

Differentiating (2.11) we obtain,

$$\frac{\partial}{\partial x_n} \sum_{\alpha, \beta=1}^{n-1} g_0^{\alpha\beta}(x_0, t_0) \xi_\alpha \xi_\beta = \frac{\partial}{\partial x_n} \sum_{\alpha, \beta=1}^{n-1} g_1^{\alpha\beta}(x_0, t_0) \xi_\alpha \xi_\beta.$$

Continuing this argument inductively we conclude that

$$\frac{\partial^j}{\partial x_n^j} \varphi^0 = \frac{\partial^j}{\partial x_n^j} \varphi^1 \text{ at } (x_0, t_0, \xi, \tau) \quad \forall j.$$

Thus the Taylor series of  $g^0$  and  $g^1$  coincide at  $(x_0, t_0)$  in these coordinates.

We may therefore extend  $g_0 = g_1$  to  $\Omega^c$  such that both are smooth and both are euclidean outside a ball.

**(2.15) Proposition.** *Let  $g_0, g_1$  the smooth Riemannian metrics on  $\bar{\Omega}$ . Assume  $\Lambda_{g_0}^h = \Lambda_{g_1}^h$ . Let  $(u_0, u_1) \in \mathcal{E}'(\mathbf{R}^n) \times \mathcal{E}'(\mathbf{R}^n)$ ,  $\text{supp } u_k \subseteq \Omega^c$ ,  $k = 0, 1$ . The solution  $v_k$  of the initial value problem*

$$\begin{aligned} \left( \frac{\partial^2}{\partial t^2} - \Delta_{g_k} \right) v_k &= 0 \text{ in } \mathbf{R}^n \times (0, T) \\ v_k|_{t=0} &= u_0 \\ \frac{\partial v_k}{\partial t}|_{t=0} &= u_1 \end{aligned}$$

satisfies  $v_0 = v_1$  in  $\Omega^c \times (0, T)$ .

**Proof.** Since solutions of the initial value problem depend continuously on the initial data, it is enough to prove the proposition for  $u_k \in C_0^\infty(\mathbf{R}^n)$  with  $\text{supp } u_k \subset \Omega^c$ ,  $k = 0, 1$ .

Let  $z$  be a solution of the initial boundary value problem

$$(2.16) \quad \begin{aligned} \left( \frac{\partial^2}{\partial t^2} - \Delta_{g_0} \right) z &= 0 \text{ in } \Omega \times (0, T) \\ z|_{t=0} = \frac{\partial z}{\partial t}|_{t=0} &= 0 \text{ in } \Omega \\ z|_{\partial\Omega \times (0, T)} &= v_1|_{\partial\Omega \times (0, T)}. \end{aligned}$$

We have

$$(2.17) \quad \frac{\partial z}{\partial \nu_{g_0}} = \Lambda_{g_0}(z|_{\partial\Omega \times (0, T)}) = \Lambda_{g_1}(v_1|_{\partial\Omega \times (0, T)}) = \frac{\partial v_1}{\partial \nu_{g_1}}$$

where  $\nu_{g_k}$  denotes the outer unit normal with respect to the metric  $g_k$ ,  $k = 0, 1$ .

Let

$$w = \begin{cases} v_1 & \text{in } \Omega^c \times (0, T) \\ z & \text{in } \Omega \times (0, T). \end{cases}$$

Then by (2.17)  $w$  satisfies

$$(2.18) \quad \left( \frac{\partial^2}{\partial t^2} - \Delta_{g_0} \right) w = 0 \text{ in } \mathbf{R}^n \times (0, T).$$

Moreover

$$(2.19) \quad \begin{cases} u|_{t=0} = \begin{cases} v_1|_{t=0} = u_0 & \text{in } \Omega^c \\ z|_{t=0} = 0 & \text{in } \Omega \end{cases} \\ \frac{\partial w}{\partial t}|_{t=0} = \begin{cases} \frac{\partial v_1}{\partial t}|_{t=0} = u_1 & \text{in } \Omega^c \\ \frac{\partial z}{\partial t}|_{t=0} = 0 & \text{in } \Omega. \end{cases} \end{cases}$$

Formulas (2.18) and (2.19) show that  $w$  solves the same initial value problem as  $v_0$ . Therefore  $w = v_0$ , which concludes the proof.

One can use the proposition above and the geometrical optics construction (2.7) to solve the wave equation with data supported outside  $\Omega^c$  (say  $u_0 = \delta_y, y \in \Omega^c, u_1 = 0$ ) to conclude that the geodesic distance function for points  $y, x \in \Omega^c$  is the same. We are going to use an alternative method which is the Hadamard parametrix construction (see Hörmander [Hö], section 12.4).

Let  $F_k(t, x, y)$  be the solution of

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \Delta_{g_k}\right)F_k &= 0, \quad k = 0, 1 \\ F_k(0, x, y) &= \delta(x - y), \quad y \in \Omega^c \\ \frac{\partial F_k}{\partial t}(0, x, y) &= 0. \end{aligned}$$

Then, assuming that the exponential map for each of the metrics  $g_k$  is a global diffeomorphism near  $\bar{\Omega}$  (i.e. no caustics in a neighborhood of  $\Omega$ ), we may write

$$(2.20) \quad F_k(t, x, y) = \sum_{j=0}^N A_j^k(x, y)(t^2 - (s_k(x, y))^2)_+^{-j+\frac{1}{2}(n-1)} + F_N^k$$

where  $F_N^k \in C^{N+1-\lfloor\frac{1}{2}(n-1)\rfloor}(\mathbf{R}_t \times \mathbf{R}_x^n \times \mathbf{R}_y^n)$  and  $A_j^k \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n), k = 0, 1$ . Here  $s_k(x, y)$  denotes the geodesic distance between  $x$  and  $y$  in the metric  $g_k, k = 0, 1$ . The distributions

$$(t^2 - (s(x, y))^2)_+^\lambda = \begin{cases} \frac{(t^2 - (s(x, y))^2)^{-\lambda}}{\Gamma(1-\lambda)} & \text{for } t^2 > (s(x, y))^2 \\ 0 & t^2 < (s(x, y))^2 \end{cases}$$

are defined for  $\text{Re } \lambda \ll 0$  and have an analytic continuation to  $\lambda \in \mathbf{C}$ .

Now from proposition (2.15) we know that if  $\Lambda_{g_0}^h = \Lambda_{g_1}^h$ , then  $F_0(t, x, y) = F_1(t, x, y)$  in  $\Omega^c$ , for  $t > 0$ . Therefore, comparing the most singular terms in (2.20) we conclude that

$$s_0(x, y) = s_1(x, y) \quad \forall x \in \Omega^c, y \in \Omega^c.$$

Thus we have proved

**Theorem 2.11.** *Let  $g_0$  and  $g_1$  be Riemannian metrics with  $\Lambda_{g_0}^h = \Lambda_{g_1}^h$ . Then if the exponential map is a global diffeomorphism in  $\bar{\Omega}$  for  $g_k, k = 0, 1$  and  $s_k(x, y)$  denotes the geodesic distance from  $x$  to  $y$  in the metric  $g_k$ , we have*

$$s_0(x, y) = s_1(x, y) \quad \forall x, y \in \partial\Omega.$$

The inverse kinematic problem in seismology is to recover  $g$  from  $s_g(x, y), x, y \in \partial\Omega$ . Again this is not possible since if  $\psi : \bar{\Omega} \rightarrow \bar{\Omega}$  is a diffeomorphism such that  $\psi|_{\partial\Omega} = \text{Identity}$ , then  $s_{\psi \circ g} = s_g$ . As in conjecture 1, the question is whether this is the only obstruction to

uniqueness. In the rest of the section we prove that the linearized version at the euclidean metric of this conjecture is valid using again the harmonic map equation.

Let  $g_\epsilon$  be a family of Riemannian metrics in  $\Omega$   $g_\epsilon = e + \epsilon h$ , where  $e$  is the euclidean metric. We also assume that  $g_\epsilon = e$  in  $\Omega^c$ .

$$(2.22) \quad s_{g_\epsilon}(x, y) = s_e(x, y) \quad \forall \epsilon.$$

An easy computation shows that

$$(2.23) \quad \int_{\gamma(x, t, v)} h(v, v) dt = 0$$

where  $\gamma(x, t, v)$  denotes a straight line through  $x$  with direction  $v$  at time  $t$ . Formula (2.23) means that the  $X$ -ray transform of the quadratic form  $h$  vanishes in the direction  $v$ .

We recall from (1.21) that the linearization at the identity of the harmonic map equation (in the direction  $h$ ) is

$$(2.24) \quad -2 \sum_{i=1}^n \frac{\partial}{\partial x_i} (h_{i\beta}) + \frac{\partial}{\partial x_\beta} \text{tr} h = 0, \quad \beta = 1, \dots, n.$$

Integrating (2.24) along the lines with direction  $v$  yields

$$(2.25) \quad \sum_{j,\beta=1}^n \int_{\gamma(x, t, v)} v_j (h_{ij} - \frac{1}{2} \text{tr} h) w_\beta = 0$$

$$\forall w \in \mathbf{R}^n \text{ with } \langle w, v \rangle = 0.$$

Arguments similar to those at the end of section 1 show that

$$\sum_{i,j=1}^n \int_{\gamma(x, t, v)} h_{ij} w_i w_j = 0 \quad \forall w \in \mathbf{R}^n$$

proving that the  $X$ -ray transform of  $h$  is zero for all  $w$ . The  $X$ -ray transform is known to be invertible (see [He] for instance); hence we conclude that  $h = 0$ .

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## The Toda Shock Problem

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### 1. Introduction

The Toda lattice is a linear chain of particles with nearest neighbor interactions governed by a force which depends exponentially on the difference of the displacements of neighboring particles. The equations of motion are given by

$$(1) \quad \ddot{x}_n = e^{x_{n-1} - x_n} - e^{x_n - x_{n+1}} \quad n = 0, \pm 1, \pm 2, \dots$$

where  $x_n(t)$  is the displacement of the  $n^{\text{th}}$  particle at time  $t$  and a dot on  $x_n$  denotes differentiation with respect to time. We examine a lattice with uniform asymptotic properties at infinity:

$$(2) \quad x_n \rightarrow 0, \quad \dot{x}_n \rightarrow -2a \operatorname{sgn} n \quad \text{as } |n| \rightarrow \infty.$$

These are shock conditions when  $a > 0$  and rarefaction conditions when  $a < 0$ . We only address the shock problem here. We use initial data

$$(3) \quad x_n(0) = 0, \quad \dot{x}_n(0) = -2a \operatorname{sgn} n, \quad a > 0.$$

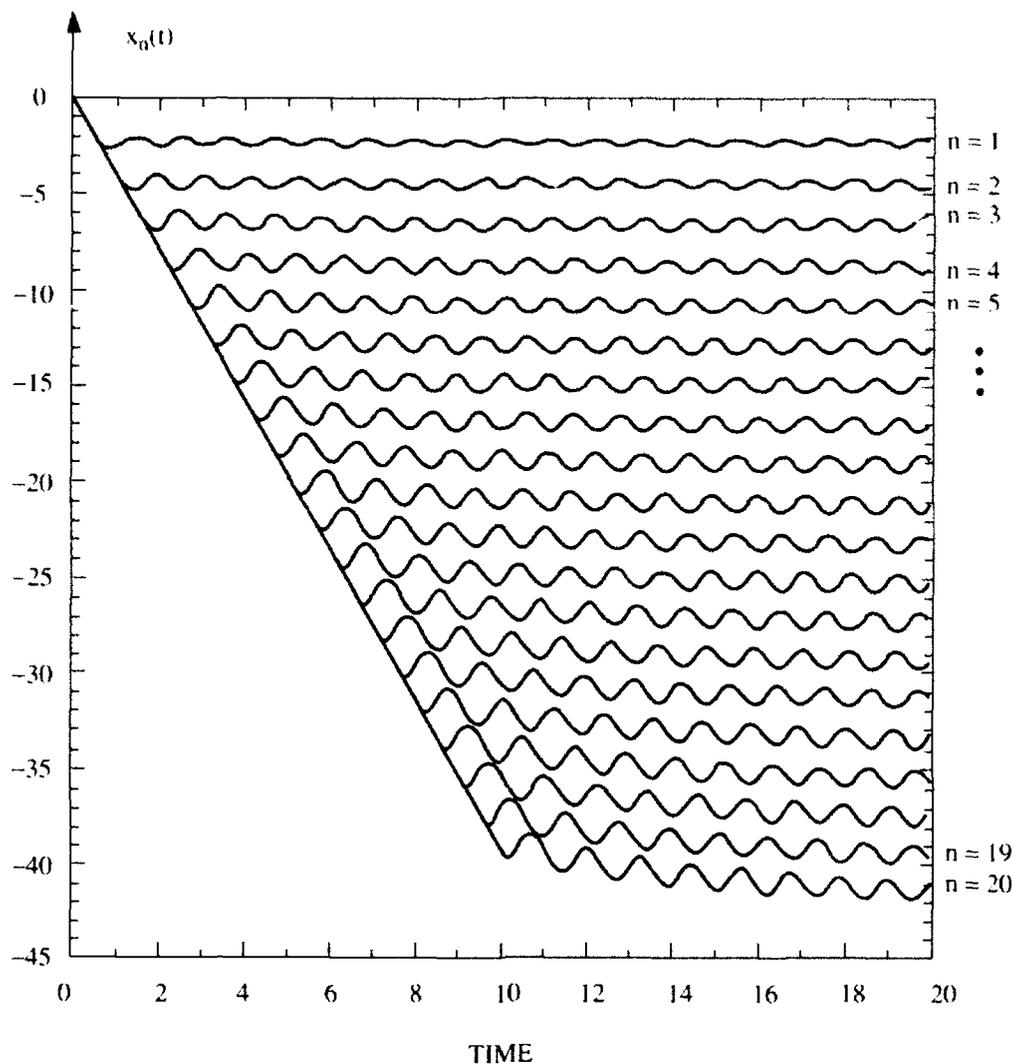
The problem is antisymmetric:  $x_n(t) + x_{-n}(t) = 0$ . Thus, it can be thought of as an initial-boundary value problem for the particles  $n = 0, 1, 2, \dots$  with the boundary condition  $x_0(t) = 0$ . By translating the system at velocity  $2a$ , we can transform it to the piston problem for the semi-infinite lattice initially at rest, forced by the zero<sup>th</sup> particle moving with constant speed  $2a$ .

We use the complete integrability of the Toda lattice to derive explicit formulae for the long-time behavior of the lattice. We assume that convergence in (2) is rapid [O]. We present results obtained by Venakides, Deift and Oba [VDO]. Our results are in exact agreement with the results of numerical experiments on the shock problem performed by Holian and Straub [HS] and by Holian, Flaschka and McLaughlin [HFM]. Particles settle into a purely periodic motion if the forcing velocity is large enough ( $a > 1$ ). We restrict our study to this case which we call supercritical. The shock at the origin sets up a wave-front traveling away from the origin at speed  $N_{max}$  (first calculated in [HFM]). There are three asymptotic regions in space-time as  $t \rightarrow \infty$ . (Space is parametrized by the particle index.) (a)  $\frac{n}{t} > N_{max}$ : Particles have only felt the shock at exponentially small levels of displacement. (b)  $N_{min} < \frac{n}{t} < N_{max}$  (the speed  $N_{min}$  is calculated explicitly). Particles travel with an  $N$ -dependent drift velocity while performing a modulated time-periodic, space-quasiperiodic motion in the lattice scales. (c)  $0 \leq \frac{n}{t} < N_{min}$ : Particles have settled to a purely time-periodic, space-periodic motion of space-period 2 when  $n \gg 1$ . When  $n$  is small the proximity to the boundary point  $x_0$  breaks the space-periodicity.

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Plot of  $x_n(t)$ ,  $n = 1, 2, \dots, 20$  for the Toda shock with  $a = 2$ .

In the case of the figure ( $a = 2$ ) an approximate evaluation gives:

$$N_{max} \approx 2.1 \quad (\text{formula 43})$$

$$N_{min} \approx .51 \quad (\text{formulas 64.65 and following remark})$$

For large  $t$ , the motion of the  $n^{\text{th}}$  particle is:

- (a) linear when  $t < \frac{n}{N_{max}} \approx \frac{n}{2.1}$ ,
- (b) modulated periodic with a drift when  $\frac{n}{2.1} < t < \frac{n}{.51}$ ,
- (c) periodic when  $\frac{n}{.51} < t$ .

We observe that the motion of the  $n^{\text{th}}$  particle changes sharply from linear to oscillatory, but very smoothly from modulated to periodic.

The local oscillatory motion in regions (b) and (c) is described in detail in terms of well-known formulae for periodic and quasiperiodic Toda waves discovered by Date and Tanaka [DT] (see figure). Recently the problem has been addressed from the point of view of modulation theory by Bloch and Kodama [BK].

2. Inverse Scattering and Lax Pairs

The Flaschka transformation

$$(4) \quad a_n = -\frac{1}{2}\dot{x}_n, \quad b_n = \frac{1}{2}e^{(x_n - x_{n+1})/2}$$

allows equations (1) to be written in Lax-pair [L] form:

$$(5) \quad \frac{dM}{dt} = BM - MB = -[M, B].$$

The tridiagonal matrices  $M$  and  $B$  are defined by:

$$(6) \quad M(t) = \begin{pmatrix} \ddots & & & & \\ & b_{n-1} & & & \\ & & a_n & & \\ & & & b_n & \\ & & & & \ddots \end{pmatrix}, \quad B(t) = \begin{pmatrix} & & & & \\ & -b_{n-1} & & & \\ & & 0 & & \\ & & & b_n & \\ & & & & \ddots \end{pmatrix}.$$

$M$  is self-adjoint and  $B$  is antisymmetric in  $\ell^2(\mathbf{Z})$ .  $M(t)$  represents the state of the lattice at time  $t$ . Our initial data (2) are given in terms of the Flaschka variables by

$$(7) \quad a_n(0) = a \operatorname{sgn} n, \quad b_n = \frac{1}{2}, \quad n = 0, \pm 1, \pm 2, \dots$$

To solve the initial value problem for (5) and (7) we employ the inverse scattering transformation (IST) which recovers the matrix  $M$  from its scattering data, (SD). The latter consist of the spectrum and the asymptotic behavior of the eigenvectors of  $M$  as  $n \rightarrow \pm\infty$ .

$$\begin{array}{ccc} M(0) & & M(t) \\ \downarrow ST & & \uparrow IST \\ SD(0) & \xrightarrow{\text{evolution of SD}} & SD(t) \end{array}$$

The scattering data are calculated in the next section by the spectral analysis of the matrix  $M(0)$ . The evolution of the scattering data is easily obtained. One verifies by direct differentiation of (9) with respect to time that the one-parameter family of unitary operators defined by [L]

$$(8) \quad \dot{U} = BU, \quad U(0) = I,$$

satisfies

$$(9) \quad U^{-1}(t)M(t)U(t) = M(0)$$

when  $M$  and  $B$  satisfy (5). Thus,  $M(t)$  is unitarily equivalent to  $M(0)$  and the spectrum of  $M(t)$  is independent of  $t$ . If  $\psi(t)$  is an eigenvector of  $M(t)$

$$(10) \quad \psi(t) = U(t)\psi(0), \quad \frac{d\psi}{dt} = \dot{U}(t)\psi(0) = B(t)U(t)\psi(0) = B(t)\psi(t).$$

In the limit  $n \rightarrow \pm\infty$ ,  $B(t) \sim \begin{pmatrix} \ddots & & & & \\ & -1/2 & & & \\ & & 0 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$ . The evolution equation for the asymptotic eigenvector, as  $n \rightarrow \pm\infty$  becomes:

$$(11) \quad \frac{d\psi}{dt} = \begin{pmatrix} \ddots & & & & \\ & -1/2 & & & \\ & & 0 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix} \psi.$$

The solution to the inverse scattering problem was discovered by Fadeev [F] who recovered the potential of the Schrödinger operator directly from its scattering data. Previously, in landmark work, Gelfand and Levitan [GL] had used the associated spectral function to recover the potential in a class satisfying much more general asymptotic conditions at infinity. Since then, the ideas of inverse scattering have been applied to many problems.

The theory of the scattering and inverse scattering transform for doubly infinite self-adjoint matrices satisfying the asymptotic relations

$$(12) \quad a_n \rightarrow a \operatorname{sgn} n, \quad b_n \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow \pm\infty.$$

with rapid convergence was derived by Oba [O]. As in the Fadeev study, the inverse problem is reduced to solving a linear integral equation, the Marcenko equation, whose distribution kernel is computed in terms of the scattering data. In the context of the Schrödinger operator, Dyson [D] has derived an explicit formula for the potential in terms of a Fredholm determinant. We derive a similar formula in the case of the Toda lattice. Our derivation, following Dyson's reasoning, is more direct than the one employed in [VDO].

### 3. The Scattering Data (supercritical case $a > 1$ )

The eigenvalue equation  $M(0)\psi = \lambda\psi$  is equivalent to the set of scalar equations

$$(13) \quad \frac{1}{2}\psi_{n-1} + a_n\psi_n + \frac{1}{2}\psi_{n+1} = \lambda\psi_n, \quad n = 0, \pm 1, \pm 2, \dots \quad a_n = a \operatorname{sgn} n.$$

Inserting  $\psi_n = z^n$  into all equations except those indexed by  $n = -1, 0, 1$ , we find that  $M(0)$  has formal eigenvectors:

$$(14) \quad \psi_n = \begin{cases} A_+ z_+^n + B_+ z_+^{-n} & \text{when } n > 0 \\ A_- z_-^n + B_- z_-^{-n} & \text{when } n < 0, \end{cases}$$

in which  $z_{\pm}$  are related to the spectral variable  $\lambda$  through the equation

$$(15) \quad \frac{1}{2}\left(z_{\pm} + \frac{1}{z_{\pm}}\right) = (\lambda \mp a), \quad |z_{\pm}| \leq 1.$$

Of the five unknown constants  $A_{\pm}, B_{\pm}$  and  $\psi_0$  three are determined from the equations (13) indexed by  $n = -1, 0, 1$ . One more constant is determined by the requirement that the eigenvector be bounded. We easily find that this can occur exactly when

$$(16) \quad \lambda \in \sigma(M) = [-a - 1, -a + 1] \cup \{0\} \cup [a - 1, a + 1]$$

The eigenvectors are real. They have multiplicity one and they satisfy the symmetry relation:

$$(17) \quad \psi_n(\lambda) = (-1)^n \psi_{-n}(-\lambda).$$

Explicitly for  $n > 0$ :

$$(18) \quad \psi_n(\lambda) = \begin{cases} \overline{S(z)}z^{-n} + S(z)z^n & \text{when } \lambda \in [a - 1, a + 1] \\ c(z)z^n & \text{when } \lambda \in [-a - 1, -a + 1] \cup \{0\}, \end{cases}$$

where  $z = z_+(\lambda)$ .



for  $\phi_{\rho-1}$  we obtain:

$$(24) \quad \begin{aligned} \phi_{\rho-1} &= 2\lambda\phi_{\rho} - 2a\phi_{\rho} - \phi_{\rho+1} = \\ &\sum_{j=\rho}^{\infty} [2Q_{\rho j}(\lambda\psi_j) - 2aQ_{\rho j}\psi_j - Q_{\rho+1,j}\psi_j] \end{aligned}$$

We now insert the relation:

$$(25) \quad \lambda\psi_j = b_{j-1}\psi_{j-1} + a_j\psi_j + b_j\psi_{j+1},$$

to obtain

$$(26) \quad \phi_{\rho-1} = 2 \sum_{j=\rho}^{\infty} Q_{\rho j} b_{j-1} \psi_{j-1} + \text{linear comb. of } \psi_j\text{'s with } j \geq \rho.$$

This proves the statement and also yields

$$(27) \quad Q_{\rho-1,\rho-1} = 2b_{\rho-1}Q_{\rho\rho}.$$

There follows

$$(28) \quad Q_{nn} = \prod_{\rho=n}^{\infty} (2b_{\rho}) = \exp \sum_{\rho=n}^{\infty} \frac{x_{\rho} - x_{\rho+1}}{2} = \exp \frac{x_n - x_{\infty}}{2}.$$

The latter statement is true provided that  $2b_{\rho}$  tends to 1 sufficiently fast as  $\rho \rightarrow +\infty$ .

In accordance with the spectral theorem, the normalized eigenvectors satisfy the relation

$$(29) \quad \int \psi_m(\lambda)\psi_n(\lambda)d\rho(\lambda) = \delta_{mn}.$$

The measure  $d\rho(\lambda)$  is absolutely continuous with respect to Lebesgue measure on the continuous spectrum and is atomic at the eigenvalue  $\lambda = 0$ . It can be determined exactly by Stone's formula. This will not be needed for our calculation. We code the scattering data into the matrix

$$(30) \quad F_{\rho\ell} = \int \phi_{\rho}(\lambda)\phi_{\ell}(\lambda)d\rho(\lambda).$$

Relation (29) combined with  $\psi_i \sim \phi_i$  as  $i \rightarrow +\infty$  imply

$$(31) \quad F_{\rho\ell} \sim \delta_{\rho\ell} \quad \text{when } \rho, \ell \rightarrow \infty.$$

We insert (22) into (30) and use (29). For equations (32) only, we use the convention that repeated indices are summed over. The matrix  $Q_{\rho\ell}$  is upper triangular. We have

$$(32) \quad \begin{aligned} F_{\rho\ell} &= \int \phi_{\rho}(\lambda)\phi_{\ell}(\lambda)d\rho(\lambda) = \\ (\text{summation convention}) &= \int Q_{\rho i}Q_{\ell j}\psi_i(\lambda)\psi_j(\lambda)d\rho(\lambda) = \\ &= Q_{\rho i}Q_{\ell j} \int \psi_i(\lambda)\psi_j(\lambda)d\rho(\lambda) = \\ &= Q_{\rho i}Q_{\ell j}\delta_{ij} = Q_{\rho j}Q_{\ell j}. \end{aligned}$$

Let  $\mathbf{F}_n$  denote the matrix  $F_{\rho\ell}$  truncated to have entries  $\rho, \ell \geq n + 1$ . The upper triangular matrix  $\mathbf{Q}_n$  is defined in a similar way.

We have

$$(33) \quad \mathbf{F}_n = \mathbf{Q}_n \mathbf{Q}_n^T, \quad \det \mathbf{F}_n = \{\det \mathbf{Q}_n\}^2.$$

For the rapid convergence of initial data which we assume as  $n \rightarrow \pm\infty$ , the matrix  $\mathbf{Q}$  can be shown to equal the identity plus a trace class matrix, thus the determinants are well-defined. Recalling that  $\mathbf{Q}_n$  is triangular we obtain

$$(34) \quad \det \mathbf{F}_n = \left\{ \prod_{\rho=n+1}^{\infty} Q_{\rho\rho} \right\}^2$$

Combining (34) and (28) we obtain

$$(35) \quad x_n = x_\infty + \ell n \frac{\det \mathbf{F}_{n-1}}{\det \mathbf{F}_n}.$$

This solves the inverse scattering problem.

We calculate the matrix  $\mathbf{F}_n$ .

$$(36) \quad F_{\rho\ell} = \int_{a-1}^{a+1} (S z^{-\rho} + \bar{S} z^\rho)(S z^{-\ell} + \bar{S} z^\ell) d\rho(\lambda) + \int_{-a-1}^{0+} c^2 z^{\rho+\ell} d\rho(\lambda)$$

Recalling that  $|S|^2 = 1$  we obtain

$$(37) \quad F_{\rho\ell} = \int_{a-1}^{a+1} (z^{\rho-\ell} + z^{\ell-\rho}) d\rho(\lambda) + \int_{a-1}^{a+1} (S^2 z^{-\rho-\ell} + \bar{S}^2 z^{\rho+\ell}) d\rho(\lambda) + \int_{-a-1}^{0+} c^2 z^{\rho+\ell} d\rho(\lambda).$$

where the upper limit  $0+$  means that the atom at zero is included. The first integral equals  $\delta_{\rho\ell}$  (Kronecker delta). Indeed, for fixed difference  $\rho - \ell$  take  $\rho$  and  $\ell$  to tend to  $+\infty$ . The last two integrals tend to zero. The statement follows directly from (31).

**Ansatz I.** We can neglect the second integral in (37).

The integral in question is oscillatory and decays as  $t \rightarrow +\infty$  while the third integral grows exponentially (cf. relation (19)). This does not constitute a rigorous argument since it is still conceivable that small changes in the entries of the matrix  $\mathbf{F}_n$  can produce a large change in the determinant of  $\mathbf{F}_n$ . Nevertheless, this step of our calculation is supported by the fact that our solution is in full agreement with the results of numerical experiments.

It is important to note that the region of validity of our Ansatz ranges from a negative value of  $\frac{z}{t}$  to  $+\infty$ . For more negative values of  $\frac{z}{t}$ , the solution obtained through the Ansatz is not spatially antisymmetric and thus obviously wrong. It is not surprising that the region of validity of the Ansatz is not antisymmetric. The use of inverse scattering which favors one of the two spatial directions breaks the antisymmetry of the original problem.

Since our initial data are antisymmetric we only need to calculate for  $\frac{z}{t} \geq 0$ , thus the region of validity of the Ansatz suffices for our purpose. In the case of nonantisymmetric initial data we would have to obtain the solution by doing inverse scattering from the right for positive  $n$ 's and from the left for negative  $n$ 's. Using the Ansatz we write:

$$(38) \quad F_{\rho\ell} = \delta_{\rho\ell} + \int_{-a-1}^{0+} c^2(z, t) z^{\rho+\ell} d\rho(\lambda).$$

Denoting the integral in (38) by  $\tilde{F}_{\rho\ell}$  we have

$$(39) \quad \det \mathbf{F}_n = \det(I + \tilde{\mathbf{F}}_n) = \det(\delta_{\rho\ell} + \tilde{F}_{\rho\ell})_{\rho, \ell \geq n+1}$$

By our statement following (33) the matrix  $\tilde{\mathbf{F}}_n$  belongs to the trace class.

### 5. Calculation of the Determinant

We expand the determinant of  $(I + s\tilde{\mathbf{F}}_n)$  in powers of  $s$  and then set  $s = 1$ . We obtain

$$(39a) \quad \det(\tilde{\mathbf{F}}_n) = 1 + \sum_{k=1}^{\infty} d(n, k),$$

where

$$d(n, k) = \frac{1}{k!} \sum_{m_k=n+1}^{\infty} \cdots \sum_{m_1=n+1}^{\infty} \det(\tilde{\mathbf{F}}_{\rho\ell})_{\rho,\ell=m_1,\dots,m_k}.$$

Inserting the value of  $\tilde{\mathbf{F}}_{\rho\ell}$ , using the multilinear dependence of the determinant on its columns and interchanging summation and integration we obtain

$$(40) \quad d(n, k) = \frac{1}{k!} \int_{-a-1}^{0+} \cdots \int_{-a-1}^{0+} \left( \prod_{i=1}^k c_i^2 \right) \sum_{m_k=n+1}^{\infty} \cdots \sum_{m_1=n+1}^{\infty} \begin{vmatrix} z_1^{m_1+m_1} & z_1^{m_1+m_2} & \cdots \\ z_2^{m_2+m_1} & z_2^{m_2+m_2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix} d\rho(\lambda_1) \cdots d\rho(\lambda_k)$$

where  $c_i = c(z_i, t)$ .

By factoring  $z_i^{m_i}$  from the  $i^{\text{th}}$  row and multiplying it into the  $i^{\text{th}}$  column we write the determinant in the integrand as  $\det(z_i z_j)^{m_j}$  where  $i, j = 1, \dots, k$ . Using the multilinear dependence of the determinant on its columns we can perform the summations.

$$\text{Multiple Sum} = \det\left(\frac{(z_i z_j)^{n+1}}{1 - z_i z_j}\right) = \left(\prod_{i=1}^k z_i^{2n+2}\right) \det\left(\frac{1}{1 - z_i z_j}\right) \quad i, j = 1, \dots, k.$$

By exploiting the facts that (i) the determinant in (40) vanishes when  $z_i = z_j$  and (ii) the denominator of the expanded determinant has known form we obtain

$$(41) \quad \det\left(\frac{1}{1 - z_i z_j}\right) = \frac{\prod_{i < j = 1, \dots, k} (z_i - z_j)^2}{\prod_{i, j = 1, \dots, k} (1 - z_i z_j)}.$$

Finally absorbing  $\left\{\frac{z^2}{1-z^2} c^2(z, 0)\right\}$  into the measure  $d\rho(\lambda)$  which becomes  $d\tilde{\rho}(\lambda)$  and utilizing the symmetry of the integrand we have

$$(42) \quad d(n, k) = \int_{-a-1 \leq \lambda_1 < \dots < \lambda_k \leq 0+} \cdots \int \exp\left\{\sum_{j=1}^k [n\ell n z_j^2 + (z_j - \frac{1}{z_j})t]\right. \\ \left. + \sum_{j=1}^k \sum_{\substack{i=1 \\ i \neq j}}^k \ell n \left| \frac{z_i - z_j}{1 - z_i z_j} \right| \right\} d\tilde{\rho}(\lambda_1) \cdots d\tilde{\rho}(\lambda_k).$$

### 6. Long-Time Behavior

We consider the limit  $t \rightarrow \infty$  while  $\frac{n}{t} \sim N \geq 0$  where  $N$  is held fixed. The double sum in (42) has negative terms and has order  $O(k^2)$ . The  $z_i$ 's can be chosen so that the single sum is positive and has order  $O(k\ell)$  provided  $N\ell n z^2 + z - \frac{1}{z}$  is positive for some subinterval of the

lower spectral band. This happens exactly when  $N$  is in the interval  $[0, N_{max})$  where

$$(43) \quad N_{max} = \frac{\sqrt{a(a+1)}}{\ln(\sqrt{a} + \sqrt{a+1})}.$$

In this region of  $N$ , the maximum of the integrand will arise when the single and double sums in (42) have the same order of magnitude, in other words when  $k = O(t)$ . The exponent then will have order  $O(t^2)$ .

The main contribution to the integral for  $d(n, k)$  will arise at the points which maximize the exponent. For a given  $k$  let there be a local maximum of the exponent at point  $(\lambda_1, \dots, \lambda_k)$ . We recall that  $\lambda_i$  is related to the  $z_i$  of expression (42) by relation (15) with  $z_i = z_+$ ,  $\lambda_i = \lambda$ . The domain of the  $\lambda_i$ 's is the union  $[-a-1, -a+1] \cup \{0\}$ . All the  $\lambda_i$ 's are distinct since  $\lambda_i = \lambda_j$  for  $i \neq j$  makes a term of the double sum in (42) equal to  $-\infty$ . Thus, we recognize two types of maximizing points, one in which all of the  $\lambda_i$ 's are in  $[-a-1, -a+1]$  and one in which  $k-1$  of the  $\lambda_i$ 's lie in the band  $[-a-1, -a+1]$  and the  $k^{th}$  one equals zero. We postulate one maximizer of each type. The postulate is confirmed by the subsequent analysis (section 7). We label the two maximizers  $(\lambda'_1, \dots, \lambda'_k)$  and  $(\lambda''_1, \dots, \lambda''_{k-1}, 0)$  where the  $\lambda'$ 's and  $\lambda''$ 's are in  $[-a-1, -a+1]$ . We denote their contributions to  $d(n, k)$  by  $A_k \exp \max_{\lambda_i \in [-a-1, -a+1]} E_k(\lambda_1, \dots, \lambda_k)$  and  $B_k \exp \max_{\lambda_i \in [-a-1, -a+1]} E_k(\lambda_1, \dots, \lambda_{k-1}, 0)$  respectively. Here  $E_k$  is the expression for the exponent in (42) and  $A_k$  and  $B_k$  are coefficients which have the physical units of the product measure. Thus

$$(44) \quad \det \mathbf{F}_n \sim \sum_k \{ A_k \exp \max_{\lambda_i \in [-a-1, -a+1]} E_k(\lambda_1, \dots, \lambda_k) + B_{k+1} \exp \max_{\lambda_i \in [-a-1, -a+1]} E_{k+1}(\lambda_1, \dots, \lambda_k, 0) \}.$$

Following Lax and Levermore [LL] we represent the points  $(\lambda_1, \dots, \lambda_k)$  by atomic measures

$$(45) \quad (\lambda_1, \dots, \lambda_k) \longleftrightarrow \frac{\pi}{t} \sum_{i=1}^k \delta(\lambda - \lambda_i) = d\Psi_k.$$

Defining the functions:

$$(46) \quad f(\lambda) = \frac{n}{t} \ell n z^2(\lambda) + z(\lambda) - \frac{1}{z(\lambda)}$$

$$(47) \quad L(\lambda, \mu) = \begin{cases} \frac{1}{\pi} \ell n \left| \frac{z(\lambda) - z(\mu)}{1 - z(\lambda)z(\mu)} \right| & \text{when } \mu \neq \lambda \\ = 0 & \text{when } \mu = \lambda \end{cases}$$

we write the exponent in (42) as

$$(48) \quad E_k = \frac{t^2}{\pi} \left\{ \int f d\Psi_k + \int \int L(d\Psi_k \times d\Psi_k) \right\}.$$

We describe our procedure for the maximization of the first term in (44). The maximization of the second term is done as a perturbation of the first term.

First we approximate the atomic measure  $d\Psi_k$  by an absolutely continuous measure  $\nu(\lambda)d\lambda$  satisfying a

- (1) positivity condition  $\nu \geq 0$
- (2) mass conservation (quantum) condition

$$(49) \quad \frac{t}{\pi} \int \nu(\lambda)d\lambda = k = \text{integer.}$$

We obtain the quadratic functional

$$(50) \quad E(\psi) = \frac{t^2}{\pi} \left\{ \int_{-a-1}^{-a+1} f\psi d\lambda + \int_{-a-1}^{-a+1} \int_{-a-1}^{-a+1} L(\lambda, \mu)\nu(\lambda)\nu(\mu)d\lambda d\mu \right\}$$

which approximates  $E_k$ . Using the standard notation for the inner product in  $L^2[-a-1, -a+1]$  and denoting by  $L$  the integral operator with distribution kernel  $L(\lambda, \mu)$  we write

$$(50a) \quad E(\nu) = \frac{t^2}{\pi} [(f, \nu) + (L\nu, \nu)].$$

It can be estimated that

$$(51) \quad \max_{\lambda_i \in [-a-1, -a+1]} E_k(\lambda_1, \dots, \lambda_k) = \max_{\psi \in \mathcal{A}_k} E(\psi) + k \ln k + O(k) \quad \text{when } k = O(t), t \rightarrow \infty.$$

$\mathcal{A}_k$  is the set of measurable functions on  $[-a-1, -a+1]$  satisfying the positivity condition and the quantum condition. The "measure coefficient"  $A_k$  in (44) can also be estimated

$$(52) \quad A_k = e^{-k \ln k + O(k)}.$$

We observe that the terms  $k \ln k$  in (51) and (52) cancel in (53).

$$(53) \quad \begin{aligned} A_k \exp \max_{\lambda_i \in [-a-1, -a+1]} E_k(\lambda_1, \dots, \lambda_k) &= \\ &= \exp \left\{ \max_{\psi \in \mathcal{A}_k} E(\psi) + O(k) \right\} = \\ &= \exp[E(\psi_k) + O(k)], \end{aligned}$$

where  $\psi_k$  solves the maximization problem in the right hand side of (51).

We now turn to the second term in the series (44). To the measure (45) we adjoin an additional atom at  $\lambda = 0$ . A perturbation calculation gives

$$(54) \quad \begin{aligned} B_{k+1} \exp \max_{\lambda_i \in [-a-1, -a+1]} E_{k+1}(\lambda_1, \dots, \lambda_k, 0) &= \\ &= A_k \exp \left\{ \max_{\lambda_i \in [-a-1, -a+1]} E_k(\lambda_1, \dots, \lambda_k) + t[f(0) + 2L\psi_k(0)] + r_k \right\} \\ &\text{where } r_k = O(1). \end{aligned}$$

The quantity  $f(0) + 2L\psi_k(0)$  is the first variation of the functional  $(f, \psi) + (L\psi, \psi)$  evaluated at zero, i.e. at the position of the extra atom.

Inserting (53) and (54) into (44) we obtain

$$(55) \quad \det \mathbf{F}_n \sim 1 + \sum_k \left\{ \exp[E(\psi_k) + O(k)] \right\} \left\{ 1 + \exp[t(f(0) + 2L\psi_k(0)) + r_k] \right\}$$

where  $\psi_k$ , constrained by the positivity and quantum conditions, maximizes the functional (50a). Formula (55) is rigorous as long as Ansatz I of section 4 is valid. We outline the procedure of maximization.

7. The Maximization Problem

We insert

$$(56) \quad \psi = \psi^* + \frac{1}{t}\psi^{\bar{}} + \frac{1}{t^2}\psi^{\equiv} + \dots$$

into (50a) to obtain

$$(57) \quad E(\psi) = \frac{t^2}{\pi}[(f, \psi^*) + (L\psi^*, \psi^*)] + \frac{t}{\pi}(f + 2L\psi^*, \psi^{\bar{}}) + \frac{1}{\pi}[(f + 2L\psi^*, \psi^{\equiv}) + (L\psi^{\bar{}}, \psi^{\equiv})] + \dots$$

1. We determine  $\psi^*$  by maximizing the leading order  $O(t^2)$  term subject to the positivity condition. The variational condition for the leading order problem is

$$(58) \quad \begin{aligned} f(\lambda) + 2L\psi^*(\lambda) &= 0 & \text{if } \psi^*(\lambda) > 0 \\ &< 0 & \text{if } \psi^*(\lambda) = 0. \end{aligned}$$

When  $\psi^*(\lambda) = 0$  the inequality is not a priori strict. It is shown to be strict once the problem is solved. The solution is unique:  $L$  can be shown to be strictly negative definite.

2. We maximize the  $O(t)$  order term  $(f + 2L\psi^*, \psi^{\bar{}})$  (for the  $\psi^*$  obtained from the leading order problem) subject to (a) positivity:  $\psi^{\bar{}}(\lambda) \geq 0$  if  $\psi^*(\lambda) = 0$  (b) quantum condition  $\frac{t}{\pi} \int_{-a-t}^{-a+1} \psi^*(\lambda)d\lambda + \frac{1}{\pi} \int_{-a-1}^{-a+1} \psi^{\bar{}}(\lambda)d\lambda = k$ . Using (58) we easily see that the  $O(t)$  term in (57) is maximized when

$$(58a) \quad \psi^{\bar{}}(\lambda) = 0 \quad \text{if } \psi^*(\lambda) = 0.$$

3. We maximize the  $O(1)$  term with respect to  $\psi^{\bar{}}$  and  $\psi^{\equiv}$ . The argument of the previous paragraph gives  $\psi^{\bar{}}(\lambda) = 0$  if  $\psi^*(\lambda) = 0$  which implies  $(f + 2L\psi^*, \psi^{\bar{}}) = 0$ . By looking at higher orders we can show  $\psi^{\equiv} \equiv \psi^{\equiv} \equiv \dots \equiv 0$ . Continuing on the  $O(1)$  term, we maximize  $(L\psi^{\bar{}}, \psi^{\equiv})$  subject to the quantum condition on the support of  $\psi^*(\lambda)$ . The variational conditions are

$$(59) \quad \begin{aligned} L\psi^{\bar{}}(\lambda) &= \frac{t}{\pi} \quad \text{when } \lambda \in \text{supp } \psi^* \quad (t : \text{Lagrange multiplier}) \\ \psi^{\bar{}}(\lambda) &= 0 \quad \text{when } \lambda \in \text{supp } \psi^* \\ \frac{1}{\pi}(\psi^{\bar{}}, 1) + \sigma t - k &= 0 \quad \text{quantum condition} \end{aligned}$$

$$(60) \quad \text{where } \sigma = \frac{1}{\pi} \int_{-a-1}^{-a+1} \psi^*(\lambda)d\lambda, \quad 1(\lambda) = \begin{cases} 1 & \text{when } \lambda \in \text{supp } \psi^* \\ 0 & \text{when } \lambda \in \text{supp } \psi^{\bar{}}. \end{cases}$$

The solutions are parametrized by  $k \in \mathbb{Z}$ .

The variational equations (58) and (59) are reduced to Riemann-Hilbert problems through the observation:

$$(61) \quad \frac{1}{2}z(\lambda) \left( \frac{dz(\lambda)}{d\lambda} \right)^{-1} \frac{d}{d\lambda} L(\lambda, \mu) = \sqrt{(\mu - a)^2 - 1} \times \underbrace{\frac{1}{\lambda - \mu}}_{\text{kernel of Hilbert transform}}$$

The solutions to (58) and (59) are obtained in terms of elliptic integrals.

We list the results and we refer to [VDO] for the proofs. When  $0 \leq N \leq N_{min}$  ( $N_{min} < N_{max}$  is defined in the paragraph following (65)).

$$(62) \quad \psi^*(\lambda) = \frac{-iP_0(\lambda)}{R_0(\lambda)} \quad \lambda \in [-a-1, -a+1]$$

where

$$(63) \quad R_0(\lambda) = [(\lambda+a+1)(\lambda+a-1)(\lambda-a+1)(\lambda-a-1)]^{1/2}, \quad R_0(\lambda) > 0 \quad \text{as } \lambda \rightarrow +\infty$$

$$(64) \quad P_0(\lambda) = \lambda^2 + N\lambda + c_0,$$

and  $c_0 < 0$  is uniquely determined, independently of  $N$ , by the relation

$$(65) \quad \int_{-a+1}^{a-1} \frac{P_0(\lambda)}{R_0(\lambda)} d\lambda = 0.$$

This solution is valid only when  $N$  ranges from zero (we only consider  $N \geq 0$ ) to the value  $N_{min}$  at which  $P_0(-a+1) = 0$ . As  $N$  increases from the value  $N_{min}$ , one of the roots of  $P_0$  enters the interval  $[-a-1, -a+1]$  and the positivity of  $\psi^*$  is violated. The support of  $\psi^*$  for  $N > N_{min}$  recedes to  $[-a-1, \gamma(N)]$ , where  $-a-1 < \gamma < -a+1$ . We have for  $N_{min} < N < N_{max}$ ,

$$(66) \quad \begin{aligned} \psi^*(\lambda) &= \frac{-iP(\lambda)}{R(\lambda)} & \text{when } \lambda \in [-a-1, \gamma] \\ &= 0 & \text{otherwise} \end{aligned}$$

where

$$(67) \quad R(\lambda) = [(\lambda+a+1)(\lambda-\gamma)(\lambda-a+1)(\lambda-a-1)]^{1/2}, \quad R(\lambda) > 0 \quad \text{when } \lambda \rightarrow +\infty$$

$$(68) \quad P(\lambda) = \lambda^2 + \left(N + \frac{-a+1-\gamma}{2}\right)\lambda + c.$$

The real constants  $\gamma$  and  $c$  are determined from:

$$(69) \quad P(\gamma) = 0, \quad \int_{\gamma}^{a-1} \frac{P(\lambda)}{R(\lambda)} d\lambda = 0.$$

As  $N$  increases further,  $\gamma$  decreases. At the value  $N$  equal

$$(70) \quad N_{max} = \frac{\sqrt{a(a+1)}}{\ln(\sqrt{a} + \sqrt{a+1})}.$$

$\gamma$  attains the value  $-a-1$ , in which case  $\psi^*(\lambda) \equiv 0$ . When  $N > N_{max}$  we have in (39)  $\det \mathbf{F}_n \sim 1$ . The value  $\frac{n}{l} = N_{max}$  is the speed of the progressing front.

The first variation of the leading order functional is

$$(71) \quad J(\lambda) = f(\lambda) + 2L\psi^*(\lambda) = 2 \int_{\gamma}^{a-1} \frac{P(\lambda')}{R(\lambda')} d\lambda' \quad \text{when} \quad -a-1 < \lambda < \gamma$$

$$= 2 \int_{\lambda}^{a-1} \frac{P(\lambda')}{R(\lambda')} d\lambda' \quad \text{when} \quad \gamma < \lambda < a-1.$$

If  $\lambda \in [-a-1, \gamma]$  ( $\gamma = -a+1$  when  $0 \leq N \leq N_{min}$ ):

$$(72) \quad \bar{\psi}_k(\lambda) = \frac{-iE_k}{R(\lambda)}, \quad L\bar{\psi}_k = E_k \int_{\gamma}^{a-1} \frac{d\lambda'}{R(\lambda')} = \text{constant}.$$

The constants  $E_k$ , where  $k$  is an integer, are determined from the quantum condition

$$(73) \quad \frac{iE_k}{\pi} \int_{-a-1}^{\gamma} \frac{d\lambda}{R(\lambda)} = \sigma t - k, \quad (\sigma \text{ defined by (60)}).$$

### 8. The Local Structure of the Waveform

The appearance of the elliptic curve  $R(\lambda)$  in the previous section is significant. Explicit periodic and quasiperiodic solutions of the Toda lattice have been derived in terms of the theta function corresponding to an elliptic curve by Date and Tanaka [DT].

For fixed  $N$ , we can identify the solution described by (55) with the solutions obtained by Date and Tanaka. Thus, (55) is a modulated (varying with  $N$ ) quasiperiodic solution of the Toda lattice. The equations (69) which give the dependence of  $\gamma$  on  $N$  can be properly thought of as modulation equations. When  $0 < N \leq N_{min}$  there is no modulation:  $\gamma = -a+1$ . We carry out the identification in the latter region of  $N$  and we omit the identification in the modulating region  $N_{min} < N < N_{max}$ . We describe the behavior at the boundary  $n = 0$ .

We use the results stated in the previous section to calculate explicitly when  $0 < N < N_{min}$ . We obtain:

$$(74) \quad \frac{1}{\pi} (f, \psi^*) = N^2 \ln(4a) - 4aN + \text{constant}.$$

$$(75) \quad \sigma = -\frac{1}{2}N + \omega, \quad \omega = \frac{-i}{\pi} \int_{-a-1}^{-a+1} \frac{\lambda^2 + c_0}{R_0(\lambda)} d\lambda = \text{pure real}.$$

$$(76) \quad L\bar{\psi}_k(0) = \frac{1}{2} \pi i \tau \left( \frac{n}{2} - \omega t + k \right), \quad \tau = \frac{\int_{\gamma}^{a-1} \frac{d\lambda}{R(\lambda)}}{\int_{-a-1}^{\gamma} \frac{d\lambda}{R(\lambda)}} = \text{pure imaginary}.$$

$$(77) \quad f(0) + 2L\psi^*(0) = -j \frac{n}{t}, \quad j = \ell n a.$$

$$(78) \quad (L\bar{\psi}_k, \bar{\psi}_k) = i\tau \pi^2 (\sigma t - k)^2 < 0.$$

We insert these in (55). We use the following Ansatz which is natural but which we have not succeeded in proving.

**Ansatz II:** *the  $O(k)$  term in (55) has an asymptotic expansion in  $k$  which is valid at least up to constants ( $k$  to the zero power). Similarly  $r_k \sim \tau$  where  $\tau$  is a constant.*

Given the Ansatz, the  $O(k)$  term in (55) would only affect our calculation up to a phase shift and up to a constant term added to  $x_n$ . We neglect the  $O(k)$  term and obtain

$$(79) \quad \det \mathbf{F}_n(t) \sim \left\{ \exp\left[\frac{n^2}{2} \ln(4a) - 2ant\right] \right. \\ \left. \times \sum_{k=1}^{\infty} \left\{ 1 + \exp\left[-jn + r + \left(k + \frac{n}{2} - \omega t\right)\pi i\tau\right] \right\} \exp\left[i\pi\tau\left(k + \frac{n}{2} - \omega t\right)^2\right] \right\}.$$

We can replace the lower summation limit  $k = 1$  with  $k = -\infty$ , since the main contribution to the series comes from  $k$ 's which satisfy  $k \sim \omega t - \frac{n}{2} = (\omega - \frac{N}{2})t = \sigma t$ , where  $\sigma > 0$  (see (60)). We define the function

$$(80) \quad \rho(\eta) = \sum_{k=-\infty}^{\infty} e^{\pi i\tau(k+\eta)^2}.$$

This is reduced readily to the standard theta function,

$$(81) \quad \theta(z) = \sum_{k=-\infty}^{\infty} e^{2\pi i k z + \pi i\tau k^2}$$

through the relation

$$\rho(\eta) = e^{i\pi\tau\eta^2} \theta(\tau\eta).$$

In terms of the  $\rho$ -function we write after a straightforward calculation

$$(82) \quad \det \mathbf{F}_n(t) \sim \left\{ \exp\left[\frac{n^2}{2} \ln(4a) - 2ant\right] \right\} \\ \times \left[ \rho\left(\frac{n}{2} - \omega t\right) + e^{-jn - \frac{\pi i\tau}{4} + r} \rho\left(\frac{n}{2} - \omega t + \frac{1}{2}\right) \right].$$

How can we derive the boundary condition  $x_0(t) \equiv 0$  from this formula? We have

$$(83) \quad x_0(t) = -2at + \ln \frac{\det \mathbf{F}_{-1}(t)}{\det \mathbf{F}_0(t)}.$$

We insert the expression (82) for  $\det \mathbf{F}_n(t)$  using the notation  $\rho = \rho(-\omega t)$  and  $\rho_{1/2} = \rho(-\omega t - \frac{1}{2})$  and recalling that  $\rho(\eta + 1) = \rho(\eta)$ . After a very short calculation we obtain

$$(84) \quad x_0(t) = \frac{1}{2} \ln 4a + \ln \frac{\rho_{1/2} + \rho e^{j - \frac{\pi i\tau}{4} + r}}{\rho + e^{-\frac{\pi i\tau}{4} + r} \rho_{1/2}}.$$

Taking the remark following (78) into account we write

$$(84a) \quad x_0(t) = \text{const.} + \ln \frac{\rho_{1/2} + \rho e^{j - \frac{\pi i\tau}{4} + r}}{\rho_{1/2} + \rho e^{\frac{\pi i\tau}{4} - r}}.$$

The condition  $x_0(t) = \text{independent of } t$ , is satisfied if and only if

$$(85) \quad j - \frac{\pi i\tau}{2} + 2r = 0, \quad (\text{recall } j = \ln a)$$

which determines  $r$ . It would be very satisfying to verify this by direct calculation, without relying on the symmetry of the problem. At present, this delicate calculation is unfortunately beyond our reach.

To arrive at a final formula for  $x_n(t)$  we insert the value of  $r$  which satisfies (85) into (82). We insert the latter into (35). We determine the unknown constant in the local mean value of  $x_n(t)$  so that  $x_0(t) = 0$ . We obtain after a short calculation

$$(86) \quad x_n(t) = -n \ln 4a + \frac{1}{2} \ln a + \ln \frac{\Delta(n-1, t)}{\Delta(n, t)}.$$

where

$$(87) \quad \Delta(n, t) = \rho\left(\frac{n}{2} - \omega t\right) + a^{-(n+\frac{1}{2})} \rho\left(\frac{n}{2} - \omega t + \frac{1}{2}\right), \quad \left|\frac{n}{t}\right| < N_{min}.$$

By the antisymmetry of our problem the formula is valid not only in the region  $0 \leq \frac{n}{t} < N_{min}$  but also in the region  $-N_{min} < \frac{n}{t} \leq 0$ . When  $|n| \gg 1$ , either the first ( $n > 0$ ) or the second term ( $n < 0$ ) dominate in (87). The solution in this case is identified with a space-periodic, time-periodic Toda wave. As expected by the antisymmetry of the problem the  $\rho$  functions in the two terms are out of phase by a half-period. When  $|n|$  is small (i.e. near the boundary particle  $n = 0$ ) both terms in (87) must be taken into account.

The formula for  $x_n(t)$  can then be identified with the formula obtained by Date and Tanaka if we take the latter with spectrum

$$[-a - 1, -a + 1] \cup [-\varepsilon, \varepsilon] \cup [a - 1, a + 1]$$

and let  $\varepsilon \rightarrow 0$ , in which case we obtain our degenerate structure

$$[-a - 1, -a + 1] \cup \{0\} \cup [a - 1, a + 1].$$

Alternatively we can obtain our solution from a periodic solution with spectrum  $[-a - 1, -a + 1] \cup [a - 1, a + 1]$  which is Darboux transformed and an eigenvalue is inserted at the origin.

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