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Ternary Weak-Signal Detection in Non-Gaussian Noise: A Preliminary Analysis for "H_0: N vs H_1: N + S_1 vs H_2: N + S_2" With Independent Sampling

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PREFACE

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13. ABSTRACT *(Maximum 200 words)*

A general analysis of the Ternary Class (M = 2): H_0: N vs H_1: S_1 + N vs H_2: S_2 + N of signal detection problems is presented, for completely general signals, i.e., both broad- and narrow-band, deterministic or random, in generalized (i.e., non-Gaussian) noise, in the limiting threshold régime. This includes optimum threshold algorithms and system performance, as measured by the appropriate error and detection probabilities. The present treatment, however, is subject to the following constraints: (1) independent noise sampling; (2) ambient noise models, i.e., noise independent of the signals; (3) uniform cost functions, e.g., C_0 (> 0) for errors, and $C_1 = 0$ for correct decisions. Under these conditions, only three principal parameters are needed: $\sigma 1^2$, $\sigma 2^2$ = signal detection parameters (= "output $(S/N)^2$") and the correlation coefficient ρ_{12} (= ρ) between the two (threshold) test statistics (or detection "algorithms") Z_1, Z_2, apart from the a priori probabilities (q, p_1, p_2) of the presence of noise alone, S_1, and S_2. Next steps, to extend the treatment to the general case (M = 3): H_1: N + S_1, vs H_2: S_2 + N vs H_3: S_3 + N, and to include correlated noise samples, are noted.

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LIST OF PRINCIPAL SYMBOLS

A_k = detector bias

$\alpha_k^{(0)}$ = (conditional) error probs, k = 1,2: "noise alone" = true state

$a_{on}^{(m)}$ = (normalized) signal amplitudes

$\beta_k^{(0)}, \beta_2^{(1)}$, etc. = (conditional) error probs.

C_0 = cost of errors

$\varepsilon, \varepsilon_0$ = signal epochs

$g^{(k)*}$ = opt. threshold algorithms (k = 1,2)

$\gamma_{0,1,2}$ = decisions

H_0, H_1, H_2 = Hypothesis states

I_c = incomplete Γ-function

$I^{(2)}$ = Basic 2nd-order probability integral, equation (A.1-1).

J = MN = total no. space x time samples

J( ) = equation (A.1-5a) et seq., a definite integral

$K_0^{(1),(2)}$ = thresholds

$L^{(2)}, L^{(4)}$ = noise statistics, equations (2-12c), (2-13b)

l_j = equation (2-12c) nonlinear functions

$\Lambda_{1,2}$ = likelihood ratios

$\lambda_{(\,)}$ = costs

M = no. of signal classes; total no. of spatial samples

μ_k = ratio of *a priori* probs., k = 1,2

m = mth spatial sample (sensor)

N = noise; total no. of time samples

n = nth time sample

ω_d = Doppler (angular) frequency

$P_D^{(k)}$ = probs. of correct detection

$p_D^{(k)}$ = (conditional) prob. of correct detection

ψ = total background noise intensity, equation (2-12c)

$\phi^{(m)}$ = mth order derivative of error function

Q_2, q_2 = 2nd-order PDFs, noise alone

R_M^* = Bayes risk

ρ, ρ_{12} = correlation coeff. for $\overline{Z_1 Z_2}$

s_n^m = (normalized) signal sample

σ_k^2 = detection parameter, $k = 1,2$

$\Delta\tau_m$ = array time delay, equation (2-14b)

$\theta_j^{(k)}$, $\theta_{m,n}^{(k)}$ = (normalized) signals, $k = 1,2$

Θ = error function

w_2 = PDF of random variables (x_1,x_2)

Z_1, Z_2 = data-dependent part of detector, see equation (2-12)

z, $z_{1,2}$ = normalized random variables, equation (3-1)

TERNARY WEAK-SIGNAL DETECTION IN NON-GAUSSIAN NOISE: A PRELIMINARY ANALYSIS FOR "H_0: N vs H_1: N + S_1 vs H_2: N + S_2" WITH INDEPENDENT SAMPLING

1. INTRODUCTION

An important extension of the usual binary detection theory involving generalized, or non-Gaussian noise (see reference 1, for example) is the *ternary case* involving two signals (M = 2) in noise, as well as noise alone. The basic hypothesis states are H_0: N, "noise alone," vs H_1: N + S_1, "signal 1 in noise" vs H_2: N + S_2, "signal 2 in noise." This is a special case of the general M-ary detection model, described in section 23.1 and outlined in (3), (b) of section 23.1 of reference 2 for the cases M = 2,3. (Also see reference 3 for some recent ternary detection analysis involving purely Gaussian noise, where effective erasure of magnetic tapes is considered.)

The present study provides new results: (1) a canonical formulation for these H_0 vs H_1 vs H_2 cases, for arbitrary signals, e.g., broadband as well as narrowband, deterministic or purely random, with (2) spatial as well as temporal sampling—thus including the potential for adaptive beam forming or generalized "matched field processing"[1] here, as well as the usual purely temporal cases, involving preformed beams; and (3) *threshold reception* in generalized noise, which in turn may be fully described by Class A and B models in most instances. (See references 4 through 6. For a review of these latter in related telecommunications applications, see references 7 and 8.)

The limitations of the present analysis are twofold: (1) restrictions to independent noise sampling, and (2) the simple, but reasonable cost assignments that provide the same costs to "failure," i.e., wrong decision, and identical (but lesser costs, necessarily) to "success," or correct decisions. The latter is usually not a serious restriction in most cases, and it greatly simplifies the analytic specification of the decision regions (see equation (23.20) and figure 23.3 of reference 2 vis-à-vis the case (b) in section 23.1 cited above). The former simplification, however, can lead to noticeably suboptimum performance vis-à-vis correlated noise sampling, when the latter can or should be applied. (This limitation will be removed in a subsequent study, along the lines of references 9 and 10.)

[1] This term is used for the spatial analogue of the earlier concept of "matched filter processing" for the temporal representations of the signal (output of receiver array or preformed beam). This is a local concept, and not a global one, in the sense of the more recent "Matched Field Processing" (MFP) or passive signal locations (PSL) developed elsewhere (see reference 17, for example).

The generic application of the present analysis is to those situations in signal detection where one of the signals, i.e., S_1, is the "desired" signal, and S_2 embodies an unwanted signal. For the current application to (active or passive) underwater acoustic systems, S_1 can represent a target return for an emitting source whose presence must be determined, while S_2 is a possible, confusing or undesired ambient emission, a false target or decoy. At the same time, the non-Gaussian character of the background noise arises from a variety of physical mechanisms: (1) isolated, undesired signal sources, e.g., shipping, biological emitters, other man-made and natural phenomena, etc.; and (2) distributed sources, such as reverberation, ice noise, surface effects, e.g., waves, rain, etc. Reverberation noise, or scattering, is by definition signal dependent, as opposed to "ambient noise," whose sources are outside the control of the transmitter in question (see reference 11). Ignore for the present the signal-dependent nature of reverberation, remembering, nevertheless, that its detailed statistical structure depends on the original, injected signal, as well as on the particular scattering mechanism, e.g., ocean-air interfaces, ocean bottoms, etc. In any case, regardless of the mechanism involved, the canonical models here are applicable not only in the present underwater acoustical context, but in the more usual, analogous telecommunications case, as well (see reference 12).

This paper is organized as follows: section 2 provides (1) a general formulation of the ternary "two-signal" (M=2) detection cases, for the simple but common cost assignments C_0 (> 0): "incorrect", and C = 0: "correct;" and (2) specialization to threshold signals and independently sampled noise fields. Section 3 gives the evaluations of the associated performance probabilities in the threshold régime. Various limiting and special cases are discussed in section 4, which also serve to verify the general results of section 3. The paper concludes in section 5 with a short summary of the scope and limitations of the present analytical model, and includes a series of next steps to be pursued in the mathematical and numerical analyses of the general ternary case. Numerical results, requiring alternative series development, are reserved for a subsequent report (see reference 13).

2. TERNARY DETECTION: H0: N vs H_1: N + S_1 vs H_2: N + S_2: GENERAL THEORY WITH EQUAL COST ASSIGNMENTS

Here the ternary signal detection problem is considered when H_0: N, "noise alone," is an alternative state. The analysis is general at this stage, except for a usually noncritical assignment of decision costs: incorrect decisions are penalized equal costs C_0 (> 0), while correct decisions are charged zero costs (see section 23.1(c) of reference 2). However, there may be unequal *a priori* probabilities of signal presence (or absence) in the received data, e.g., $P_2 = (q, p_1, p_2) \geq 0$, $q+p_1+p_2 = 1$ for the present case of two possible signal classes (S_1) and (S_2). (For an account of the general M-ary theory, $M \geq 2$, see chapter 23 of reference 2. See also references 14 and 15, where the M-ary detection formulation is used for signal classification in Gaussian noise.)

2.1 THE DECISION PROCESS: (M = 2)

With the cost assignments (see pp. 1031 et seq. in reference 2),

$$\left.\begin{array}{l} \left.\begin{array}{l} C_l^{(o)} = C_o;\ l = 1,2 \\ C_l^{(k)} = C_o;\ l = k \end{array}\right\} \text{"failure"} \\ C_k^{(k)} = 0;\ l = k = 1,2;\ C_o^{(o)} = 0 \end{array}\right\}, \quad \begin{array}{c} \text{and from} \\ \text{equation} \\ (23.7)\ \therefore \end{array} \quad \left.\begin{array}{l} \lambda_l^{(o)} \equiv C_l^{(o)} - C_0^{(o)};\ l = 1,2\ (= \text{M here}) \\ \lambda_k^{(k)} \equiv C_k^{(k)} - C_o^{(k)}\ (< 0);\ k = l = 1,2 \\ \lambda_k^{(l)} \equiv C_k^{(l)} - C_o^{(l)};\ l \neq k\ (= 1,2) \end{array}\right\}; \tag{2-1}$$

thus,

$$\left.\begin{array}{l} \lambda_2^{(o)} = C_2^{(o)} - C_o^{(o)} = C_o \\ \lambda_1^{(o)} = C_1^{(o)} - C_o^{(o)} = C_o \end{array}\right\} \left.\begin{array}{l} \lambda_2^{(1)} = C_2^{(1)} - C_o^{(1)} = 0 \\ \lambda_1^{(1)} = C_1^{(1)} - C_o^{(1)} = -C_o \end{array}\right\} \left.\begin{array}{l} \lambda_2^{(2)} = C_2^{(2)} - C_o^{(2)} = -C_o \\ \lambda_1^{(2)} = C_1^{(2)} - C_o^{(2)} = 0 \end{array}\right\}, \tag{2-2}$$

with the *thresholds*

$$\left\{K_o^{(1)} = \frac{\lambda_2^{(o)}\lambda_1^{(2)} - \lambda_1^{(o)}\lambda_2^{(2)}}{\Delta}\right.;\ \left\{K_o^{(2)} = \frac{\lambda_1^{(o)}\lambda_2^{(1)} - \lambda_1^{(1)}\lambda_2^{(o)}}{\Delta}\right.;\ \Delta \equiv \lambda_1^{(1)}\lambda_2^{(2)} - \lambda_2^{(1)}\lambda_1^{(2)} = C_o^{(2)};$$

$$\therefore \quad = 1 \qquad\qquad = 1\,. \tag{2-3}$$

The associated *decision rules* (see equations (23.20a) through (23.20c) of reference 2)) are:

$$\boxed{\begin{array}{ll} \textit{decide}\ \gamma_0\ \textit{if} & \Lambda_{1,2}(\mathbf{V}) \le 1;\ (\Lambda \ge 0); \\ \textit{decide}\ \gamma_1\ \textit{if} & \Lambda_1(\mathbf{V}) \ge \Lambda_2(\mathbf{V}) \text{ and } \Lambda_1(\mathbf{V}) \ge 1; \\ \textit{decide}\ \gamma_2\ \textit{if} & \Lambda_2(\mathbf{V}) \ge \Lambda_1(\mathbf{V}) \text{ and } \Lambda_2(\mathbf{V}) \ge 1, \end{array}} \tag{2-4}$$

which are sketched in figure 2-1. (The simple geometry is a consequence of the choice of simple costs (see equation 2-2).) Figure 2-1b shows the corresponding decision regions for

$$x_k = \log \Lambda_k,\ k = 1,2\ , \tag{2-5}$$

viz.,

$$\left.\begin{array}{l} \textit{decide}\ \gamma_o\!:\ x_1, x_2 \le 0\ ;\ \textit{decide}\ \gamma_1\!:\ x_1 \ge x_2\ ;\ x_1 \le 0\ ; \\ \textit{decide}\ \gamma_2\!:\ x_2 \ge x_1\ ;\ x_2 \ge 0 \text{ (see equation (2-4))} \end{array}\right\}. \tag{2-6}$$

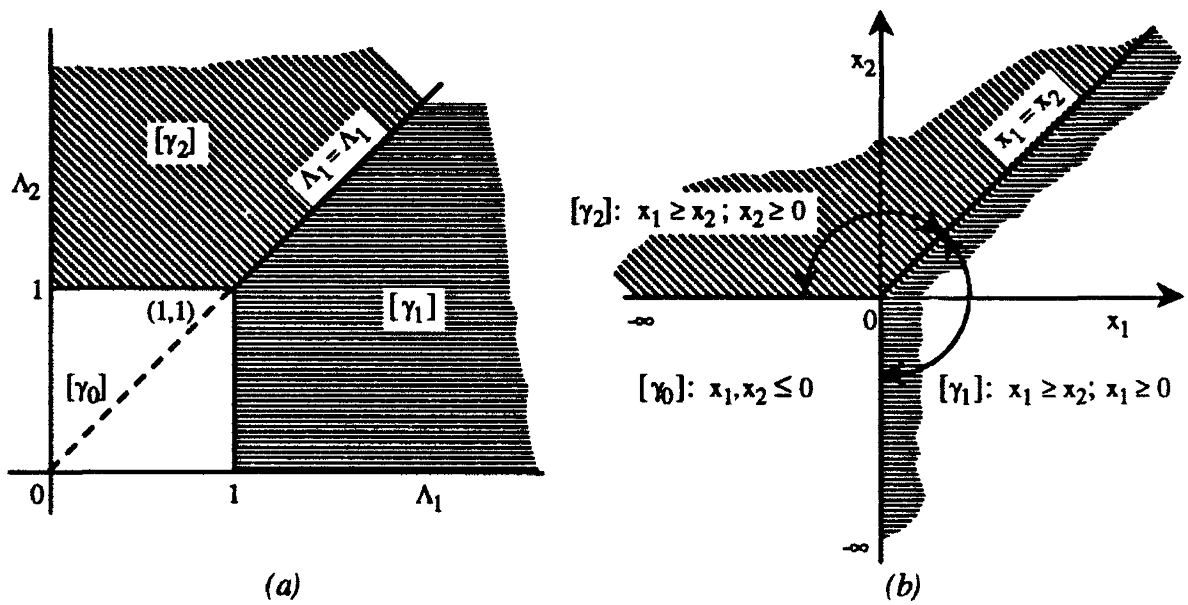

Figure 2-1. Decision Regions: γ_0, γ_1, γ_2, and Boundaries

The equalities in the decision rule relations (see equations 2-4 and 2-6) define the boundaries of the decision regions (γ_j), j = 0,1,2 here[2].

2.2 BAYES RISK

As seen in equation (23.37) of reference 2, specialized to M = 2 (k = 0: noise only), the *Bayes Risk*, or *Minimum Average Cost*, of decisions is given by

$$R_M^* = R_{oM} + q\sum_{k=1}^{2}\lambda_k^{(o)}\alpha_k^{(o)*} + \sum_{k=1,l=0}^{2}{}' p_k\lambda_l^{(k)}\left\langle\beta_l^{(k)*}\right\rangle_k + \sum_{k=0}^{2}p_k\lambda_k^{(k)}\left\langle\eta_k^{(k)*}\right\rangle_k\ , \qquad (2\text{-}7a)$$

where

$$\left.\begin{aligned}\left\langle\eta^{(k)*}\right\rangle_k = 1-\sum_{j=0}^{2}{}'\left\langle\beta_j^{(k)*}\right\rangle_k\ , \quad \therefore \left\langle\eta_1^{(1)}\right\rangle_1 &= 1-\left\langle\beta_o^{(1)*}\right\rangle_1-\left\langle\beta_2^{(1)*}\right\rangle_1\ ;\\ \left\langle\eta_2^{(2)}\right\rangle_1 &= 1-\left\langle\beta_o^{(2)*}\right\rangle_2-\left\langle\beta_1^{(2)*}\right\rangle_k\end{aligned}\right\} \qquad (2\text{-}7b)$$

for the present ternary case. Using the cost assignments of equation (2-1) yields the ternary Bayes risk here

$$R_2^* = qC_o\left(\alpha_1^{(o)*}+\alpha_2^{(o)*}\right)+C_o\left[p_1\left(\left\langle\beta_o^{(1)*}\right\rangle_1+\left\langle\beta_2^{(1)*}\right\rangle_1\right)+p_2\left(\left\langle\beta_o^{(2)*}\right\rangle_2+\left\langle\beta_1^{(2)*}\right\rangle_2\right)\right]. \qquad (2\text{-}8)$$

with $\Re_{o2} = (p_1+p_2)C_o$, $q+p_1+p_2=1$. For a check consider the binary case $M=2$, S_1 vs S_2 ; $p_1+p_2=1$:

$$\begin{aligned}\therefore R_2^* &= \Re_{o2}-C_o\left[p_1\left(1-\left\langle\beta_2^{(1)*}\right\rangle_1\right)+p_2\left(1-\left\langle\beta_2^{(1)*}\right\rangle_2\right)\right]\\ &= C_o\left[\left(p_1\left\langle\beta_2^{(1)*}\right\rangle_1\right)+\left(p_2\left\langle\beta_1^{(2)*}\right\rangle_2\right)\right],\end{aligned} \qquad (2\text{-}9a)$$

[2] If different costs are assigned to the various "successful" and "unsuccessful" (or incorrect) decisions, then the simple decision rules, as in equation 2-4 above, are generalized to *sums* of likelihood ratios, rather than ratios, with the resulting great increase in the analytic complexity of the decision process and performance evaluation.

which is correct. Also, for the binary "on-off" cases $p_1 = q$, $p_2 = p$, $\Re_{oM} = pC_o$,so that

$$R_2^*\Big|_{on\text{-}off} = qC_o\alpha_1^{(o)*} + pC_o\left\langle\beta_o^{(1)*}\right\rangle_1 , \tag{2.9b}$$

which is likewise correct.

Rather than express performance in terms of the error probabilities, it can be more convenient to use the (unconditional) probabilities, $\left(P_D^*\right)$, of correct decisions, when it comes to their evaluation. With

$$\left.\begin{array}{l} P_D^{(1)*} \equiv p_1\left(1-\left\langle\beta_o^{(1)*}\right\rangle_1 - \left\langle\beta_2^{(1)*}\right\rangle_1\right), \ = p_1 p_D^{(1)*} ; \\ P_D^{(2)*} \equiv p_2\left(1-\left\langle\beta_o^{(2)*}\right\rangle_2 - \left\langle\beta_1^{(2)*}\right\rangle_2\right), \ = p_2 p_D^{(2)*} ; \\ P_D^{(o)*} \equiv q\left(1-\alpha_1^{(o)*} - \alpha_2^{(o)*}\right), \ = q p_D^{(o)*} \end{array}\right\} , \tag{2-10}$$

where $P_D^{(j)*}$ are the *conditional* probabilities of correct decisions, the Bayes risk (see equation (2-8)) can be written more compactly:

$$\boxed{R_{M=2}^* = C_o\left(1 - \sum_{j=0}^{2} P_D^{(j)*}\right) = C_o\left(1 - \sum_{j=0}^{2} p_{(j)} p_D^{(j)*}\right),} \qquad p_o = q . \tag{2-11}$$

Note that in the limiting cases, $S_1, S_2 \to \infty$, $p_D^{(j)*} \to 1$, and therefore $P_D^{(j)*} \to p_j$, , $\sum_0^2 p_j = 1$; therefore $R_{M=2}^* = 0$, as expected. Conversely, when $S_1, S_2 \to 0$, $p_D^{(j)*} = p_j/2$; therefore $P_D^{(j)*} \to p_j/2$, and therefore $R_{M=2}^* \to C_o/2$ at the other extreme; for these cost assignments, see equation (2-1).

2.3 CANONICAL THRESHOLD DETECTION ALGORITHMS, S_1 AND S_2

From reference 16, for each signal class S_1 and S_2, the following canonical[3] *coherent* and *incoherent* threshold detection algorithms exist for *independent sampling of the noise field*,[4] but with generalized signals and noise.

2.3.1 Coherent Detection, <s> ≠ 0: (k = 1,2)

$$\boxed{\log \Lambda_{coh}^{(k)} \doteq g_{J\text{-}coh}^{(k)*} = A_k^{(coh)} + Z_k(\mathbf{x})_{coh}} , \tag{2-12a}$$

where

$$\left. \begin{aligned} A_k^{(coh)} \equiv \log \mu_k - \sigma_{coh}^{(k)^2} \ , \ Z_k(\mathbf{x})_{coh} \equiv -\sum_j^J \left\langle \theta_j^{(k)} \right\rangle l(x_j) \ , \\ \sigma_{coh}^{(k)^2} \equiv L^{(2)} \sum_j^J \left\langle \theta_j^{(k)} \right\rangle^2 \end{aligned} \right\} . \tag{2-12b}$$

Here, as before, k = 1,2 refers to the generalized (and normalized) signals $\theta^{(k)}$, while j = (m,n) represents a space-time sample. Also,

$$l(x_j) \equiv \frac{d}{dx} \log w_1(x|H_o)\Big|_{x=x_j} = l_j \ ; \ L^{(2)} \equiv \left\langle l^2 \right\rangle_{H_o} = \int l^2 w_1(x|H_o)\, dx \ ; \ \mu_k = \frac{p_k}{q} \ , \tag{2-12c}$$

with

$$\theta_j = a_{o\,n}^{(m)} s_n^{(m)} = a_o^{(m)}(t_n) s^{(m)}(t_n) \ ; \ a_{o\,n}^{(m)} = A_o^{(m)} \frac{(t_n)}{\sqrt{\Psi}} \ ; \ \left\langle s_n^{(m)^2} \right\rangle = 1 \ ; \ \Psi = I + \Psi_G \tag{2-12d}$$

---

[3] "Canonical" here means that the *forms* of these algorithms, and their associated performance parameters, remain invariant of the particular physical application, although, of course, the *numerical* values will depend on the specific problem at hand.

[4] This latter is equivalent to treating the noise field as spectrally "white" in both time and space. As noted in section 1, this sampling constraint can lead to serious degradation of performance vis-à-vis algorithms which properly include at least the dominant first-order correlations of the noise when this is physically possible (see references 9 and 10). This situation will be considered in a subsequent study.

for the normalized signal, with actual amplitude, $A_{on}^{(m)}$: m = mth-sensor or "spatial sample" in the received field, while n refers to t_n, the nth time sample. Here I = intensity of the non-Gaussian noise component, with ψ_G the intensity of the (frequently) much smaller background Gaussian noise contribution. Local stationarity and homogeneity is assumed over the bounded sampling region ε_J = MN, m = 1,...,M ; n = 1,...,N . For *coherent detection*, signal epoch is known and equals ε_0, so that $\langle s_n^{(m)} \rangle_{\varepsilon=\varepsilon_0} \neq 0$, and hence $\langle \theta_j \rangle \neq 0$.

2.3.2 *Incoherent Detection,* $\langle s \rangle = 0$ *:* $(k = 1, 2)$

Similarly, from reference 16,

$$\boxed{\log \Lambda_{inc}^{(k)} \doteq g_{J\text{-}inc}^{(k)*} = A_k^{(inc)} + Z_k(\mathbf{x})_{inc}\ ,} \tag{2-13a}$$

where

$$\left.\begin{aligned} &A_k^{(inc)} \equiv \log \mu_k - \frac{1}{2}\sigma_{inc}^{(k)^2}\ ;\ Z_k(\mathbf{x})_{inc} \equiv \frac{1}{2!}\sum_{jj'} \left(l_j l_{j'} + l_{j'}' \delta_{jj'} \right) \left\langle \theta_j^{(k)} \theta_{j'}^{(k)} \right\rangle ; \\ &\sigma_{inc}^{(k)^2} = \frac{1}{4}\sum_{jj'} \left[\left(L^{(4)} - 2L^{(2)^2} \right) \delta_{jj'} + 2L^{(2)^2} \right] \left\langle \theta_j^{(k)} \theta_{j'}^{(k)} \right\rangle^2 \ ;\ \delta_{jj'} = \delta_{mm'}\delta_{nn'}\ ; \\ &l_j' = \frac{d}{dx} l \Big|_{x_j}\ ;\ L^{(4)} \equiv \left\langle \left(l' + l^2 \right)^2 \right\rangle_{H_0} = \int \left(\frac{w_1''}{w_1} \right)^2 w_1(x|H_0)\, dx \end{aligned}\right\} . \tag{2-13b}$$

Also,

$$\left\langle \theta_j^{(k)} \theta_{j'}^{(k)} \right\rangle = \left(\overline{a_o^2}\, \hat{m}_{|n-n'|}^{(m,m')} \rho_{|n-n'|}^{(m,m')} \right)_k\ ,\ \text{with}\ \hat{m}_{|n-n'|}^{(m,m')}\Big|_k \equiv \left(\frac{\left\langle a_{on}^{(m)} a_{on'}^{(m')} \right\rangle}{\overline{a_o^2}} \right)_k\ ,\ k = 1, 2\ , \tag{2-13c}$$

$$\rho_{|n-n'|}^{(m,m')} \equiv \left\langle s_n^{(m)} s_{n'}^{(m')} \right\rangle,\ \text{with}\ \left\langle s_n^{(m)} \right\rangle = 0\ \text{for}\ \textit{incoherent detection}\ .$$

The narrowband signals considered here may be written

$$s_n^{(m)} = \sqrt{2} \cos\left[(\omega_o + \omega_d)(t_n - \varepsilon + \Delta\tau_m) - \phi_n \right],\ t_n = n\Delta t\ , \tag{2-14a}$$

in which

$$\left.\begin{array}{lll} f_o & = & \text{"carrier" or central frequency of the signal spectrum;} \\ \omega_d & = & (2\pi f_d) = \text{Doppler, often consisting of a determininistic } (f_{oD}) \\ & & \text{and a random component } (f_{od}); \\ \varepsilon & = & \text{signal epoch} = \varepsilon_o \text{, known for coherent observation ;} \\ & & \text{(usually) uniformly distributed over a carrier cycle,} \\ & & 1/f_o \text{, for incoherent observation;} \\ \phi_n & = & \phi(t_n) = \text{a possible phase modulation: for example,} \\ & & \text{PSK , where } \phi_n = \pi n \text{ , } t_n < t < t_{n+1} \text{ , etc.} \end{array}\right\}, \qquad \text{(2-14b)}$$

where one or more of the signal parameters are different, each $k = 1,2$. The quantity $\Delta\tau_m^{(k)}$ is the net path delay from the m^{th} sensor to some selected reference point associated with the array, viz.:

$$\Delta\tau_m = \left(\hat{\mathbf{i}}_{ok} - \hat{\mathbf{i}}_{oR}\right) \bullet \frac{\mathbf{r}_m}{c_o} = \left(\nu_{ok} - \nu_{oR}\right) \bullet \frac{\mathbf{r}_m}{f_{ok}} \ ; \ \left(f_o \lambda_o\right)_k = c_o \ ; \ \therefore \nu_k = \frac{\hat{\mathbf{i}}}{\lambda_{ok}} \ , \qquad \text{(2-14c)}$$

where $\hat{\mathbf{i}}_{ok}$ and $\hat{\mathbf{i}}_{oR}$ are respectively the unit vectors in the direction of the (incoming k^{th}) signal wavefront and the main axis of the beam formed by summing the outputs of the (receiving) sensors (Σ_m) ; $\mathbf{r}_m$ is the vector distance of the m^{th} element of the array to the reference point; c_0 and λ_{ok} are respectively the speed of (group) propagation and the (central) wavelength of the (k^{th}) signal in the medium.

With *incoherent detection* $\left[(\varepsilon \neq \varepsilon_o) \ , \ \langle s \rangle_\varepsilon = 0 \ , \text{ or } \langle \theta \rangle = 0\right]$, the signal's autocovariance $\langle \theta_j \theta_{j'} \rangle$, instead of the waveforms, are dealt with (see equation (2-13c)).

The carrier or rapidly varying portion of $\langle \theta_j \theta_{j'} \rangle$ becomes

$$\left.\begin{array}{l} \rho_{|n-n'|}^{(m,m')}\Big|_k = \left\{ \exp\left[-\dfrac{\left[\Delta\omega_d (n-n')\Delta t\right]^2}{2} \right] \cos\left[\omega_o (n-n')\Delta t + k_o \left(\hat{\mathbf{i}}_o - \hat{\mathbf{i}}_{oR}\right) \bullet \Delta \mathbf{r}_{mm'} \right] \right\}_k \ , \\ \left(k_o = \dfrac{2\pi}{\lambda_o} = \dfrac{\omega_o}{c_o} \right)_k \end{array}\right\}, \ \text{(2-15)}$$

where $\Delta\omega_d$ is a measure of the "Doppler spread," if any, assumed to be Gaussian, while the $[\phi_n]$ are independent over n here, and $\Delta\mathbf{r}_{m,m'} = \mathbf{r}_m - \mathbf{r}_{m'}$. Clearly, directing the beam so that $\hat{\mathbf{i}}_{oR} \rightarrow \hat{\mathbf{i}}_{ok}$ maximizes ρ, as expected, while uncertainty as to Doppler $(\Delta\omega_d > 0)$ and signal direction $(\hat{\mathbf{i}}_o)$ degrades ρ and hence performance (see references 12 and 16).

Since appropriate delays and weightings for each sensor output can be inserted here, there is an opportunity for *adaptive beam-forming*. The space-time algorithms in equations (2-12) and (2-13) provide (threshold) optimal "rules" for so doing, to the extent permitted by learning during data acquisition (≡ "adaptive processing") and/or *a priori* knowledge of the signal and noise parameters and statistics, e.g., $\overline{a_o^2}$, $\overline{a^2}$, ε_o, $w_1(x|H_0)$, etc., required to implement these algorithms. This adaptive beam-forming and the associated temporal processing (t_n) is called *decision matched-field processing* (see footnote 1), which is optimal from the point of view of the general criterion of decision theory and Bayes risk. This is more general than the usual criterion of matched-field processing for estimation, or detection, based on maximizing array output on the basis of signal-to-noise ratio[5].

In many instances, the receiving beam is preset or *preformed* with no option of adaptive beam forming. Then, formally, the spatial index in the above algorithms is dropped: $j \rightarrow n$, $J \rightarrow N$, and $M = 1$, with $\Delta\tau_m^{(k)} \rightarrow 0$; only time processing is permitted. This is the usual situation in most electromagnetic (EM) telecommunication applications, where array $L \lesssim$ wavelength, but not in underwater acoustics, where $L >> \lambda_o$, and thus the structure of the noise and signal fields permits its exploitation for improved performance.

2.4 PERFORMANCE PROBABILITIES: TERNARY (M = 2)

At this point there is a transform to the new random variables:

$$\mathbf{Z} = \mathbf{U} - \mathbf{A}\ ;\ \mathbf{A} = (A_k)\ ,\ \mathbf{U} = \left(\log \Lambda^{(k)}\right)\ ;\ \mathbf{Z} = (Z_k)\ ,\ k = 1,2\ . \tag{2-16}$$

[5] When dealing with parameter estimation, the appropriate decision-theoretic extensions must be used (see reference 2, chapter 21, for example) based on maximum likelihood estimation (MLE), or least mean-square error (LMSE) criteria, which follow, in turn, from the Bayes risk theory by appropriate choice of cost function (see section IX and the appendix of reference 16, for example). (See also reference 17 for an alternative approach in underwater acoustics for estimating source location.)

Next, use equations (5) and (6) of problem 23.2 in reference 2, to write the various conditional error probabilities $(M = 2)$[6] :

$$\alpha_1^{(0)*} \equiv \beta_0^{(1)*} = \int_{-A_1}^{\infty} dZ_1 \int_{-\infty}^{Z_1+A_1-A_2} dZ_2 Q_2(\mathbf{Z}) \, ; \; \alpha_2^{(0)*} \equiv \beta_2^{(0)*} = \int_{-A_2}^{\infty} dZ_2 \int_{-\infty}^{Z_2+A_2-A_1} dZ_1 Q_2^{(1)}(\mathbf{Z}) \, ; \tag{2-17a}$$

$$\left\langle \beta_0^{(1)*} \right\rangle = \int_{-\infty}^{-A_1} dZ_1 \int_{-\infty}^{-A_2} dZ_2 P_2^{(1)}(\mathbf{Z}) \, ; \; \left\langle \beta_2^{(1)*} \right\rangle = \int_{-A_2}^{\infty} dZ_2 \int_{-\infty}^{Z_2+A_2-A_1} dZ_1 P_2^{(1)}(\mathbf{Z}) \, ; \tag{2-17b}$$

$$\left\langle \beta_0^{(2)*} \right\rangle = \int_{-\infty}^{-A_1} dZ_1 \int_{-\infty}^{-A_2} dZ_2 P_2^{(2)}(\mathbf{Z}) \, ; \; \left\langle \beta_1^{(2)*} \right\rangle = \int_{-A_1}^{\infty} dZ_1 \int_{-\infty}^{Z_1+A_1-A_2} dZ_2 P_2^{(2)}(\mathbf{Z}) \, . \tag{2-17c}$$

These relations are also readily established from figure 2.1b from the definitions of the error probabilities. Here Q_2 and $P_2^{(k)}$ are the PDFs of $\mathbf{Z}$, (see equation (2-16)), to be determined presently for the canonical threshold detection algorithms in equations (2-12) and (2-13) and in section 3.

The forms of the probabilities of correct detection follow in similar fashion from equation (2-10):

$$P_D^{(0)*} = q \int_{-\infty}^{0} dU_1 \int_{-\infty}^{0} dU_2 Q_2(\mathbf{U}) \, , \; \mathbf{U} = (U_1, U_2) \text{ (see equation (2-16))} \, , \tag{2-18a}$$

$$= q p_D^{(0)*} = q \int_{-\infty}^{-A_1} dZ_1 \int_{-\infty}^{-A_2} dZ_2 Q_2(\mathbf{Z}) \, , \tag{2-18b}$$

and

$$P_D^{(1)*} = p_1 \int_0^{\infty} dU_1 \left(\int_{-\infty}^{0} dU_2 + \int_0^{U_1} dU_2 \right) P_2^{(1)}(\mathbf{U}) = p_1 \int_0^{\infty} dU_1 \int_{-\infty}^{U_1} dU_2 P_2^{(1)}(\mathbf{U}) \, , \tag{2-19a}$$

$$= p_1 P_D^{(1)*} = p_1 \int_{-A_1}^{\infty} dZ_1 \int_{-\infty}^{Z_1+A_1-A_2} dZ_2 P_D^{(1)}(\mathbf{Z}) \, . \tag{2-19b}$$

[6] Note the (inadvertent) omissions of $-A_1$ and $-A_2$ in the limits of $<\beta_2^{(1)*}>$ and $<\beta_1^{(2)*}>$ in equation (A.1-14c) of reference 3 and correspondingly, in $p_D^{(1)*}$ and $p_D^{(2)*}$ (see equations (A.1-16b) and (A.1-17b) of reference 3). The consequences of this are noted in appendix B here.

Similarly,

$$\boxed{\begin{aligned} P_D^{(2)*} &= p_2\int_0^{\infty} dU_2\left(\int_{-\infty}^{0} dU_1 + \int_0^{U_2} dU_1\right)P_2^{(2)}(U) = p_2\int_0^{\infty} dU_2 \int_{-\infty}^{U_2} dU_1 P_2^{(2)}(U)\,, && \text{(2-20a)} \\ &= p_2 P_D^{(2)*} = p_2 \int_{-A_2}^{\infty} dZ_1 \int_{-\infty}^{Z_2+A_2-A_1} dZ_1 P_D^{(2)}(Z)\,. && \text{(2-20b)} \end{aligned}}$$

As measures of performance, the $P_D^{(j)*}$ or $p_D^{(j)*}$ $(\text{where } j = 0,1,2)$ can be used directly or in conjunction with equation (2-11) for the Bayes risk.

3. EVALUATION OF THE PERFORMANCE PROBABILITIES: TERNARY (M = 2)

The first task is to obtain the PDFs Q_2 and $P_2^{(k)}$, of the (here) threshold detection algorithms in equations (2-12) and (2-13), and then to determine the various decision probabilities in equations (2-17) through (2-20) associated with this canonically formulated ternary detection problem. Two critical features permit comparatively simple analytic results, expressed at worst by convergent series of known, tabulated functions. These features are: (1) the asymptotic normality (A-N) of the algorithms $g^*_{coh,\,inc}$ in equations (2-12) and (2-13) by the Central Limit Theorem; (2) this ternary situation, which involves only *two* (M = 2) signal classes in addition to noise alone, which fact is reflected in the double, rather then triple, integrals representing the desired decision probabilities.

3.1 THE NORMALIZED PDFs OF Z

The A-N nature of g^*, because of the weak-signal constraint, allows the needed PDF of the Zs, now in normalized form, to be written as

$$\boxed{\begin{aligned} w_1(\mathbf{z}|H_\eta) &\doteq \frac{\exp\left[-\dfrac{\left(z_1^2 - 2z_1z_2\rho + z_2^2\right)}{2(1-\rho_2)}\right]}{2\pi\sqrt{1-\rho_2}}\ ,\ (\eta = 0,1,2) \\ &= \begin{bmatrix} q_2(\mathbf{Z})\ ,\ \eta = 0\ , \\ p_2^{(k)}(\mathbf{z})\ ,\ \eta = k = 1,2\ , \end{bmatrix} = p_2(\mathbf{Z})\ . \end{aligned}} \tag{3-1a}$$

See section 7.2 in reference 2, where specifically

$$\begin{matrix} k = 1,2 \\ (\eta = 0,1,2) \end{matrix} : \left\{ \begin{aligned} & z_k \equiv \frac{Z_k - \langle Z_k \rangle_\eta}{\sigma_k}\ ,\ \langle Z_k \rangle_\eta \equiv \langle Z_k(\mathbf{x}) \rangle_{H_\eta}\ ;\ \langle\ \rangle_\eta \equiv \int_{-\infty}^{\infty} w_1(\mathbf{x}|H_\eta)\,d\mathbf{x}\ ; \\ & \rho \equiv \rho_{12}^{\eta} \equiv \left[\left(Z_1 - \langle Z_1 \rangle_1\right)\left(Z_2 - \langle Z_2 \rangle_2\right)_\eta\right]\ ; \\ & \sigma_k^{(\eta)^2} \equiv \mathrm{var}_{H_\eta} Z_k = \left[\left(Z_k - \langle Z_k \rangle_\eta\right)^2\right]_\eta \end{aligned} \right\}. \tag{3.1b}$$

Note that $\rho_2^{(1)}(z) = \rho_2^{(2)}(z)$ in this normalized form (see equation (3-1a)).

In sections 3.1.1 and 3.1.2, the various moments that are summarized in equation (3-1b) are evaluated using the results of appendixes A-1 and A-2 of reference 18.

3.1.1 Coherent Detection

$$\left.\begin{aligned}
&\langle Z_k \rangle_0 = 0 \text{ (see equation (A.1-13b))} ; \\
&\sigma_{k\text{-coh}}^{(0)^2} = L^{(2)} \sum_j \langle \theta_j^{(k)} \rangle \equiv \sigma_{k\text{-coh}}^2 \text{ (see equation (A.2-14))} ; \\
&(\sigma_1 \sigma_2)_{coh} \rho_{12}^{(0)} = \langle Z_1 Z_2 \rangle_{H_0} = L^{(2)} \sum_j \langle \theta_j^{(1)} \rangle \langle \theta_j^{(2)} \rangle = (\sigma_1 \sigma_2)_{coh} \rho_{coh}
\end{aligned}\right\} : H_0 , \qquad (3\text{-}2)$$

since $\langle l_j l_{j'} \rangle_{H_0} = L^{(2)} \delta_{jj'}$.

Similarly, $(\eta = 1,2)$:

$$\left.\begin{aligned}
&\langle Z_k \rangle_{1,2} = L^{(2)} \sum_j \langle \theta_j^{(k)} \rangle^2 + O\left(\overline{\theta^4}\right) \text{ (see equation (A.2-3) , } \doteq \sigma_{k\text{-coh}}^2 ; \\
&\sigma_k^{(\eta)^2} = L^{(2)} \sum_j \langle \theta_j^{(k)} \rangle^2 + O\left(\overline{\theta^4}\right) \text{ (see equation (A.2-3) , } \doteq \sigma_{k\text{-coh}}^2
\end{aligned}\right\} H_{1,2} : k = 1,2 \; ;(3\text{-}3)$$

and $$\sigma_1 \sigma_2 \rho_{12}^{(\eta)} \doteq L^{(2)} \sum_j \langle \theta_j^{(1)} \rangle \langle \theta_j^{(2)} \rangle + O\left(\overline{\theta^4}\right) \doteq (\sigma_1 \sigma_2)_{coh} \rho_{coh} .$$

3.1.2 Incoherent Detection

$$\left.\begin{aligned}
&\langle Z_k \rangle_0 = 0 \text{ (see equation (A.2-25)} ; \\
&\sigma_{k\text{-inc}}^{(0)^2} = \text{equation (2-13b) exactly} ; \\
&\left(\sigma_1 \sigma_2 \rho_{12}^{(0)}\right)_{inc} = \frac{1}{4} \sum_{jj'} \left[\left(L^{(4)} - 2L^{(2)^2} \right) \delta_{jj'} + 2L^{(2)^2} \right] \langle \theta_j^{(1)} \theta_{j'}^{(1)} \rangle \langle \theta_j^{(2)} \theta_{j'}^{(2)} \rangle \\
&\qquad = (\sigma_1 \sigma_2)_{inc} \rho_{inc} \text{ , exactly.}
\end{aligned}\right\} ; \qquad (3\text{-}4)$$

For the alternative hypotheses (k=1,2) ,

$$\left.\begin{array}{l}\langle Z_k\rangle_{1,2} \doteq \sigma^2_{k-inc} = \text{equation (2.13b)} \; ; \; \sigma^{(\eta)^2}_{k-inc} \doteq \sigma^2_{k-inc} \text{ (see pp. 226 - 232)} \; ; \\ \left(\sigma_1^{(\eta)}\sigma_2^{(\eta)}\rho_{12}^{(k)}\right)_{inc} \doteq (\sigma_1\sigma_2)_{inc}\rho_{inc} \text{ (see equation (3 - 4))}\end{array}\right\} : H_{1,2} \, . \qquad (3\text{-}5)$$

From equations (2-12d) and (2-13b),

$$\boxed{A_k^{(coh)} \equiv \log\mu_k - \frac{\sigma^2_{k-coh}}{2} \; ; \; A_k^{(inc)} \equiv \log\mu_k - \frac{\sigma^2_{k-inc}}{2} \; ; \; k = 1,2 \, ,} \qquad (3\text{-}6)$$

using equations (3-2) through (3-5) for the detailed structures.

Accordingly, to summarize, equation (3-16) can be compactly expressed as

$$\boxed{\left\{\begin{array}{l} H_0 : z_k = \left(\dfrac{Z_k}{\sigma_k}\right)_{coh} \; ; \; \overline{Z_k} = 0 \; ; \; \rho_{coh} = L^{(2)}\displaystyle\sum_j \frac{\langle\theta_j^{(1)}\rangle\langle\theta_j^{(2)}\rangle}{(\sigma_1\sigma_2)_{coh}} \; ; \\ \sigma_k^2 = \sigma^{(k)^2}_{coh} = L^{(2)}\displaystyle\sum_j \langle\theta_j^{(k)}\rangle^2 \; ; \\ H_k : z_k = \left(\dfrac{Z_k - \sigma_k^2}{\sigma_k}\right)_{coh} \; ; \; \overline{Z_k} \doteq \sigma^2_{k-coh} \; ; \; \rho_{coh} = \rho_{coh}|H_0 \; ; \end{array}\right\} : \textit{coherent} \, ;} \qquad (3\text{-}7a)$$

and

$$\boxed{\left\{\begin{array}{l} H_0 : z_k = \left(\dfrac{Z_k}{\sigma_k}\right)_{inc} \; ; \; \overline{Z_k} = 0 \; ; \; \rho_{inc} = \dfrac{1}{4}\displaystyle\sum_{jj'}\left[\left(L^{(4)} - 2L^{(2)^2}\right)\delta_{jj'} + 2L^{(2)^2}\right] ; \\ \bullet \dfrac{\langle\theta_j^{(1)}\theta_{j'}^{(1)}\rangle\langle\theta_j^{(2)}\theta_{j'}^{(2)}\rangle}{(\sigma_1\sigma_2)_{inc}} \; ; \; H_k : z_k = \left(\dfrac{Z_k - \sigma_k^2}{\sigma_k}\right)_{inc} \; ; \\ \overline{Z_k} \doteq \sigma^2_{k-inc} = \dfrac{1}{4}\displaystyle\sum_{jj'}\left[\left(L^{(4)} - 2L^{(2)^2}\right)\delta_{jj'} + 2L^{(2)^2}\right] \\ \bullet \langle\theta_j^{(k)}\theta_{j'}^{(k)}\rangle^2 \; ; \; \rho_{inc} = \rho_{inc}|H_0 \, . \end{array}\right\} : \textit{incoherent} \, ;} \qquad (3\text{-}7b)$$

In the subsequent analysis, the subscripts "coh" and "inc" on z_k , σ_k , etc. will be dropped unless this results in ambiguity.

3.2 THE PERFORMANCE PROBABILITIES: (M = 2)

Using the transformations of equations (3-1b) on the unnormalized (Gaussian) PDFs for Z_1 and Z_2 (see equation (7.9) of reference 2), gives the equation (3-1a), as noted above, and allows $Q_2(\mathbf{Z})$ and $P_2^{(k)}(\mathbf{Z})$ in equations (2-7) through (2-20) to be replaced by $q_2(\mathbf{z})$ and $p_2^{(k)}(\mathbf{z})$, respectively (see equation 3-1)). The results for equations (2-17) through (2-20) now take the explicit forms:

$$\alpha_1^{(0)*} \equiv \beta_0^{(1)*} = \int_{-A_1/\sigma_1}^{\infty} dz_1 \int_{-\infty}^{(\sigma_1 z_1 + A_1 - A_2)/\sigma_2} dz_2 q_2(\mathbf{z})\,;\ \alpha_2^{(0)*} \equiv \beta_2^{(0)*} = \int_{-A_2/\sigma_2}^{\infty} dz_2 \int_{-\infty}^{(\sigma_2 z_2 + A_2 - A_1)/\sigma_1} dz_1 q_2(\mathbf{z})\,; \tag{3-8a}$$

$$\left\langle \beta_0^{(1)*} \right\rangle = \int_{-\infty}^{(-A_1-\sigma_1^2)/\sigma_1} dz_1 \int_{-\infty}^{(-A_2-\sigma_2^2)/\sigma_2} dz_2 p_2^{(1)}(\mathbf{z})\,;\ \left\langle \beta_2^{(1)*} \right\rangle = \int_{-A_2/\sigma_2}^{\infty} dz_2 \int_{-\infty}^{(\sigma_2 z_2 + A_2 - A_1 + \sigma_2^2 - \sigma_1^2)/\sigma_1} dz_1 p_2^{(1)}(\mathbf{z})\,; \tag{3-8b}$$

$$\left\langle \beta_0^{(2)*} \right\rangle = \int_{-\infty}^{(-A_1-\sigma_1^2)/\sigma_1} dz_1 \int_{-\infty}^{(-A_2-\sigma_2^2)/\sigma_2} dz_2 p_2^{(2)}(\mathbf{z})\,;\ \left\langle \beta_1^{(2)*} \right\rangle = \int_{(-A_1)/\sigma_1}^{\infty} dz_1 \int_{-\infty}^{(\sigma_1 z_1 + A_1 - A_2 + \sigma_1^2 - \sigma_2^2)/\sigma_2} dz_2 p_2^{(2)}(\mathbf{z})\,. \tag{3-8c}$$

Note that since $p_2^{(k)}(\mathbf{z}) = p_2(\mathbf{z})$ in equation (3-1), the subscripts 1 and 2 can be interchanged in the limits of all the error probability integrals to obtain the $\left\langle \beta^{(\ )*} \right\rangle$s in equation (3-8).

The same can be done for equations (2-18) through (2-20) in normalized form (without the respective *a priori* probability factors):

$$p_D^{(0)*} = \int_{-\infty}^{-A_1/\sigma_1} dz_1 \int_{-\infty}^{-A_2/\sigma_2} dz_2 q_2^{(2)}(\mathbf{z})\,;\ p_D^{(1)*} \int_{(-A_1-\sigma_1^2)/\sigma_1}^{\infty} dz_1 \int_{-\infty}^{(\sigma_1 z_1 + A_1 - A_2 + \sigma_1^2 - \sigma_2^2)/\sigma_2} dz_2 p_2^{(1)}(\mathbf{z})\,; \tag{3-9a}$$

$$p_D^{(2)*} = \int_{(-A_2-\sigma_{21}^2)/\sigma_2}^{\infty} dz_2 \int_{-\infty}^{(\sigma_2 z_2 + A_2 - A_1 + \sigma_2^2 - \sigma_1^2)/\sigma_1} dz_1 p_2^{(2)}(\mathbf{z})\,;\ \text{and } p_2^{(k)}(\mathbf{z}) = p_2(\mathbf{z}) \text{ (see equation (3-1))}\,. \tag{3-9b}$$

3.2.1 Probabilities of Correct Detection

Next,

$$\left.\begin{aligned} q_2(\mathbf{z})\,,\ p_2^{(k)}(\mathbf{z}) &= w_2\left(\mathbf{z}|H_n\right) \text{ (see equation (3-1)}\,, \\ &= w_2\left(x_1, x_2\right) \text{ (see equation (A-1) in reference 19)} \end{aligned}\right\}, \tag{3-10}$$

and all have the same analytic form (see equation (A-1)), so the basic result of I = equation (A-6c), with equation (A–8b) and special cases, is to evaluate all the decision probabilities in equations (3-8) and (3-9). These results are specifically:

$$p_D^{(0)*} = \frac{1}{2\sqrt{2\pi}} \left\{ \begin{array}{l} \left[J\left[\frac{-A_1}{\sigma_1} \middle| \frac{-\rho}{\sqrt{2(1-\rho^2)}}, \frac{\frac{-A_2}{\sigma_2}}{\sqrt{2(1-\rho^2)}} \right] - J\left[-\infty \middle| \frac{-\rho}{\sqrt{2(1-\rho^2)}}, \frac{\frac{-A_2}{\sigma_2}}{\sqrt{2(1-\rho^2)}} \right] \right] \\ +J\left[\frac{-A_1}{\sigma_1} \middle| \frac{-\rho}{\sqrt{2(1-\rho^2)}}, \infty \right] - J\left[-\infty \middle| \frac{\rho}{\sqrt{2(1-\rho^2)}}, \infty \right] \end{array} \right\}. \quad \text{(3-11a)}$$

Explicitly, the following series may be used $\left(\text{for } |\rho| < 1/\sqrt{2}\right)$[7]

$$\boxed{\therefore p_D^{(0)*} = \frac{1}{4} \left\{ \begin{array}{l} \left[1 + \Theta\left(\frac{-A_1}{\sqrt{2}\sigma_1} \right) \right] \left[1 + \Theta\left(\frac{-A_2}{\sqrt{2(1-\rho^2)}\sigma_2} \right) \right] \\ + \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{2}\rho}{\sqrt{1-\rho^2}} \right)^n \frac{\Gamma\left(\frac{n+1}{2}\right)}{n!} \phi^{(n-1)}\left(\frac{-A_2}{\sigma_2\sqrt{1-\rho^2}} \right) \\ \left[I_c\left(\frac{A_1^2}{2\sigma_1^2}, \frac{n+1}{2} \right) - 1 \right] \end{array} \right\},} \quad \text{(3-11b)}$$

where $I_c(a;\nu) \equiv \left(1/\Gamma(\nu)\right)\int_0^a y^{\nu-1}\, e^{-y}\, dy$, $\nu > 0$: *incomplete Γ - function* (see reference 22) and $\phi^{(m)}(x) \equiv d^m\left(e^{-x^2/2}/\sqrt{2\pi}\right) dx^m$ is tabulated also (see reference 19). Here $\Theta(x) = \left(2/\sqrt{\pi}\right)\int_0^x e^{-t^2} dt = \operatorname{erf} x$ is the familiar *error* function of references 20 and 21.

[7] For $|\rho| < 1/\sqrt{2}$, another series is needed. This is developed in reference 13, allowing a complete numerical evaluation for $-1 \le \rho \le 1$.

Further inspection of equation (3-1) in equation (3-9a) shows that if $\rho = 0$, then $w_2 = w_1(z_1)w_1(z_2)$, i.e., w_2 is symmetrical, and so equation (3-11b) reduces to the much simpler relation

$$p_D^{(0)*}\Big|_{\rho=0} = \frac{1}{4}\left\{\left[1+\Theta\left(\frac{-A_1}{\sqrt{2}\sigma_1}\right)\right]\left[1+\Theta\left(\frac{-A_2}{\sqrt{2}\sigma_2}\right)\right]\right\}, \tag{3-12}$$

whereas, if $\rho \neq 0$, then $p_D^{(0)*}$ is unsymmetrical (in A_1/σ_1 and A_2/σ_2), as expected, since w_2 is also unsymmetrical. In the limits $\rho \to \pm 1$, $w_2(\mathbf{z}) = w_1(z_1)\delta(z_2 \mp z_1) = w_1(z_2)\delta(z_1 \mp z_2)$, and equation (3-9a) must be evaluated directly to get

$$p_D^{(0)*}\Big|_{\rho=\pm 1} = \frac{1}{2}\left[1+\Theta(B_{min})\right] \; ; \; B_{min} = \min\left(\frac{-A_1}{\sqrt{2}\sigma_1}, \frac{-A_2}{\sqrt{2}\sigma_2}\right), \tag{3-13}$$

(where min(-5,-2)=-5 , etc.). (For the cases σ_1, $\sigma_2 \to 0$, see equation (3-17).)

The conditional probabilities $p_D^{(k)*}$ (in equation (3.9b)) are found in the same way. Equation (A-6c) becomes

$$p_D^{(1)*} = \frac{1}{2\sqrt{2\pi}}\left\{\begin{array}{l} J\left[\infty \left| \dfrac{\frac{\sigma_1}{\sigma_2}-\rho}{\sqrt{2(1-\rho^2)}} \right., \dfrac{A_1 - A_2 + \sigma_1^2 - \sigma_2^2}{\sigma_2\sqrt{2(1-\rho^2)}}\right] \\ -J\left[\dfrac{-A_1-\sigma_1^2}{\sigma_1} \left| \dfrac{\frac{\sigma_1}{\sigma_2}-\rho}{\sqrt{2(1-\rho^2)}} \right., \dfrac{A_1 - A_2 + \sigma_1^2 - \sigma_2^2}{\sigma_2\sqrt{2(1-\rho^2)}}\right] \\ +J\left[\infty \left| \dfrac{-\rho}{\sqrt{2(1-\rho^2)}} \right., \infty\right] - J\left[\dfrac{-A_1-\sigma_1^2}{\sigma_1} \left| \dfrac{\rho}{\sqrt{2(1-\rho^2)}} \right., \infty\right] \end{array}\right\}, \tag{3-14a}$$

which reduces $\left(\text{for } |\rho| < 1/\sqrt{2}\right)$ (see footnote 7) to

$$\boxed{p_D^{(1)*} = \frac{1}{4}\left\{\begin{array}{l}\left[1+\Theta\left(\frac{\sigma_1^2+A_1}{\sqrt{2}\sigma_1}\right)\right]\left[1+\Theta\left(\frac{\sigma_1^2-\sigma_2^2+A_1-A_2}{\sqrt{2}\sigma_2\sqrt{1-\rho^2}}\right)\right] \\ +\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left[\frac{\sqrt{2}\left(\frac{\sigma_1}{\sigma_2}-\rho\right)}{\sqrt{1-\rho^2}}\right]^n \frac{\Gamma\left(\frac{n+1}{2}\right)}{n!}\phi^{(n-1)}\left(\frac{A_1-A_2+\sigma_1^2-\sigma_2^2}{\sigma_2\sqrt{1-\rho^2}}\right) \\ \bullet\left[1-I_c\left[\left(\frac{A_1+\sigma_1^2}{\sqrt{2}\sigma_1}\right)^2, \frac{n+1}{2}\right]\right]\end{array}\right\}} \quad (3\text{-}14b)$$

Note that for $\sigma_1 \to \infty$, $(\sigma_2 < \infty)$, $\left(A_1+\sigma_1^2\right)/\sigma_1 \to \sigma_1/2 \to \infty$, etc., $I_c \to 1$, and $p_D^{(1)*} \to 1$, as required: S_1 is surely detected (conditionally on being present). In this limit $(\sigma_1 \to \infty)$, equation (3-9b) becomes also $p_D^{(1)*} = \int\int_{-\infty}^{\infty} p_2^{(1)}(z)\, dz = 1$, as a further check. Since $p_D^{(k)*}$ is symmetrical, e.g., $1 \to 2$, $2 \to 1$ in equation (3-9b), it can be written explicitly $\left(|\rho| < 1/\sqrt{2}\right)$ (see footnote 7):

$$p_D^{(2)*} = \frac{1}{4}\left\{\begin{array}{l}\left[1+\Theta\left(\frac{\sigma_2^2+A_2}{\sqrt{2}\sigma_2}\right)\right]\left[1+\Theta\left(\frac{\sigma_2^2-\sigma_1^2+A_2-A_1}{\sqrt{2}\sigma_1\sqrt{1-\rho^2}}\right)\right] \\ +\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left[\frac{\sqrt{2}\left(\frac{\sigma_2}{\sigma_1}-\rho\right)}{\sqrt{1-\rho^2}}\right]^n \frac{\Gamma\left(\frac{n+1}{2}\right)}{n!}\phi^{(n-1)}\left(\frac{A_2-A_1+\sigma_2^2-\sigma_1^2}{\sigma_1\sqrt{1-\rho^2}}\right) \\ \bullet\left[1-I_c\left[\left(\frac{A_2+\sigma_2^2}{\sqrt{2}\sigma_2}\right)^2, \frac{n+1}{2}\right]\right]\end{array}\right\}. \quad (3\text{-}15)$$

Again, as expected, $p_D^{(2)*} \to 1$ as $\sigma_2 \to \infty$.

3.2.1.1 Special Cases. When $\rho = 0$, then $p_D^{(k)*}$, $k = 1,2$, does not reduce to the symmetrical form of equation (3-12) for $p_D^{(0)*}$, because of the asymmetries in signal level: $\sigma_1 \neq \sigma_2$. Furthermore, note that for σ_1, $\sigma_2 \to 0$, since $\phi^{(m)}(\pm\infty) = 0$, $m \geq 0$:

$$p_D^{(k)*} = \lim_{\sigma\to 0} \frac{1}{4}\left\{\left[1+\Theta\left(\frac{\log\mu_k}{\sqrt{2}\sigma}\right)\right]\left[1+\Theta\left(\frac{\log\left(\frac{p_k}{p_j}\right)}{\sqrt{2}\sigma}\right)\right]\right\}, \; j,k = 1,2 \; ; \; j \neq k . \tag{3-16a}$$

Several cases can be distinguished:

$$\left.\begin{array}{lll}
\mu_{1,2} > 1 \; ; & \frac{p_{1,2}}{p_{2,1}} > 1 : & p_D^{(k)*} = 1 \; ; \\
 & \frac{p_{1,2}}{p_{2,1}} < 1 : & = 0 \; ; \\
\mu_{1,2} < 1 \; ; & \frac{p_{1,2}}{p_{2,1}} <=> 1 : & = 0 \; ; \\
\mu_1 = \mu_2 : p_1 = p_2 \; ; & p_{1,2} > q : & = \frac{1}{2} \; ; \\
 & p_{1,2} < q : & = 0 \; ; \\
p_1 = p_2 = q = \frac{1}{3} \; ; & \mu_1 = \mu_2 = 1 : & = \frac{1}{4}
\end{array}\right\} ; \; k = 1,2 \; . \tag{3-16b}$$

For the noise-alone case in equation (3-11b), since $\phi^{(m)}(\pm\infty) = 0$, $m \geq 0$:

$$p_D^{(0)} = \lim_{\sigma_{1,2}\to 0} \frac{1}{4}\left\{\left[1+\Theta\left(\frac{-\log\mu_1}{\sqrt{2}\sigma_1}\right)\right]\left[1+\Theta\left(\frac{-\log\mu_2}{\sqrt{2}\sigma_2}\right)\right]\right\}, \tag{3-17a}$$

and therefore

$$
\left.\begin{array}{lrl}
\mu_{1,2} > 1: & p_D^{(0)*} & = 0 ; \\
\mu_{1,2} < 1: & & = 1 ; \\
\mu_{1,2} > 1 \,,\; \mu_{2,1} < 1: & & = 0 ; \\
\mu_1 = \mu_2 > 1: & & = 0 ; \\
\mu_1 = \mu_2 < 1: & & = 1 ; \\
\mu_1 = \mu_2 = 1 \,,\; \therefore\; p_{1,2} = q = \frac{1}{3}: & & = \frac{1}{4} ; \\
\mu_{1,2} < 1 \,;\; \mu_{1,2} = 1: & & = \frac{1}{2}
\end{array}\right\} . \tag{3-17b}
$$

3.2.2 Probabilities of Error

The error probabilities of equation (3-8) can be calculated in similar fashion with the help of equation (A-6c). The general results for equation (3-8a) are

$$
\alpha_1^{(0)*} = \frac{1}{2\sqrt{2\pi}} \left\{
\begin{array}{l}
J\left[\infty \left| \dfrac{\dfrac{\sigma_1}{\sigma_2} - \rho}{\sqrt{2\left(1-\rho^2\right)}} \,,\, \dfrac{A_1 - A_2}{\sigma_2\sqrt{2\left(1-\rho^2\right)}} \right.\right] \\
-J\left[\dfrac{-A_1}{\sigma_1} \left| \dfrac{\dfrac{\sigma_1}{\sigma_2} - \rho}{\sqrt{2\left(1-\rho^2\right)}} \,,\, \dfrac{A_1 - A_2}{\sigma_2\sqrt{2\left(1-\rho^2\right)}} \right.\right] \\
+J\left[\infty \left| \dfrac{\rho}{\sqrt{2\left(1-\rho^2\right)}} \,,\, \infty \right.\right] - J\left[\dfrac{-A_1}{\sigma_1} \left| \dfrac{-\rho}{\sqrt{2\left(1-\rho^2\right)}} \,,\, \infty \right.\right]
\end{array}
\right\} \equiv \beta_1^{(0)*} . \tag{3-18a}
$$

The series form $\left(\text{for } |\rho| < 1/\sqrt{2}\right)$ of equation (3-18a) is

$$\therefore \alpha_1^{(0)*} = \frac{1}{4}\left\{\begin{array}{l} \left[1-\Theta\left(\frac{A_1}{\sqrt{2}\sigma_1}\right)\right]\left[1+\Theta\left(\frac{A_1-A_2}{\sigma_2\sqrt{2(1-\rho^2)}}\right)\right] \\ +\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{\sqrt{2}\left(\frac{\sigma_1}{\sigma_2}-\rho\right)}{\sqrt{1-\rho^2}}\right]^n \Gamma\left(\frac{n+1}{2}\right)\phi^{(n-1)}\left(\frac{A_1-A_2}{\sigma_2\sqrt{1-\rho^2}}\right) \\ \bullet\left[1-I_c\left[\left(\frac{A_1^2}{2\sigma_1^2}+\frac{n+1}{2}\right)\right]\right] \end{array}\right\}, \tag{3-18b}$$

$|\rho| < 1/\sqrt{2}$ (see footnote 7) .

Note that

$$\alpha_1^{(0)*}\Big|_{\substack{\sigma_1\to\infty \\ \sigma_2<\infty}} = 0 \text{ , while } \alpha_2^{(0)*}\Big|_{\substack{\sigma_2\to\infty \\ \sigma_1<\infty}} = \frac{1}{2}\left[1-\Theta\left(\frac{-\log\mu_1+\frac{\sigma_1^2}{2}}{\sqrt{2}\sigma_1}\right)\right]. \tag{3-18c}$$

For $\alpha_2^{(0)*}\left(\equiv \beta_2^{(0)*}\right)$, the subscripts $1 \to 2$ and $2 \to 1$ are exchanged in equations (3-18a) and (3-18b).

Next, for equation (3-8b),

$$\left\langle\beta_1^{(0)*}\right\rangle = \frac{1}{2\sqrt{2\pi}}\left\{\begin{array}{l} J\left[\frac{-A_1-\sigma_1^2}{\sigma_1}\middle|\frac{-\rho}{\sqrt{2(1-\rho^2)}}, \frac{-A_2-\sigma_2^2}{\sigma_2\sqrt{2(1-\rho^2)}}\right] \\ -J\left[-\infty\middle|\frac{-\rho}{\sqrt{2(1-\rho^2)}}, \frac{-A_2-\sigma_2^2}{\sigma_2\sqrt{2(1-\rho^2)}}\right] \\ +J\left[\frac{-A_1-\sigma_1^2}{\sigma_1}\middle|\frac{\rho}{\sqrt{2(1-\rho^2)}}, \infty\right] - J\left[-\infty\middle|\frac{\rho}{\sqrt{2(1-\rho^2)}}, \infty\right] \end{array}\right\} \equiv \beta_0^{(2)*}, \tag{3-19a}$$

and

$$
\therefore \left\langle \beta_0^{(1)*} \right\rangle = \frac{1}{4} \left\{ \begin{array}{l} \left[1 + \Theta\left(\dfrac{-A_2 - \sigma_2^2}{\sigma_2 \sqrt{2(1-\rho^2)}} \right) \right] \left[1 + \Theta\left(\dfrac{-A_1 - \sigma_1^2}{\sigma_1 \sqrt{2(1-\rho^2)}} \right) \right] \\ + \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n!} \left(\dfrac{\sqrt{2}\rho}{\sqrt{1-\rho^2}} \right)^n \Gamma\left(\dfrac{n+1}{2} \right) \phi^{(n-1)}\left(\dfrac{-A_2 - \sigma_2^2}{\sigma_2 \sqrt{1-\rho^2}} \right) \\ \bullet \left[I_c \left[\left(\dfrac{(A_1 + \sigma_1^2)^2}{2\sigma_1^2} + \dfrac{n+1}{2} \right) - 1 \right] \right] \end{array} \right\} . \qquad (3\text{-}19b)
$$

$|\rho| < 1/\sqrt{2}$ (see footnote 7) .

Since $p_2^{(1)} = p_2^{(2)} = p_2(z)$ (see equation (3-1)), here $\left\langle \beta_0^{(2)*} \right\rangle = \left\langle \beta_0^{(1)*} \right\rangle$ also (from equations (3-8b) and (3-8c))[8].

Similarly, for $\left\langle \beta_2^{(1)*} \right\rangle$ and $\left\langle \beta_1^{(2)*} \right\rangle$

$$
\left\langle \beta_2^{(1)*} \right\rangle = \frac{1}{2\sqrt{2\pi}} \left\{ \begin{array}{l} J\left[\infty \left| \dfrac{\frac{\sigma_2}{\sigma_1} - \rho}{\sqrt{2(1-\rho^2)}} \right. , \dfrac{A_2 - A_1 + \sigma_2^2 - \sigma_1^2}{\sigma_1 \sqrt{2(1-\rho^2)}} \right] \\ -J\left[\dfrac{-A_2}{\sigma_2} \left| \dfrac{\frac{\sigma_2}{\sigma_1} - \rho}{\sqrt{2(1-\rho^2)}} \right. , \dfrac{A_2 - A_1 + \sigma_2^2 - \sigma_1^2}{\sigma_1 \sqrt{2(1-\rho^2)}} \right] \\ +J\left[\infty \left| \dfrac{\rho}{\sqrt{2(1-\rho^2)}} \right. , \infty \right] - J\left[\dfrac{-A_2}{\sigma_2} \left| \dfrac{-\rho}{\sqrt{2(1-\rho^2)}} \right. , \infty \right] \end{array} \right\} , \qquad (3\text{-}20a)
$$

---

[8] In fact, the equality $\left\langle \beta_0^{(1)*} \right\rangle = \left\langle \beta_0^{(2)*} \right\rangle$ here (see equation (3-8b), with $p_2^{(k)}(z) = p_2(z)$, each k, allows the interesting equivalence:

$$
\sum_{n=1}^{\infty} \frac{1}{n!} a^n \Gamma\left(\frac{n+1}{2} \right) \phi^{(n-1)}\left(\frac{X_1 \sqrt{2}\rho}{a} \right) \left[I_o\left(\frac{X_2^2}{2} , \frac{n+1}{2} \right) - 1 \right] =
$$

$$
\sum_{n=1}^{\infty} \frac{1}{n!} a^n \Gamma\left(\frac{n+1}{2} \right) \phi^{(n-1)}\left(\frac{X_2 \sqrt{2}\rho}{a} \right) \left[I_o\left(\frac{X_1^2}{2} , \frac{n+1}{2} \right) - 1 \right] .
$$

and

$$
\therefore \left\langle \beta_2^{(1)*} \right\rangle = \frac{1}{4}\left\{
\begin{aligned}
&\left[1-\Theta\left(\frac{-A_2}{\sqrt{2}\sigma_2}\right)\right]\left[1+\Theta\left(\frac{A_2-A_1+\sigma_2^2-\sigma_1^2}{\sqrt{2}\sigma_1\sqrt{2\left(1-\rho^2\right)}}\right)\right] \\
&+\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{\sqrt{2}\left(\frac{\sigma_2}{\sigma_1}-\rho\right)}{\sqrt{1-\rho^2}}\right]^n \Gamma\left(\frac{n+1}{2}\right)\phi^{(n-1)}\left(\frac{A_2-A_1+\sigma_2^2-\sigma_1^2}{\sigma_1\sqrt{1-\rho^2}}\right) \\
&\bullet\left[1-I_c\left(\frac{A_2^2}{2\sigma_2^2}+\frac{n+1}{2}\right)\right]
\end{aligned}
\right\}, \qquad \text{(3-20b)}
$$

$|\rho| < 1/\sqrt{2}$ (see footnote 7) .

For $\left\langle \beta_1^{(2)*} \right\rangle$ *, the subscripts* $1 \to 2$ *and* $2 \to 1$ *in equation (3-20b) can be interchanged (compare equations (3-8b) vs (3-8c).* Observe that as σ_1 or $\sigma_2 \to \infty$, these (conditional) error probabilities (if signal 1 or 2 is actually present) all vanish as expected from physical considerations (see section 4).

3.2.2.1 Special Cases. In addition, as σ_1 or $\sigma_2 \to 0$, (see equations (3-16) and (3-17)), there are a variety of special results, depending on the *a priori* probabilities q , p_1 , and p_2 . These are determined from equations (3-18) through (3-20), which become, when $(\sigma = \sigma_1 \, , \, \sigma_2) \to 0$:

$$
\left.
\begin{aligned}
\alpha_1^{(0)*} &\Rightarrow \lim_{\sigma\to 0}\frac{1}{4}\left\{\left[1+\Theta\left(\frac{\log\mu_1}{\sqrt{2}\sigma}\right)\right]\left[1+\Theta\left(\frac{\log\left(\frac{\mu_1}{\mu_2}\right)}{\sqrt{2}\sigma}\right)\right]\right\} = \beta_1^{(0)*} && \text{(3-21a)} \\
\alpha_2^{(0)*} &\Rightarrow \lim_{\sigma\to 0}\frac{1}{4}\left\{\left[1+\Theta\left(\frac{\log\mu_2}{\sqrt{2}\sigma}\right)\right]\left[1+\Theta\left(\frac{\log\left(\frac{\mu_2}{\mu_1}\right)}{\sqrt{2}\sigma}\right)\right]\right\} = \beta_2^{(0)*} && \text{(3-21b)}
\end{aligned}
\right\} \sigma_1 \, , \, \sigma_2 = \sigma \to 0 \, .
$$

As before (see equation (3-16b), several cases can be distinguished:

$$\left.\begin{array}{llll}
\mu_{1,2} > 1\,; & \dfrac{p_{1,2}}{p_{2,1}} > 1: & \alpha_1^{(0)*} & = 1\,; \\
 & \dfrac{p_{1,2}}{p_{2,1}} < 1: & & = 0\,; \\
\mu_{1,2} < 1\,; & \dfrac{p_{1,2}}{p_{2,1}} <=> 1: & & = 0\,; \\
\mu_1 = \mu_2\,, \therefore p_1 = p_2\,; & p_{1,2} > q: & & = \dfrac{1}{2}\,; \\
 & p_{1,2} < q: & & = 1\,; \\
p_1 = p_2 = q = \dfrac{1}{3}\,,\ \therefore & \mu_1 = \mu_2 = 1: & & = \dfrac{1}{4}
\end{array}\right\} \sigma \to 0\,. \qquad (3\text{-}21c)$$

(Note that, in fact, equations (3-21a) and (3-21b) are the same as equations (3-16a) and (3-16b).)

In a similar way, when $\sigma_1\,,\ \sigma_2 \to 0$, then equation (3-19b) reduces to

$$\left\langle \beta_0^{(1)*} \right\rangle \Rightarrow \lim_{\sigma \to 0} \frac{1}{4}\left\{\left[1 + \Theta\left(\frac{-\log\mu_2}{\sqrt{2}\sigma}\right)\right]\left[1 + \Theta\left(\frac{-\log\mu_1}{\sqrt{2}\sigma}\right)\right]\right\} \Leftarrow \left\langle \beta_0^{(2)*} \right\rangle . \qquad (3\text{-}22a)$$

The various cases here are therefore

$$\left.\begin{array}{lll}
\mu_{1,2} > 1: & \left\langle \beta_0^{(1)*} \right\rangle = \left\langle \beta_0^{(2)*} \right\rangle & = 0\,; \\
\mu_{1,2} < 1: & & = 1\,; \\
\mu_1 > 1\,,\ \mu < 1: & & = 0\,; \\
\left.\begin{array}{l} \mu_1 > 1\,,\ \mu_2 = 1 \\ \mu_2 > 1\,,\ \mu_1 = 0 \end{array}\right\}: & & \left.\begin{array}{l} = 0 \\ = 0 \end{array}\right\}; \\
\left.\begin{array}{l} \mu_1 = \mu_2 > 1 \\ \mu_1 = \mu_2 < 1 \end{array}\right\}: & & \left.\begin{array}{l} = 0 \\ = 1 \end{array}\right\}; \\
p_1 = p_2 = q\,,\ \therefore \mu_1 = \mu_2 = 1: & & = \dfrac{1}{4}\,; \\
\left.\begin{array}{l} \mu_1 = 1\,;\ \mu_2 < 1\,,\ \therefore p_1 = q \\ \mu_2 = 1\,;\ \mu_1 < 1\,,\ \therefore p_2 = q \end{array}\right\}: & & \left.= \dfrac{1}{2}\right\}
\end{array}\right\} \sigma \to 0\,. \qquad (3\text{-}22b)$$

Finally, for $\langle\beta_2^{(1)*}\rangle$ and $\langle\beta_1^{(2)*}\rangle$, from equation (3-20b), $(\sigma_{1,2} \to 0)$:

$$\langle\beta_2^{(1)*}\rangle \Rightarrow \lim_{\sigma \to 0} \frac{1}{4}\left\{\left[1+\Theta\left(\frac{\log\mu_2}{\sqrt{2}\sigma}\right)\right]\left[1+\Theta\left(\frac{\log\left(\frac{\mu_2}{\mu_1}\right)}{\sqrt{2}\sigma}\right)\right]\right\}, \qquad \text{(3-23a)}$$

and

$$\langle\beta_1^{(2)*}\rangle \Rightarrow \lim_{\sigma \to 0} \frac{1}{4}\left\{\left[1+\Theta\left(\frac{\log\mu_1}{\sqrt{2}\sigma}\right)\right]\left[1+\Theta\left(\frac{\log\left(\frac{\mu_1}{\mu_2}\right)}{\sqrt{2}\sigma}\right)\right]\right\}. \qquad \text{(3-23b)}$$

Again, note that since equations (3-23a) and (3-23b) are the same as equations (3-21b) and (3-21a), respectively, the various limiting cases of equation (3-21c) apply directly here as well.

4. ADDITIONAL SPECIAL CASES

Further special cases of interest arise when one of the (processed) signals is indefinitely large, e.g., when $\sigma_1 \to \infty$ and $\sigma_2 < \infty$, or $\sigma_2 < \infty$. See equations (3-11b), (3-14b), and (3-15) for the $p_D^{(\,)*}$s and equations (3-18b), (3-19b) and (3-20b) for the following results.

4.1 $(\sigma_{1,2} \to \infty\,;\ \sigma_{2,1} < \infty)$

For the probabilities of correct detection,

$$\left\{\begin{array}{ll}
p_D^{(0)*}\Big|_{\sigma_1\to\infty} = \dfrac{1}{2}\left[1+\Theta\left(\dfrac{-A_2}{\sqrt{2}\sigma_2}\right)\right] = \left(1-\alpha_1^{(0)*}-\alpha_2^{(0)*}\right)_{\sigma_1\to\infty},\ \sigma_2<\infty\,; & (4-1a)\\
p_D^{(0)*}\Big|_{\sigma_2\to\infty} = \dfrac{1}{2}\left[1+\Theta\left(\dfrac{-A_1}{\sqrt{2}\sigma_1}\right)\right] = \left(1-\alpha_1^{(0)*}-\alpha_2^{(0)*}\right)_{\sigma_2\to\infty},\ \sigma_1<\infty\,; & (4-1b)\\
p_D^{(1)*}\Big|_{\sigma_1\to\infty} = 1,\ \sigma_2<\infty\,;\ p_D^{(1)*}\Big|_{\sigma_2\to\infty} = 0,\ \sigma_1<\infty\,; & (4-2a)\\
p_D^{(2)*}\Big|_{\sigma_1\to\infty} = 0,\ \sigma_2<\infty\,;\ p_D^{(2)*}\Big|_{\sigma_2\to\infty} = 1,\ \sigma_1<\infty & (4-2b)
\end{array}\right\}.$$

Because of the signal level asymmetries $\sigma_1 \neq \sigma_2$, $\sigma_1/\sigma_2 \to \infty$, and $\rho \neq \pm 1$, $p_D^{(0)*}$ may be expected to be nonvanishing, even when one of the signals is effectively infinite: there is always a nonzero, non-unity probability of falsely deciding signal #1 or #2, as is evident from equations (4-1a) and (4-1b). On the other hand, as in equations (4-2a) and (4-2b), $p_D^{(k)*}\Big|_{\sigma_k\to\infty} = 1$, $p_D^{(k)*}\Big|_{\sigma_j\to\infty} = 0$, $j \neq k$, and $\sigma_k < \infty$, as expected; very (∞) strong signals are correctly decided, probability 1, with the alternative $p_D^{(j)} = 0$, by exclusion.

In a similar way, for the error probabilities:

$$\left.\begin{aligned}\alpha_1^{(0)*}\Big|_{\substack{\sigma_1\to\infty\\ \sigma_2<\infty}} = 0\ ;\ \alpha_1^{(0)*}\Big|_{\substack{\sigma_2\to\infty\\ \sigma_1<\infty}} = \frac{1}{2}\left[1-\theta\frac{-A_1}{\sqrt{2}\sigma_1}\right];\quad &(4\text{-}3a)\\ \alpha_2^{(0)*}\Big|_{\substack{\sigma_2\to\infty\\ \sigma_1<\infty}} = 0\ ;\ \alpha_2^{(0)*}\Big|_{\substack{\sigma_1\to\infty\\ \sigma_2<\infty}} = \frac{1}{2}\left[1-\theta\frac{-A_2}{\sqrt{2}\sigma_2}\right]\quad &(4\text{-}3b)\end{aligned}\right\};$$

and, from equation (3-19b),

$$\left\langle\beta_0^{(1)*}\right\rangle\Big|_{\substack{\sigma_1\to\infty\\ \sigma_2<\infty}} = 0 = \left\langle\beta_0^{(2)*}\right\rangle\ ;\ \left\langle\beta_0^{(1)*}\right\rangle\Big|_{\substack{\sigma_2\to\infty\\ \sigma_1<\infty}} = 0\ ; \tag{4-4}$$

and, from equation (3-20b),

$$\left\langle\beta_2^{(1)*}\right\rangle\Big|_{\substack{\sigma_1\to\infty\\ \sigma_2<\infty}} = 0\ ;\ \left\langle\beta_2^{(1)*}\right\rangle\Big|_{\substack{\sigma_2\to\infty\\ \sigma_1<\infty}} = 0\ ; \tag{4-5a}$$

and

$$\left\langle\beta_1^{(2)*}\right\rangle\Big|_{\substack{\sigma_1\to\infty\\ \sigma_2<\infty}} = 0\ ;\ \left\langle\beta_1^{(2)*}\right\rangle\Big|_{\substack{\sigma_2\to\infty\\ \sigma_1<\infty}} = 0\ . \tag{4-5b}$$

Equations (4-4) and (4-5) combined in equation (2-10) yield equations (4-2a) and (4-2b), as required. Thus, whenever the true state is "noise alone" H_0: N, as in equations (4-3a) and (4-3b), and the non-infinite signal is decided, the associated decision error probability is nonvanishing. Whenever the true state is H_1: $S_1 + N$ or H_2: $S_2 + N$, and either signal is infinite, the error probabilities vanish. Both effects are expected: in the former case, there is always a nonzero probability of incorrect decisions because one of the signals is finite; in the latter case, the infinite signal ensures a correct decision or "signal," and hence zero probability of the alternative decisions.

4.2 THE CASES $\rho = \pm 1$

Here, as in equation (3-12), $w_2(\mathbf{z}) = w_1(z_1)\delta(z_2 \mp z_1) = w_1(z_2)\delta(z_1 \mp z2)$, so that evaluating equation (3-9) directly gives (the earlier result of equation (3-13)):

$$p_D^{(0)*}\Big|_{\rho=\pm1} = \frac{1}{2}\left[1+\Theta(B_{min})\right] \,;\; B_{min} \equiv \min\left(\frac{-A_1}{\sqrt{2}\sigma_1} \,,\; \frac{-A_2}{\sqrt{2}\sigma_2}\right) ; \tag{4-6a}$$

$$= \frac{1}{2}\left[1+\Theta\left(\frac{\sigma}{2\sqrt{2}}\right)\right]_{\rho=1} \;;\; = \Theta\left(\frac{\sigma}{2\sqrt{2}}\right)_{\rho=-1} \,,$$

along with

$$\left.\begin{aligned} p_D^{(1)*}\Big|_{\rho=\pm1} &= \Theta\left[\frac{A_1-A_2+\sigma_1^2-\sigma_2^2}{\sqrt{2}(\sigma_2-\sigma_1)}\right]-\Theta\left(\frac{-A_1-\sigma_1^2}{\sqrt{2}\sigma_1}\right);\; && \frac{-A_1-\sigma_1^2}{\sigma_1}<\frac{A_1-A_2+\sigma_1^2-\sigma_2^2}{\sigma_2-\sigma_1}\,, \qquad (4-6b) \\ &= 0\,, && \frac{-A_1-\sigma_1^2}{\sigma_1}>\frac{A_1-A_2+\sigma_1^2-\sigma_2^2}{\sigma_2-\sigma_1} \qquad (4-6c) \end{aligned}\right\};$$

and

$$\left.\begin{aligned} p_D^{(2)*}\Big|_{\rho=\pm1} &= \Theta\left[\frac{A_2-A_1+\sigma_2^2-\sigma_1^2}{\sqrt{2}(\sigma_1-\sigma_2)}\right]-\Theta\left(\frac{-A_2-\sigma_1^2}{\sqrt{2}\sigma_2}\right);\; && \frac{-A_2-\sigma_2^2}{\sigma_2}<\frac{A_2-A_1+\sigma_2^2-\sigma_1^2}{\sigma_1-\sigma_2}\,, \\ &= 0\,, && \frac{-A_2-\sigma_2^2}{\sigma_2}>\frac{A_2-A_1+\sigma_2^2-\sigma_1^2}{\sigma_1-\sigma_2} \end{aligned}\right\}. \tag{4-6d}$$

In a similar fashion, from equation (3-8):

$$\left.\begin{aligned} \alpha_1^{(0)*}\left(=\beta_1^{(0)*}\right)\Big|_{\rho=\pm1} &= \Theta\left[\frac{A_1-A_2}{\sqrt{2}(\sigma_2-\sigma_1)}\right]-\Theta\left(\frac{-A_1}{\sqrt{2}\sigma_1}\right);\; && \frac{A_1-A_2}{\sigma_2-\sigma_1}>\frac{-A_1}{\sigma_1}\;; \\ &= 0\,, && \frac{A_1-A_2}{\sigma_2-\sigma_1}<\frac{-A_1}{\sigma_1} \end{aligned}\right\}; \tag{4-7a}$$

$$\left.\begin{aligned} \alpha_2^{(0)*}\Big|_{\rho=\pm1} &= \Theta\left[\frac{A_2-A_1}{\sqrt{2}(\sigma_1-\sigma_2)}\right]-\Theta\left(\frac{-A_2}{\sqrt{2}\sigma_2}\right);\; && \frac{A_2-A_1}{\sigma_1-\sigma_2}>\frac{-A_2}{\sigma_2}\;; \\ &= 0\,, && \frac{A_2-A_1}{\sigma_1-\sigma_2}<\frac{-A_2}{\sigma_2} \end{aligned}\right\}; \tag{4-7b}$$

$$\beta_0^{(1)*}\left(=\beta_0^{(2)*}\right)\Big|_{\rho=\pm1} = \frac{1}{2}\left[1+\Theta(C_{min})\right] \,;\; C_{min} \equiv \left(\frac{-A_1-\sigma_1^2}{\sigma_1} \,,\; \frac{-A_2-\sigma_2^2}{\sigma_2}\right); \tag{4-7c}$$

$$\left\langle \beta_2^{(1)*} \right\rangle\Big|_{\rho=\pm 1} = \Theta\left[\frac{A_2 - A_1 + \sigma_2^2 - \sigma_1^2}{\sqrt{2}(\sigma_1 - \sigma_2)}\right] - \Theta\left(\frac{-A_2}{\sqrt{2}\sigma_2}\right);\quad \frac{A_2 - A_1 + \sigma_2^2 - \sigma_1^2}{(\sigma_1 - \sigma_2)} > \frac{-A_2}{\sigma_2}\,;$$
$$= 0\,,\qquad \frac{A_2 - A_1 + \sigma_2^2 - \sigma_1^2}{(\sigma_1 - \sigma_2)} < \frac{-A_2}{\sigma_2}\quad ;\text{(4-7d)}$$

and

$$\left\langle \beta_1^{(2)*} \right\rangle\Big|_{\rho=\pm 1} = \Theta\left[\frac{A_1 - A_2 + \sigma_1^2 - \sigma_2^2}{\sqrt{2}(\sigma_2 - \sigma_1)}\right] - \Theta\left(\frac{-A_1}{\sqrt{2}\sigma_1}\right);\quad \frac{A_1 - A_2 + \sigma_1^2 - \sigma_2^2}{(\sigma_2 - \sigma_1)} > \frac{-A_1}{\sigma_1}\,;$$
$$= 0\,,\qquad \frac{A_1 - A_2 + \sigma_1^2 - \sigma_2^2}{(\sigma_2 - \sigma_1)} < \frac{-A_1}{\sigma_1}\quad .\text{(4-7e)}$$

4.3 THE CASE $\rho = 0$

When $\rho = 0$, the situation of orthogonal signals exists in the coherent cases: $\bar{\theta}_n^{(1)} = \sqrt{2}\bar{a}_o \cos\, \omega_o(t_n - \varepsilon_o)$; $\bar{\theta}_n^{(2)} = \sqrt{2}\bar{a}_o \sin\, \omega_o(t_n - \varepsilon_o)$; $\varepsilon_o = (\pi/2\omega_o\ ,\ 3\pi/2\omega_o)$ or $(0, \pi/\omega_o)$ for (maximum) amplitudes for proper choice of epoch ε_o at the receiver, e.g., $\bar{\theta}_n^{(1)}\bar{\theta}_n^{(2)} = \rho_{coh} = 0$. With *incoherent detection* $\left(\bar{\theta}^{1,2} = 0\right)$, $\left\langle \theta_n^{(1)}\theta_{n'}^{(1)} \right\rangle \cong \delta_{nn'}$, etc., for $\rho = 0$. In any case, when $\rho = 0$, the general results in equations (3-11b), (3-14b), and (3-15) are directly modified on setting $\rho \to 0$ therein, as in equation (3-12). Equations (3-14) and (3-15) do not reduce further, unless $\sigma_1 = \sigma_2$, in which case

$$p_D^{(1)*}\Big|_{\substack{\rho=0 \\ \sigma_1=\sigma_2=\sigma}} = \frac{1}{4}\left[1 + \Theta\left(\frac{\sigma^2 + A_1}{\sqrt{2}\sigma}\right)\right]\left[1 + \Theta\left(\frac{\log\left(\frac{\mu_1}{\mu_2}\right)}{\sqrt{2}\sigma}\right)\right], \tag{4-8a}$$

and

$$p_D^{(2)*}\Big|_{\substack{\rho=0 \\ \sigma_1=\sigma_2=\sigma}} = \frac{1}{4}\left[1 + \Theta\left(\frac{\sigma^2 + A_2}{\sqrt{2}\sigma}\right)\right]\left[1 + \Theta\left(\frac{\log\left(\frac{\mu_2}{\mu_1}\right)}{\sqrt{2}\sigma}\right)\right], \tag{4-8b}$$

with $p_D^{(0)*}$ here given by equation (3-12).

4.4 REDUCTION TO THE BINARY CASES: H_1 vs H_2 AND H_1 vs H_0

The more general results (in section 3.2) readily reduce to the binary cases upon dropping the inappropriate error probabilities and regarding q as $q \to 0$, e.g., $\log(\mu_1, \mu_2) \to \infty$ with $p_1 + p_2 = 1$, etc., now. Thus, for *binary signals*:

H_1: $S_1 + N$ vs H_2: $S_2 + N$:

$$p_D^{(1)*} = \frac{1}{2}\left[1 + \Theta\left(\frac{\frac{\sigma_1^2 - \sigma_2^2}{2} + \log \mu_{12}}{\sqrt{2}\sigma_2\sqrt{1-\rho^2}}\right)\right]; \log \mu_{12} = \log\left(\frac{p_1}{p_2}\right); \tag{4-9a}$$

and

$$p_D^{(2)*} = \frac{1}{2}\left[1 + \Theta\left(\frac{\frac{\sigma_2^2 - \sigma_1^2}{2} + \log \mu_{21}}{\sqrt{2}\sigma_1\sqrt{1-\rho^2}}\right)\right]; \log \mu_{21} = \log\left(\frac{p_2}{p_1}\right). \tag{4-9b}$$

Equations (4-9a) and (4-9b) are alternate forms to the more familiar results (see section 20.4-5, problem 20.12 of reference 2), which latter are given in terms of $\log(\Lambda_2/\Lambda_1) = \log \Lambda_{12}$ rather than $\log \Lambda_2$ vs $\log\Lambda_1$, as is done here.

Also in the "on-off" cases H_0: N vs H_1: $S_1 + N$, now $\rho = 0$, $\sigma_2 \to 0$, so that equations (3-18b) and (3-14a) reduce to

$$\alpha_1^{(0)*} = \frac{1}{2}\left\{1 - \Theta\left(\frac{\sigma_1}{2\sqrt{2}} - \frac{\log \mu_1}{\sqrt{2}\sigma_1}\right)\right\} \ (\equiv \alpha^*), \ (K = 1 \text{ here}); \tag{4-10a}$$

and

$$p_D^{(1)*} = \frac{1}{2}\left\{1 + \Theta\left(\frac{\sigma_1}{2\sqrt{2}} + \frac{\log \mu_1}{\sqrt{2}\sigma_1}\right)\right\} = \frac{1}{2}\left\{1 + \Theta\left[\frac{\sigma_1}{\sqrt{2}} - \Theta^{-1}\left(1 - 2\alpha_1^{(0)*}\right)\right]\right\}, \tag{4-10b}$$

which are well-known results (K=1 here) (see equations (6.6) and (6.7a) in reference 12, for example).

5. CONCLUSIONS

The principal new features of this preliminary investigation are the explicit development of *threshold* Ternary Detection Theory for the two-signal cases: H_0: N vs H_1: $S_1 + N$ vs H_2: $S_2 + N$, when the noise is (additive) non-Gaussian and when the signals are themselves unrestricted. In short, this is a canonical theory, invariant in the form of the analytic results. Both threshold algorithms and detection performance are derived. Special cases of interest are then readily obtained by specializing these general results, as sections 3 and 4 attest.

However, the treatment is not completely general in that

1. Threshold signals, not signals at all levels of intensity, are considered;

2. A restricted, though important, class of cost assignments, which determines the decision regions, is employed (see section 2.1);

3. Independent noise samples are assumed, in both space and time;

4. The analysis is confined to the two-signal cases (M = 2), with "noise alone" as the third alternative (see section 2);

5. The noise is ambient and thus signal-independent; and

6. For numerical results, series valid only for $|\rho| < 1/\sqrt{2}$ are obtained here. Reference 13 contains the complete range for $-1 \geq \rho \leq 1$.

Under these constraints only three parameters (other than the *a priori* probabilities, q, p_1, and p_2) are required for a complete analysis: the *signal detection parameters* σ_1^2, σ_2^2, from equations (3-2) and (3-3), and the *correlation coefficient* $\rho_{1\,2}(=\rho)$ relating to the two detection algorithms, Z_1 and Z_2, from equation (2-16), as shown by the results of section 3.

Restriction 3 can be removed, as a result of recent work (references 9 and 10). The extension to the full ternary case H_1: $S_1 + N$ vs H_2: $S_2 + N$ vs H_3: $S_3 + N$, item 4 under restriction 2 above, appears analytically possible though tedious. Restriction 5 can also be lifted, as noted in reference 11 and references therein. On the other hand, analytical results for general signal levels, and/or general cost assignments are not achievable, and computational procedures must then be invoked for specific cases. Accordingly, next steps in the development of the ternary detection theory here are:

1. Extension of $(M = 2)$ threshold cases to include noise correlation (see references 9 and 10);

2. Numerical evaluation of the general results for $(M = 2)$ obtained in section 3, for both the "correct" detection and error probabilities, for a usefully broad spectrum of values for the three parameters involved, viz., σ_1^2, σ_2^2, and ρ (see section 3.1). This is done in reference 13;

3. Extension of the present results to signal-dependent noise, which is particularly important in active régimes, where reverberation is dominant; and

4. Formal extension to the full threshold signal case $(M = 3)$, at least to the point of identifying ways of analytic evaluation of the triple integrals (with 3rd-order Gaussian integrands) specifying the performance probabilities.

Finally, there remains the application of these general results to specific problems in sonar and related "active" as well as "passive" situations.

APPENDIX A
EVALUATION OF TERNARY PROBABILITY INTEGRALS

In the analysis developed in the preceding text for the important ternary case H_0: N vs H_1: $S_1 + N$ vs H_2: $S_2 + N$, the general probability integral is encountered:

$$I^{(2)} \equiv \int_{a_1}^{b_1} dx_1 \int_{a_2}^{cx_1+b_2} dx_2 w_2(x_1,x_2), \tag{A-1}$$

where, specifically (as in section 7.2 of reference 19),

$$w_2(x_1,x_2) = \frac{\exp\left\{-\dfrac{x_1^2 - 2x_1x_2\rho_{1\,2} + x_2^2}{2(1-\rho_{1\,2}^2)}\right\}}{2\pi\sqrt{1-\rho_{1\,2}^2}}, \quad |\rho_{1\,2}| \le 1, \tag{A-2a}$$

in which x_1, x_2 are the normalized (i.e., "standardized") random variables

$$\left.\begin{aligned} & x_1 = \frac{X_1 - \overline{X}_1}{\sigma_1}\ ;\ x_2 = \frac{X_2 - \overline{X}_2}{\sigma_2}\ ;\ \rho_{1\,2} = \overline{x_1 x_2} = \text{covariance of } x\ ; \\ & \sigma_1^2 = \text{var}\, X_1 = \overline{(X_1 - \overline{X}_1)^2}, \text{ etc.} \end{aligned}\right\}. \tag{A-2b}$$

An analytic reduction of equation (A-1) is sought from which numerical results can be obtained, using tabulated functions, as well as various analytic limits. It appears that in the present general form of equation (A-1), the best, i.e., formally simplest, results that can be obtained involve at most a single series of tabulated functions. Begin with

$$\int_{a_2}^{cx_1+b_2} w_2(x_1,x_2)dx_2$$

$$= \frac{\exp\left\{\dfrac{-x_1^2}{2(1-\rho_{1\,2}^2)}\right\}}{2\pi\sqrt{1-\rho_{1\,2}^2}} \int_{a_2}^{cx_1+b_2} \exp\left\{\frac{-x_2^2}{2(1-\rho_{1\,2}^2)} + \frac{x_1x_2\rho_{1,2}}{(1-\rho_{1\,2}^2)}\right\} dx_2\ ;$$

$$= \frac{\exp\left\{\dfrac{-x_1^2}{2(1-\rho_{1\,2}^2)}\right\}}{2\pi\sqrt{1-\rho_{1\,2}^2}} \bullet \exp\left\{\frac{x_1^2\rho_{1\,2}^2}{2(1-\rho_{1\,2}^2)}\right\} \int_{a_2}^{cx_1+b_2} \exp\left\{-\left(\frac{\sqrt{A}x_2 - Bx_1}{2\sqrt{A}}\right)^2\right\} dx_2\ ;$$

$$A = \frac{1}{2}\left(1 - \rho_{1\,2}^2\right) \; ; \; B = \frac{\rho_{1\,2}}{\left(1 - \rho_{1\,2}^2\right)} \; ;$$

$$= \frac{e^{-x_1^2/2}}{2\pi\sqrt{1-\rho_{1\,2}^2}} \frac{1}{\sqrt{A}} \int_{-Bx_1/2\sqrt{A}+\sqrt{A}a_2}^{x_1\left(\sqrt{A}c_1 - B/2\sqrt{A}b_2\right)+\sqrt{A}b_2} e^{-z^2}\,dz \; ;$$

$$\therefore \int_{a_2}^{cx_1+b_2} w_2(x_1, x_2)\,dx_2 = \frac{e^{-x_1^2/2}}{\sqrt{2\pi}} \bullet \frac{1}{2}\left\{\Theta\left[\frac{x_1(c_1 - \rho_{1\,2}) + b_2}{\sqrt{2\left(1-\rho_{1\,2}^2\right)}}\right] + \Theta\left[\frac{x_1\rho_{1\,2} - a_2}{\sqrt{2\left(1-\rho_{1\,2}^2\right)}}\right]\right\}, \tag{A-3a}$$

where

$$\Theta(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \tag{A-3b}$$

is the familiar error integral in section 7.1 of reference 19; for associated tables, see references 20 and 21.

The next step is to consider the evaluation of integrals of the form

$$I' = \int_{a_1}^{b_1} dx_1\, e^{-x_1^2/2}\Theta(x_1A_1 + B_1) = \int_0^{b_1} dx_1\, e^{-x_1^2/2}\Theta(x_1A_1 + B_1) - \int_0^{a_1} dx_1\, e^{-x_1^2/2}\Theta(x_1A_1 + B_1), \tag{A-4}$$

which reduces to the consideration of the generic relation

$$\boxed{J \equiv \int_0^a e^{-z^2/2}\Theta\left(\hat{A}z + \hat{B}\right) dz \,,} \tag{A-5a}$$

or its equivalent

$$J \equiv \frac{e^{\hat{B}_1^2/2\hat{A}^2}}{\hat{A}} \int_{\hat{B}}^{a\hat{A}+\hat{B}} \exp\left\{-\frac{y^2}{2\hat{A}} + \frac{\hat{B}y}{\hat{A}}\right\}\Theta(y)\,dy\,, \tag{A-5b}$$

on change of variable $\hat{A}z + \hat{B} = y$. Thus, using J (from equation (A-5a)) in equation (A-4) (from (A-3)) for $I^{(2)}$ in equation (A-1):

$$I^{(2)} = \frac{1}{\sqrt{2\pi}} \left\{ \begin{array}{l} \frac{1}{2}\left[\int_0^{b_1}(A_1x + B_1)\, e^{-x^2/2}dx - \int_0^{a_1}(A_1x + B_1)\, e^{-x^2/2}dx\right] \\ + \frac{1}{2}\left[\int_0^{b_1}\Theta(A_1'x + B_1')\, e^{-x^2/2}dx - \int_0^{a_1}\Theta(A_1'x + B_1')\, e^{-x^2/2}dx\right] \end{array} \right\}, \tag{A-6a}$$

with

$$A_1 = \frac{c - \rho_{12}}{\sqrt{2(1-\rho_{12}^2)}} \ , \ B_1 = \frac{b_2}{\sqrt{2(1-\rho_{12}^2)}} \ ; \ A_1' = \frac{\rho_{12}}{\sqrt{2(1-\rho_{12}^2)}} \ , \ B_1' = \frac{-a_2}{\sqrt{2(1-\rho_{12}^2)}} \ , \qquad \text{(A-6b)}$$

or, using equation (A-5a), the equivalent is obtained:

$$\boxed{\therefore I^{(2)} = \frac{1}{2\sqrt{2\pi}}\left\{J(b_1|A_1,B_1) - J(a_1|A_1,B_1) + J(b_1|A_1',B_1') - J(a_1|A_1',B_1')\right\} .} \qquad \text{(A-6c)}$$

Accordingly, the task here is to evaluate the integral J (from equation A-5a). Start with the expansion of $\Theta(\hat{A}z + \hat{B})$ about $\hat{B}$, viz.:

$$\Theta(\hat{A}z + \hat{B}) = \Theta(\hat{B}) + \sum_{n=1}^{\infty} \Theta^n(\hat{B}) \frac{(\hat{A}z)^n}{n!} . \qquad \text{(A-7a)}$$

Here, with $d\hat{B} = dx/\sqrt{2}$ and $\hat{B} = x/\sqrt{2}$,

$$\Theta^n(\hat{B}) = \frac{2}{\sqrt{\pi}}\left(\frac{d^{n-1}}{d\hat{B}^{n-1}} e^{-\hat{B}^2}\right) = 2^{1+n/2} \frac{d^{n-1}}{dx^{n-1}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} = 2^{1+n/2} \phi^{(n-1)}(x = \hat{B}\sqrt{2}) \ , \ n \geq 1 \ , \qquad \text{(A-7b)}$$

in which $\phi^m(x) \equiv (1/\sqrt{2\pi})(d^m/dx^m)e^{-x^2/2}$ is tabulated (see reference 22), as a form of error function and its mth-order derivatives, here expressed as a normalized Gauss PDF $w_1(x) = e^{-x^2/2}/\sqrt{2\pi}$, $m = 0$. Applying equation (A-7) to equation (A-5a) results in

$$J(a|\hat{A},\hat{B}) = \int_0^a e^{-z^2/2}\left\{\Theta(\hat{B}) + \sum_{n=1}^{\infty} \frac{(2\hat{A})^n}{n!}\left(\frac{z}{\sqrt{2}}\right)^n \phi^{(n-1)}(\sqrt{2}\hat{B})\right\} dz \ , \qquad \text{(A-8a)}$$

which becomes finally[9]

$$\boxed{J(a|\hat{A},\hat{B}) = \sqrt{\frac{\pi}{2}}\Theta\left(\frac{a}{\sqrt{2}}\right)\Theta(\hat{B}) + \sqrt{2}\sum_{n=1}^{\infty}\frac{(2\hat{A})^n}{n!}\Gamma\left(\frac{n+1}{2}\right)I_c\left(\frac{a^2}{2} \ ; \ \frac{n+1}{2}\right)\phi^{(n-1)}(\sqrt{2}\hat{B}) .} \ \text{(A-8b)}$$

---

[9] It is shown in reference 13 that the series, based on equation (A-7a), etc., converges only for $|\hat{A}| < 1/\sqrt{2}$. A complete treatment of $(|\rho| \geq 1/\sqrt{2})$ is provided in reference 13, along with numerical results.

Here

$$I_c(\beta;\nu) \equiv \frac{1}{\Gamma(\nu)} \int_0^\beta y^{\nu-1} e^{-y}\, dy \,, \ \nu = 0 \ ; \ \textit{incomplete } \Gamma\textit{-function} \,, \tag{A-9a}$$

This latter is tabulated also (see reference 20). Here use

$$\left. \begin{aligned} &\int_0^{\pm a} e^{-z^2/2}\, dz = \pm\sqrt{\frac{\pi}{2}}\,\Theta\!\left(\frac{a}{\sqrt{2}}\right); \\ &\int_0^{\pm a} e^{-z^2/2}\left(\frac{z}{\sqrt{2}}\right)^n dz = \frac{1}{\sqrt{2}} \int_0^{a^2/2} y^{[((n+1)/2)]-1} e^{-y}\, dy = \frac{\Gamma\!\left(\frac{n+1}{2}\right)}{\sqrt{2}}\, I_c\!\left(\frac{a^2}{2}\ ;\ \frac{n+1}{2}\right) \end{aligned} \right\} . \tag{A-9b}$$

Some limiting values of $J\left(a\middle|\hat{A},\hat{B}\right)$ are readily found:

$$J\left(\pm\infty\middle|\hat{A},\hat{B}\right) = \pm\sqrt{\frac{\pi}{2}}\,\Theta\left(\hat{B}\right) + \sqrt{2}\sum_{n=1}^{\infty} \frac{\left(2\hat{A}\right)^n}{n!}\,\Gamma\!\left(\frac{n+1}{2}\right)\phi^{(n-1)}\left(\sqrt{2}\hat{B}\right),\ \left|\hat{A}\right| \le \frac{1}{\sqrt{2}}\ , \tag{A-10a}$$

and

$$J\left(\pm a\middle|\infty,\hat{B}\right) = \int_0^{\pm a} e^{-z^2/2} dz = \pm\sqrt{\frac{\pi}{2}}\,\Theta\!\left(\frac{a}{\sqrt{2}}\right) = J\left(\pm a\middle|\hat{A},\infty\right), \tag{A-10b}$$

from equation (A-5a) directly, since $\Theta(\infty) = 1$:

$$J\left(\pm\infty\middle|\infty,\hat{B}\ ,\text{ or } \hat{A},\infty\ ,\text{ or } \infty,\infty\right) = \pm\sqrt{\frac{\pi}{2}}\,. \tag{A-10c}$$

As a check, let $b_1, b_2 = \infty$ and $a_1, a_2 = -\infty$. I=1, as required, since w_2 is indeed normalized, so that $\iint_{-\infty}^{\infty} w_2 dx_1 dy_2 = 1$. Also,

$$J\left(0\middle|\hat{A},\hat{B}\right) = 0\ ; \tag{A-11a}$$

$$J\left(a\middle|0,\hat{B}\right) = \sqrt{\frac{\pi}{2}}\,\Theta\!\left(\frac{a}{\sqrt{2}}\right)\Theta\left(\hat{B}\right); \tag{A-11b}$$

$$J(a|0,0) = 0\ ; \tag{A-11c}$$

and

$$J\left(a\middle|\hat{A},0\right)=\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n\left(\sqrt{2}\hat{A}\right)^{2n+1}}{(2n+1)}\,{}_1F_1\left(\frac{a^2}{2}\;;\,n+1\right), \tag{A-11d}$$

where $\phi^{(2n)}(0)=\left((-1)^n(2n)!/2^n n!\right)\sqrt{2\pi}$ (see equation (A.1.23) in reference 19),

$$\therefore J\left(\infty\middle|\hat{A},0\right)=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\sqrt{2}\hat{A}\right)=\int_0^{\infty}e^{-z^2/2}\Theta\left(\hat{A}z\right)dz\,. \tag{A-11e}$$

In addition, a further check on the accuracy of the general result (equation (A-6c) is obtained by considering the following special case:

$$I_1^{(2)}\equiv\int_{-0}^{\infty}\int w_2(x_1,x_2)dx_1dx_2=\left\{\begin{array}{l}\dfrac{1}{2\sqrt{2\pi}}\left\{\begin{array}{l}J\left[\infty\middle|\dfrac{-\rho}{\sqrt{2(1-\rho^2)}},\infty\right]-J\left[0\middle|\dfrac{\rho}{\sqrt{2(1-\rho^2)}},\infty\right]\\+J\left[\infty\middle|\dfrac{-\rho}{\sqrt{2(1-\rho^2)}},0\right]-J\left[0\middle|\dfrac{\rho}{\sqrt{2(1-\rho^2)}},0\right]\end{array}\right\}\\=\dfrac{1}{4}+\dfrac{1}{2\pi}\tan^{-1}\left[\dfrac{\rho}{\sqrt{(1-\rho^2)}}\right]=\dfrac{1}{4}\left\{1+\dfrac{2}{\pi}\tan^{-1}\left[\dfrac{\rho}{\sqrt{(1-\rho^2)}}\right]\right\}\end{array}\right\}, \tag{A-12a}$$

where $J(0|\,,\,)=0$, and where equations (A-10c) and (A-11e) have been used. Limiting cases of equation (A-12) here are

$$\rho=1:\ w_2(x_1,x_2)=w_1(x_1)\delta(x_2-x_1)\quad\therefore I_1=\frac{1}{2}\,, \tag{A-12b}$$

and

$$\rho=0:\ w_2(x_1,x_2)=w_1(x_1)w_1(x_2)\quad\therefore I_1=\left(\frac{1}{2}\right)\bullet\left(\frac{1}{2}\right)=\frac{1}{4}\,, \tag{A-12c}$$

as expected. When $0<|\rho|<1$, then I_1 obeys $1/4<I_1<1/2$.

There remains the case where $\hat{A}$ is large $\left(\gtrsim 3\right)$ but not such that $z\hat{A}>>3$, $z\sim 0$.

Then the integeral for J is split into two parts, viz.:

$$\left.\begin{aligned} J &= \left[\int_0^{z_0} + \int_{z_0}^{a(\geq z_0)}\right] e^{-z^2/2}\Theta\left(\hat{A}z+\hat{B}\right)dz\,,\ \hat{A}z_0 \gtrsim 3 \ \therefore \Theta\left(\hat{A}z_0\right) = 1 - O\left(10^{-6}\right);\\ &\doteq \int_0^{z_0} e^{-z^2/2}\Theta\left(\hat{A}z+\hat{B}\right)dz + \int_{z_0}^{a} e^{-z^2/2}dz \end{aligned}\right\}; \tag{A-13}$$

$$\left.\begin{aligned} \therefore J\left(0\middle|\hat{A},\hat{B}\right)\Big|_{\hat{A}z_0\geq 3} &\doteq J\left(\frac{3}{\hat{A}}\middle|\hat{A},\hat{B}\right) + \sqrt{\frac{\pi}{2}}\left[\Theta\left(\frac{a}{\sqrt{2}}\right) - \Theta\left(\frac{3}{\sqrt{2}\hat{A}}\right)\right];\\ & a \geq z_0 \text{ (see equation (A-9b) ;}\\ &\doteq J\left(\frac{3}{\hat{A}}\middle|\hat{A},\hat{B}\right),\ \text{since } \Theta\left(\frac{a}{\sqrt{2}}\right) < 10^{-6}\ ,\ \text{etc.} \end{aligned}\right\}. \tag{A-14}$$

Accordingly, with equation (A-8b) applied to equation (A-6b), (comparatively) simple analytic expressions have been obtained for the general probability integral I (see equation (A-1)), in terms of the extensively tabulated functions $\Theta(x)$ in equation (A-3a), $\phi^{(m)}(x)$ in equation (A-7b), et seq., and I_c in equation (A-9a), namely, the error functions, related derivatives, and the incomplete Γ-function. Computer evaluation of J and I is then straightforward.

APPENDIX B
CORRECTIONS FOR SOME RESULTS IN REFERENCE 3

In a recent work (reference 3), which among other things considered the special ternary case (M = 2) with purely (correlated) Gaussian noise and completely known signals S_1, S_2 ($\neq S_1$), i.e., coherent detection, at all signal levels, the limits on the integral expressions for the various decision probabilities were inadvertently incompletely specified. The qualitative nature of the results were not affected, but the quantitative expressions were. The correct analytic forms are given here.

Begin with equations (A.1-14a) and (A.1-14c), respectively, of reference 3:

$$\left.\begin{array}{l}\text{Add in the limits } -A_2 \text{ to } y_1 + A_1\,;\ -A_1 \text{ to } y_2 + A_2\,; \\ \text{Add in the limits } -A_1 \text{ to } y_2 + A_2\,;\ -A_2 \text{ to } y_1 + A_1\end{array}\right\}. \tag{B-1}$$

Also, for equation (A.1-16b),

$$P_D^{(1)*} = p_1 \int_{-A_1}^{\infty} dy_1 \int_{-\infty}^{y_1 + A_1 - A_2} dy_2 P_2^{(1)}(y)\,, \tag{B-2a}$$

and, for equation (A.1-17b),

$$P_D^{(2)*} = p_2 \int_{-A_2}^{\infty} dy_1 \int_{-\infty}^{y_2 + A_2 - A_1} dy_2 P_2^{(2)}(y)\,. \tag{B-2b}$$

The results of these limit modifications are that equation (A.1-33b) becomes $\left(\text{with } \Phi \equiv \tilde{s}k^{-1}s\right)$:

$$\left.\begin{array}{l} P_D^{(1)*} = \dfrac{p_1}{2}\left[1 + \Theta\left(B_{\max}^{(1)}\right)\right], \\ B_{\max}^{(1)} \equiv \max\left[\dfrac{a_o\sqrt{\Phi}}{2\sqrt{2}} + \dfrac{\log\mu_1}{\sqrt{2}a_o\sqrt{\Phi}}\ ,\ \dfrac{a_o\sqrt{\Phi}}{\sqrt{2}} - \dfrac{\log\left(\dfrac{\mu_2}{\mu_1}\right)}{\sqrt{2}a_o\sqrt{\Phi}}\right] \end{array}\right\}, \tag{B-3}$$

while equation (A.1-33c) is replaced by equation (A.1-14a) (see equations B-1, etc.).

Similarly, for equation (A.1-34b):

$$\left.\begin{aligned} &P_D^{(2)*} = \frac{p_2}{2}\left[1+\Theta\left(B_{max}^{(2)}\right)\right], \text{ with} \\ &B_{max}^{(2)} \equiv \max\left[\frac{a_o\sqrt{\Phi}}{2\sqrt{2}} + \frac{\log\mu_2}{\sqrt{2}a_o\sqrt{\Phi}}\ ,\ \frac{a_o\sqrt{\Phi}}{\sqrt{2}} - \frac{\log\left(\frac{\mu_1}{\mu_2}\right)}{\sqrt{2}a_o\sqrt{\Phi}}\right] \end{aligned}\right\}, \tag{B-4}$$

In the same way, for equation (A.1-35):

$$R_{M=2}^{*} = \frac{C_o}{2}\left[1 - q\Theta\left(\frac{\Gamma}{2\sqrt{2}} - \frac{\min_k \mu_k}{\sqrt{2}\Gamma}\right) - p_1\Theta\left(B_{max}^{(1)}\right) - p_2\Theta\left(B_{max}^{(2)}\right)\right],\ \Gamma \equiv a_o\sqrt{\Phi}\ ; \tag{B-5}$$

so that, for equation (A.1-36),

$$R_{M=2}^{*}\Big|_{p_j=1/3} = \frac{C_o}{2}\left[1 - \frac{1}{3}\Theta\left(\frac{\Gamma}{2\sqrt{2}}\right) - \frac{2}{3}\Theta\left(\frac{\Gamma}{\sqrt{2}}\right)\right], \tag{B-6}$$

which gives, for equation (A.1-27),

$$\Delta R_{M=2}^{*} = \frac{C_o}{6}\left[\Theta\left(\frac{\Gamma}{2\sqrt{2}}\right) + 2\Theta\left(\frac{\Gamma}{\sqrt{2}}\right)\right]. \tag{B-7}$$

Finally, equation (A.1-38) becomes, for $q = p_1 = p_2 = 1/3$:

$$P_e^{*} = \frac{1}{2}\left[1 - \frac{1}{3}\Theta\left(\frac{\Gamma}{2\sqrt{2}}\right) - \frac{2}{3}\Theta\left(\frac{\Gamma}{\sqrt{2}}\right)\right]. \tag{B-8}$$

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