AD-A264 080

AR-008-256

SUBMARINE INTERNAL WAVES

(2)

E.O. TUCK

MRL-GD-0049

JANUARY 1993

APPROVED

FOR PUBLIC RELEASE

MATERIALS RESEARCH LABORATORY

DSTO

Submarine Internal Waves

E.O. Tuck

MRL General Document
MRL-GD-0049

Abstract

Details are provided of research directed towards the development of the computer program WAKE for predicting the internal waves produced by the passage of a submarine in a density stratified ocean. The research was undertaken within DSTO during the period February-March 1992. The report takes the form of a series of appendices (presented in chronological order), which provide a record of the progress of the research during that period. Results are presented which indicate that a representative submarine produces internal waves which have a velocity magnitude of about one millimeter per second.

DSTO MATERIALS RESEARCH LABORATORY

Published on behalf of

Professor E.O. Tuck
University of Adelaide

by

Materials Research Laboratory
Cordite Avenue, Maribyrnong
Victoria, 3032 Australia

Telephone: (03) 246 8111
Fax: (03) 246 8999

AR No. 008-256

SUBMARINE INTERNAL WAVES

by
E.O. Tuck
Applied Mathematics Department
University of Adelaide
S.A., Australia

A report on a research project carried out during February-March 1992 at DSTO-Salisbury in collaboration with the following DSTO staff: D. Cartwright, R. Webster, G. Furnell, A. Legg, D. Madurasinghe.

Abstract

Details are provided of research directed towards the development of the computer program WAKE for predicting the internal waves produced by the passage of a submarine in a density stratified ocean. The research was undertaken within DSTO during the period February-March 1992. The report takes the form of a series of appendices (presented in chronological order), which provide a record of the progress of the research during that period. Results are presented which indicate that a representative submarine produces internal waves which have a velocity magnitude of about one millimeter per second.

Summary

In late 1991. I agreed to undertake an investigation for DSTO of submarine wakes with emphasis on the internal waves generated by the passage of the submarine through a stratified ocean. The primary task was to compute these internal waves for a given submarine. and this task was completed in the period February-March 1992. Research assistance was provided by Graham Furnell and Tony Legg of DSTO. and programming assistance by Michael Carroll of Ebor Pty Ltd. Additional advice and guidance was provided by David Cartwright and Dan Madurasinghe of DSTO.

This report is in the form of a collection of Appendices. These are presented in chronological order. giving a record of progress as the project unfolded. Not all of the Appendices are of equal importance. and some are preliminary or tentative. in part superseded by subsequent work. All are neveretheless included. to give an accurate impression of the character of the project.

Appendix 1 is the most important. setting the style and background for the project. This Appendix was actually written before the main project commenced. Appendix 8 is also very important. since the project follows similar pathways to the Canadian effort headed by Dawson and Hughes. There are many things that could not be done or done sufficiently thoroughly in the very short time available. Some of these could make good topics for further studies. Appendix 13 is a partial list of these topics. Appendix 15 is a very preliminary report of final results. It had to be left till the very last. and is presented in the form mostly of graphs with hand-written annotation. A more complete summary of results may be presented later.

The bottom line is that a submarine at representative depths and speeds in a representative stratification makes internal waves of a velocity magnitude of about one millimeter per second. This sounds small. and is. although (because of the long wavelength and large period - about 20 minutes - of the internal wave) it does correspond to actual particle displacements of the order of about one metre. These results are of the order of magnitude of those reported elsewhere. They may or may not be detectable. at least indirectly. e.g. via bunching of capillaries. Our program allows parametric study of various submarine shapes. sizes. depths and speeds. and of various stratifications.

Contents

Appendix 1: 10 February 1992

Source in Stratified Fluid

Some preliminary notes on internal wave generation, actually prepared by EOT prior to commencement of project.

1. Basic equations.

Let the fluid velocity vector be $\mathbf{q} = (U + \phi_x, \phi_y, w)$ where $\phi = \phi(x, y, z)$ and $w = w(x, y, z)$ are small. That is, the flow is a small perturbation to the uniform stream U in the $+x$ direction. The above representation implies that the z-component of the vorticity is zero, but other components are not: the flow is not irrotational, and ϕ is not a velocity potential.

Now the continuity equation is

$$\phi_{xx} + \phi_{yy} + w_z = 0 \tag{1.1}$$

and there is a second equation of motion for a stratified medium with density $\rho = \rho(z)$, of the form

$$w_{xx} - \phi_{xxz} = \frac{\rho'}{\rho}(\phi_{xx} + \kappa w) \tag{1.2}$$

where $\kappa = g/U^2$.

[I am not too sure about the reference status of this equation. *Later insertion: In fact, I now believe that it is well documented and derived in C.-S. Yih's 1965 book, see bibliography, Appendix 2.* It seems to be used by people such as Keller, but without derivation. I have derived it myself in two quite independent ways, getting the same answer. First by linearising the Euler equation - straightforward and boring. Second by considering an ocean consisting of many layers of uniform density fluid with standard Kelvin linearised free boundary conditions at the interfaces, then letting the layer thicknesses tend to zero. Incidentally, the so-called Boussinesq approximation seems to be to drop the term on the right in ϕ_{xx}, but I see no reason to do that.]

Our task is to solve these equations in $z < 0$ subject to the free surface condition

$$\phi_{xx} + \kappa w = 0 \tag{1.3}$$

at $z = 0$, and (for an infinitely deep ocean) $\phi, w \to 0$ as $z \to -\infty$. Note that the free surface condition (1.3) follows from (1.2) by requiring the coefficient of the density gradient to vanish, which is as it must be, since the free surface is just a step discontinuity in density.

Actually, we don't want to solve (1.1) everywhere; instead we want to allow a source at $(0, 0, -h)$. Hence we should replace the zero on the right hand side by a delta function $\delta(x).\delta(y).\delta(z+h)$. Equivalently, we need solutions of (1.1) possessing suitable singularities at this point, see below.

2. Fourier Decomposition

Write

$$\phi = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\alpha x + i\beta y}\Phi(z;\alpha,\beta)\,d\alpha d\beta \tag{2.1}$$

and

$$w = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\alpha x + i\beta y}W(z;\alpha,\beta)\,d\alpha d\beta \tag{2.2}$$

Then (with $k^2 = \alpha^2 + \beta^2$ and using a dash for d/dz), (1.1) gives $W' = k^2\Phi$ while (1.2) gives

$$W - \Phi' = \frac{\rho'}{\rho}(\Phi - \frac{\kappa}{\alpha^2}W) \tag{2.3}$$

Then, eliminating Φ, we have an ODE for $W = W(z)$, namely

$$k^2W - W'' = \frac{\rho'}{\rho}(W' - \sigma W) \tag{2.4}$$

or

$$(\rho W')' - (k^2\rho - \sigma\rho')W = 0 \tag{2.5}$$

where

$$\sigma = \kappa k^2/\alpha^2 = \kappa \sec^2\theta \tag{2.6}$$

with $\alpha = k\cos\theta$.

Equation (2.5) is a straightforward variable-coefficient second-order ODE, and is to be solved subject to suitable free surface and bottom conditions. The former is just the Fourier transform of (1.3), which leads to

$$W' = \sigma W \tag{2.7}$$

at $z = 0$. The bottom condition is best expressed by assuming that the density is uniform below some level H, in which case the solution to (2.5) for $z < -H$ is proportional to e^{kz}. Hence in that range

$$W' = kW \tag{2.8}$$

and the appropriate bottom condition is then that (2.8) also holds at $z = -H$ for solutions in $z \geq -H$. Of course if there were an actual flat impermeable sea floor at $z = -H$, (2.8) would be replaced by $W = 0$, but we shall use (2.8).

Non-trivial solution of the ODE (2.5) subject to both the free surface condition (2.7) and the bottom condition (2.8) is not possible in general, but is possible (at fixed σ) for special values of the wave number k, i.e. for eigenvalues $k = K_j(\sigma)$, $j = 0, 1, 2, \ldots$ There is always at least one eigenvalue $k = K_0(\sigma) = \sigma$, where the solution is just proportional to $W = e^{kz}$, as is immediately apparent from the form (2.4) of the ODE, even if the density is an arbitrary non-constant function of z. If ρ is constant, this is the only solution, and is the usual Kelvin surface wave. If ρ is not constant, there may be more solutions $j = 1, 2, \ldots$

and these are what are called internal waves. The relationship $k = K_j(\sigma)$ is a form of the dispersion relationship for the j'th internal wave.

3. Solution Method for Sources

Although there are many direct methods for solving the above eigenvalue problem if all that we needed was the dispersion relation $k = K_j(\sigma)$, the following indirect method is preferable if we need solutions for sources.

Suppose $W = W_1(z)$ and $W = W_2(z)$ are two separate solutions of (2.5), defined as follows. For W_1, we satisfy the bottom boundary condition (2.8) but not the free surface boundary condition (2.7), and normalise (arbitrarily) the value of W_1 at the bottom, obtaining an initial value problem to be solved upward in $z > -H$, starting with the initial conditions

$$\begin{aligned} W_1(-H) &= e^{-kH} \\ W_1'(-H) &= ke^{-kH} \end{aligned} \tag{3.1}$$

at $z = -H$.

Similarly, W_2 satisfies the free surface condition (2.7) but not the bottom condition (2.8), and is normalised (arbitrarily) to the starting value -1 at $z = 0$. Then we solve downward for $W_2(z)$ as an initial value problem in $z < 0$, starting with the initial conditions

$$\begin{aligned} W_2(0) &= -1 \\ W_2'(0) &= -\sigma \end{aligned} \tag{3.2}$$

at $z = 0$.

Each of W_1 and W_2 always exists for any choice of k, σ, for reasonable $\rho(z)$, and is readily computed by any standard ODE-solver numerical package. Unless k and σ are connected by the dispersion relation, W_1 and W_2 are linearly independent of each other, but if it happens that $k = K_j(\sigma)$, they must be proportional to each other, since then and only then can both boundary conditions be satisfied by the same function.

The Wronskian $W_1W_2' - W_2W_1'$ must be inversely proportional to the density $\rho(z)$, a standard property of variable-coefficient equations like (2.5), so we can write for all z

$$W_1(z)W_2'(z) - W_2(z)W_1'(z) = D\rho(-h)/\rho(z) \tag{3.3}$$

for some constant $D = D(k, \sigma)$, the value of the Wronskian at $z = -h$. Now consider the following discontinuous solution of (2.5), namely:

$$W_0(z) = \begin{cases} -W_2'(-h)W_1(z), & \text{if } z < -h; \\ -W_1'(-h)W_2(z) & \text{if } -h < z < 0. \end{cases} \tag{3.4}$$

[Note that W_0, like W_1 and W_2, depends implicitly on the parameters k and σ as well as the coordinate z, and will be displayed as $W_0(z; k, \sigma)$ whenever it is necessary to indicate that dependence.] The function $W_0(z)$ satisfies both boundary conditions (2.7) and (2.8), by construction. Its derivative is continuous across $z = -h$, whereas its value jumps by

$W_0(-h+0) - W_0(-h-0) = D$. Hence $W(z) = W_0(z)/D$ is the required discontinuous solution for a unit source, which must have a unit-magnitude jump across $z = -h$.

The important point is that $D = 0$ when $k = K_j(\sigma)$. That is, linear independence fails when the Wronskian vanishes. Hence $D(k, \sigma) = 0$ is an implicit form of the dispersion relation, and that relation can be determined by numerical solution for W_1 and W_2, and hence for $D(k, \sigma)$. The solution W_0 is well defined and bounded for all k, σ values. Hence the source solution W_0/D is singular, with a pole wherever $D = 0$, i.e. wherever $k = K_j(\sigma)$.

[Figure 1.1 is a sample plot of the Wronskian $D(k, \sigma)$ (see (3.3) at $z = -h$ versus k at various σ. Note the value where $D = 0$. See Figure 6.1 for the corresponding dispersion relation curves $k = K_j(\sigma)$.]

4. Pole Avoidance and Free Waves

The solution now found for the vertical velocity w is given by the Fourier integral (1.5), which after a change to polar wave numbers by $\alpha = k\cos\theta, \beta = k\sin\theta$ becomes

$$w(x, y, z) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty} e^{ik(x\cos\theta + y\sin\theta)}\,\frac{W_0(z; k, \kappa\sec^2\theta)}{D(k, \kappa\sec^2\theta)}\,k\,dk\,d\theta \tag{4.1}$$

However, since $D = 0$ at $k = K_j(\kappa\sec^2\theta)$, we must distort the path of k-integration to avoid the poles where $D = 0$. This procedure is the same for all poles $j = 0, 1, 2, 3, \ldots$, and is well established for the Kelvin surface wave pattern $j = 0$. Namely, if $\cos\theta > 0$, distort the path above the pole, and if $\cos\theta < 0$, distort it below the pole.

The reason for this choice is clear if we consider rotation of the path of k-integration through $\pm 90°$. If $x < 0$, and this rotation is done through $\pm 90°$ when $\cos\theta \gtrless 0$, the pole is not crossed during the process of this rotation, and the resulting integral along the imaginary k-axis contains an exponentially decaying factor. Hence w tends to zero rapidly as $x \to -\infty$, and there are no waves upstream, as required by the radiation condition.

Once this decision about pole avoidance is made, our task of determining the flow from a source is in principle completed, and it "only" remains to evaluate the double integral (4.1), with W_0 and D known by solution of the ODE (2.5) (for all values of k and $\sigma = \kappa\sec^2\theta$). This is a mammoth task, bearing in mind that only recently has it even been considered feasible to do routine computations for unstratified fluids, where the ODE part of the task is eliminated.

However, the main far-field ($x \to +\infty$) contribution is from the poles. This contribution can be estimated by rotating the path of k-integration in the opposite direction from that for $x < 0$. In that case, one passes across the poles, so picking up a contribution from their residues, before arriving at an integral on the imaginary k-axis which again tends to zero rapidly, as $x \to +\infty$. Hence the dominant terms in w as $x \to +\infty$ are the contributions from the poles, namely

$$w_F(x, y, z) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} 2\pi i\,\mathrm{sgn}(\cos\theta)\sum_j e^{ik(x\cos\theta + y\sin\theta)}\,\frac{W_0(z; k, \kappa\sec^2\theta)}{D_k(k, \kappa\sec^2\theta)}\,k\,d\theta \tag{4.2}$$

where $k = K_j(\kappa\sec^2\theta)$.

Note that the denominator D_k in (4.2) is the partial k-derivative of the Wronskian $D(k,\sigma)$, evaluated at the point $k = K_j(\sigma)$ where $D = 0$, and this quantity must be evaluated numerically with precision as part of the over-all computational task. The above expression can be simplified slightly to a real integral on $(0, \pi/2)$, namely

$$w_F = -\frac{2}{\pi}\sum_j \int_0^{\pi/2} \sin(K_j x\cos\theta)\cos(K_j y\sin\theta)\frac{W_0}{D_k} K_j \, d\theta \tag{4.3}$$

The expressions (4.2) or (4.3) are single integrals with respect to θ, and obviously an order of magnitude easier to evaluate than the double integral (4.1). In the ship-wave example, w_F is usually called the "free-wave" contribution, and we retain that terminology here. Although free waves are only a far-field approximation, the ship-wave experience is that they approximate the full flow field very well, as close as one or two ship lengths astern.

5. Stationary Phase

Although I advocate evaluation of the integral (4.3) without further approximation, it is possible to make the usual stationary phase approximation for large $r = \sqrt{(x^2 + y^2)}$ observing that (4.3) is the sum of real parts of integrals of the form

$$\int F(\theta)e^{ir\Psi(\theta)}\, d\theta \tag{5.1}$$

where the amplitude function $F(\theta)$ is well behaved, and the phase function is

$$\Psi(\theta) = K(\theta)\cos(\theta - \gamma). \tag{5.2}$$

where $\gamma = \arctan(y/x)$ is the polar angle in the horizontal plane, and $K(\theta) = K_j(\kappa \sec^2\theta)$.

For large r, the main contribution to the integral (5.1) is from the neighbourhood of stationary values of θ, namely those where $\Psi'(\theta) = 0$, which satisfy

$$\tan(\theta - \gamma) = K'(\theta)/K(\theta) \tag{5.3}$$

For most reasonable $K(\theta)$ there are two roots θ of (5.3) for each $|\gamma|$ less than a certain upper bound γ_0, and none above that value. The two roots correspond to transverse (smaller θ) and diverging (larger θ) waves which are observed for $|\gamma| < \gamma_0$, and there are no waves for $|\gamma| > \gamma_0$. The waves tends to be greatest in magnitude near $|\gamma| = \gamma_0$. Determination of the value of γ_0 is one of the important tasks, and this value is different for each internal wave mode j. If we re-write (5.3) as

$$\gamma = \theta - \arctan(K'(\theta)/K(\theta)) \tag{5.4}$$

we see that the upper bound for $|\gamma|$ must occur when $d\gamma/d\theta = 0$, which leads to a condition involving the second derivative $K''(\theta)$ that is not worth writing down. For general $K(\theta)$ it is best to simply compute $\gamma(\theta)$ from (5.4) and note its maxima or minima.

For example, the Kelvin surface wave $j = 0$ has $K_0 = \sigma$, so $K(\theta) = \kappa \sec^2\theta$, and (5.4) becomes

$$\gamma = \theta - \arctan(2\tan\theta) \tag{5.5}$$

which is negative for positive θ, with a minimum given by $-\gamma = \gamma_0 = \arctan(1/2\sqrt{2}) \approx 19.5^\circ$ when $\theta = \arctan(1/\sqrt{2}) \approx 36^\circ$. This value of γ_0 is the famous Kelvin ship wave angle, and applies at any speed for any moving object at any submergence, producing waves on the surface of an infinitely deep fluid, even if the fluid density is non-constant.

But the internal waves $j = 1, 2, 3, \ldots$ will have a different γ_0, and there seems to be evidence that it is smaller, i.e. that the internal wave wakes are narrower than Kelvin wakes.

6. Constant Density

The special case of constant density is worth giving in full. Then there is only the one mode $j = 0$, and we can make the following identifications

$$W_1(z) = e^{kz} \tag{6.1}$$

$$W_2(z) = -\cosh(kz) - \frac{\sigma}{k}\sinh(kz) \tag{6.2}$$

$$D(k, \sigma) = k - \sigma \tag{6.3}$$

$$W_0(z; k, \sigma) = \frac{k+\sigma}{2} e^{k(z-h)} + \frac{k-\sigma}{2}\,\mathrm{sgn}(z+h)\,e^{-k|z+h|} \tag{6.4}$$

and

$$w_F = -\frac{2}{\pi}\kappa^2 \int_0^{\pi/2} \sin(\kappa x \sec\theta)\cos(\kappa y \sec^2\theta \sin\theta) e^{\kappa(z-h)\sec^2\theta} \sec^4\theta \, d\theta \tag{6.5}$$

which is well known.

(Figure 1.2 shows computations from (6.5) by D. Madurasinghe.)

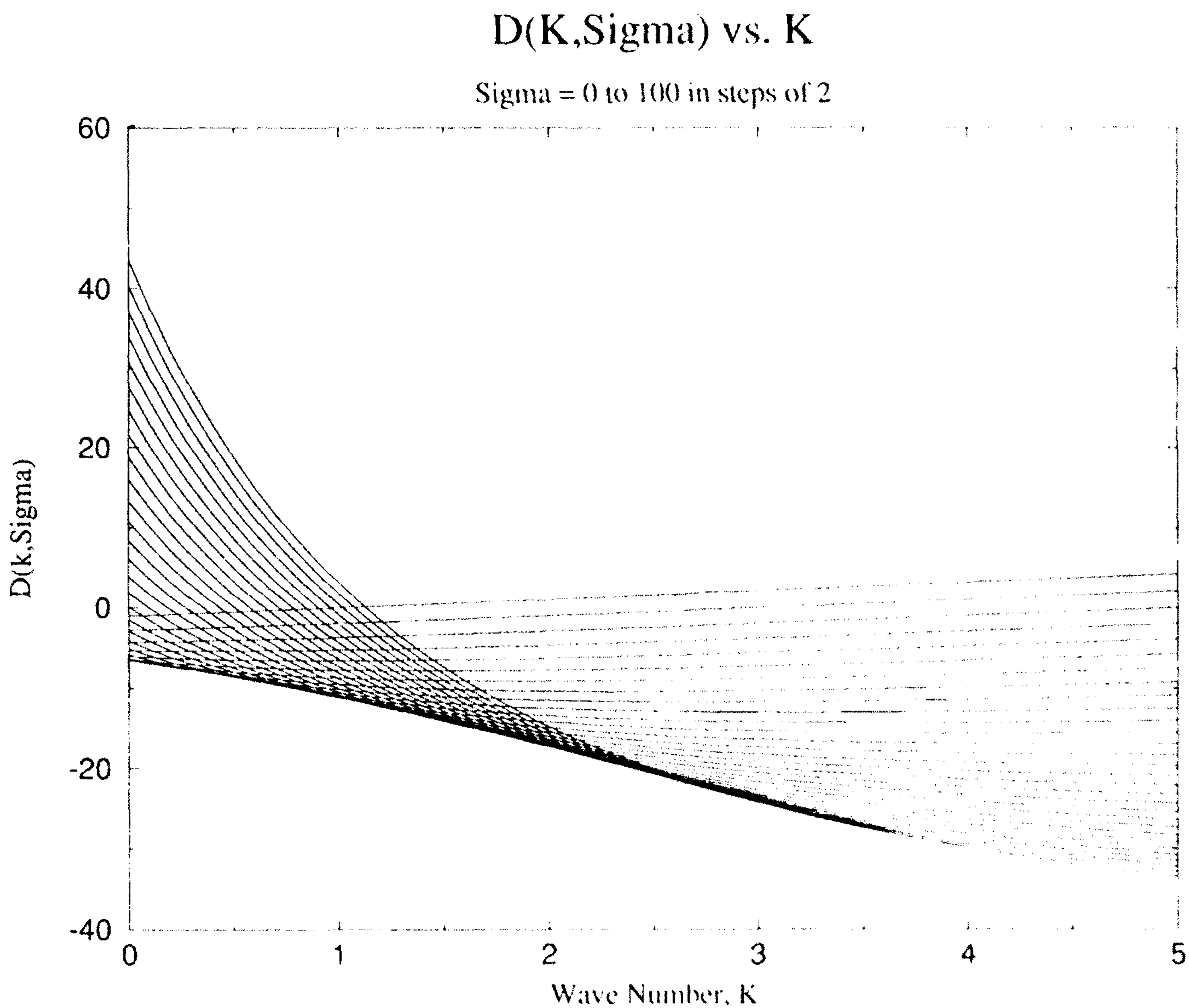


Figure 1.1: Sample graph of Wronskian $D(k,\sigma)$.

Figure 1.2: Surface waves computed by D. Madurasinghe.

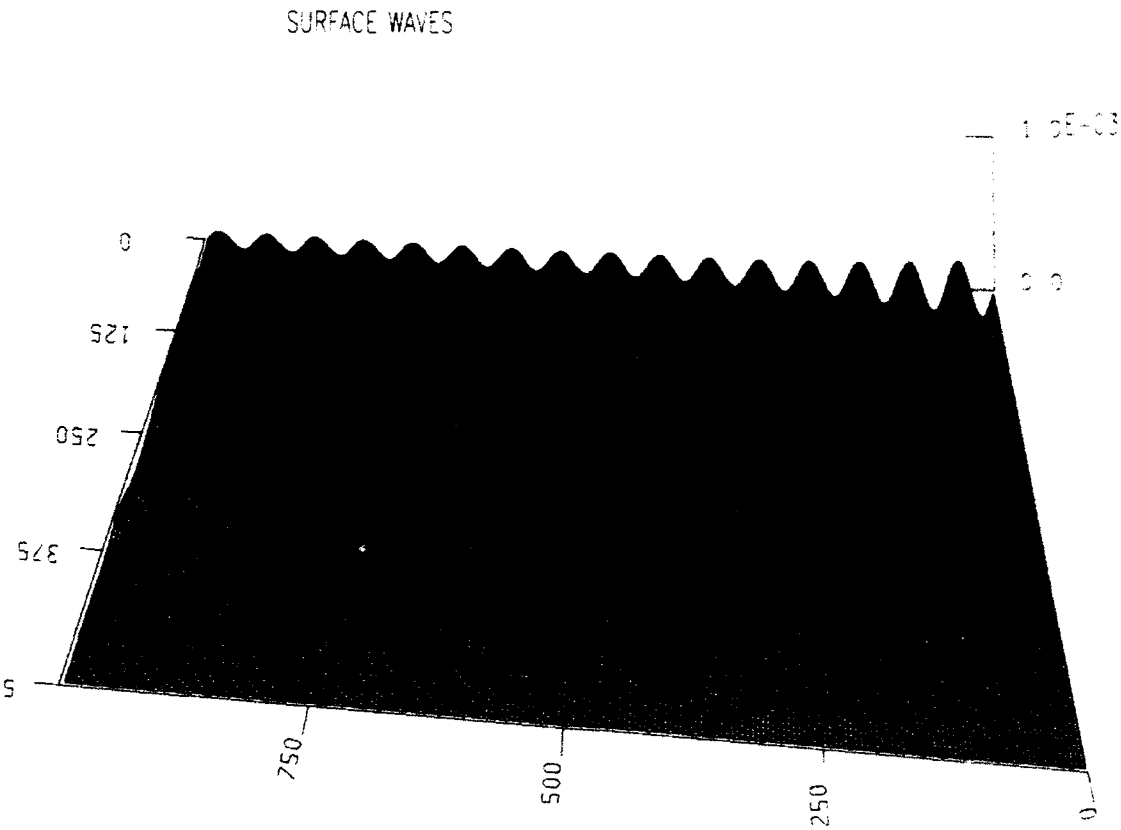
$$w_r = \mathrm{Re}\int_{-\pi/2}^{\pi/2} F(\theta)\, e^{-ik\sec^2\theta(x\cos\theta + y\sin\theta)}\, d\theta = \mathrm{Re}\int_{-\pi/2}^{\pi/2} F(\theta)\, e^{-i\varphi(R,\theta,\beta)}\, d\theta$$

Where

$$F(\theta) = -\frac{ik^2\sec^4\theta}{\pi}\, e^{k(z-h)\sec^2\theta}$$

$$\varphi(R,\theta,\beta) = Rk\sec^2\theta\cos(\theta-\beta)\ ;\ \beta = \tan^{-1}\left(\frac{y}{x}\right),\quad R^2 = x^2 + y^2$$

Input: z=0.0, h= 50 meters, k=0.1 (~36 km/h) Density=Const.



Appendix 2: 10 February 1992

Bibliography on Internal Waves and Wakes

Brown E.D., "SAIC report on infinite-depth solitary waves" see also a *J. Fluid Mech* paper in about 1990.

Dawson T.W., "The DREP internal wave normal mode model theoretical background", DREP T.M. 88-7, Canada, 1987

Gilbreath H.E. and Brandt A., "Experiments on the generation of internal waves in a stratified fluid", *A.I.A.A. J.* **23** (1985) 693.

Gray. *Phys. Fluids* **26** (1983) 2919.

Hudimac A.A. "Ship waves in a stratified ocean", *J. Fluid Mech.* **11** (1961) 229.

Ichiye. *Pure and Applied Geophys.* **119** (1981) 294.

Hall R.E. et al. "Short wave Kelvin wakes(?)", SAIC report 87 1794 1987(?).

Hughes B.A. "Surface wave wakes and internal wave wakes produced by surface ships", *15th ONR Symp Naval Hydro.* 1986.

Keller J.B. Levy D.M. and Ahluwalia D.S., "Internal and surface wave production in a stratified fluid", *Wave Motion* **3** (1981) 215.

Keller J.B. and Munk W., "Internal wave wakes of a body in a stratified fluid" *Phys. Fluids* **13** (1970) 1425.

Lighthill M.J. "Waves in Fluids", Cambridge, 1978.

Miles J. W., "Internal waves generated by a horizontally moving source", *J. Geophys Fluid Dyn.* **2** (1971) 63.

Miloh T. and Tulin M., "A theory of dead water phenomena", *17th ONR Symp Naval Hydro.* 1988.

Phillips O.M., "The Dynamics of the Upper Ocean", Cambridge, 1977

Reed A.M., Beck R.F., Griffin O.M. and Peltzer R.D., "Hydrodynamics of remotely sensed surface ship wakes", *Trans. Soc. Nav. Arch. Mar. Eng.* Nov. 1990.

Sharman R.D. and Wurtele M.G., "Ship waves and lee waves" *J. Atmos. Sci.* **40** (1983) 396.

Tuck E.O., Collins J.I and Wells W.H., "On ship wave patterns and their spectra", *J. Ship Res.*, March 1971.

Tulin M. and Miloh T., "Ship internal waves in a shallow thermocline: The supersonic case", *18th ONR Symp Naval Hydro* Ann Arbor, Michigan, 1990.

Tulin M., Miloh T., Yao P.W and Yao. Y., "Near field internal waves generated by a ship in a shallow thermocline", *6th Int. Workshop on Water Waves and Floating Bodies* Wood's Hole, Mass., 1991.

Yih C.-S., "Dynamics of Non-homogeneous Fluids", MacMillan 1965

Appendix 3: 18 February 1992

Dispersion Relations

[Note: In part improved by later work, see Appendix 6 to follow.]

Our concern here is very much with dispersion relations for internal waves, namely relations between wave speed c and wave number k. For convenience, we use instead of c a quantity proportional to its reciprocal square, namely

$$\sigma = g/c^2$$

where g is gravity. Then we need a connection between k and σ, e.g. $k = K(\sigma)$. One of our first tasks is to compute this relation for a given density distribution. Because this numerical task obscures some qualitative features, this report summarises some such features.

The empirical dispersion relation attributed to Phillips (1977) by Tulin and Miloh (1990) can be written

$$\frac{\Delta\rho}{\rho}\sigma = k\,[\coth(kH) + 1 + k\epsilon]$$

This contains as its leading term the finite depth dispersion relation

$$\sigma = k\coth(kH)$$

for water waves in a fluid of constant depth H, but modified with gravity reduced in the ratio of the relative density change across the thermocline. The parameters are the depth H from the free surface to the beginning of the thermocline and the thickness ϵ of the thermocline.

[Figure 3.1 is a sample of the $k - \sigma$ relationship. It is not based on Phillips formula, but is similar.]

The small and large-k limits are of interest. For small k, we have

$$\sigma = \frac{\rho}{\Delta\rho}\,[H^{-1} + k + O(k^2)]$$

so if σ_1 is the value of σ for the first internal wave at $k = 0$ then we estimate

$$\sigma_1 = \frac{\rho}{\Delta\rho}H^{-1}$$

and then for $\sigma > \sigma_1$ (but close to it) we have

$$k = \frac{\Delta\rho}{\rho}(\sigma - \sigma_1) + O(\sigma - \sigma_1)^2$$

This applies only to the first internal wave mode. Phillips doesn't specify it, but I believe for the second mode, $\sigma_2 = 9\sigma_1$ approximately, and more generally that the j th mode starts at $\sigma = \sigma_j$ where approximately $\sigma_j = (2j + 1)^2\sigma_1$. That is, I expect that

$$k = c_j(\sigma - \sigma_j) + O(\sigma - \sigma_j)^2$$

for some c_j.

The large-k limit is

$$k = \left(\frac{\Delta\rho}{\rho\epsilon}\right)^{1/2}\sigma^{1/2}$$

That is, σ and k grow large together, with k varying as the square root of σ. The coefficient of proportionality varies like the square root of the density difference, and also like the inverse square root of the thermocline thickness ϵ, but does not depend on the thermocline depth H.

I am rather interested in corrections to this, especially those which discriminate between modes. I am inclined to think that $\sigma^{1/2}$ needs to be replaced by $(\sigma - \sigma_j^*)^{1/2}$ for some σ_j^* which is not the same as σ_j though it must increase (like σ_j does) as the square of the mode number j. Also, I think that in general there should be an additive term, so the general large-k expansion should be

$$k = C(\sigma - \sigma_j^*)^{1/2} + k_j$$

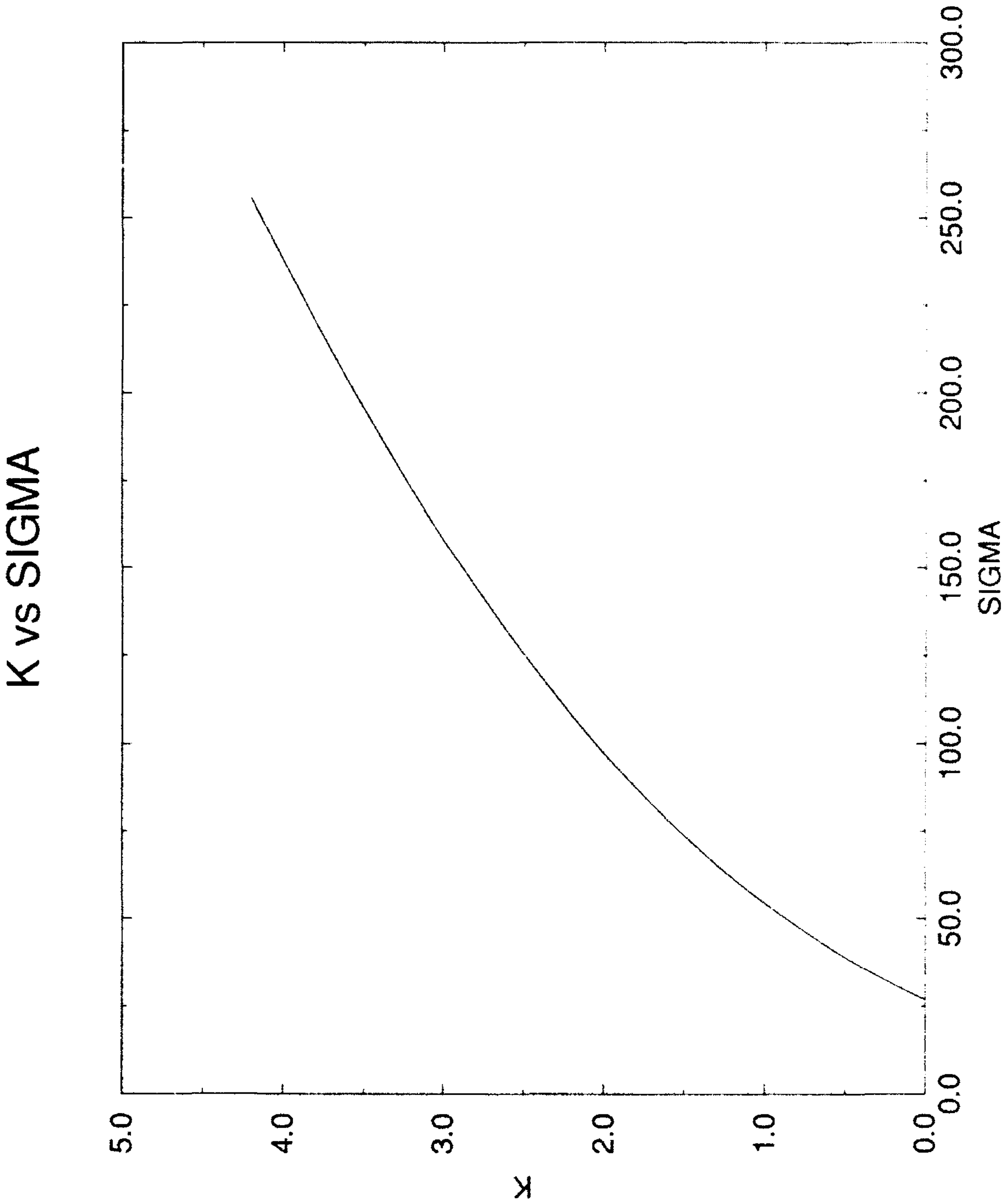
where C is a universal coefficient independent of mode number, and σ_j^* and k_j depend on mode number.

One cannot rely on the Phillips formula for this type of additional information since it assumes a special form for the density distribution – as evidenced from the fact that the results depend on parameters like H and ϵ which are only really meaningful in the context of that distribution. So a useful piece of analytic or semi-numerical work would be filling in some of these details for a general density profile. That is, what parameters of the profile do quantities like σ_j, σ_j^*, k_j, c_j, C, etc. depend upon?

The Phillips empirical formula is an example of what one can do with a so-called Boussinesq approximation. This makes use of the fact that the density variations are small in absolute terms. It only takes them into account when multiplied by the gravity parameter g. Then one can also use a large-σ approximation. In effect, one lets σ tend to infinity and ν tend to zero, while retaining the product $\sigma.\nu$. There are some simplifications to the general ODE problem, though not too many. I think there is a lot could be done of this nature.

Note that the general Phillips formula has this Boussinesq character. My discussion above pretended that $\Delta\rho$ was arbitrary, but in fact the Phillips formula is only valid if it is small, in which case σ is *always* large, of the order of the inverse square root of $\Delta\rho$, whereas k is not necessarily small, ranging from 0 to ∞ in general.

Figure 3.1: Typical dispersion relation for mode 1.



Appendix 4: 27 February 1992

Program A

The program for submarine internal wave wakes is at a point of development where we need to standardise it in some way. For definiteness, let us call the present version Program A.

Program A takes as input a file containing values of the density $\rho(z)$. It needs no other input. It solves the ODE (2.5) of Appendix 1, using the Runge-Kutta-Nystrom method. *Note: All equation numbers relate to that Appendix.*

The solution is done only once for each k, σ, namely yielding the $W_1(z)$ function which satisfies initial conditions (3.1) at the bottom $z = -H$. When we reach the top $z = 0$, we immediately evaluate

$$D(k, \sigma) = W_1'(0) - \sigma W_1(0)$$

which happens to be the same as that defined by equation (3.3) when $z = 0$ and $h = 0$.

Thus we have a function subroutine for $D(k, \sigma)$. This is now used only to find its zeros. That is, we use the secant method to solve the equation $D(k, \sigma) = 0$. This can be done either by finding the value of σ at fixed k or (as in the report) by finding the value of $k = K(\sigma)$ at fixed σ. Each internal wave is represented by a separate function $K_j(\sigma), j = 0, 1, 2, \ldots$ and we can find each solution separately numerically. However, we probably are only interested in $j = 1$.

Once $K(\sigma)$ is determined, the only thing the program presently does is to plot it.

Program structure

Presently a collection of modules. Namely a main program "SEARCH", and subprograms "DET", "FIND INTERVAL" "RTSEC", "DENSITY".

These are linked by files called MAKEFILE, COMP etc. which do some things Mike should write up (2-3 paras). There is also a library called "K S GRAPH" and some other bits and pieces.

The important subprogram is "DET". This does the ODE solving, by calling a subroutine called "NYST", which has a subroutine called "RHSF" for the right hand side of the 2nd order ODE.

Program B+

Over the next 6 days we need to extend this program. First (and this is done already) we need to add determination of $W_2(z)$. This is done by duplicating and extending "DET". Keep "DET" almost as it is, it is still needed. Write a new subroutine "WFUNCTS" which computes both $W_1(z)$ and $W_2(z)$. This is all it does, and it outputs these functions and their derivatives. It could also compute the Wronskian $W_1(z)W_2'(z) - W_2(z)W_1'(z)$ if we wanted to do that in this subroutine, though perhaps that is better left for later.

Importantly, all the above is done with *only* the density ρ as input, together with the parameters k, σ. (And some numerical or tolerance parameters?). We do not yet have to specify anything about the submarine, in particular not about its depth h of submersion.

Now we specify h. (It has to be one of the z values for which we have done the computations of W_1, W_2). Then we use (3.4) to set up $W_0(z)$. That should be a subroutine all of its own.

Now turn to (4.3). That contains a function

$$A(z,\theta) = W_0(z)K/D_k$$

which we must compute. The θ dependence comes about because all 3 quantities on the right depend on k and/or $\sigma = \kappa \sec^2\theta$.

Here W_0 is just computed, and $K(\sigma)$ is found by Program A, and D_k is the partial derivative of D with respect to k at fixed σ, evaluated at $k = K(\sigma)$. I think the subprogram "DET" can be modified to yield D_k, since differences of D are already used in the secant method. Note that we only need W_0 and D_k with $k = K(\sigma)$, not with a general k and σ.

Simultaneously (Graham) we need to push the program forward on a couple of other fronts, namely to go from the $K(\sigma)$ determination (subprogram "DET") toward kinematic aspects of the wave pattern (i.e. those not needing to know the actual wave-generating efficiency of the submarine) One immediate objective is the Kelvin stationary phase angle, thus the γ versu θ information of equation (5.4). Other interesting matters are generalisations of what is in the Tuck, Collins and Wells (1971) Journal of Ship Research paper. In particular, generalisations of the frequencies (Fig 6) observed by Fourier transforming a one-dimensional cut, and the "ridge" or expected location of peaks in the double Fourier transform of a two-dimensional record (Fig 7).

Appendix 5: 2 March 1992.

EOT Summary

Here is my summary of the submarine wake program so far.

My full-time involvement began 10 February 1992 and is due to end 27 March 1992. The initial team with full-time involvement consisted of Tuck. Furnell. Legg and Madurasinghe. from DSTO. plus Carroll on contract from Ebor as programmer. Part time involvement or overseeing role is provided by Cartwright. Haack. Webster. Marwood. etc.

Division of responsibility initially seemed to be Tuck for general ideas. Furnell for long-term DSTO carrying on of these ideas after 27 March (and hence close liason with Tuck before 27 March). Legg for input and (especially graphic) output considerations. Madurasinghe for the numerical integration end phase of the numerical work. Carroll does programming directly as required by Tuck.

The team immediately seemed too large to me. Hence at my suggestion. the immediate working team was reduced to Tuck. Furnell. Carroll. at least for the first few weeks. This report summarises progress by that team.

For the first two weeks. the team was mostly feeling its way and getting to know each other. I arranged to transfer some very crude computer programs here. and Carroll started to modify these. He was at first looking at ways to improve the ODE solving aspect. using non-uniform grids. but this proved a non-useful approach. especially since we intend to use a uniform density grid. so was abandonned in favour of retaining the Runge-Kutta-Nystrom method in my original program.

I also brought some Macintosh programs which used graphic input and output techniques. These were of interest mainly to indicate to all members of the team what sort of thing we wanted.

At this stage. Furnell was mostly concerned with parallel development of analytic models for step-wise exponential density. ma y to use as checks on the general program.

On 19 February. I gave a seminar. After this. the focus of the work was clearer. and the reduced team set out with more specific tasks. Carroll cleaned up the program as it presently stood. and this was designated "Program A" on 27 February. This program takes a given density profile as input data. and computes the dispersion relation $k = K(\sigma)$ of the internal waves for that stratification. It does so by solving the ODE (2.5) for $W_1(z)$ (numbers refer to the Tuck preliminary note "Source in Stratified Fluid". Appendix 1). Note that it does not solve for W_2 (see (3.2)) yet. The Wronskian $D(k.\sigma)$ is computable at $z = 0$ (for submarine depth $h = 0$) (see (3.3)) without the need to compute W_2. Then the equation $D(k.\sigma) = 0$ is solved for k at each fixed σ by the secant rule.

This dispersion relation is of independent interest. We have devoted some time to understanding the nature of the k versus σ curves for idealised and actual stratifications. For realistic actual stratifications. the σ values are very large. typically measured in thousands. when k is of the order of unity. The first internal wave starts with $k = 0$ at some $\sigma = \sigma_1$ value of this large size. then k increases monotonically with σ. eventually becoming large when σ is large. asymptotically like the square root of σ. Similarly. the second internal wave starts with an even higher value of $\sigma = \sigma_2 \approx 9\sigma_1$. and has a k value always smaller than that of the first wave. etc.

There are many consequences for the final form of the internal waves of this character to the dispersion relation, and Furnell is pursuing some of these for realistic and idealised stratifications. These include the nature of the Kelvin wake pattern, in particular its angle (narrow wakes, mostly diverging waves, etc.), and some aspects of Fourier-analysed detection as in the Tuck, Collins, Wells (1971) paper. Note that this work can proceed even without a knowledge of the wave-amplitude generation capability of the particular body (submarine) making the waves.

The next phase of development of the program to determine actual wave amplitudes is to develop Program A further, to compute W_2, W_0 (see (3.4)) and a quantity which appears in (4.3) but is not given a specific symbol there, namely

$$A(z;\sigma) = kW_0(z;k,\sigma)/D_k(k,\sigma)$$

Actually as written, this quantity appears to depend on k too, but we only want its value when k satisfies the dispersion relation $k = K_j(\sigma)$. This A is the amplitude of the wave generated by a unit source at the current value of the wave-speed parameter σ. Note that if the actual wave-speed is c then $\sigma = g/c^2$. Also note that since $c = U\cos\theta$ where U is the submarine speed at an angle θ to the wave direction, this confirms (2.6), namely $\sigma = \kappa\sec^2\theta$, with $\kappa = g/U^2$. Ultimately we shall consider $A = A(\theta)$ and integrate (4.3) with respect to θ, so summing up internal waves of all directions.

Note that there is a separate contribution to the integral (4.3) for each internal wave mode j. Program A simply computes as many internal waves j as we specify. But we shall mostly only be concerned with one or at most two.

An important ingredient in the determination of A is the partial derivative D_k of the Wronskian with respect to k at fixed σ. This is the slope of the D versus k plot as it crosses $D = 0$. All three ingredients k, W_0, D_k of A are presently being computed, and computation of A is being done, though at time of writing insufficient checks have been made on it.

Next step (assuming that A for a source is computable) is to move from a source to a submarine. First we do a Rankine ovoid, namely a source-sink pair separated by a distance of the order of the submarine length. When that length L is large compared to the maximum radius R, the required source strength is (according to slender body theory) $U\pi R^2$. That is, we take the unit source result, already computed, multiply it by U times the maximum cross section area πR^2, and subtract the same thing shifted by a distance L aft.

In fact, doing this explicitly in (4.3) gives a result where the "sin" is replaced by a "cos", and the "A" for the unit source is multiplied by

$$\left[U\pi R^2\right] \cdot \left[2\sin\left(\frac{1}{2}K(\sigma)L\cos\theta\right)\right]$$

the first square bracket being the source strength, and the second the source-sink separation factor. An interesting formal special case is the limit as $L \to 0$ and $R \to \infty$ with R^2L bounded, which yields a submerged sphere, with the "sin" of the second square bracket replaced by its argument. This may be a useful test case to compute first.

Appendix 6: 5 March 1992

Short wave limit of dispersion relation

Make the Boussinesq approximation and write $\nu = -\rho'/\rho$. Then the ODE (2.5) of Appendix 1 becomes

$$W'' + [\sigma\nu(z) - k^2]\,W = 0$$

There is no hope of a solution satisfying reasonable top and bottom conditions unless the coefficient in square brackets above takes *at least some* positive values. After all, if it is wholly negative the solutions are exponential-like, and can only have one zero, whereas when the coefficient is positive, the solutions are sinusoidal-like, and can have lots of zeros. [I believe that there is a theorem to that effect for ODEs, but it is intuitively obvious anyway.]

Now if we let $k \to \infty$ at fixed σ, eventually the coefficient must become wholly negative for all z. Hence there will be no solution satisfying the boundary conditions. Therefore, σ must increase with k at such a rate as to keep at least some positive values for the coefficient. This will happen last (as we increase k at fixed σ) at those z values where $\nu(z)$ is greatest.

Hence suppose the maximum value of $\nu(z)$ is ν_m and that it occurs at $z = z_m$. Then it is clear that as k and σ become large together,

$$\sigma\nu_m - k^2 \to 0$$

or

$$k \to \sqrt{\sigma\nu_m}$$

Thus the large-k limit of the dispersion relation depends only on the behaviour of the stratification at the depth where the density is varying the most.

We can improve on this estimate as follows. Suppose in the neighbourhood of the maximum density gradient, the stratification is of the form

$$\nu(z) \approx \nu_m - \mu(z - z_m)^2$$

for some positive constant μ. This is a quite general relationship since $\nu(z)$ must have a maximum at $z = z_m$. Then suppose $-\epsilon$ is the correction to k^2, i.e.

$$k^2 = \sigma\nu_m - \epsilon$$

Then, writing $t = z - z_m$, the above ODE becomes approximately (locally to $t = 0$)

$$\ddot{W} + [\epsilon - \sigma\mu t^2]\,W = 0$$

and must be solved subject to $W \to 0$ as $t \to \pm\infty$.

This is a standard Hermite or parabolic cylinder function ODE, and for example has the simple exponential solution

$$W = e^{-\epsilon t^2/2}$$

if $\epsilon^2 = \sigma\mu$. More generally there are solutions which are polynomials of degree $j - 1$ times the above exponential, providing

$$\epsilon^2 = (2j - 1)^2\sigma\mu$$

This is the j'th internal wave.

Thus as $\sigma \to \infty$,

$$k^2 = \sigma\nu_m - (2j - 1)\sqrt{\sigma\mu} + O(1)$$

or

$$k = \sqrt{\sigma\nu_m} - (j - \frac{1}{2})\sqrt{\frac{\mu}{\nu_m}} + O(\sigma^{-1/2})$$

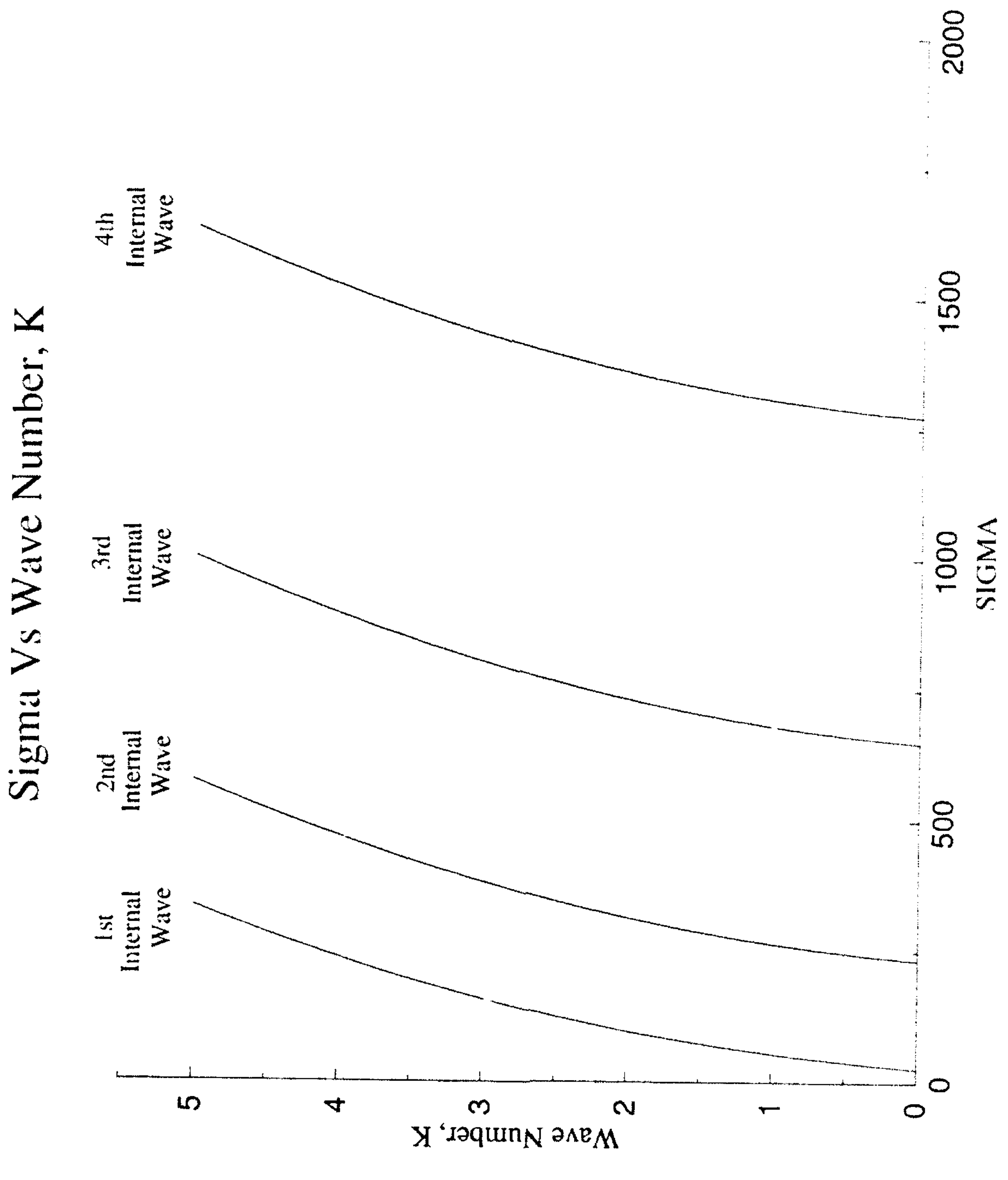
Hence (asymptotically) each mode is obtained from the previous one just by a constant down shift in k of magnitude

$$\Delta k = \sqrt{\frac{\mu}{\nu_m}}$$

This answers one or two of the questions in Appendix 3. In particular, in the last equation of that appendix, $C = \sqrt{\nu_m}$ and $k_j = \Delta k$ (independent of mode number j). I don't know what σ_j^* is: that is a higher order effect for smooth stratifications such that $\nu(z)$ has a simple maximum at an interior point $z = z_m$. Experience with nonsmooth stratifications (e.g. stepwise constant $\nu(z)$) or with those where the maximum density gradient occurs at an end point, or with empirical formulae like Phillips's, tends to suggest erroneously that an apparent rightshift in σ is the leading-order correction, rather than an apparent downshift in k.

[Figure 6.1 gives samples of our computation of the $k - \sigma$ curves.

Figure 6.1: Typical dispersion relation – several modes.



Appendix 7: 11 March 1992.

What submarine?

Most of our work so far has been on the waves made by a single isolated point source of unit strength. The "final" result is equation (4.3) of Appendix 1, namely

$$w = -\frac{2}{\pi}\int_0^{\pi/2} \sin(kx\cos\theta)\cos(ky\sin\theta)A(k)d\theta$$

where $A(k) = W_0 k/D_k$.

At this stage (Program B) we are computing $A(k)$ successfully for the unit source, for various stratifications and source depths.

Note that the above form of (4.3) does not distinguish modes j, the results being assumed computed separately for each mode and summed at the end. I have just written k for $K_j(\sigma)$, assuming that if the above integral is being carried out with respect to the theta variable, then we work out $\sigma = \kappa\sec^2\theta$, then call upon our program to evaluate the corresponding k, etc.

A serious alternative being considered at the moment is to convert directly from a θ integral to a k integral, i.e. set $d\theta = (d\theta/dk)dk$, working out

$$\frac{d\theta}{dk} = 1/\left[\frac{dk}{d\sigma}.2\kappa\sec^3\theta\sin\theta\right]$$

which demands output from the program of the slope $dk/d\sigma$ of the dispersion relation, which is not too hard to get.

Importantly, the integral with respect to θ self-truncates at its lower end, indeed only occupies a small range near $\theta = \pi/2$ for realistic small stratifications. That is, the generating amplitude A is identically zero for $\sigma < \sigma_j$ (the lowest value of σ for each mode), and σ_j is a large number, of the order of thousands. Hence the actual lower limit of integration is $\theta = \theta_j$ where $\sigma_j = \kappa\sec^2\theta_j$ or

$$\theta_j = \frac{\pi}{2} - \arcsin\sqrt{\frac{\kappa}{\sigma_j}}$$

which is very close to $\pi/2$ unless κ is very large (slow sub, but that will make small waves anyway).

This is all well known, and commented upon by most authors. It means that internal waves are largely diverging rather than transverse, i.e. their crests are parallel to the sub's track. But there are a number of important consequences.

First note what happens when we make the change of variable from θ to k. Then the range of k is fully from 0 to ∞ (corresponding to $\theta = \theta_j$ to $\pi/2$). Meanwhile the x-wise wave number $k\cos\theta$ is going from 0 to a finite upper bound, whose magnitude (see Appendix 6) can be shown to be $\sqrt{\kappa\nu_m}$, a small quantity. The y-wise wave number is essentially k itself, and goes from 0 to ∞. The integral converges subject to only mild restrictions on A because of the fact that $d\theta/dk$ tends to zero like k^{-2} (which also follows from Appendix 6 after some manipulation).

Consider first the Rankine ovoid representation of a real submarine of length L and volume V. According to slender body theory, this is a cylinder of uniform cross-section area $V/L = \pi R^2$. This holds so long as $R << L$.

But now let us examine what happens when we start to evaluate the integral for the waves. As discussed at the end of Appendix 5, all that happens relative to the source integral is that the $\sin(kx\cos\theta)$ gets replaced by a "cos", and the amplitude factor A is multiplied by

$$2\sin(\frac{L}{2}k\cos\theta)$$

We just showed that the largest value that can be taken by the x-wise wave number $k\cos\theta$ is the small quantity $\sqrt{\kappa\nu_m}$. Hence the argument of the above sin function can never exceed $\sqrt{\kappa\nu_m}L/2$.

A rough estimate for realistic stratifications is $\nu_m = 10^{-7}\mathrm{m}^{-1}$, so this quantity comes to about $0.005L/U$ where the submarine length L is in metres and the speed U in metres per second. I believe that this will be small for all realistic submarine lengths and speeds, at least small enough for its sine to be reasonably replaced by itself.

If that is done, what results is the same as if we had let $L \to 0$. In other words, all submarines of the same volume generate the same internal waves, irrespective of their length/diameter ratio! Hence we can assume the submarine is a sphere, which is generated by a dipole.

The actual effect is to replace the sine in (4.3) by a cosine and to multiply the amplitude function A by the x-wise wavenumber $k\cos\theta$, and the final result by just the product of the speed and volume of the submarine UV. Yet another way to see this is to note that the waves due to a dipole are proportional to the x-derivative of the waves due to a source.

In any case, we should compute some waves due to dipoles and (separately) some due to Rankine ovoids of the same volume, and see if there is any difference. Dipoles first.

Appendix 8: 12 March 1992.

A review of the DREP report by T.W. Dawson

The 1988 Canadian report DREP TM88-7 by T.W. Dawson is very relevant to our project. This note is a summary of some important features of it.

Overview

DREP TM88-7 is about 90 pages long, and very detailed. It is an impressive document in its accuracy and coverage. It reads like a Ph.D. thesis, and I would rate it as deserving of a Ph.D. if it was one. On that basis and others, I assess its cost as about 3 high-level man-years, though this could be conservative, since there are references to supporting work that was obviously done in parallel. The cost of the research directly reported in TM88-7 alone must therefore exceed a half million dollars; if the supporting work is also included, perhaps the total cost would run to several million dollars. Compare our (approximately) 6-week times 4-person effort!

The good and bad news is that this report is highly relevant to our project and uses a comparable methodology. I do not believe that we will be able to come up with anything significantly better than the Canadians. Hence one serious alternative strategy would have been simply to buy in their finished product. If that was not possible, we could have attempted to duplicate it using TM88-7 as our guide. I hope to make it clear that the latter would have been infeasible in the time available.

That is probably the bad news. The good news is that we are on the right track in our independent approach. In particular, I am absolutely amazed that my preliminary note "Source in a stratified fluid" (call it EOT92-1 here; it is reproduced as Appendix 1 of the final report) which was written without benefit of studying TM88-7 first, describes a methodology that is quite like Dawson's. There are many other things one could do, and some in our group thought we might be doing them, ranging up to supercomputer Navier-Stokes solutions. But I chose a more conservative approach, and so did Dawson. We could both be wrong, of course, but there is some experimental evidence at the end of TM 88-7 that Dawson's results are good, which is another piece of good news.

Summary of TM88-7

1. "Introduction." A nice summary of the problem. Note (p1) the discussion of two separate generating mechanisms, "wake collapse" and "hull effect", and an important remark that the methodology allows both mechanisms to be studied, although the illustration is only for the second. This also applies to our approach. The approximations and assumptions (p.2) are as I would make, and (to my surprise, since I thought few others had even heard of it!) Dawson also chose a Rankine ovoid model of the submarine.

2. "Model and basic equations." This repeats derivations that are for example given in Yih's (1965) textbook (see bibliography, Appendix 2). On p.8 the Boussinesq approximation is introduced. I do not propose to use this approximation explicitly, though I am sure that it is a good approximation for realistic stratifications. However, as a practical matter, it does not provide any numerical simplification to the ODE that has to be solved for the vertical distribution of velocity, and hence need not be made except as an analytic tool. The comparison is between my EOT92-1 equation (2.5) and Dawson's TM88-7 equation (3.4) (subject to (3.2)). My equation reduces to Dawson's if $(\rho W')'$ is replaced by $\rho W''$, which is the Boussinesq approximation, valid in the limit of small stratification. Note that his $N^2 = -g\rho'/\rho$, and my $\sigma = g/c^2$.

3. "General solution of the basic equations." One difference between our approaches is that Dawson uses rigid lid conditions $W = 0$ both at the "free surface" $z = 0$ and the "bottom" $z = d$ (his z is downward). I use neither. My free surface is a genuine free one, where $W' = \sigma W$, and my "bottom" is not really a bottom, but rather a level where the density becomes uniform, and is assumed uniform for ever below in an infinite depth ocean. Neither of these differences is too important in practice. The apparent rigidity of the free surface is consistent with the Boussinesq approximation; the bottom condition is probably not too important in practice unless the submarine is close to it.

Dawson's equation (3.11) looks familiar to me. It is closely connected with my EOT92-1 (4.3), though arising very differently. In both cases, the formula involves in its denominator, the partial derivative with respect to a wave parameter, of the Wronskian between two solutions of the fundamental homogeneous ODE. These solutions are in general linearly independent (so the Wronskian is non-zero) but are of main interest to us when dependent (so the Wronskian is zero). In that case, the wave number k and my parameter σ (equivalently the wave speed c) are connected numerically by the dispersion relation. Dawson avoids actual differentiation of the Wronskian by some manipulation around p. 13, but I don't see any problem doing it by brute force numerical differentiation.

The summation over modes in (3.11) and throughout TM88-7 represents the major point of departure between the philosophies of Dawson and me (and I think a number of others are with me on this). TM88-7 adopts the view that one must compute *many* internal waves. There are infinitely many. They are ordered with respect to speed, there being a maximum-speed mode $j = 1$ (minimum σ in my notation), then a slower mode $j = 2$ (larger σ) etc. Roughly speaking, the speeds go down like the inverse of the mode number j (σ goes up like j^2). Since σ is already rather large even for the first mode $j = 1$, the actual σ numbers for higher modes pretty quickly get huge. It is my view that almost all that one wants to know about internal wave generation is available by assuming that only the first mode is generated. We can calculate several modes if we want to; the results are in any case just additive, so we can test this hypothesis easily; I wonder if Dawson did?

4. "Uniformly translating source: frequency inversion." Dawson has assumed a general source distribution doing somewhat general things up to now. Now he specialises to steady horizontal translation, as I have done from the outset, and first does an isolated source.

5. "Extraction of steady-state fields." More specialisation.

6. "Evaluation of k_r integrals". It appears that Dawson is ambitious enough to try to evaluate the whole thing. That is, in my terminology, he evaluates not only the free wave single-integral contribution from the residues at the poles $k = K_j(\sigma)$, but also the double integral local disturbances near the source. As I say on p. 4 of EOT92-1, this is a truly "mammoth task". I can only have admiration for Dawson for attempting it and more. Perhaps we in the ship-hydrodynamic community have been too conservative, but I do have my doubts about whether one can really do it with accuracy and efficiency. J.N. Newman has been quite scathing in the past about some attempts. Anyway, if one is mainly interested in the far field, the free-wave integral EOT92-1 (4.3) should suffice. Here I am calling upon decades of ship-hydrodynamic experience, with a slight worry that some of this experience might not translate directly to internal waves.

7. "Numerical considerations". This is a very long and detailed chapter. It describes Dawson's technique for solving the ODE, his (3.4), equivalent to EOT92-1 (2.5). Namely, simply represent the ocean by many layers, in each of which the density is assumed to vary exponentially, so N^2 is constant. Fit these layers together with suitable continuity conditions, and you have a tool for ODE solution.

Much of this chapter is taken up with concern for the accuracy of this technique related to things like "evanescent modes" (complex eigenvalues), and close approach of the real eigenvalues for one mode to those for another.

I believe to a certain extent these worries are an artefact of the method Dawson is using to solve the ODE, though they are a warning to us also. We are using a direct numerical solution method of Runge-Kutta type (not for any reason other than ease of programming) and I don't think we are seeing any similar difficulties.

Figure 2 on page 38 is very important. It provides a benchmark density distribution (indirectly via the Brunt-Vaisala frequency $N = \sqrt{g\nu}$). We are using it to recover the actual density ρ by integration and exponentiation, and then inputting that to our program. This is a bit silly, since in effect our program just differentiates ρ immediately to give ν again, but is justifiable on convenience grounds. After some effort, I think we have quite good data for ρ at intervals of 10 metres, as used by Dawson in his Figure 2, and also interpolated to 5 metre intervals. This density data is shown as a Figure here.

Dawson's Figure 3 on p.41 is one that we should be able to reproduce. It is in effect the dispersion relation plot, equivalent to our k versus σ plots. Indeed there is considerable equivalence. Dawson's variable s is proportional to our σ, and (see his eq. (.13)) his p is proportional to k^2/σ. So, apart from some possible confusion over the proportionality coefficients, once we find k versus σ, we have his Figure 3.

Of course, there are 21 modes on his Figure 3; we are rather more modestly thinking of computing about 3 modes! One difficulty I find with Figure 3 is that I can't tell which mode is which. The lowest curves should be the earliest modes, but the first 3 of these are incomplete. The logarithmic scale doesn't help. Anyway, the sooner we attempt this comparison the better.

The rest of Chapter 7 is special numerical techniques, and perhaps not too interesting. Figure 7 (p.64) starts to be interesting again, and all the remaining Figures should be reproducable by us.

8. "Wake examples". Here on p.71 Dawson introduces the Rankine ovoid idea, and gives a concrete example with submarine-like dimensions. He then gives final results for that example, at several speeds and depths of submersion, in the stratification of his Figure 2. Figures 12-14 are 3D plots which we should be able to reproduce of the surface currents. However, the actual scale of these plots is not easy to pick up, though there are Tables of extreme values provided.

In the caption to Figure 12, the "local" effect near the submarine is noted. We can't reproduce that, since we neglect the local effect via the double integral term in the wave field. If the local effect is as small as that shown in Figure 12, this should not be too much of a worry.

9. "Comparison with measured data". Figure 16 is a very interesting comparison between computation and experiment, the experimental internal waves being ship-generated. The agreement is very encouraging, both qualitatively and quantitatively. We could not directly reproduce this without more information, presumably obtainable from Reference 1, which is a DREP report of 1985. I would like to know how many internal wave modes were significant, and whether "local" effects (the double integral terms) were significant.

10.+ "Concluding remarks, Reference and Distribution Lists." Useful. The US reports under the name Milder seem likely to be interesting. There are some interesting inclusions (Cartwright, Reed) and omissions (Tulin) in the distribution list.

Appendix 9: 13 March 1992

Direct θ-integration

After some thought, I have decided that (at least for the time being) we will do the θ integration directly, i.e. without the change of variable to k canvassed in Appendix 7. The alternative is still worth considering, but let's go with θ-integration first.

Hence the vertical velocity made by a Rankine ovoid of length L and maximum radius R moving at speed U is given by

$$w/U = -\frac{2V}{\pi}\int_{\theta_m}^{\pi/2}\cos(kx\cos\theta)\cos(ky\sin\theta)\left[\frac{\sin(k\cos\theta L/2)}{L/2}\right]A(k)d\theta$$

where $V = \pi R^2 L$ is its volume.

The lower limit of integration is θ_m, which is the minimum θ value, corresponding to the minimum $\sigma = \sigma_m$ value for each (separate) internal wave mode j. Namely

$$\theta_m = \frac{\pi}{2} - \arcsin\sqrt{\frac{\kappa}{\sigma_m}}$$

We assume for validity of this formula that $\kappa < \sigma_m$, which is not a severe requirement since σ_m is numerically large. This is what Tulin and Miloh (1990: see bibliography, Appendix 2) call (misleadingly) the "supersonic" range. This just means that the submarine is moving faster than the fastest internal wave, namely faster than about one metre per second. Absence of small-θ contributions means absence of transverse waves. If the submarine is moving so slowly that this condition is violated ("subsonic" motion in the sense of Tulin and Miloh 1990), the generated wave amplitude is probably too low to detect anyway. However, if we did need to compute it, we would just set $\theta_m = 0$, since then all θ values occur, and there are transverse waves.

Since σ_m is large, θ_m is very close to $\pi/2$. The range of integration above is therefore a very narrow one with respect to θ. I think this is good news for convergence of the integration. Actually, that convergence is also illustrated by the k transformation, which leads to an integration on an infinite k-range, but with k^{-2} rate of decay of the integrand. In some sense this decay rate with respect to k is equivalent to the small θ-range.

Traditionally (for surface waves) we are worried in evaluating this sort of integral about the effect on our integration method of a rapidly-varying character to the integrand. This comes from the trigonometric functions when x and y are large, strictly when kx and ky are large. That occurs when we are in the far field of the disturbance, strictly many wavelengths from it.

There are several reasons to suspect that this won't be a problem in our case. First, the internal wavelengths are much greater than those for surface waves, so at a fixed (perhaps large) x, y, the value of kx, ky may not be so large. Second, there is an actual upper bound on the coefficient of x, namely $k\cos\theta < \sqrt{\kappa\nu_m}$ (see Appendix 7) which is small. Hence there cannot be a rapidly varying character to the contribution from the factor in x, no matter how large x is. Since θ is close to $\pi/2$, the y factor is essentially $\cos(ky)$, and will be rapidly varying when y is large. However, we are rather less interested in large y (far to the side of the submarine) than we are in large x (far astern of the submarine). Finally, there is only one point θ of stationary phase (so long as $\kappa < \sigma_m$) and that is close to $\pi/2$ (of course since all θ values have that property.) If we were to be worried about the neighbourhood of points of stationary phase, we would be worrying about the neighbourhood of $\theta = \pi/2$ where there is no contribution at all from the x factor. The last argument is probably facile: but anyway, all I am arguing is that we should press ahead doing the integration without worrying too much, and the proof of the pudding is in the eating.

In Appendix 7, it was argued that, as generators of internal waves, all submarines of the same volume are indistinguishable, irrespective of shape and length. I think this is an important principle that we can check, most directly for the Rankine bodies by varying their length/diameter ratio $L/(2R)$. In particular, in the limit as $kL \to 0$ (note: length tending to zero relative to wavelength, not necessarily relative to diameter), we recover the internal waves generated by a sphere, where the quantity in square brackets is simply replaced by $k\cos\theta$, and the result is independent of L. It would be very interesting to check this by computations at a sequence of L values.

Some comment on dimensions is appropriate here. Obviously the output w/U must be dimensionless. The volume factor outside the integral has dimensions of length cubed, so the integrand must have dimensions of length to the power -3. This is true since it is not hard to see that A has dimensions of length to the power -2 (see Appendix 1, below (3.4): W_0/D is dimensionless).

I suggest that we use consistent SI dimensions, i.e. input and use L in metres etc. Up till now we have normalised all variables so that the apparent depth of the ocean H is unity. I think this is no longer appropriate in the production program, so we should input the actual depth H in metres, etc. Since all results are proportional to the submarine volume V, that factor need not be inputted. If it is omitted, we are computing the wave made by a submarine of volume one cubic metre: a submarine of volume 1000 cubic metres makes 1000 times that wave. Otherwise, all variables are in true SI units.

Appendix 10: 17 March 1992

Density data

As seen in Appendix GF (a preliminary note by Graham Furnell), the data is not always presented to us directly in the form of density versus depth $\rho(z)$. It may be salinity or temperature profiles, or it may be speed of sound data or even Brunt-Vaisala frequencies whose square is proportional to our intermediate variable $\nu(z)$.

Generally though, we expect to see a $\rho(z)$ that increases with depth (anything else being statically unstable) and our results only depend on the ratio between two densities, so without loss of generality we can assume that the density at the free surface is normalised to unity. Then it will increase with depth and the actual increases are generally less than one percent.

The action takes place in the first few hundreds of metres. There may be a mixing zone for the first few tens of metres where ρ is essentially constant, then perhaps a relatively rapid increase with a peak density gradient in the next few tens of metres (the thermocline), then less dramatic increases in density, with a levelling off at about 200-300 metres to a final density of $\rho = 1.0015$ to 1.0070, (times the free surface value) at least in the examples I have seen.

There are interesting examples where the Brunt-Vaisala frequency $N(z)$ is quoted in radians per second. [Sometimes it is given in cycles per hour, which must be first converted by multiplying by $2\pi/3600$]. One way to proceed with these would be to put this data directly into our program. Instead, however, it is possible to recover the actual density profile from $N(z)$.

Namely, since

$$N^2 = -g\rho'/\rho = g\nu$$

then (normalised to unity at $z = 0$)

$$\rho(z) = \exp(-\int_0^z N^2(t)/g\,dt)$$

We have done this integration for a few sample data sets. The most important is Dawson's Figure 2, reproduced here as Figure 10.1. The consequent density curve is Figure 10.2. A decision has to be made as to where to cut off the data. Dawson shows N as non-zero even down to depths of the order of kilometres. It is a good question whether this data is believable, but in any case, the actual values of N^2 are small, of the order of hundreds of times less than the maximum of N^2, so the density is only increasing very slowly at great depth. We have to do some numerical experimentation to verify this, but I would expect that we would be making very little error if we were to assume that the density was constant whenever z is greater than about 300 metres in the conditions of Dawson's data, and also for some other examples we have seen. There are some cases where there does seem to be significant variation down to order of a kilometre, but I doubt its significance for our problem.

Figure 10.1: Dawson's Brunt-Vaisala frequency.

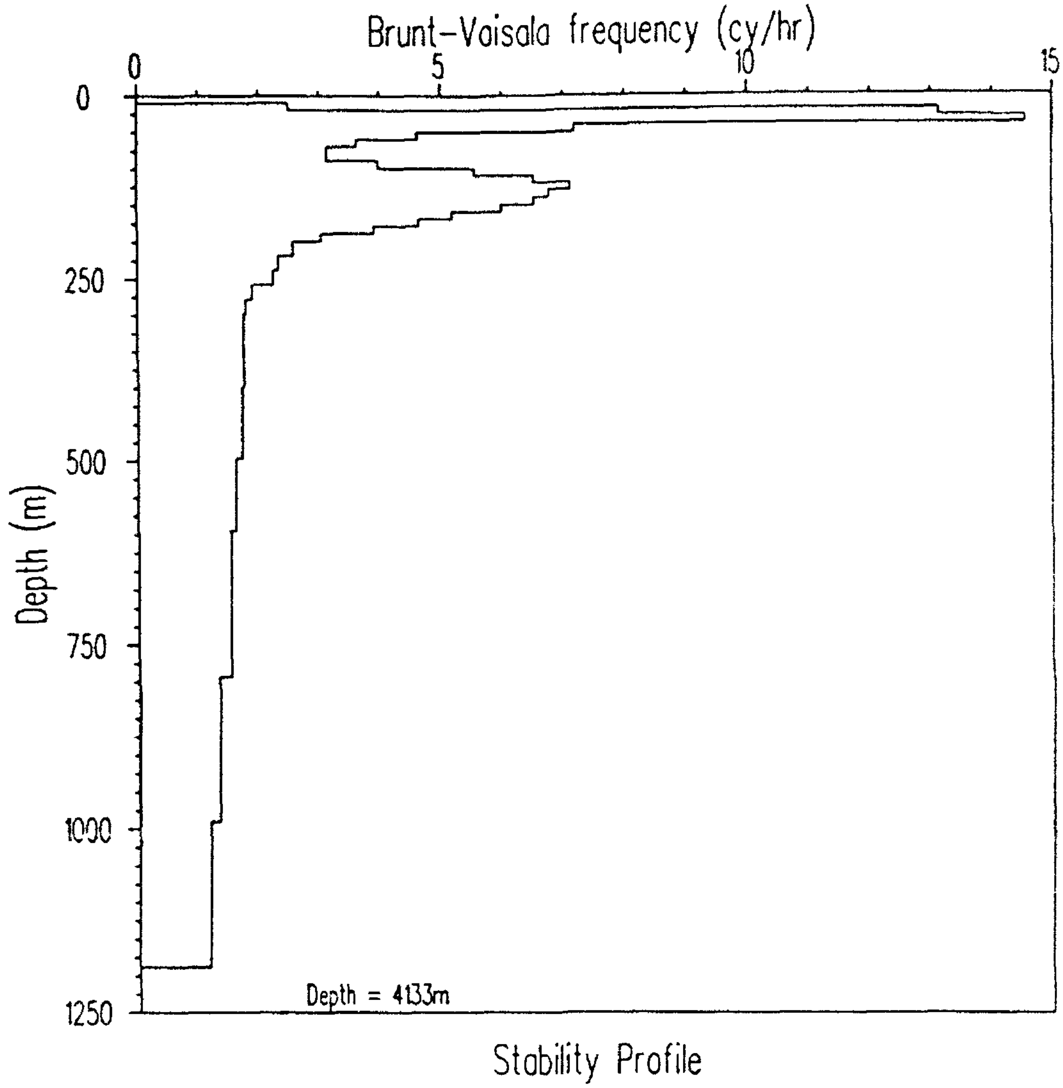


Figure 2. The layered Brunt-Vaisala frequency profile used for the examples in this report. It is based on a smoothed version of measured data taken at Ocean Station P during the summertime.

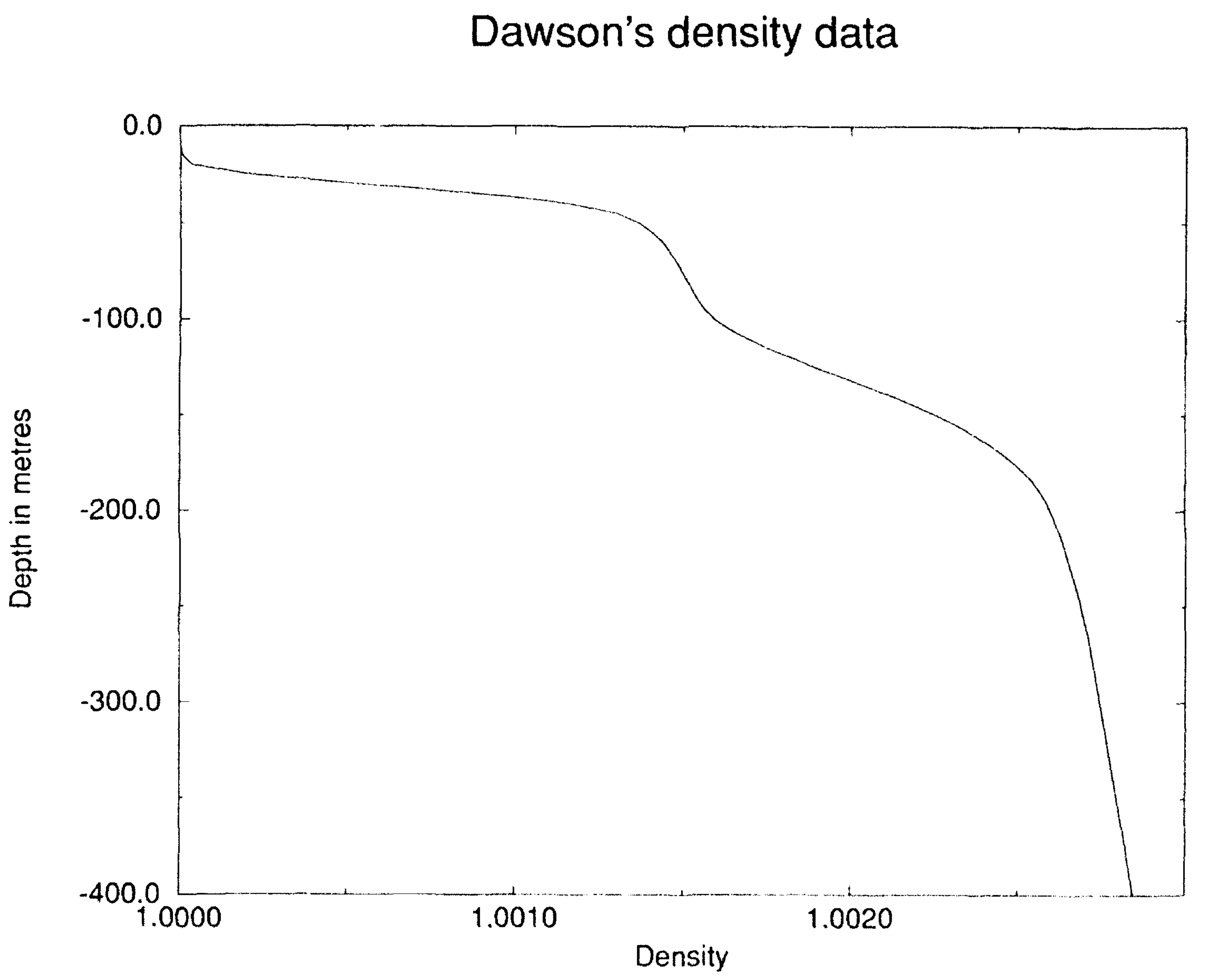


Figure 10.2: Re-constructed Dawson density profile.

Appendix 11: 20 March 1992

Early output

We are now in a position to produce some preliminary output quantities that can and in some cases have been checked against existing results.

In the first place, regarding just the kinematics, namely the dispersion relation, we have checked a number of Dawson's results, using our own re-construction (Figure 10.2) of his density profile. That reconstruction is a potential source of error, but the results seem pretty good, so I think our reconstruction must be OK.

For example, Figure 11.1 reproduces Dawson's Figure 3, as anticipated in Appendix 8. We have to pick a value for "N^2_{max}", as used in TM88-7, p. 37. According to Figure 2 of TM88-7, N_{max} = 14.6 cycles per hour ; at least that is the largest value of the Brunt-Vaisala frequency displayed on that step-like Figure, and we believe it is likely that that was the value used by Dawson for his calculations leading to Figure 3. Interestingly, however, our first choice was to "smooth" the steps in Figure 2, which leads to a "better" estimate N_{max} = 15.9 cycles per hour, since there is a very sharp maximum that is not well captured by the steps in Figure 2. The result was to shift all our results by a small constant amount, which (on the logarithmic scale used) corresponded to the approximately 20% factor in N^2_{max}.

In any case, with what we believe was Dawson's choice of N_{max}, we are always spot on his graph for modes 1 and 2. Dawson gives 21 modes, but only incomplete graphs for the most important first 3 of these. We show (dashed) the complete graphs for modes 1,2 and 3: those for modes [illegible] 2 merge smoothly with Dawson's and extend Dawson's results correctly toward p [illegible], i.e. toward $k = 0$

The sam[illegible]ue for mode 3 with the results shown in Figure 11.1, which was based on use of density data extending down to about a kilometre depth. However, it is worth noting that our first computations using data extending down to only 500 metres depth diverged from Dawson's mode-3 curve near its bottom end. This is because mode 3 has a maximum at about 600-800 metres, and is not adequately captured by 500 metre data. However, the actual generated amplitude (see the sample curve of Figure 11.7) is smaller by a factor of about 10 than the corresponding mode 1 and 2 signal, reflecting the fact that a submarine at about 100 metres depth is not an efficient exciter of a mode whose maximum is much deeper.

As of this date, we are rapidly producing graphs of $A(z, \theta)$. These give a very good picture of dependence on parameters like submarine depth, mode number, for a particular stratification. We are presently using Dawson's, but that is a bit weird, and we should shift to some more realistic stratifications. The Figures to follow are preliminary indications of behaviour of $A(z)$ for various σ (equivalently k) values. The first 4 Figures show mode-1, for various submarine depths. Figure 11.6 is a mode-2 for depth 100 metres, and Figure 11.7 mode-3 for depth 100 metres.

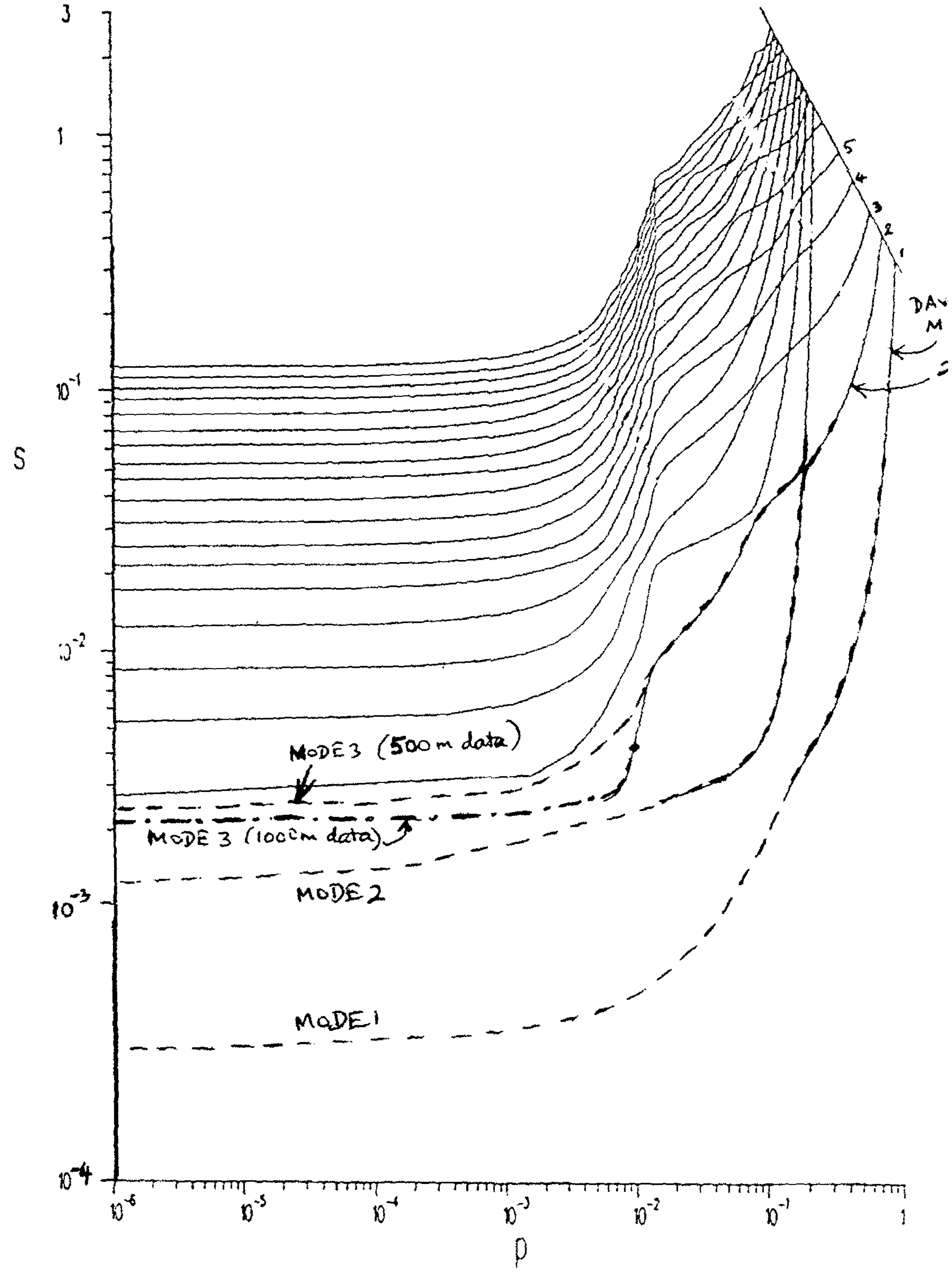


Figure 11.1: Comparison of our (dashed) dispersion relation with Dawson.

Figure 11.3: Mode-1 at $h = 100$.

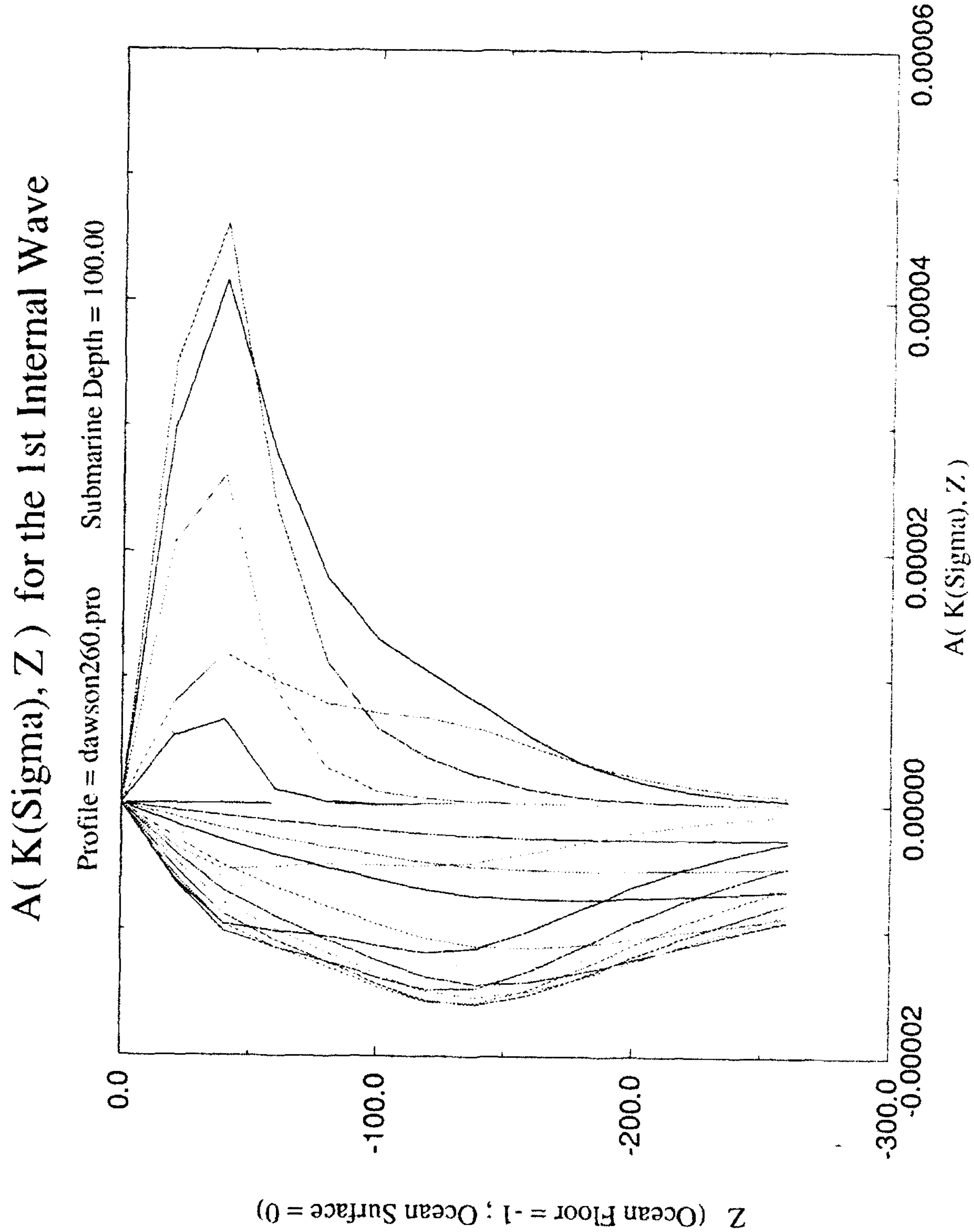


Figure 11.2: Mode-1 at $h = 50$.

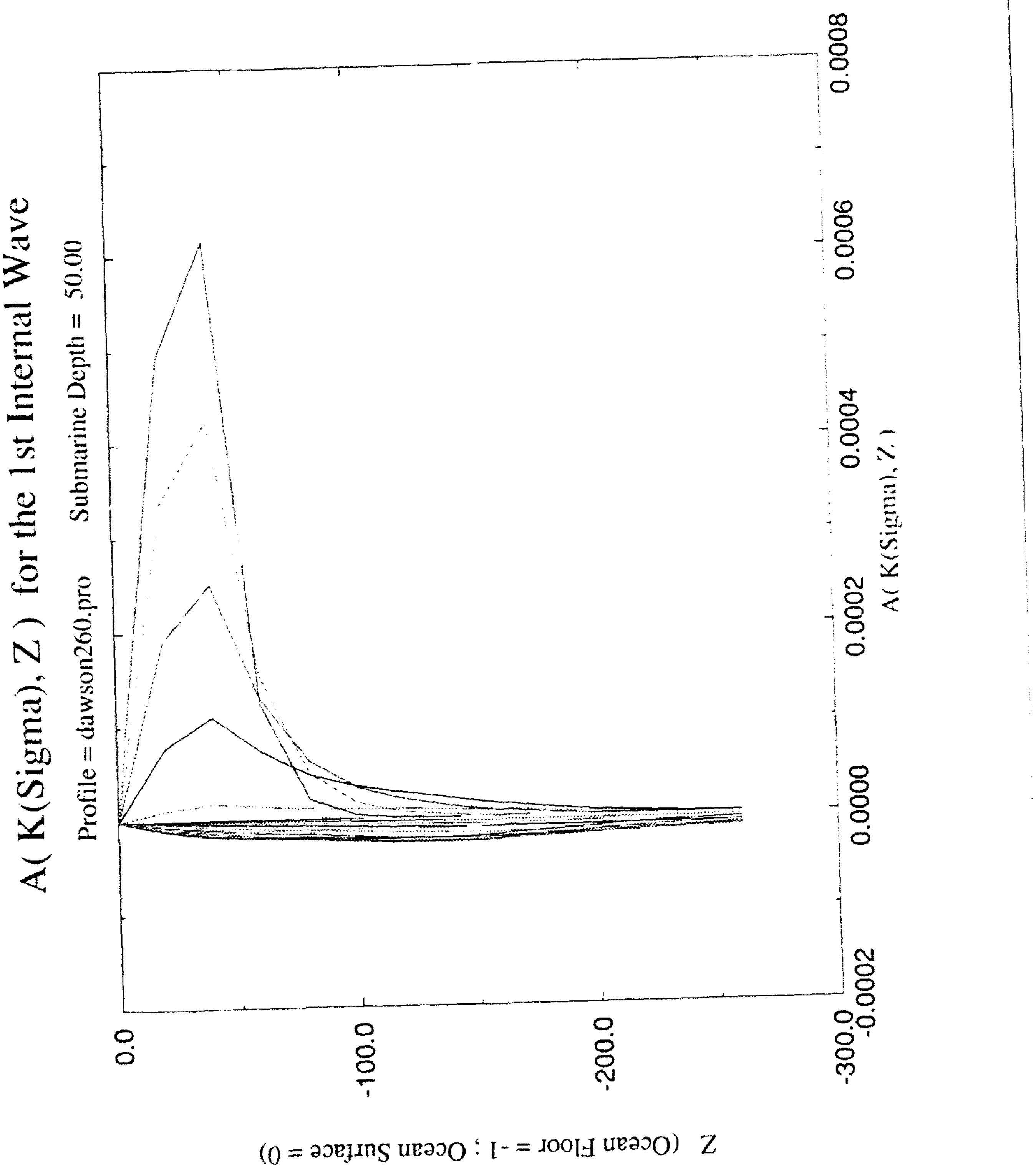


Figure 11.4: Mode-1 at $h = 150$.

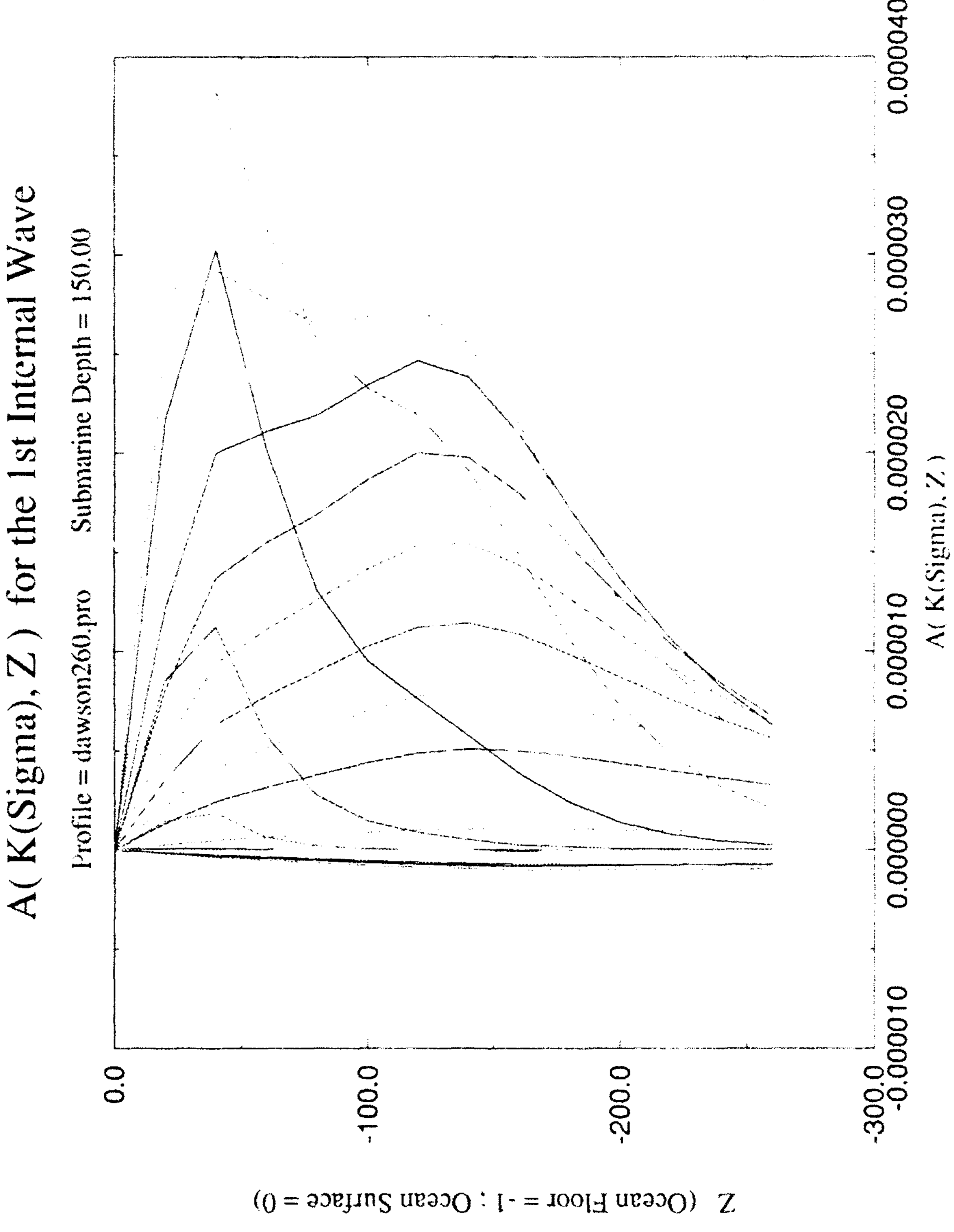


Figure 11.5: Mode-1 at $h = 200$.

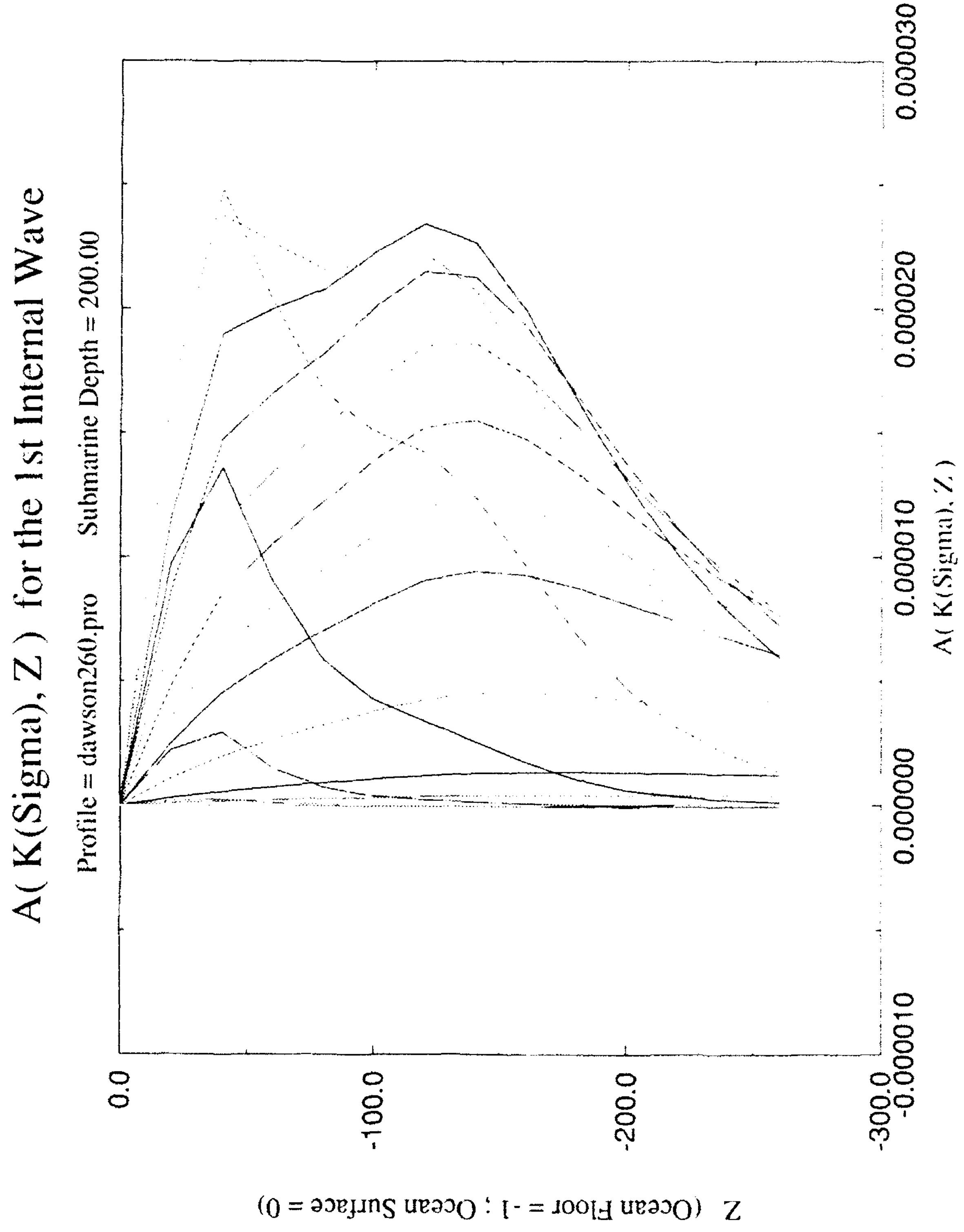


Figure 11.6: Mode-2 at h = 100.

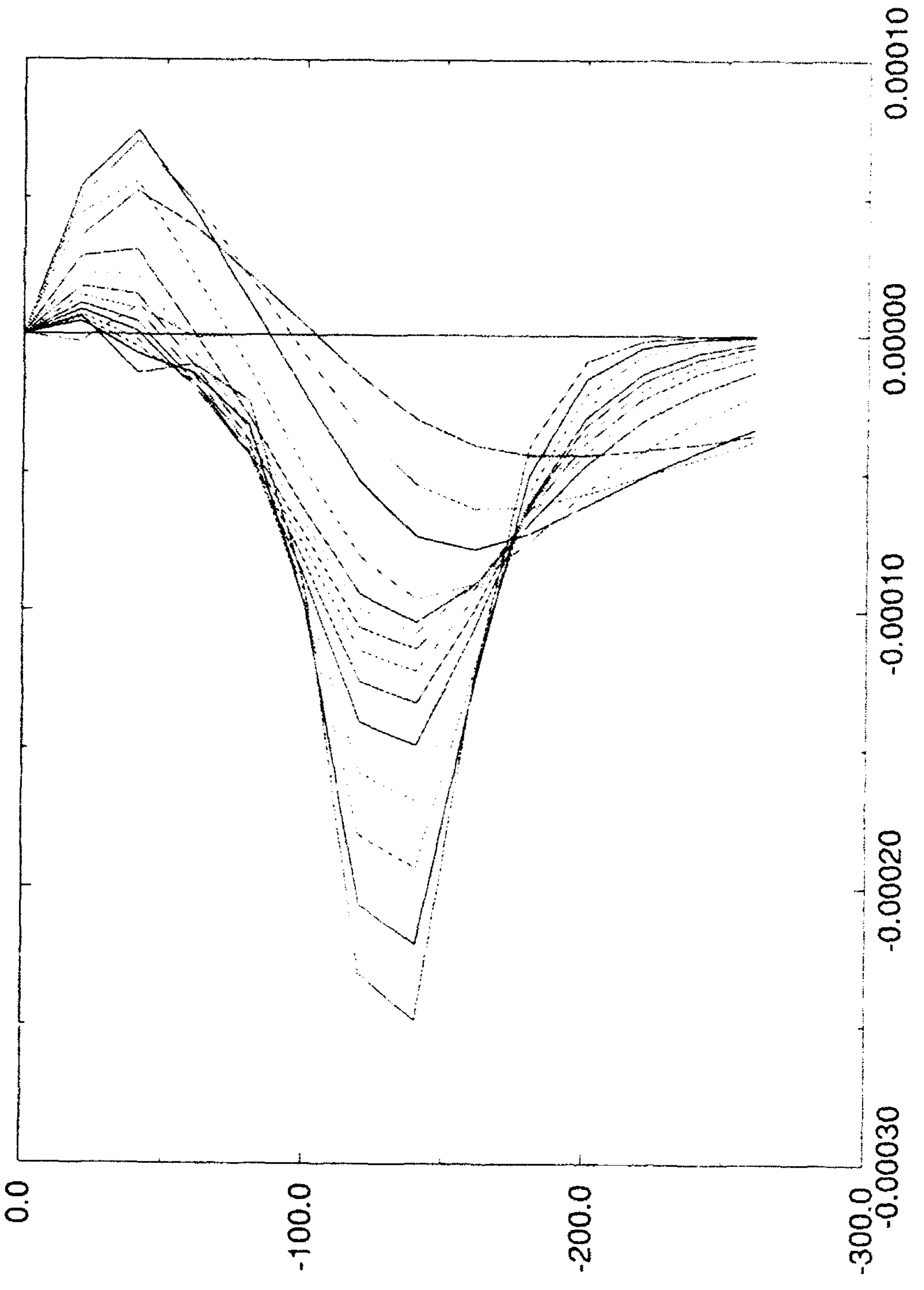


Figure 11.7: Mode-3 at $h = 100$ (260-metre data).

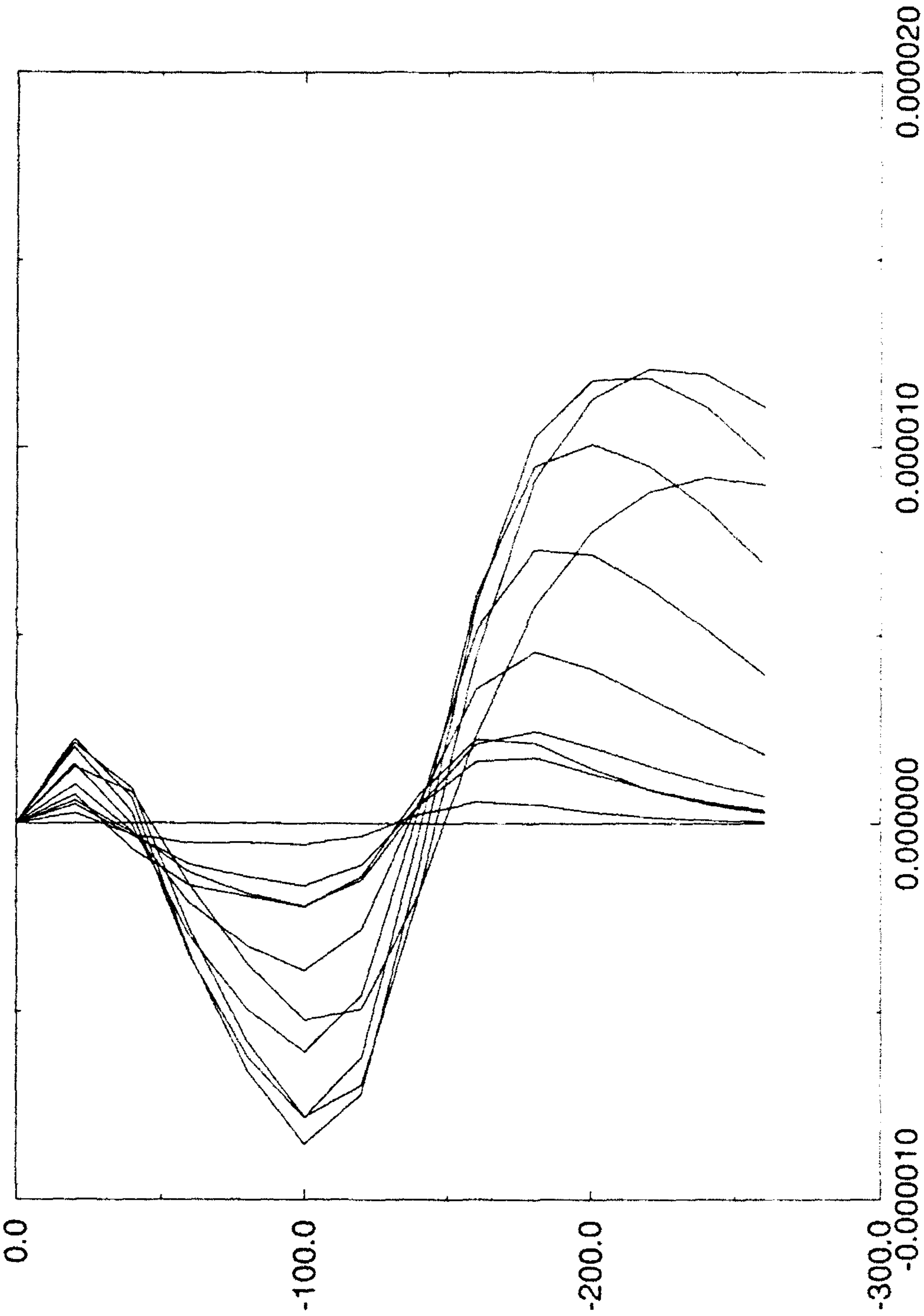
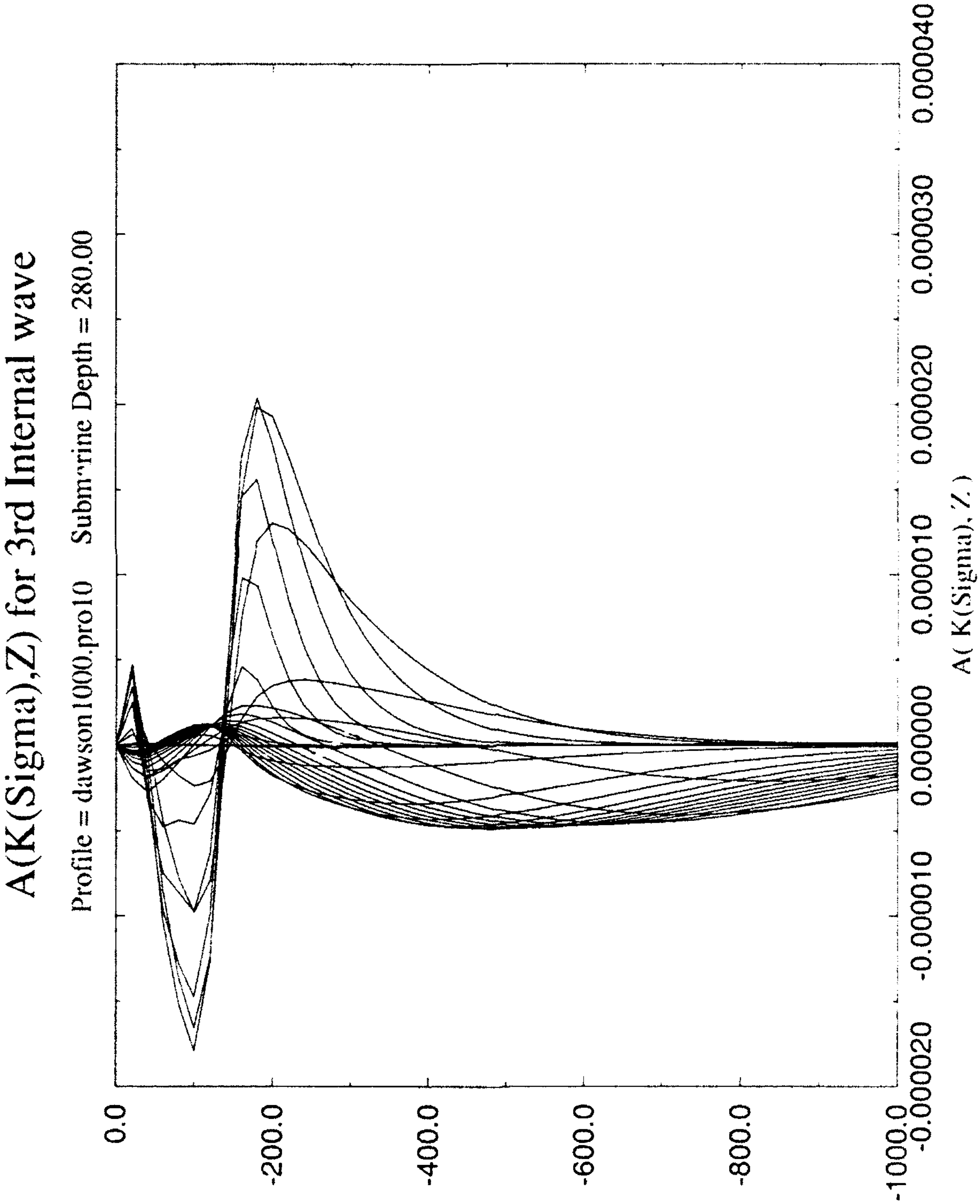


Figure 11.8: Mode-3 at $h = 280$ (1120-metre data).



Appendix 12: 23 March 1992.

Other components of velocity, and large wavelength.

For reasons associated with the exact linear theory (see Appendix 1) we have concentrated on the vertical component w so far. A somewhat disappointing aspect of that concentration is that the value of this component is very small at the free surface. It is not actually zero, but appears so on the plots, relative to the values at depths of the order of 100 metres or so.

In fact, it is consistent with the Boussinesq approximation that $w \approx 0$ at the free surface, a result that can be seen by noting the linear free surface condition $W' = \sigma W$ in the limit of large σ.

From the practical point of view, the other two velocity components u, v are more significant, and their values are at least as great in order of magnitude at the free surface as they are at depth. These two components are the gradient of a potential with Fourier transform Φ available from the equation above (2.3) of Appendix 1, namely $\Phi = k^{-2}W'$. Hence as well as the leading equation of Appendix 9, namely,

$$w/U = -\frac{2V}{\pi}\int_{\theta_m}^{\pi/2} \cos(kx\cos\theta)\cos(ky\sin\theta)\left[\frac{\sin(k\cos\theta.L/2)}{L/2}\right] A(z;k)\,d\theta$$

we have

$$u/U = \frac{2V}{\pi}\int_{\theta_m}^{\pi/2} \sin(kx\cos\theta)\cos(ky\sin\theta)\left[\frac{\sin(k\cos\theta.L/2)}{L/2}\right] B(z;k)\cos\theta\,d\theta$$

and

$$v/U = \frac{2V}{\pi}\int_{\theta_m}^{\pi/2} \cos(kx\cos\theta)\sin(ky\sin\theta)\left[\frac{\sin(k\cos\theta.L/2)}{L/2}\right] B(z;k)\sin\theta\,d\theta$$

where $B = A'/k = W_0'/D_k$, the prime as usual meaning $\partial/\partial z$.

That is, to compute these velocity components, just substitute the z derivative W_0' for the function W_0, and delete the factor k in the original definition $A = kW_0/D_k$ to give a new amplitude function B. Then replace A by B in the integrals, and for u include a factor $-\cos\theta$ and for v a factor $-\sin\theta$. Finally, the x and y factors are swapped appropriately between cosines and sines. The result is no more difficult to compute than w.

Estimates of orders of magnitude are interesting. First note that u is the disturbance x-component of velocity due to the submarine, the total being $U + u$ in a frame of reference moving with the submarine. Then, at least in the far "supercritical" speed range when $\kappa << \sigma_j$ and hence θ_m is close to $\pi/2$, we must have $u << v$, since then $\cos\theta << \sin\theta$. Thus the lateral velocity disturbance should be more significant than the along-track disturbance. This does seem to be in rough agreement with Dawson's results.

At the same time, there are a couple of arguments suggesting that v and w are of the same order of magnitude. Recalling that w is essentially zero at the free surface whereas v is not, this means that the value of v at the free surface may be comparable in magnitude with that of w at depths like 100 metres. In the first place, we may consider that the important values of the wavelength $2\pi/k$ are comparable with significant depth scales for rates of change of the density stratification. Hence the order of magnitude of k is comparable to that for $\partial/\partial z$. Hence B and A should have comparable magnitudes; and thus so do v and w.

Another related intuitive idea is expressed by the attached Figure 12.1, which is not new, but appears in other publications on internal waves generated by steadily moving bodies. Namely, when we have neglected u, continuity in the cross-flow plane demands that a snapshot of the flow field takes the form of closed loops.

I do not believe that this means that fluid particles necessarily undergo such loops; there would not in any case be time for particles to move round such loops as the submarine passed. This matter calls for further study. But nevertheless one gets the impression from such pictures that there must be a rough balance between the magnitudes of v and w.

Wave-like behaviour occurs in each of the components v, w as functions of y, and these waves are likely to be nearly out of phase. Hence when w is at its greatest upward value at depth, meanwhile v should be nearly zero. In between two such values of y there will be a near-zero point for w (at depth; w is always near zero at the free surface). At that value of y, then v will be near maximum, and that will be especially true at the free su face itself, etc.

Figure 12.1 applies to the first internal wave mode only, and the loops extend down to the effective end of stratification for that mode. A similar picture must hold for higher modes, e.g. Figure 12.2 for mode 2, with two loops in the depth direction. These loops are pure speculation. We can compute them, though, and should do so ASAP.

It is always well to keep in mind that the general length scale of interest for these internal waves is that of the stratification, namely the order of about one hundred metres. This scale starts life as a scale in the z-direction, and always correctly indicates that scale, so that any disturbance penetrates up and down over these rather large distances with ease. In particular, whatever internal wave is present at the surface will also be present at several metres depth with essentially no change in amplitude, and will only actually attenuate at depths of the order of many hundreds of metres.

Meanwhile, this scale is also determining the horizontal length scale of the wave. Other factors (such as the u, v disparity above) suggest some disparity between its manifestation in the x and y directions. Roughly speaking, I believe that the y-wise wavelength preserves the depth scale of about a hundred metres. But the x-scale may be stretched to several hundreds of metres, so that the internal wave appears to be propagating mainly sideways to the submarine's track: a mainly "diverging" wave pattern. In any case, these are very long waves, compared for example to surface waves made by a ship (or even a submarine), which have wavelengths of the order of only several metres, nearly two orders of magnitude smaller.

Internal waves are thus qualitatively unlike ship waves; they could be said to look more like "currents". They are unlikely to be observed directly, but rather via their influence on other observable quantities like capillary waves. Indeed, one (perhaps controversial?) suggestion has been that these roughness-like elements of the micro-scale free surface pattern become bunched together (higher?) where there is an up-current in Figure 12.1, i.e where w has a (positive) maximum, and become lower in magnitude where there is a (negative) minimum of w. Hence on images which detect this roughness, there may be a visible indirect image of the internal wave.

Figure 12.1: Sketch of conjectured mode-1 cross-sections.

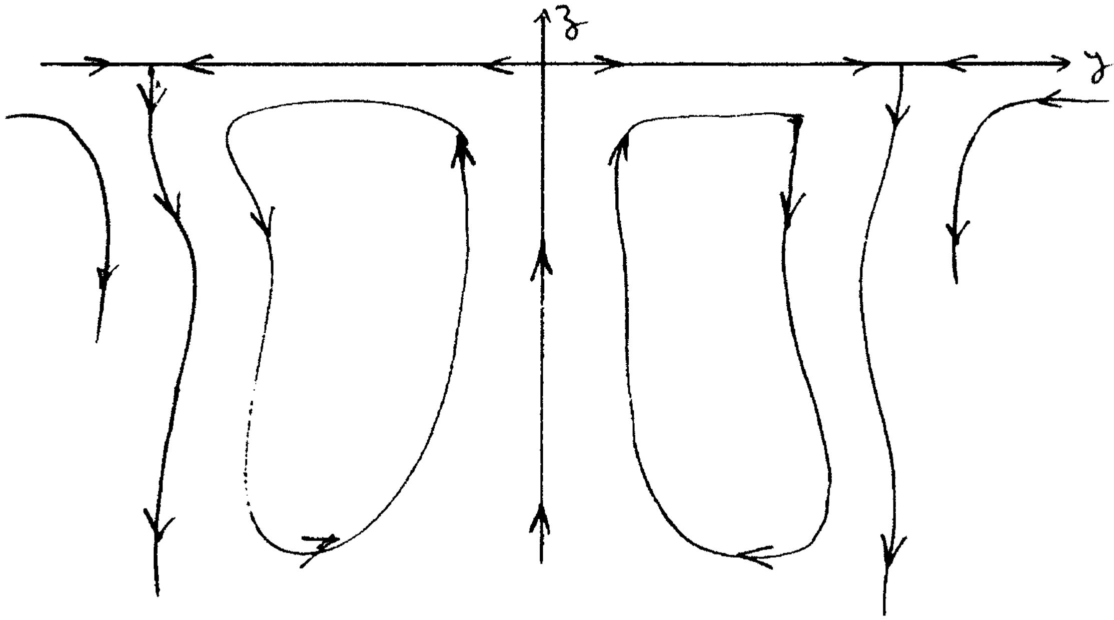
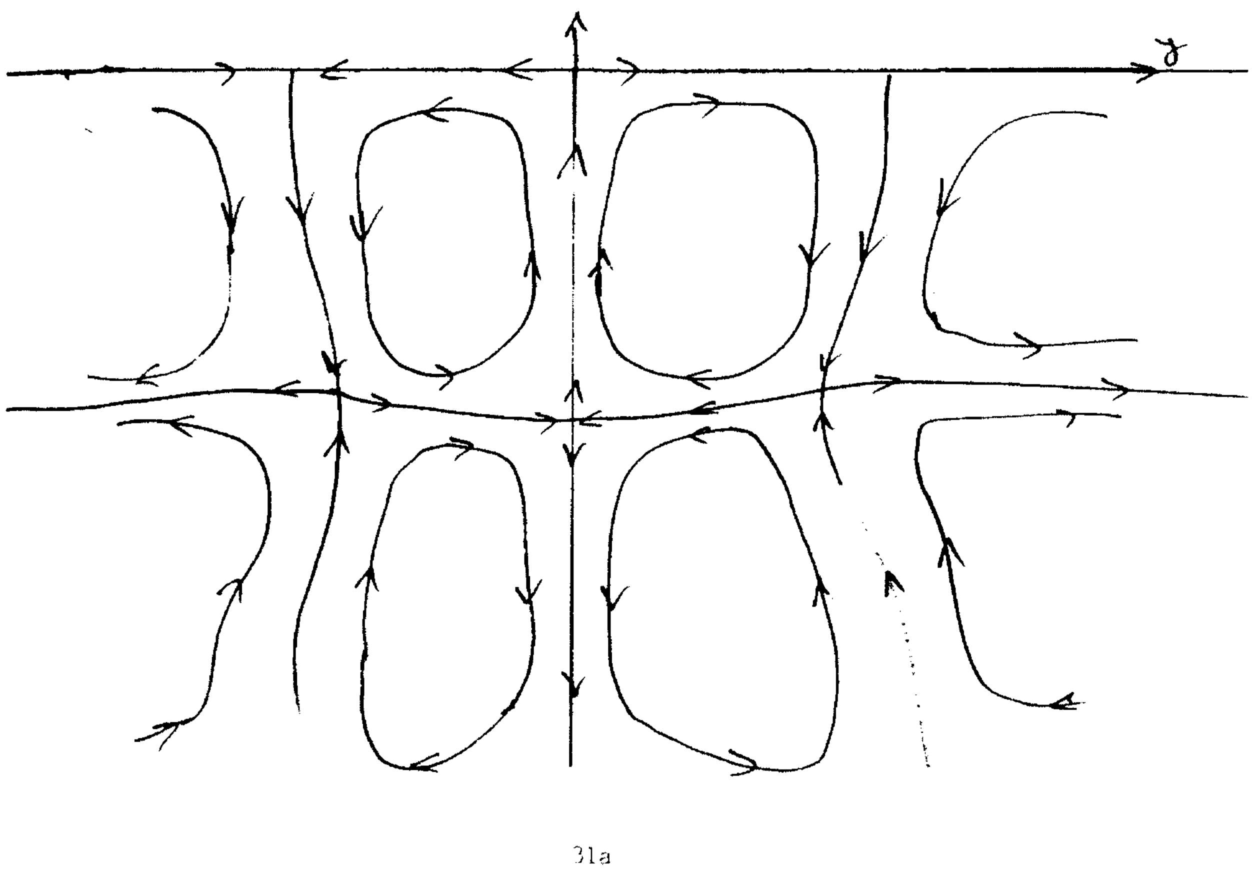


Figure 12.2: Sketch of conjectured mode-2 cross-sections.



Appendix 13: 24 March 1992

Further effort

1. Mode number effects.

I would have liked to do more comparative work on the influence of the various internal wave modes. Does the generating efficiency decrease significantly with mode number? How many modes does one in practice have to consider? How does the numerical accuracy vary with mode number – since there is more structure with respect to depth in the higher modes, do we need a finer numerical grid for them? The present program is rather primitive in the way that it picks out the particular mode that is desired. It could be improved in that respect.

2. Fourier outputs.

The quantity $A(k)$ on which we have spent most time is a measure of wave-generating efficiency in the space of the wave-number k. However, our use of it has been directed toward subsequent integration to give the actual point-wise flow field $w(x, y, z)$. There are other possibilities. In particular, there may be ways to detect $A(k)$ itself, in effect. Some of these ideas are contained in the Tuck, Collins, Wells (1971) paper (see Appendix 2), and essentially I am suggesting that the present work be combined with that.

3. Kelvin angles

Another related matter that we gave some thought to early in the project but never found the time to pursue further, is the simple Kelvin stationary phase angle. This is discussed in Appendix 1, and there is no reason why the γ versus θ curves could not be computed routinely as additional output from our program. There is a separate curve for each internal wave mode and (for internal waves as distinct from surface waves in water of infinite depth) a distinct curve for every separate speed of the disturber. I believe that the results may be in some way similar to those for the Kelvin waves in shallow water, at supercritical speeds, for which (as part of my seminar) I drew up the attached curve.

4. Other stratifications

We have given perhaps unwarranted attention to a particular stratification, namely that used by Dawson. The only reason for this is so that we can compare with some of Dawson's results. But that stratification is not typical, and is perhaps somewhat bizarre. There are many other more typical stratifications that should be tried, including some of direct interest for the present application to detection. A related numerical matter is the question of how deep to truncate the data. We have used data from Dawson of up to 480 metres, but this seems unnecessary; only the first 300 metres is really significant, but more testing is needed.

5. Dispersion relations

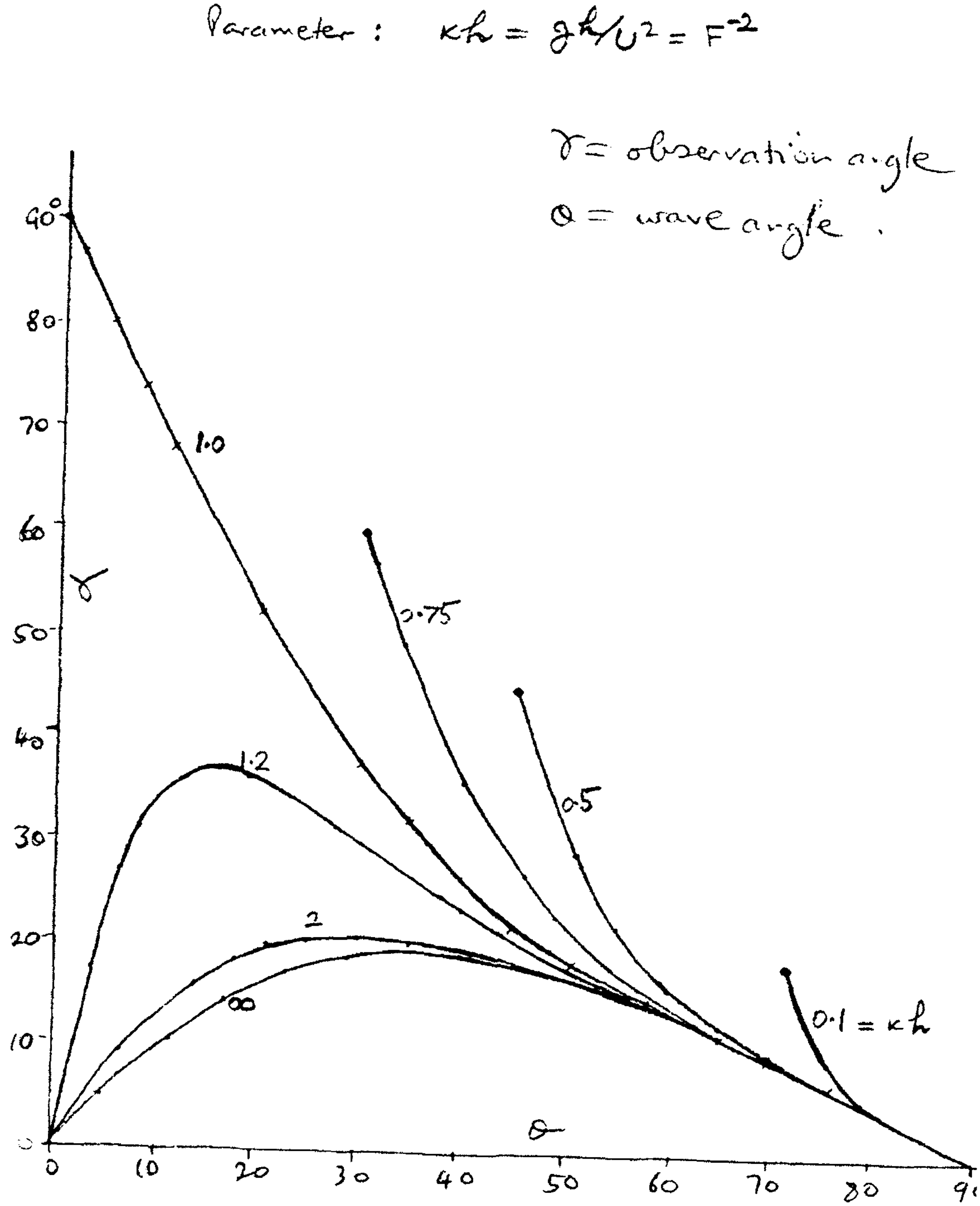
Appendices 3 and 6 contain ideas for analytic work of a novel nature on the character of the dispersion relations for internal waves. Just what features of the stratification do the parameters of the dispersion relation depend upon? Phillips's formula suggests things about the separate dependence on depth and thickness of the thermocline, but this depends on a particular assumption about the stratification, and the whole idea of a "depth" or a "thickness" or even a "thermocline" is not necessarily meaningful for a general stratification. The ideas in these Appendices could form the basis for publishable research on internal waves. In particular, the nature of the large-σ (or k) limit could be clarified using ideas from Appendix 6. This applies not only to the dispersion relation, it is possible to use the method of Appendix 6 to determine all our output parameters in the short-wave limit, but this has not yet been done. There is also a numerical element to this: our program has difficulty in distinguishing modes in the short-wave limit (see 1 above). If it could be guided by the asymptotic analytic work, it might be able to do that job more systematically.

6. Graphical output

There has not been time to do as good a display job as we would have liked on the output data. Several options were investigated briefly, e.g. the Sun Vision package, Matlab, etc. We shoudd be able to produce quite nice views of the flow field, including "rubber sheet" plots, perspective views from any direction, etc. An important view of the output is the "cellular" or "looped" structure in the cross-flow (x=constant) plane, discussed in Appendix 11. We have not had time to prepare accurate plots of this structure. Another form of output not pursued is the actual wave displacement, e.g. particle motions.

7. Other output and detection matters. Indirect effects of the internal wave field should be given more time. These include electromagnetic effects, effects on capillary wave size and location, and other determinants of the radar return.

Figure 13.1: Kelvin angles for finite-depth surface waves.



Appendix 14: 26 March 1992

EOT post-mortem

"Post mortem" is not an accurate description of the fate of this project, I hope! There is much more that can and should be done.

In the beginning, everyone involved had a lot to learn about wakes and internal waves. If we knew more to begin with, we could have avoided some pitfalls. We might in fact have chosen to do little of what was eventually done, relying instead for example on the Canadian study (Appendix 8), which adopts a similar methodology.

I believe that would have been a mistake. It was necessary for our own computer program to be developed, in order that we could build up our own knowledge and intuition. This has now been done. It is possible that our program is better than those of previous investigators. It is certainly comparable, and is producing results of comparable quality. This in less than 2 months at a very small total cost.

Internal waves are not very big. Some of us would like to have seen waves with velocities of the order of metres per second instead of millimetres per second. It is not for me to judge whether these are measureable or detectable signals. Certainly it appears that detection must be indirect, and that has also been appreciated by a number of previous investigators.

At the beginning, it was felt that more of my time might be spent on non-internal wave aspects of the wake. This has not proved possible in a direct way. However, indirectly we are covering some parts of the general problem. In a crude sense, the viscous wake can be interpreted as a "tail" added to the stern of the submarine, in which case one can get information about its effect by studying a "non-balanced" source-sink pair in the Rankine-ovoid model. Immediate properties of the actual turbulent wake are another matter.

From the personal point of view, this has been a most challenging and interesting project. I thank David Cartwright for organising it, and my colleagues while working on it, Graham Furnell, Tony Legg and Michael Carroll. I have not been easy to work with, being ever conscious of the limited time available for me to contribute. Now that this time is up, I hope that the work can be carried on within DSTO.

Appendix 15: 27 March 1992. (informal)

Final results.

This section of the report is written separately from the rest, in part because of last minute interruptions to the computing system that prevented its incorporation into the main report, and in part because the same interruptions prevented us from obtaining (in time) as high a quality of final output as we would have preferred. It is planned that a separate report will be produced by Graham Furnell when a complete set of fully-checked final results are obtained.

By "final" output, we mean actual computations of the flow field (u, v, w) components, as functions of the spatial co-ordinates (x, y, z). This "only" requires numerical integration of the integrals appearing in Appendix 12, since we now are confident of our ability to compute accurately the generation amplitudes A and B.

However, there are pitfalls in that numerical integration task. The main difficulty is that when x and y are large, the integrand of this θ-integration is rapidly varying. This is no surprise, of course: it is the basis for Kelvin's stationary phase argument. There are several methods to account for it numerically. One (see Tuck, Collins and Wells 1971) is to mimic the method of stationary phase. This was used by Madurasinghe in producing Figure 1.2.

Another more brute-force method is to use many values of θ in the discretisation. However, we don't want or need to compute A and B at many θ (hence σ and k) values, since these are not very rapidly varying functions, though expensive to compute. It is only the sine and cosine functions involving x and y that are rapidly varying. Hence a good procedure, also tried by Madurasinghe, is to interpolate within (say) 30 computed values of A, while dividing the integration range of the θ integral into many thousands of sub-intervals.

We have had time to do neither of these things yet. Hence the results attached are only reliable for relatively small values of x and y. In particular, it is not possible to demonstrate any asymptotic decay as $x \to \infty$ or $y \to \infty$ unless this rapid oscillation is accounted for numerically, so our results display spurious oscillations that do not decay at infinity.

The Figures attached show the lateral velocity $v(x, y, 0)$ at the free surface, as a function of y for fixed x. They were computed for a submarine of length $L = 100$ metres and radius $R = 5$ metres, travelling at $U = 2.5$ metres/second at depth $h = 35$ metres in an ocean with the stratification of Dawson's report. These results were computed using Simpson's rule on either 30 or 60 intervals.

For example, the first two figures are at $x = 500$, which is probably already too large, with 30 and 60 θ-points respectively. The 30-point output is smoother, for reasons not entirely appreciated at this time, and there is general order of magnitude agreement. There is good quantitative agreement between these curves about the major minimum of about -0.00075 (nearly a millimetre per second) at about $y = 25$ metres and (perhaps) a subsequent maximum of $+0.0002$ at about $y = 100$ metres, but this quantitative agreement vanishes for larger y, and the results are no longer to be believed at these y values.

We have computed some other output at more reasonable (smaller) values of r, generating waves of the order of one millimetre per second, but at the time of writing have not been able to plot it. Plots will be attached if produced before the end of this day

Figure 15.1: $v(500, y, 0)$ using 30 θ's.

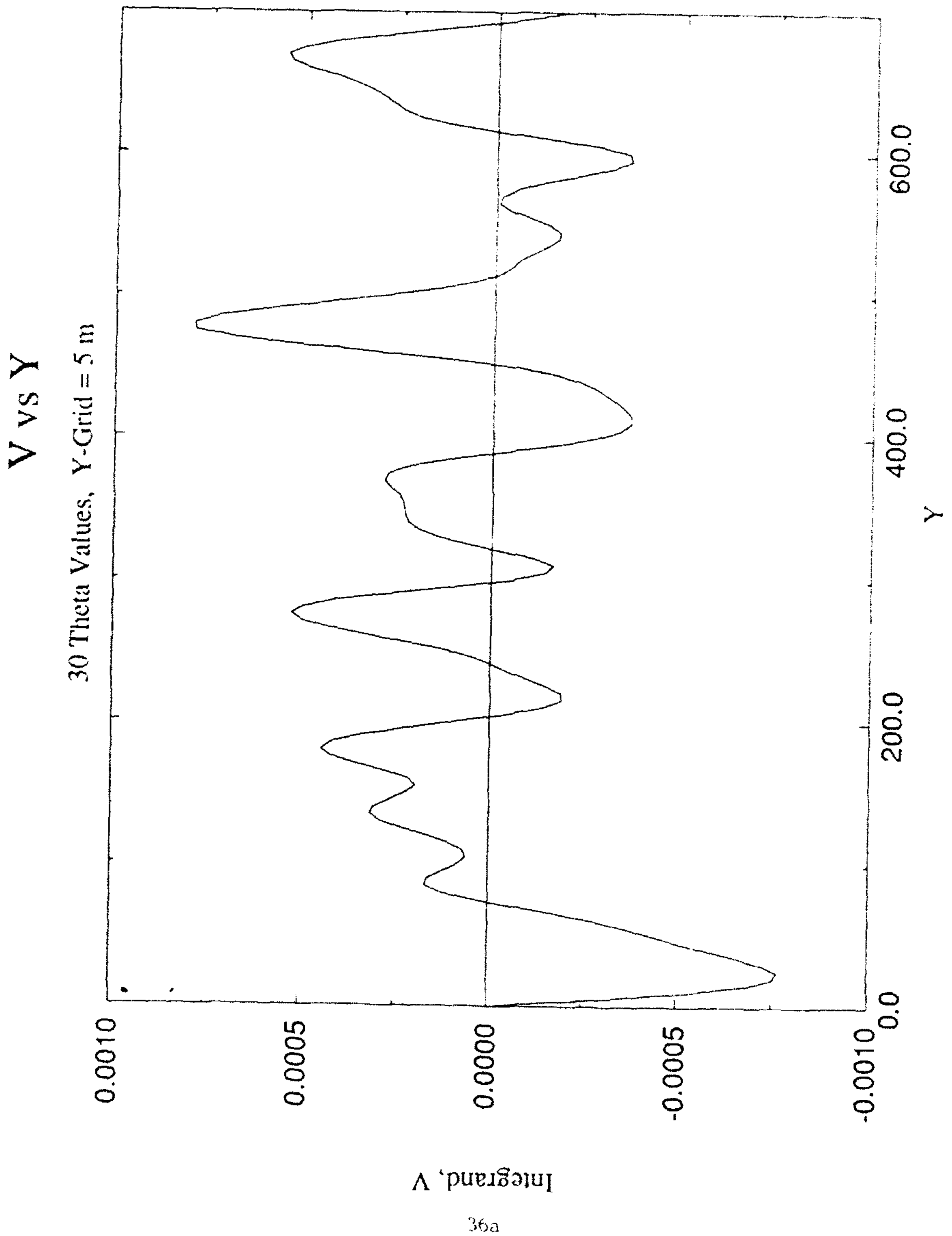


Figure 15.2: $v(500, y, 0)$using 60 θ's.

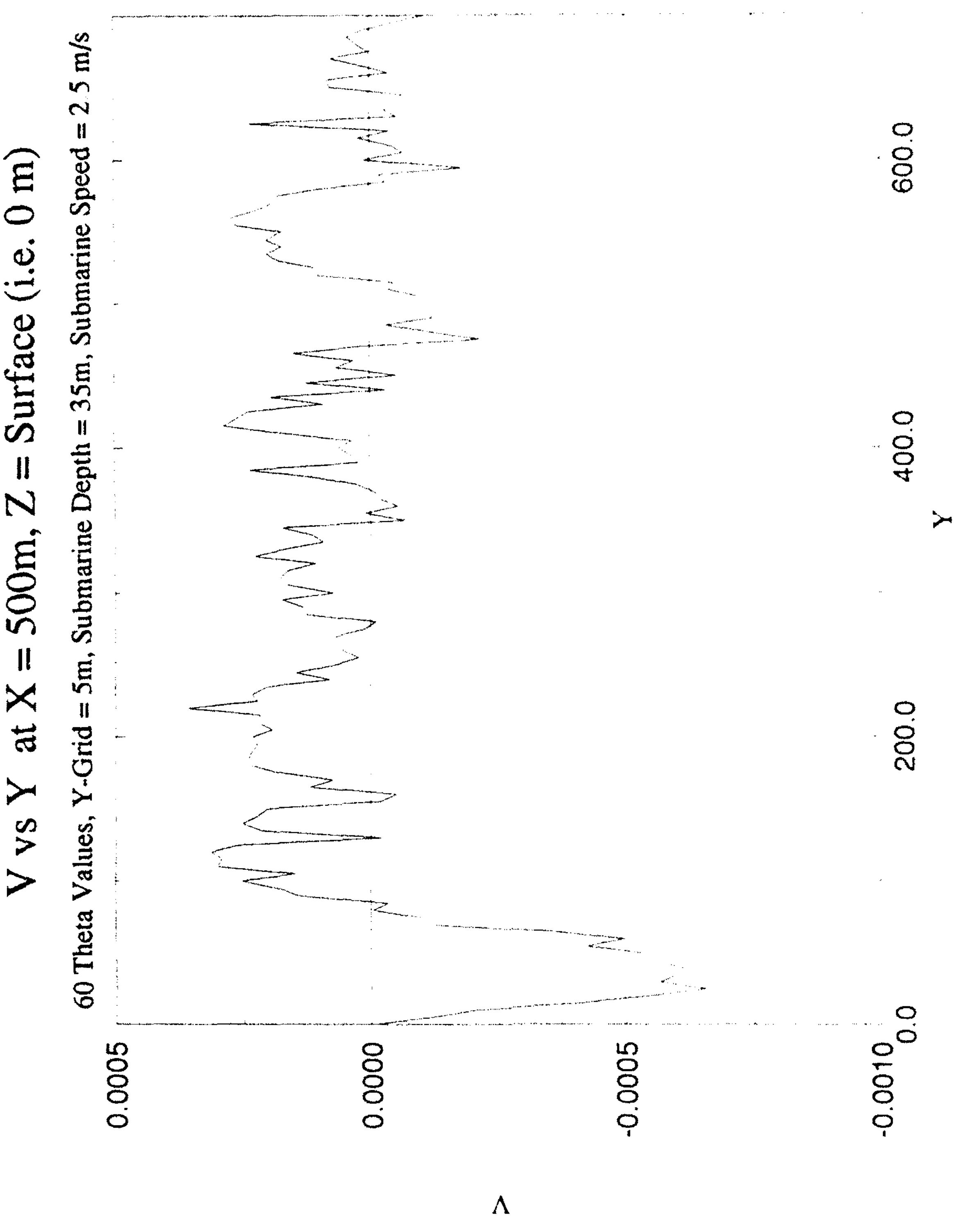


Figure 15.3: $v(x, y, 0)$versus y for $x = 0$ to 100 metres.

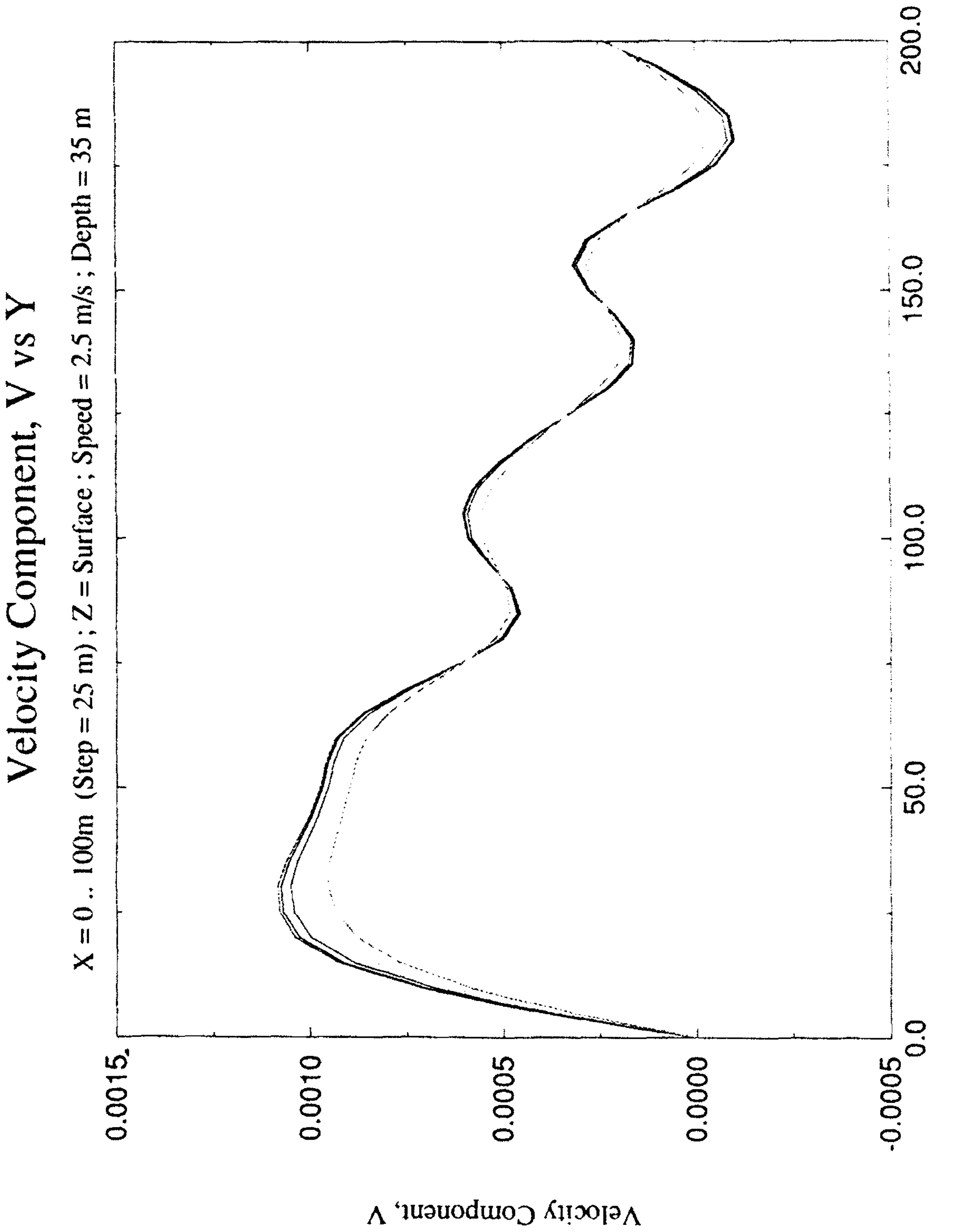


Figure 15.4: $v(x, y, 0)$ versus y for x = 100 to 300 meters.

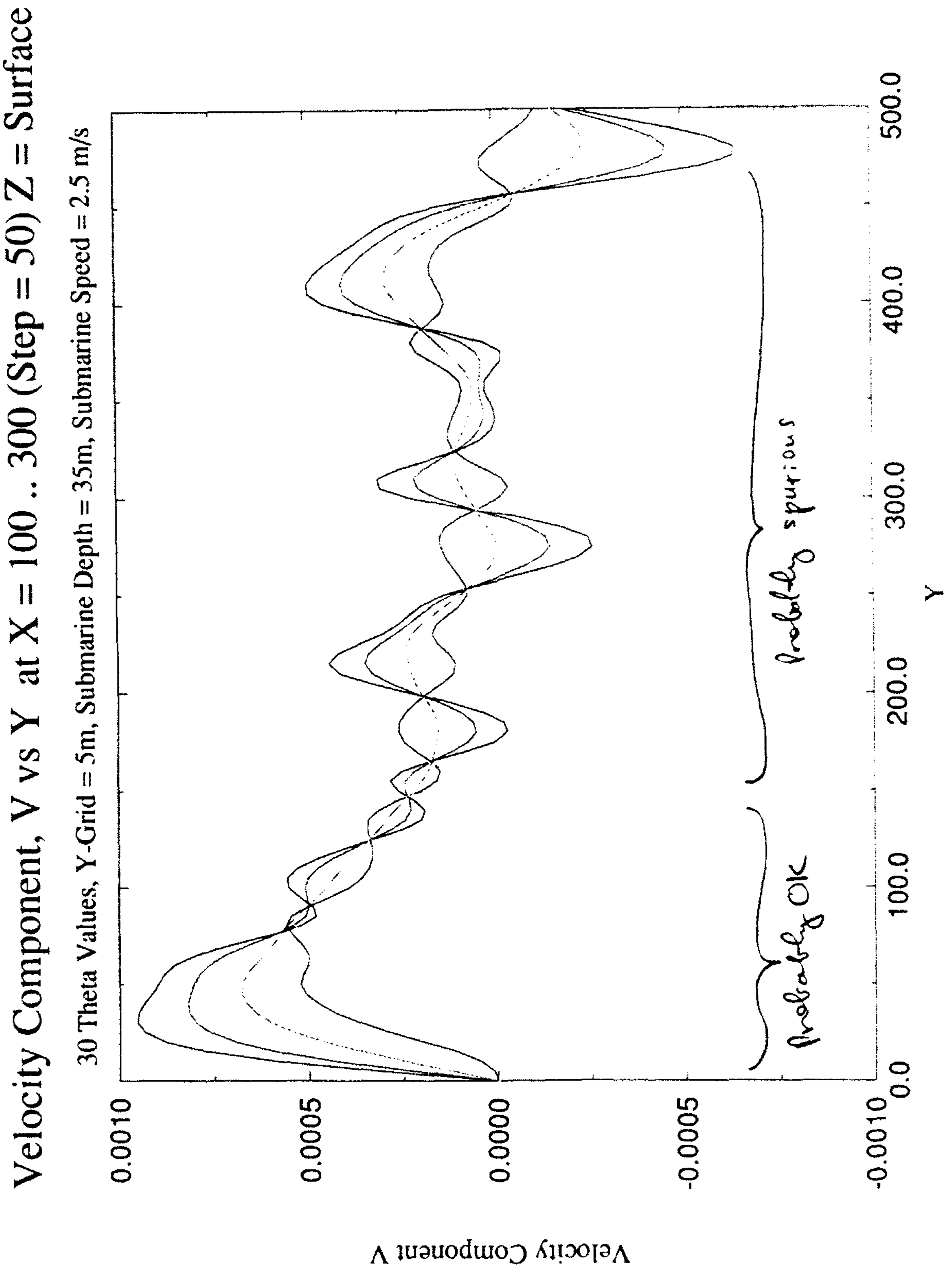


Figure 15.5: $u(x, y, 0)$ versus y for $x = 0$ to 100 meters.

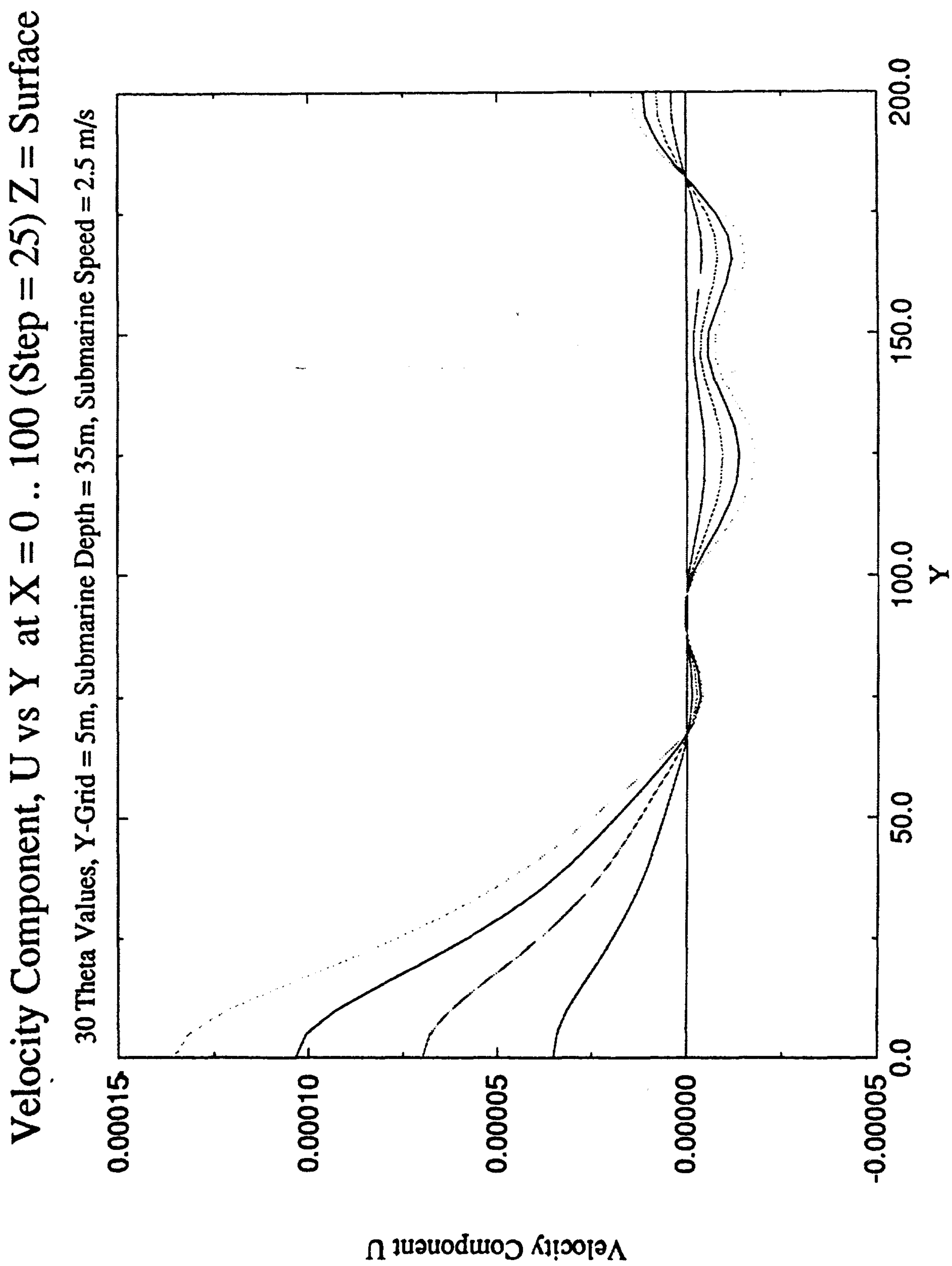


Figure 15.6. As 15.3 but w instead of v, and evaluated at $z = -30$. ($w \simeq 0$ at the FS $z = 0$)

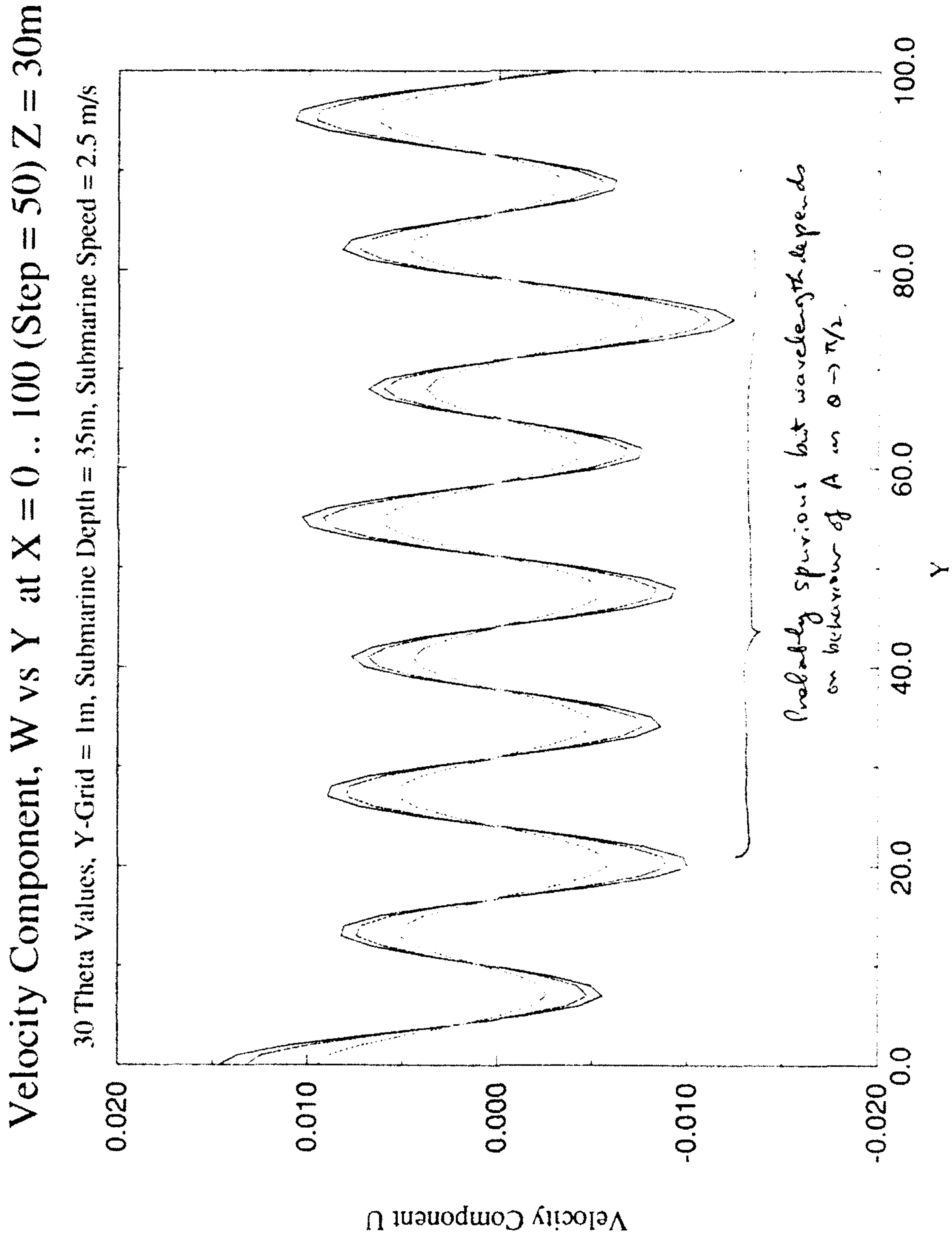


Figure 15.7: $v(x, y, 0)$ versus y for $x = 0$ to 100 meters. Speed $U = 10\ ms^{-1}$. Note *smaller* wave! (and almost independent of x – to be expected, see formula: more and more *diverging* waves as $U \to \infty$).

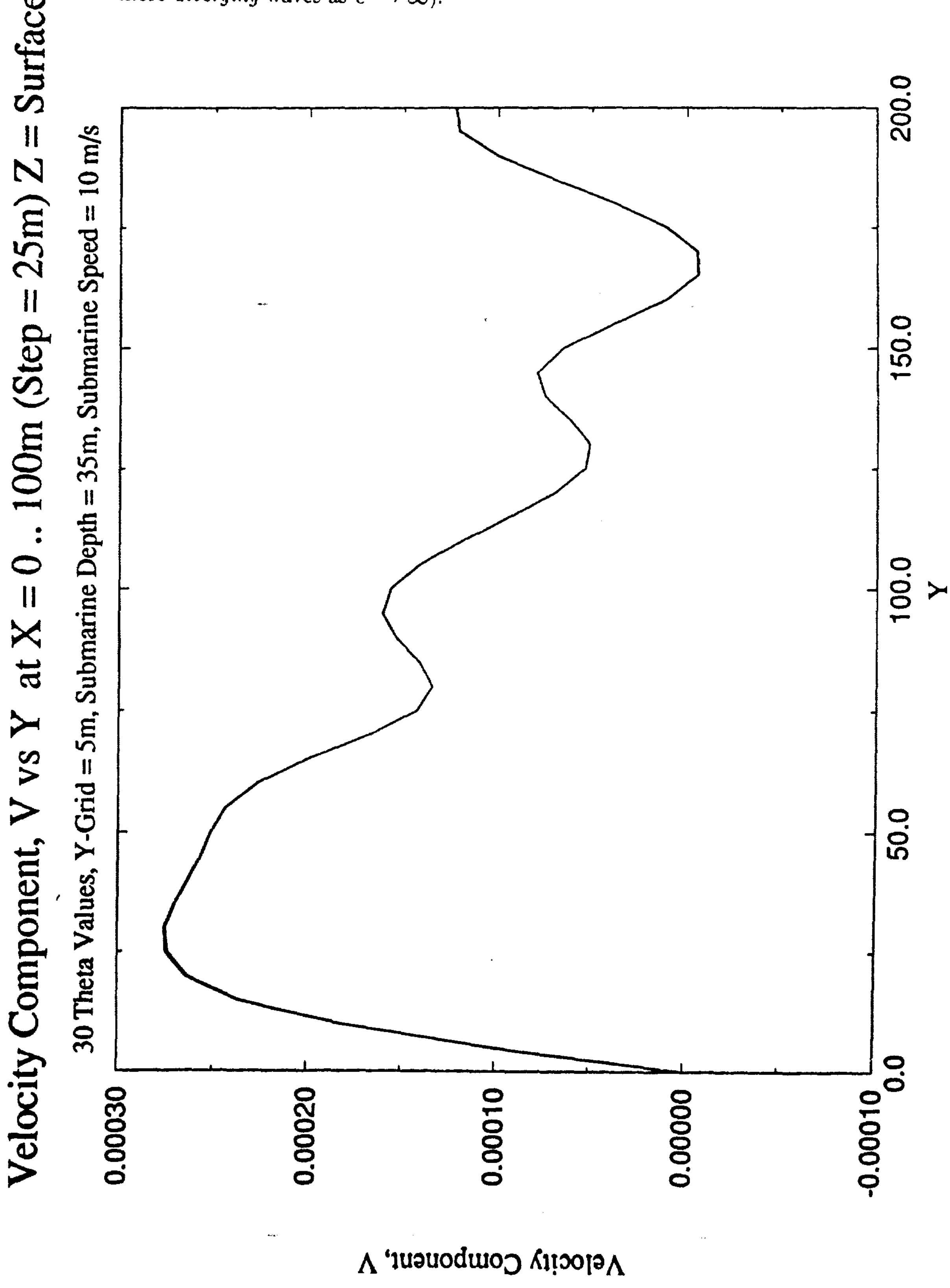
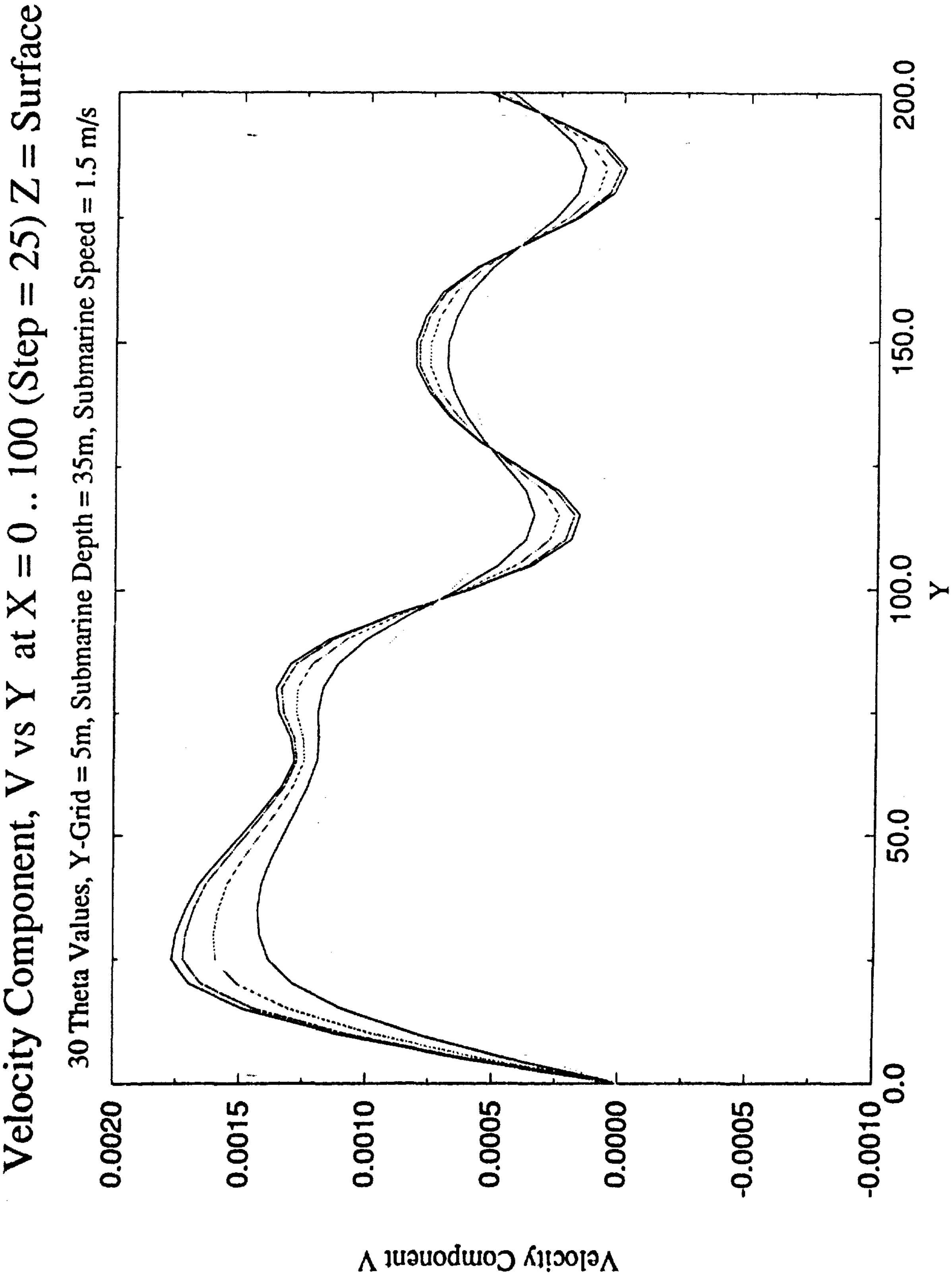


Figure 15.8: $v(x, y, 0)$ versus y for $x = 0$ to 100 meters. Speed $U = 1.5\ ms^{-1}$. Note *bigger* wave! ("Critical speed" where $\kappa = \sigma_1 \simeq 4.7$ is $U = 1.5$!) i.e. this is when $\theta_{min} = 0$.



REPORT NO.	AR NO.	REPORT SECURITY CLASSIFICATION
MRL-GD-0049	AR-008-256	Unclassified

TITLE

Submarine internal waves

AUTHOR(S)
E.O. Tuck

CORPORATE AUTHOR
DSTO Materials Research Laboratory
PO Box 50
Ascot Vale Victoria 3032

REPORT DATE	TASK NO.	SPONSOR
January, 1993	NAV 92/040	RAN
FILE NO.	**REFERENCES**	**PAGES**
G6/4/8-4453	18	65

CLASSIFICATION/LIMITATION REVIEW DATE

CLASSIFICATION/RELEASE AUTHORITY
Chief, Maritime Operations Division

SECONDARY DISTRIBUTION

Approved for public release

ANNOUNCEMENT

Announcement of this report is unlimited

KEYWORDS

Wave Speed
Dispersion Relations
Computer Modelling
Wave Wakes
Density Profiles

ABSTRACT

Details are provided of research directed towards the development of the computer program WAKE for predicting the internal waves produced by the passage of a submarine in a density stratified ocean. The research was undertaken within DSTO during the period February-March 1992. The report takes the form of a series of appendices (presented in chronological order), which provide a record of the progress of the research during that period. Results are presented which indicate that a representative submarine produces internal waves which have a velocity magnitude of about one millimeter per second.

Submarine Internal Waves

E.O. Tuck

(MRL-GD-0049)

DISTRIBUTION LIST

Director, MRL
Chief, Maritime Operations Division
Research Leader, Airborne Sonar
Professor E.O. Tuck, University of Adelaide
MRL Information Service

Chief Defence Scientist (for CDS, FASSP, ASSCM) (1 copy only)
Director, Surveillance Research Laboratory
Director (for Library), Aeronautical Research Laboratory
Director, Electronics Research Laboratory
Head, Information Centre, Defence Intelligence Organisation
OIC Technical Reports Centre, Defence Central Library
Officer in Charge, Document Exchange Centre (8 copies)
Army Scientific Adviser, Russell Offices
Air Force Scientific Adviser, Russell Offices
Navy Scientific Adviser, Russell Offices - data sheet only
Scientific Adviser, Defence Central
Director-General Force Development (Land)
Senior Librarian, Main Library DSTOS
Librarian, MRL Sydney
Librarian, H Block
UK/USA/CAN ABCA Armies Standardisation Rep. c/- DGAT (8 copies)
Librarian, Australian Defence Force Academy
Counsellor, Defence Science, Embassy of Australia - data sheet only
Counsellor, Defence Science, Australian High Commission - data sheet only
Scientific Adviser to DSTC, C/- Defence Adviser - data sheet only
Scientific Adviser to MRDC, C/- Defence Attache - data sheet only
Head of Staff, British Defence Research and Supply Staff (Australia)
NASA Senior Scientific Representative in Australia
INSPEC: Acquisitions Section Institution of Electrical Engineers
Head Librarian, Australian Nuclear Science and Technology Organisation
Senior Librarian, Hargrave Library, Monash University
Library - Exchange Desk, National Institute of Standards and Technology, US
Exchange Section, British Library Document Supply Centre
Periodicals Recording Section, Science Reference and Information Service, UK
Library, Chemical Abstracts Reference Service
Engineering Societies Library, US
Documents Librarian, The Center for Research Libraries, US

Dr D. Richardson, MOD
Dr J. Ternan, MOD
Dr A. Theobald, MOD
Dr J. Vrbancich, MOD
Dr J. Smelt, MOD
Dr R. Webster, MOD
Dr G. Furnell, MOD
Chief, Microwave Radar Division, SRL
Chief, Optoelectronics Division, SRL
Research Leader, Surveillance Systems, SRL
Dr G. Haack, MRD, SRL
Dr C. Anderson, MRD, SRL
Dr D. Madurasinghe, MRD, SRL
Dr D. McDonald, MRD, SRL