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A CLASS OF PLANAR WELL-COVERED GRAPHS WITH GIRTH FOUR

by

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Abstract

A well-covered graph is a graph in which every maximal independent set is a maximum independent set; Plummer introduced the concept in a 1970 paper. The notion of a 1-well-covered graph was introduced by Staples in her 1975 dissertation: a well-covered graph G is 1-well-covered if and only if G-v is also well-covered for every point v in G. Except for K_2 and C_5, every 1-well-covered graph contains triangles or 4-cycles. Thus, triangle-free 1-well-covered graphs necessarily have girth 4. We show that all planar 1-well-covered graphs of girth 4 belong to a specific infinite family, and we give a characterization of this family.

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INTRODUCTION

A set of points in a graph is independent if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the independence number of G and is denoted by $\alpha(G)$. A set of independent points which attains the maximum size is referred to as a maximum independent set. A set S of independent points in a graph is maximal (with respect to set inclusion) if the addition to S of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [14] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a well-covered graph. The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. Campbell [2] characterized all cubic well-covered graphs with connectivity at most two, and Campbell and Plummer [3] proved that there are only four 3-connected cubic planar well-covered graphs. Royle and Ellingham [16] have recently completed the picture for cubic well-covered graphs by determining all 3-connected cubic well-covered graphs.

For a well-covered graph with no isolated points, the independence number is at most one-half the size of the graph. Well-covered graphs whose independence number is exactly one-half the size of the graph are called very well-covered graphs. The subclass of very well-covered graphs was characterized by Staples [17] and includes all well-covered trees and all well-covered bipartite graphs. Independently, Ravindra [15] characterized bipartite well-covered graphs and Favaron [6] characterized the very well-covered graphs. Recently, Dean and Zito [4] characterized the very well-covered graphs as a subset of a more general (than well-covered) class of graphs.

Finbow and Hartnell [7] and Finbow, Hartnell, and Nowakowski [8] studied well-covered graphs relative to the concept of dominating sets. Finbow, Hartnell, and Nowakowski have also obtained a characterization of well-covered graphs with girth at least five [9].

A well-covered graph is 1-well-covered if and only if the deletion of any point from the graph leaves a graph which is also well-covered. A well-covered graph is in the class W_2 if and only if any two disjoint independent sets in the graph can be extended to disjoint maximum independent sets. Staples [18] showed that a well-covered graph is 1-well-covered if and only if it is in W_2. Since we will appeal mostly to the notion of extending two disjoint independent sets to disjoint maximum independent sets, henceforth we use the W_2 nomenclature instead of referring to 1-well-covered graphs.

The class of well-covered graphs contains all complete graphs and all complete bipartite graphs of the form $K_{n,n}$. The only cycles which are well-covered are C_3, C_4, C_5, and C_7. We note that all complete graphs are also in W_2, but no complete bipartite graphs (except $K_{1,1}$) are in W_2. The cycles C_3 and C_5 are the only cycles in W_2.

PRELIMINARY RESULTS

We assume that all graphs are connected, unless otherwise stated. The reader is referred to [1] for terminology and notation not defined here. Note that a disconnected graph is in W_2 if and only if each of its components is in W_2. Suppose G is well-covered, $G \neq K_1$. Let v be a point in G and consider the graph G-v. Since $G \neq K_1$, there exists a point $u \sim v$. Since G is well-covered, the point u is contained in a maximum independent set I in G. Clearly, v is not in I. Thus, I is also independent in G-v. Consequently, $\alpha(G-v) = \alpha(G)$ for any point v. Hence, from a result of Erdös and Gallai [5] it follows that $\alpha(G) \leq |V(G)|/2$. Thus, W_2 graphs inherit this bound on independence number.

Staples [18] proved that a W_2 graph cannot have an endpoint.

Theorem 1. If $G \in W_2$ and G is not complete, then $\delta \geq 2$.

In the next theorem, we prove that a point on a 4-cycle in a W_2 graph must have at least three neighbors.

Theorem 2. If $G \in W_2$ and v is a point on a 4-cycle in G, then $\deg(v) \geq 3$.

Proof. Suppose v is on the 4-cycle vabc in G. Also suppose that $\deg(v) = 2$. Then $\{v\}$ and $\{b\}$ cannot be extended to disjoint maximum independent sets in G, a contradiction since $G \in W_2$. Thus, $\deg(v) \geq 3$. []

In Theorem 3, we show that any point in a W_2 graph that is not on a triangle must be on a 5-cycle.

Theorem 3. If $G \in W_2$ and v is a point in G, then v is on either a triangle or a 5-cycle.

Proof. Suppose v is not on a triangle. Suppose also that v is not on a 5-cycle. Let $N_1 = N(v)$ and $N_2 = \{x \in V(G): d(x,v) = 2\}$. Since v is not on a triangle, then N_1 is independent. Since $\delta \geq 2$ by Theorem 1, each point in N_1 has a neighbor in N_2. For each $x \in N_1$, let $N'(x) = N(x) \cap N_2$. Pick $a_x \in N'(x)$ and let $A = \{a_x: x \in N_1\}$. If $a_x \neq a_y$, then a_x is not adjacent to a_y; otherwise, vxa_xa_yy is a 5-cycle in G containing v. Thus, A is independent. Since A dominates N_1, then it follows that A and $\{v\}$ don't extend to disjoint maximum independent sets in G. This is the desired contradiction. []

A W_2 graph can have a point of degree two or possibly two adjacent points of degree two. However, we show in the next theorem that a W_2 graph cannot have a point of degree two with each of its neighbors of degree two.

Theorem 4. If G is well-covered (and not a cycle) with a path of three consecutive points of degree two, then G is not in W_2.

Proof. Suppose a, b and c are points in G such that $a \sim b$, $b \sim c$ and deg(a) = deg(b) = deg(c) = 2. Since G is not a cycle, then a is not adjacent to c. Assume to the contrary that G is in W_2. Then by Theorem 3, the point b must lie on a 5-cycle . Suppose a 5-cycle containing b is C = xabcy. Since G is not a cycle, then either deg(x) > 2 or deg(y) > 2. Without loss of generality, assume deg(x) > 2. Let $u \sim x$ such that $u \notin C$. Then {u,c} is independent. So {u,c} and {a} don't extend to disjoint maximum independent sets in G, a contradiction.

Therefore, G is not in W_2. []

Consider a graph G which is not complete and point v in G. By deleting v and its neighbors, we obtain a subgraph of G. Specifically, we define the subgraph $G_v = G\text{-}N[v]$.

In Theorem 5, we state a *necessary* condition for a well-covered graph to be in W_2, which is proved in [13]. We will reference Theorem 5 on several occasions in this paper.

Theorem 5. If a graph G is in W_2 and G is not complete, then G_v is in W_2 for all v in G.

The girth of a graph is the size of a smallest cycle in the graph. We say a graph with no cycles has infinite girth. In [13], we prove the following theorem.

Theorem 6. If $G \in W_2$ ($G \neq K_2$ or C_5), then girth $G \leq 4$.

Hence, a W_2 graph (other than K_2 and C_5) must contain a triangle or a 4-cycle. Thus, a triangle-free W_2 graph (other than K_2 and C_5) has girth 4. In [13], we construct infinite families of W_2 graphs with girth 4. We study *planar* W_2 graphs of girth four for the remainder of this paper.

In general, a W_2 graph can have a cutpoint. However, we prove in the next theorem that a W_2 graph of girth four cannot have a cutpoint.

Theorem 7. If G is a W_2 graph of girth 4, then G is 2-connected.

Proof. Assume to the contrary that G has a cutpoint v. Let $G_1, G_2, \ldots, G_n$ be the components of G-v. By Theorem 1.20, graphs $G_1, \ldots, G_n$ are W_2 graphs. Let $N_i = N(v) \cap G_i$, for $i = 1,\ldots,n$. Since G has girth 4, then N_i is independent for all i. Since $G_i \in W_2$, there exists maximum independent sets J_i in G_i such that $J_i \cap N_i = \varnothing$, for all i. Clearly, $J = J_1 \cup \ldots \cup J_n$ is an independent set in G. Consequently, J and $\{v\}$ are disjoint independent sets in G which do not extend to disjoint maximum independent sets in G. This is a contradiction since $G \in W_2$. Hence, G is 2-connected. []

A line in a graph G is a critical line if its removal increases the independence number. A line-critical graph is a graph with only critical lines. Staples proved in [17] that a triangle-free W_2 graph is line-critical. Hence, all graphs given subsequently in this paper are line-critical.

PLANAR W_2 GRAPHS OF GIRTH FOUR

In this section, we will characterize all *planar* W_2 graphs of girth 4. For graphs drawn in the plane, we say two faces are adjacent if they share a line. If a face F contains point v, we say F is incident to v. The size of a face is the number of points it contains. We refer to the order and sizes of the faces incident to a point v as the face configuration at v.

Lebesgue [10] developed the theory of Euler contributions for planar graphs and Ore [11] and Ore and Plummer [12] used the theory to study plane graph colorings. The Euler contribution of a point v, $\phi(v)$, is defined as the quantity $\phi(v) = 1 - (1/2)\deg(v) +$

$\Sigma(1/x_i)$, where the sum is taken over all faces F_i incident to v and x_i is the size of F_i. If $|F(G)|$ denotes the number of faces in the plane graph G, then it follows that $\Sigma_v\, \phi(v) = |V(G)| - |E(G)| + |F(G)|$. Here the sum is taken over all points v in G. Since Euler's formula for plane graphs says $|V(G)| - |E(G)| + |F(G)| = 2$, then we have $\Sigma_v\, \phi(v) = 2$. Thus, $\phi(v)$ must be positive for some v in G. If $\phi(v) > 0$, we say v is a point with positive Euler contribution.

An infinite family.

The following construction allows us to build larger planar W_2 graphs of girth 4 from a given such graph. It can be verified directly from the definition of a W_2 graph that the construction indeed yields a W_2 graph.

Construction 1. Suppose G is a W_2 graph with adjacent degree two points x and y which are not on a triangle. Let $N(x) = \{u,y\}$, $N(y) = \{x,v\}$, and let a, b and c be new points. Form a new graph H with

$V(H) = V(G) \cup \{a,b,c\}$, and

$E(H) = E(G) \cup \{xa,ab,bc,cy,cu\}$. See Figure 1.

Then H is a W_2 graph with $\alpha(H) = \alpha(G) + 1$.

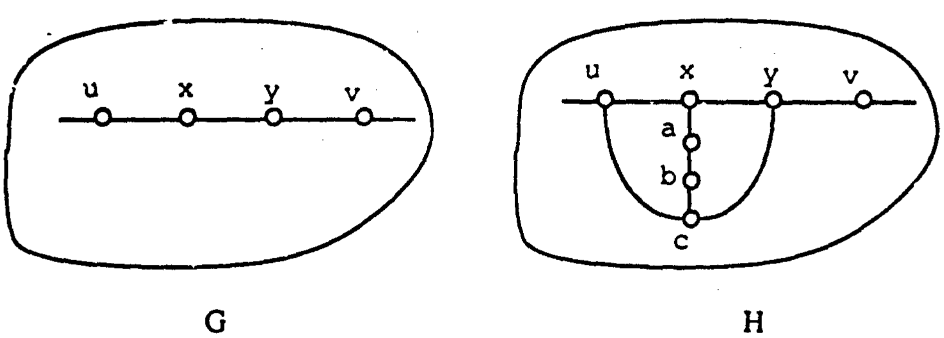

Figure 1

In Construction 1, if G is planar and has girth 4, then clearly H is also a planar W_2 graph of girth 4. In the following theorem, we recursively construct an infinite family of planar W_2 graphs of girth 4.

Theorem 8. Let $n \geq 3$ be a positive integer. Then there exists a planar W_2 graph of girth 4, denoted G_n, such that $\alpha(G_n) = n$ and $|V(G_n)| = 3n - 1$.

Proof. (By induction on n.) For $n = 3$, let G_3 be the graph on eight points given in Figure 2. Then $\alpha(G_3) = 3$ and $|V(G_3)| = 3(3) - 1$. For $k \geq 3$, let G_{k+1} be a graph obtained from G_k by the construction given in Construction 1. Assume $\alpha(G_k) = k$ and $|V(G_k)| = 3k - 1$. From the observation preceding Theorem 8, graph G_{k+1} is a planar W_2 graph of girth 4. Also, $|V(G_{k+1})| = |V(G_k)| + 3 = 3k - 1 + 3 = 3(k+1) - 1$, and $\alpha(G_{k+1}) = \alpha(G_k) + 1 = k + 1$.

Therefore, G_{k+1} satisfies the statement of the theorem. The result follows by induction. []

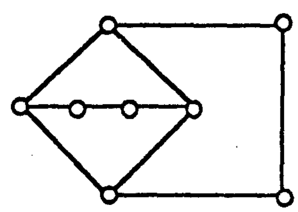

Figure 2

Now that we have an infinite family of planar W_2 graphs of girth 4, we work toward showing that all planar W_2 graphs of girth 4 are in the family in Theorem 8.

A characterization.

Since the smallest cycle in a W_2 graph of girth 4 is a 4-cycle, it is of interest to learn what we can about 4-cycles in these graphs. The next lemma will help us to determine those W_2 graphs of girth 4 which have exactly one 4-cycle.

Lemma 9. Suppose G is a W_2 graph of girth 4. Let C be a 4-cycle in G. If deg(v) = 3 for all points v in C, then G is isomorphic to the graph given in Figure 2.

Proof. Let $C = v_1v_2v_3v_4$. Assume $\deg(v_i) = 3$, for all i. Since G has girth 4, then v_1 is not adjacent to v_3 and v_2 is not adjacent to v_4. Let $u_i \sim v_i$ such that u_i is not in C, for all i. Since G has girth 4, then $u_i \neq u_{i+1}$ for all i (addition mod 4).

Suppose $u_1 = u_3$. Then $N(v_1) = N(v_3)$. So $\{v_1\}$ and $\{v_3\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Thus, $u_1 \neq u_3$ and, similarly, $u_2 \neq u_4$. So we can assume that $i \neq j$ implies $u_i \neq u_j$.

Suppose $u_1 \sim u_4$. Since $\deg(v_1) = 3$, then $\{u_4, v_3\}$ is independent and dominates $N(v_1)$. Thus, $\{u_4, v_3\}$ and $\{v_1\}$ don't extend to disjoint maximum independent sets in G, a contradiction. So u_1 is not adjacent to u_4. Similarly, u_1 is not adjacent to u_2, u_2 is not adjacent to u_3, and u_3 is not adjacent to u_4.

Since $G \in W_2$, $\deg(u_1) \geq 2$. Since $u_1 \neq u_i$, $i \neq 1$, then u_1 is not adjacent to v_i, $i \neq 1$. Thus, there exists $y \sim u_1$ such that $y \notin C$. If y is not adjacent to v_3, then $\{y, v_3\}$ and $\{v_1\}$ don't extend to disjoint maximum independent sets in G. So we assume $y \sim v_3$; that is, $y = u_3$ and $u_1 \sim u_3$. Moreover, we have shown that $\deg(u_1) = 2$. By symmetry, $\deg(u_3) = 2$.

By a symmetrical argument, $u_2 \sim u_4$ and $\deg(u_2) = \deg(u_4) = 2$. So $\deg(v_i) = 3$ for all i and $\deg(u_i) = 2$ for all i. Therefore G can be drawn in the plane as the graph given in Figure 2. []

Now we show in Theorem 10 that there is only one W_2 graph of girth 4 with exactly one 4-cycle.

Theorem 10. If G is a W_2 graph of girth 4 with exactly one 4-cycle, then G is isomorphic to the graph in Figure 2.

<u>Proof</u>. Suppose G is a W_2 graph of girth 4 with exactly one 4-cycle. Let C = abcd be the 4-cycle in G. By Theorem 5, graph $G_v \in W_2$ for all points v in G. By Theorem 2, $\deg(x) \geq 3$ for all x in C.

Suppose $\deg(x) \geq 4$ for some x in C. Without loss of generality, assume $\deg(b) \geq 4$. Let w and y be neighbors of b such that $\{w,y\} \cap \{a,c\} = \emptyset$. Since G has only one 4-cycle, then d is adjacent to neither w nor y. Consider the W_2 graph G_d. Note that b, y and w are in the same component of G_d. Since G has only one 4-cycle, then G_d has no 4-cycles. Thus, G_d is a W_2 graph with girth > 4.

By Theorem 6, each component of G_d is a line or a 5-cycle. So the component H of G_d containing b, y and w must be a 5-cycle, say H = stybw. See Figure 3.

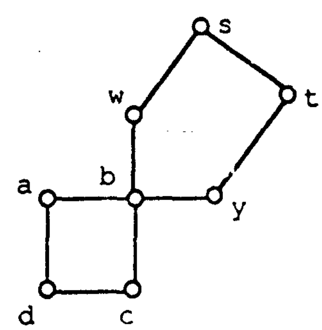

Figure 3

Since G has only one 4-cycle, then s is adjacent to neither a nor c. Thus, the points a, b, c and y are all in the same component of the W_2 graph G_s. But then the component of

G_s containing a, b, c and y is neither a line nor a 5-cycle. Since G_s is a W_2 graph with girth > 4, we obtain a contradiction via Theorem 6.

Hence, deg(b) = 3. It follows that deg(x) = 3 for all x in C. By Lemma 9, the result follows. []

In the following theorem, we prove that if a W_2 graph has a point of degree two which does not have a neighbor of degree two, then the graph is not planar. As a consequence, we prove in Corollary 12 that if a planar W_2 graph of girth 4 has points of degree two, then those points of degree two must occur in adjacent pairs.

Theorem 11. Suppose G is in W_2 and contains a point v of degree two which is not on a triangle and whose neighbors have degree ≥ 3. Then G is not planar.

<u>Proof.</u> Let N(v) = {a,b}. Since v is not on a triangle, then a is not adjacent to b. Let N_1 = N(a)-v and N_2 = N(b)-v. By Theorem 2, a 4-cycle in a W_2 graph cannot have a point of degree two. Thus, $N_1 \cap N_2 = \varnothing$. Suppose there exist points x and y such that $x \in N_1$, $y \in N_2$ and x is not adjacent to y. Then {x,y} and {v} don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Hence, $x \in N_1$ and $y \in N_2$ implies $x \sim y$. Since deg(a) ≥ 3 and deg(b) ≥ 3, then there exist points u_1 and v_1 in N_1 and points u_2 and v_2 in N_2. Since $x \sim y$ for all $x \in N_1$, for all $y \in N_2$, it follows that $u_1 \sim u_2$, $u_1 \sim v_2$, $u_2 \sim v_1$ and $v_1 \sim v_2$. Thus, G is not planar. []

Corollary 12. If G is a planar W_2 graph of girth 4 with $\delta = 2$, then the points of degree two occur as adjacent pairs.

<u>Proof.</u> Suppose v is a point of degree two in G. Since G has girth 4, by Theorem 11 it follows that v has a neighbor of degree two. By Theorem 4, the point v cannot have two neighbors of degree two. Thus, v has exactly one neighbor with degree two. Hence, the points of degree two occur as adjacent pairs in G. []

We note that it is possible for a W_2 graph of girth 4 to have a point of degree two whose neighbors have degree greater than two. The graph in Figure 4 is one such example. Moreover, using the graph in Figure 4 as a starting graph, an infinite family of W_2 graphs of girth 4 can be recursively constructed via Construction 1. Each graph in this family has a point of degree two whose neighbors have degree greater than two.

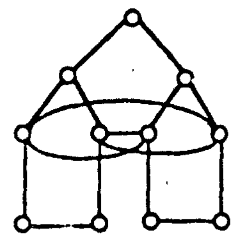

Figure 4

We return to consideration of planar W_2 graphs of girth 4 with points of degree two. Since we know degree two points occur in pairs, we consider the structure around adjacent points of degree two.

Lemma 13. Suppose G is a planar W_2 graph of girth 4 with adjacent degree two points x and y. Let $N(x) = \{u,y\}$ and $N(y) = \{v,x\}$. Then $\deg(u) = \deg(v) = 3$. Moreover, u and v have two common neighbors.

Proof. By Theorem 7, graph G is 2-connected. By Theorem 3, the points x and y are on a 5-cycle C. Then C = xyvwu. By Theorem 4, $\deg(u) \geq 3$ and $\deg(v) \geq 3$. Thus, $\deg(w) > 2$ by Theorem 11. Let $N'(w) = N(w)-\{u,v\}$.

Let $U_x = N(u)-x$. Suppose there exists some $p \in U_x$ such that p is not adjacent to v. Then $\{p,v\}$ and $\{x\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Thus, $p \in U_x$ implies $p \sim v$.

Assume that $\deg(u) > 3$.

Suppose U_x has at least two points outside C and no points inside C. Let $a \in U_x$ such that no point of U_x is in the interior of cycle uwva, and let $b \in U_x$ such that a is the only member of U_x in the interior of cycle uwvb. Since G has girth 4, then $\{a,b,w\}$ is independent (see Figure 5).

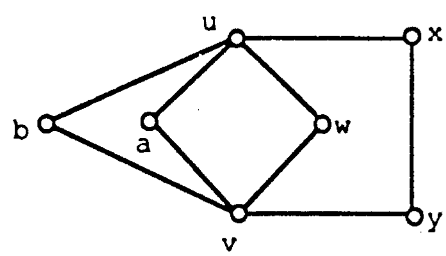

Figure 5

Suppose $z \in N'(w)$ implies $z \sim a$. Then $\{a\}$ and $\{w\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Thus, there exists some $z \in N'(w)$ such that z is not adjacent to a. Consider the graph G_z. By Theorem 5, graph $G_z \in W_2$.

Let $A_1 = \{c \sim a$: c is inside the cycle uwva$\}$ and $A_2 = \{d \sim a$: d is inside cycle uavb$\}$. If $A_1 = \varnothing$, then w is a cupoint for G, contradicting the 2-connectedness of G. Thus $A_1 \neq \varnothing$. If there exists $c \in A_1$ such that c is not adjacent to z, then a is a cutpoint for G_z. Since G_z is 2-connected by Theorem 7, we obtain a contradiction. Thus, $c \in A_1$ implies $c \sim z$. Now, if $A_2 = \varnothing$, then $\{b,z\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in G. So $A_2 \neq \varnothing$.

Let $B = \{f \sim b$: f is inside cycle uavb$\}$. Suppose there exists some $d \in A_2$ such that $d \notin B$. Then a is a cutpoint in the graph G_b, a contradiction by Theorem 7. Thus, $d \in A_2$ implies $d \in B$; that is, A_2 is contained in B. But then $\{b,z\}$ and $\{a\}$ don't extend to disjoint maximum independent sets in G.

Hence, it is not possible that U_x has at least two points outside C and no points inside C. By symmetry, we cannot have at least two points of U_x inside C and no points of U_x outside C.

If U_x has at least one point inside C, then rename w as the innermost point of U_x inside C; that is, choose w so that no point of U_x is inside the 5-cycle xyvwu. Now we can proceed as above to obtain a contradiction.

Hence, deg(u) = 3. By symmetry, deg(v) = 3. Let N(u) = {x,w,t}. From above, $p \in U_x$ implies $p \sim v$. Thus, N(v) = {y,w,t}. []

Hence, if x is a point of degree two in a planar W_2 graph of girth 4, then x must have one neighbor, say y, with degree two and one neighbor, say u, with degree three. In addition, if v is the second neighbor of y, then deg(v) = 3 and u and v have two common neighbors.

We show in the next theorem that a planar W_2 graph of girth 4 with points of degree two must be in the family constructed in Theorem 8.

Theorem 14. Suppose G is a planar W_2 graph of girth 4 with $\delta = 2$. Then G is a member of the family of graphs given in Theorem 8.

Proof. (By induction on the number of 4-cycles.) Suppose G is a planar W_2 graph of girth 4 with $\delta = 2$. Suppose G has exactly one 4-cycle. Then by Theorem 10, it follows that $G = G_3$ given in Theorem 8. Assume the inductive hypothesis: if G is a planar W_2 graph of girth 4 with $\delta = 2$ and the number of 4-cycles in G is exactly k-1 ($k \geq 2$), then G is a member of the family of graphs given in Theorem 8.

Suppose G is a planar W_2 graph of girth 4 with $\delta = 2$ and the number of 4-cycles in G is exactly k ($k \geq 2$). By Corollary 12, graph G has adjacent degree two points, say x and y. Let N(x) = {u,y} and N(y) = {v,x}. It follows from Lemma 13 that deg(u) = deg(v) = 3 and u and v have two common neighbors. So let N(u) = {x,w,a} and N(v) =

$\{y,w,a\}$. Without loss of generality, assume a is exterior to cycle xyvwu (see Figure 6). By Corollary 12, deg(w) > 2 and deg(a) > 2.

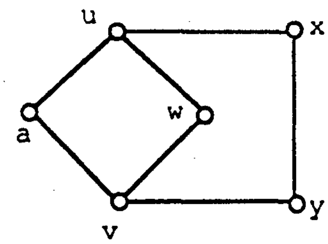

Figure 6

If there exists some point s, $s \notin \{u,v\}$, such that $s \sim w$ and $s \sim a$, then $\{s,x\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Thus, $N'(w) \cap N'(a) = \varnothing$, where $N'(w) = N(w)-\{u,v\}$ and $N'(a) = N(a)-\{u,v\}$.

Case 1. Suppose $\deg(w) \geq 4$.

Case 1.1. Also suppose $\deg(a) \geq 4$. Since we are assuming $\deg(w) \geq 4$ and $\deg(a) \geq 4$, then it follows that there exist distinct points z_1, z_2, a_1, a_2 such that $z_i \in N'(w)$ and $a_i \in N'(a)$, $i = 1, 2$.

Suppose there exist i and j such that a_i is not adjacent to z_j. Then $\{a_i,z_j,y\}$ is independent and so $\{a_i,z_j,y\}$ and $\{u\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Thus $a_i \sim z_j$ for all i and all j. But this contradicts the planarity of G.

Case 1.2. Thus we must have $\deg(a) = 3$. Consider the graph G_y. By Theorem 5, graph G_y is in W_2. In G_y, the points a and u are adjacent points of degree two. Since $N'(w) \cap N'(a) = \varnothing$, it follows that G_y has exactly one less 4-cycle than G (uwva is the only 4-cycle in G containing v). Since the number of 4-cycles in G is exactly k, then the number of 4-cycles in G_y is exactly k-1. So G_y is a planar W_2 graph of girth 4 with $\delta = 2$ and the number of 4-cycles in G_y is exactly k-1. By the inductive assumption, the graph

G_y is in the family of graphs given in Theorem 8. Then G can be obtained from G_y via the construction in Construction 1, with x, y and v playing the roles of a, b and c, respectively (see Figure 7). Thus, G is a member of the family of graphs given in Theorem 8.

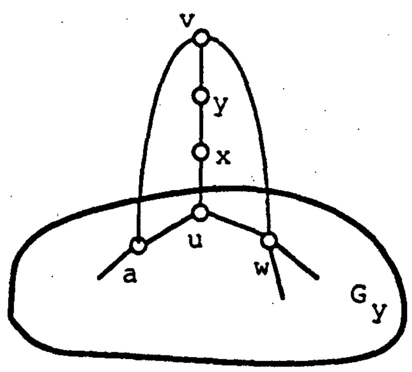

Figure 7

Case 2. Suppose deg(w) = 3. Then in the graph G_y, points w and u are adjacent points of degree two. As in Case 1.2, the number of 4-cycles in G_y is exactly k-1 and, hence, G_y is in the family of graphs given in Theorem 8. Once again, G can be obtained from G_y via the construction in Construction 1, with x, y and v playing the roles of a, b and c. Thus, G is a member of the family of graphs given in Theorem 8. []

In order to complete the discussion on planar W_2 graphs of girth 4, we will show that a planar W_2 graph of girth 4 must have points of degree two. The theory of Euler contributions will be used for this.

We need to consider possible face configurations at a point of degree three in a planar W_2 graph of girth 4. In a 2-connected planar graph, the faces incident at a point can intersect in various ways. However, for a planar W_2 graph of girth 4, the following lemma shows that adjacent faces which are incident to a point of degree three always have a line as their intersection. We omit the proof of the lemma.

Lemma 15. Suppose G is planar W_2 graph of girth 4 with $\delta \geq 3$. Suppose v is a point of degree three with $N(v) = \{u_1,u_2,u_3\}$, incident faces F_1, F_2 and F_3 (where face F_1 contains lines vu_1 and vu_2, face F_2 contains lines vu_2 and vu_3, and face F_3 contains lines vu_3 and vu_1) and positive Euler contribution, $\phi(v)$. Then $F_1 \cap F_2 = vu_2$, $F_2 \cap F_3 = vu_3$ and $F_3 \cap F_1 = vu_1$.

In Theorem 16, by considering all possible face configurations at a point v with $\deg(v) = 3$ and $\phi(v) > 0$, we conclude that a planar W_2 graph of girth 4 must have points of degree two.

Theorem 16. If G is a planar W_2 graph of girth 4, then $\delta = 2$.

Proof. Assume to the contrary that $\delta \geq 3$. Let $\phi(v)$ be the Euler contribution of point v in G. If $\deg(v) = 4$, then $\phi(v) = -1 + \Sigma(1/x_i)$, where the sum is taken over the four faces incident to v. Since G has girth 4, the largest possible value for $\Sigma(1/x_i)$ is 1, when v has face configuration (4,4,4,4). Hence, $\phi(v) \leq 0$ whenever $\deg(v) = 4$. If $\deg(v) = 5$, then $\phi(v) = -3/2 + \Sigma(1/x_i)$, where the sum is taken over the five faces incident to v. The largest possible value for $\Sigma(1/x_i)$ here is 5/4, when v has face configuration (4,4,4,4,4). Hence, $\phi(v) < 0$ whenever $\deg(v) = 5$. In fact, $\phi(v) < 0$ whenever $\deg(v) \geq 5$. Since G must have a point v with $\phi(v) > 0$ and we are assuming $\delta \geq 3$, then we must have $\delta = 3$.

So assume v is a point in G with $\deg(v) = 3$ and $\phi(v) > 0$. Then $\phi(v) = -1/2 + \Sigma(1/x_i)$, where the sum is taken over the three faces incident to v; $\phi(v) > 0$ implies that $\Sigma(1/x_i) > 1/2$. Since G has girth 4, the only possible face configurations at v are the following solutions to the Diophantine inequality $\Sigma(1/x_i) > 1/2$:

1. (4,4,n), for $n \geq 4$;
2. (4,5,n), for $5 \leq n \leq 19$;
3. (4,6,n), for $6 \leq n \leq 11$;
4. (4,7,n), for $7 \leq n \leq 9$;
5. (5,5,n), for $5 \leq n \leq 9$;
6. (5,6,n), for $6 \leq n \leq 7$.

Let $N(v) = \{u_1,u_2,u_3\}$. Let F_1, F_2 and F_3 be the faces incident to v (where face F_1 contains lines vu_1 and vu_2, face F_2 contains lines vu_2 and vu_3, and face F_3 contains lines vu_3 and vu_1). It follows from Lemma 15 that $F_1 \cap F_2 = vu_2$, $F_1 \cap F_3 = vu_1$ and $F_2 \cap F_3 = vu_3$.

Case 1. Suppose v has face configuration (4,4,n), $n \geq 4$. Let $F_1 = vu_1au_2$ and $F_2 = vu_2bu_3$. Since G has girth 4, then a is not adjacent to b. Then $\{a,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$. Therefore, (4,4,n), $n \geq 4$, cannot occur as a face configuration for v.

Case 2. Suppose v has face configuration (4,5,n), $5 \leq n \leq 19$. Let $F_1 = vu_1au_2$, $F_2 = vu_2bcu_3$ and $F_3 = vu_3df...eu_1$ (e = f when n = 5). If a is not adjacent to c, then $\{a,c\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So $a \sim c$. Similarly, $a \sim d$. Since G is planar, point b is adjacent to none of points d, e or u_1, point c is not adjacent to e, and d is not adjacent to u_2. See Figure 8.

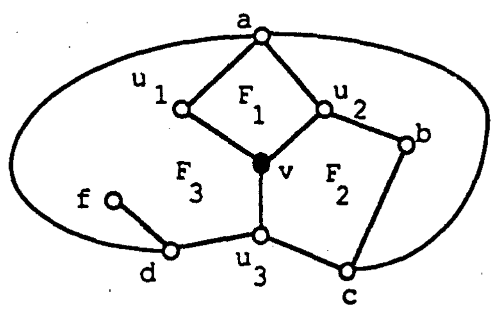

Figure 8

Suppose $\deg(u_3) \neq 3$. Then there exists $w \sim u_3$ such that $w \notin \{c,d,v\}$ and $\{w,b,e\}$ is independent. Then $\{w,b,e\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, a contradiction. Thus, $\deg(u_3) = 3$.

Let $N = N(c)-\{a,b,u_3\}$. Assume $t \in N$ implies $t \sim u_2$. Then $\{d,u_2\}$ and $\{c\}$ don't extend to disjoint maximum independent sets in G. Thus, there exists some $t \in N$ such that t is not adjacent to u_2. But then $\{t,u_2,f\}$ is independent and so $\{t,u_2,f\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

Thus, v cannot have face configuration (4,5,n), $5 \leq n \leq 19$.

Case 3. Suppose v has face configuration (4,6,n), for $6 \leq n \leq 11$. Let $F_1 = vu_1au_2$, $F_2 = vu_2bwcu_3$ and $F_3 = vu_3d...eu_1$. As in Case 2, we have $a \sim c$ and $a \sim d$. Thus, since G is planar, e is adjacent to neither b nor c. Since G has girth 4, then b is not adjacent to c. Hence, $\{b,c,e\}$ is independent and so $\{b,c,e\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, contradicting $G \in W_2$.

Thus, v cannot have face configuration (4,6,n), $6 \leq n \leq 11$.

Case 4. Suppose v has face configuration (4,7,n), $7 \leq n \leq 9$. Let $F_1 = vu_1au_2$, $F_2 = vu_2bxycu_3$ and $F_3 = vu_3dz...eu_1$. As in Case 2, we have $a \sim c$ and $a \sim d$. Also as in Case 2, $\deg(u_3) = 3$. If b is not adjacent to c, then $\{b,c,e\}$ is independent (since G is planar). Thus, $\{b,c,e\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G, a contradiction. So $b \sim c$. It follows that y is not adjacent to u_2. But then $\{y,u_2,z\}$ is independent and so $\{y,u_2,z\}$ and $\{u_3\}$ don't extend to disjoint maximum independent sets in G.

Thus, v cannot have face configuration (4,7,n), $7 \leq n \leq 9$.

Case 5. Suppose v has face configuration (5,5,n), $5 \leq n \leq 9$. Let $F_1 = vu_1abu_2$, $F_2 = vu_2cdu_3$ and $F_3 = vu_3ex...fu_1$ ($x = f$ when $n = 5$).

Case 5.1. Suppose a is not adjacent to c. If neither a nor c is adjacent to e, then $\{a,c,e\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. Thus, either $a \sim e$ or $c \sim e$.

Case 5.1.1. Suppose a ~ e. Then neither b nor d is adjacent to f. See Figure 9.

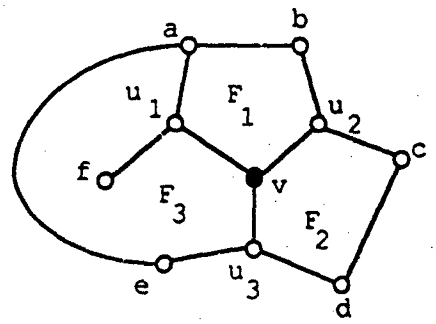

Figure 9

If b is not adjacent to d, then {b,d,f} and {v} don't extend to disjoint maximum independent sets in G. Thus, b ~ d. As in Case 2, deg(u_3) = 3. Now we can apply the argument given in Case 2 to obtain a contradiction.

Case 5.1.2. So c ~ e. If b is not adjacent to f, then {b,f,d} and {v} don't extend to disjoint maximum independent sets in G. So b ~ f. See Figure 10.

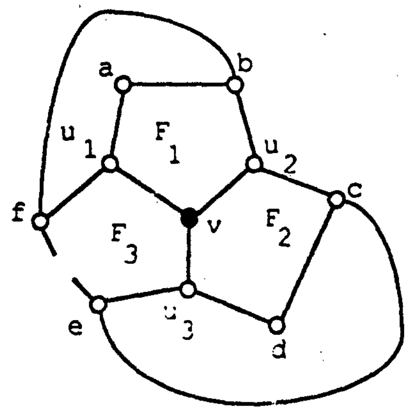

Figure 10

As in Case 2, with u_2 playing the role of u_3, we have $\deg(u_2) = 3$. Now we can apply the argument given in Case 2, with $N = N(c)-\{d,e,u_2\}$, to show that this configuration cannot occur.

Case 5.2. Hence, $a \sim c$. If f is not adjacent to d, then $\{f,d,b\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So assume $f \sim d$. See Figure 11.

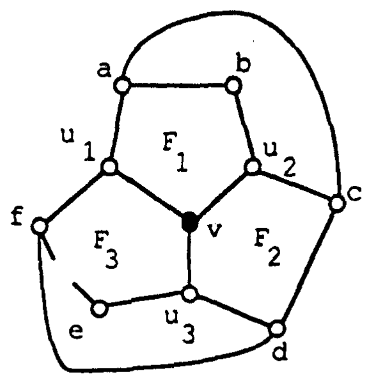

Figure 11

As in Case 2, with u_1 playing the role of u_3, we have $\deg(u_1) = 3$. Now we can apply the argument given in Case 2 to obtain a contradiction.

Thus, v cannot have face configuration (5,5,n), $5 \le n \le 9$.

Case 6. Suppose v has face configuration (5,6,n), n = 6 or 7. Let $F_1 = vu_1abu_2$, $F_2 = vu_2cxdu_3$ and $F_3 = vu_3ewyfu_1$ (w = y when n = 6). Since G has girth 4, then c is not adjacent to d.

Case 6.1. Suppose $a \sim c$. If f is not adjacent to d, then $\{b,d,f\}$ is independent and so $\{b,d,f\}$ and $\{v\}$ don't extend to disjoint maximum independent sets in G. So assume $f \sim d$. See Figure 12.

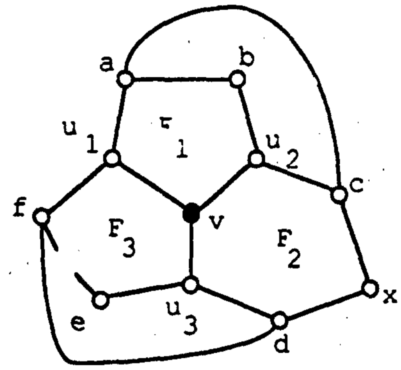

Figure 12

As in Case 2, with u_1 playing the role of u_3, we have $deg(u_1) = 3$. Now we can apply the argument given in Case 2 to obtain a contradiction.

Case 6.2. So a is not adjacent to c. If a is not adjacent to d, then {a,c,d} and {v} don't extend to disjoint maximum independent sets in G, a contradiction. So assume a ~ d.

Case 6.2.1. Suppose n = 6. Since G has girth 4, then f is not adjacent to e. Then {b,f,e} is independent and so {b,f,e} and {v} don't extend to disjoint maximum independent sets in G. So n = 6 cannot occur.

Case 6.2.2. Suppose n = 7. If f is not adjacent to d, then {c,d,f} and {v} don't extend to disjoint maximum independent sets in G. So assume f ~ d. If f is not adjacent to e, then {f,b,e} and {v} don't extend to disjoint maximum independent sets in G. So assume f ~ e. See Figure 13.

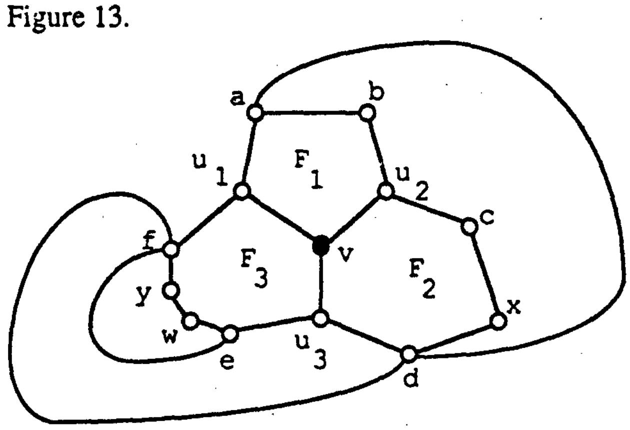

Figure 13

As in Case 2, with u_1 playing the role of u_3, we have $\deg(u_1) = 3$. Then $\{b,y,u_3\}$ is independent and so $\{b,y,u_3\}$ and $\{u_1\}$ don't extend to disjoint maximum independent sets in G. Thus, $n = 7$ cannot occur.

Hence, v cannot have face configuration (5,6,n), for $n = 6$ or 7. Thus, $\deg(v) = 3$ with $\phi(v) > 0$ leads to a contradiction in all possible cases. Therefore, $\delta \leq 2$. Since $G \in W_2$ and $G \neq K_2$, then by Theorem 1 it follows that $\delta \geq 2$. We conclude that $\delta = 2$. []

Hence, we are able to completely characterize the planar W_2 graphs of girth 4. In particular, the next corollary shows that the family of graphs in Theorem 8 is identical to the family of planar W_2 graphs of girth 4.

Corollary 17. If G is a planar W_2 graph of girth 4, then G is a member of the family of graphs given in Theorem 8.

Proof. This follows immediately from Theorem 16 and Theorem 14. []

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