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A TUTORIAL ON EXIT PUPILS AND EYE ROTATION WITH VIRTUAL IMAGE OPTICAL DISPLAYS (U)

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HUMAN ENGINEERING DIVISION

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FOR THE COMMANDER

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Exit pupils of virtual image optical displays and eye entrance pupils and how their sizes and relative positions influence light input to the eye are discussed. Equations based on simplifying assumptions about the eye and its rotational behavior are derived and graphed to describe the position of the center and edge of the eye entrance pupil after eye rotation and for the rotation angles at which exit pupil vignetting begins and at which vignetting is total. The graphs and equations are useful for descriptive purposes and for practical application in choosing, specifying or designing virtual image optical display devices. Worked out numerical examples are supplied to further aid in understanding pupils and eye rotation and how they interact. The reader is not required to have a background in optics or in college mathematics.

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PREFACE

This report was prepared in the Human Engineering Division, Crew Systems Directorate, of the Armstrong Laboratory (AL), Wright Patterson Air Force Base, Ohio. The work was performed under Project 7184, "Man-Machine Integration Technology," Task 718411, "Design Parameters for Visually-coupled Display Systems." Thanks are due to coworkers for encouragement in organizing notes into this report. The assistance of Miss Sheila Radford in preparing the manuscript is sincerely appreciated.

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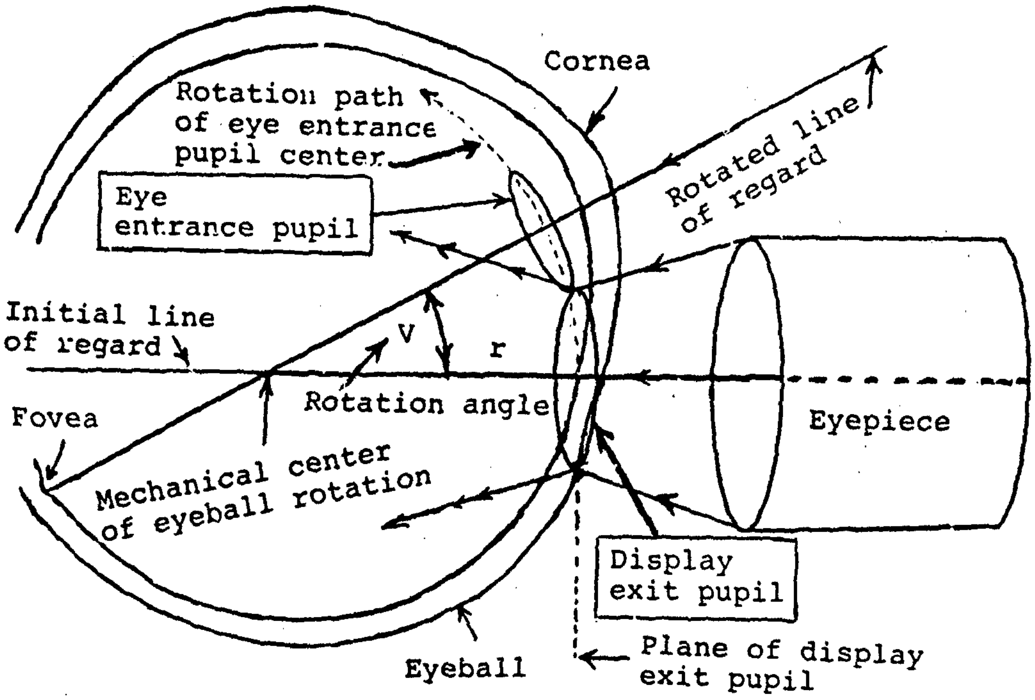
Introduction

Optical display instruments that provide exit pupils and virtual display images are widely used. They are sometimes called aerial image displays. Common examples are binoculars, telescopes, optical sights, and microscopes. Some less common examples are periscopes, helmet-mounted displays, and optical displays used in research and training simulators. Despite their wide-spread use, most users do not understand the nature of display exit pupils and how light input to the observer's eye varies with eye rotation angle and the sizes of the display exit pupil and the observer's eye entrance pupil. People are frequently frustrated when they look up available technical literature for information to help them understand exit pupils, entrance pupils and what happens with eye rotation when using display devices that have exit pupils. There is very little available literature that quantifies light loss as a function of eye rotation angle. The literature that is available is usually highly technical and understanding it often requires a background in basic optics and college mathematics. This tutorial does not require such a background.

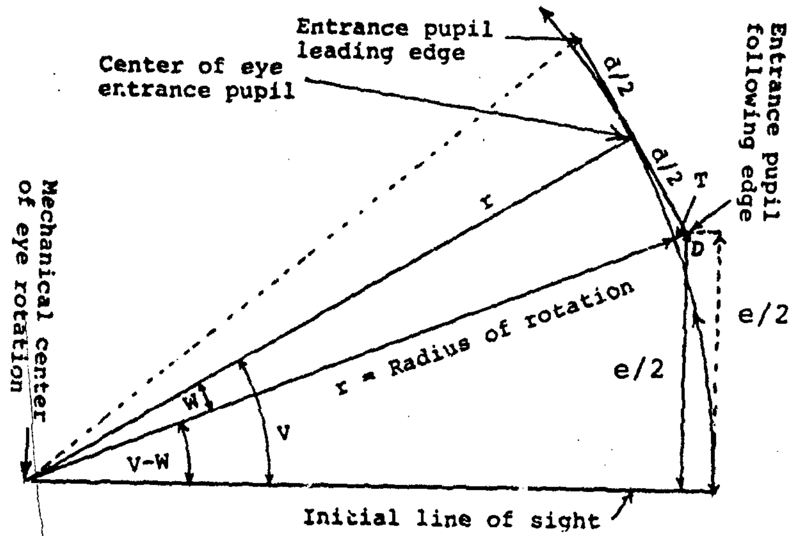
Angular rotation of the line of sight is necessary for an observer to examine image details in different parts of a display. Because the mechanical center of rotation of the eye is behind the entrance pupil of the eye, eyeball rotation moves the eye entrance pupil across the beam of light emerging from the eyepiece of the optical display device. When using optical displays that have large apparent fields of view, to view image details that are near the periphery of the display requires an observer viewing the center of the display to rotate his eyeball through a large angle. Unless the display exit pupil is large, this rotation may move the eye entrance pupil partly out of, or even entirely out of, the beam of light from the eyepiece. When the eye pupil is partly out of the exit pupil, light input to the eye is reduced. When the eye pupil is entirely out of the light beam from the eyepiece, as shown In Figure 1, the display image disappears.

When the location of the eye pupil relative to the exit pupil causes any part of the eye pupil to be out of the light beam from the eyepiece, some light is lost. This exclusion or cut off is called vignetting of the exit pupil. Figure 1 shows how eye rotation can cause vignetting.

For optical displays that are not fixed in position relative to the observer's head, corrective movements of the head, the instrument, or a combination of the two, toward or away from the eyepiece or sideways, will bring the eye entrance pupil fully into the light beam from the eyepiece. Such corrective eye movements are normally easily and quickly made, becoming automatic, so that the device user usually is not aware of making them. This occurs with users of binoculars, telescopes, and microscopes.



A. Rotated position of the eyeball at which the eye entrance pupil has moved out of the display exit pupil.



B. The optical situation when eye rotation has moved the eye entrance pupil out of the exit pupil: total vignetting begins.

Fig. 1. The optical situation when eye rotation has moved the eye entrance pupil away from an initial line of regard that was along the optical axis of the eyepiece.

Display image visibility problems due to eye position relative to the eyepiece of a display system can occur with optical display devices that are not fixed relative to the observer's head. They can occur when the observer or the examined object is in motion, especially when observing from a moving vehicle, or when observing with an optical instrument that has a small exit pupil. Due to weapon recoil, firing weapons with optical sights will cause the exit pupil to move relative to the eye, with attendant viewing problems. It may not be possible to regain a lost display image, again find the image of the target, and position a reticle on the target image as quickly as necessary. In such situations, a large exit pupil will minimize the probability of losing the display image.

It is clear, from these examples, that a large exit pupil is valuable, even when corrective eye, head, or instrument movements are possible. However, to obtain a large exit pupil, a price must be paid in volume, weight and cost of the equipment. Larger exit pupils require larger optics.

Helmet-mounted virtual image optical displays are fixed in position relative to an observer's head, hence their exit pupils are fixed in position relative to the user's eye entrance pupils. For helmet-mounted displays, the conditions of use can require display exit pupils that are much larger in diameter than the largest eye pupils that will occur under conditions of use. For such displays, as for displays not fixed in position relative to the user's head, eye rotation to view off-axis areas of a wide-angle display can rotate the eye entrance pupil partly out of the beam of light from the eyepiece, especially if the exit pupil is small. However, the main reason why exit pupils in helmet-mounted displays are much larger than eye entrance pupils is that the use conditions move the exit pupil relative to the user's head, hence relative to his eye entrance pupils. This motion can be due to motion of the scalp on the head or to movement or slipping of the helmet on the scalp. Such movement can occur from bumps to the helmet or to the user, from quick or jerky head motion from search behavior, turbulence, rough terrain, vehicle speed changes and vehicle maneuvers.

When relative movement occurs between the display exit pupil and the entrance pupil of the eye of the user of a helmet-mounted display system, head and eye movements do not return the system to a proper alignment of the pupils. A lost display image or a loss in field of view or a reduction in display image luminance is not thereby regained. In some situations, when the displayed image becomes dimmer or is lost, the display user could hold the helmet in both hands and reposition it on his head to regain normal operation. However, in many situations, such as when controlling a vehicle, the user's hands may be too busy at other tasks for the helmet user to manually reposition his helmet. For systems in which helmet position relative to the vehicle is used to automatically aim cameras, guns, or missiles, such a rough manual repositioning would not be acceptable. In such systems, the user moves his head to place a reticle upon an object of interest, and the system then aims the sensor, guns, etc. When the helmet moves with respect to the head, the alignment and accuracy of the system is upset. Manual repositioning must be followed by an alignment procedure.

It is clear, from these comments, that very large display exit pupils may be advantageous, or even necessary, for both display systems fixed relative to a user's head and those that are not fixed.

An understanding of the relationship of the relative positions and sizes of display exit pupils and eye entrance pupils to the amount of light entering the eye is useful, particularly to those who apply systems with exit pupils in research. Availability of information required to calculate the angle at which vignetting begins and the angle at which it is total is essential when the exit pupil size of optical displays that have exit pupils must be specified.

In some display systems, weight is so critical that every ounce of weight that can be safely removed must be removed. A compromise must be made between the need for large exit pupils and a requirement that the display system be small and light. An understanding of the interaction between entrance and exit pupil sizes and eye rotation angle allows realistic compromises or trade-offs to be made between exit pupil size and the weight and cost of display equipment.

This tutorial was written to assist readers to understand entrance pupils and exit pupils and how they interact with eye rotation to influence the amount of light that enters the eye from an optical display. No realistic design compromises are possible without such understanding. No previous knowledge of optics or of mathematics beyond elementary algebra, geometry, and trigonometry is required for the reader to be able to follow the text.

OBTAINING DATA ON LIGHT LOSS WITH EYE PUPIL ROTATION

Specifying or selecting optical display equipment that has exit pupils, or making optical design decisions about them, requires data on light loss that may occur from eyeball rotation, particularly for systems in which the head of the observer is fixed relative to the display equipment.

Such data could be obtained by using any one of several approaches. For example, a comprehensive data collection could be obtained by directly measuring retinal illumination on the fovea of the eye. Measurements would be taken when the line of sight is on the optical axis of the eyepiece, then at several off-axis angles for several directions of rotation. The measurements could be taken for several exit pupil diameters, eye entrance pupil diameters, and eyepiece fields of view. The diameter of the eye entrance pupil would have to be controlled. Due to the large number of independent variables, a great many measurements would be required. Special equipment would be required to measure retinal illuminance. Such a study would be a major undertaking.

A more feasible study would measure the angles of rotation away from the eyepiece optical axis when vignetting of the exit pupil began and when it was total for a range of exit pupil sizes, eye entrance pupil sizes, and display fields of views. The light falling upon the eyeball would be observed to determine when vignetting began and when it was total. It would not be necessary to measure retinal illumination.

A less ambitious study than this second one could take measurements on a schematic or model eye for the same independent variables mentioned above. Such a study would not obtain data over a range of eyeball sizes that would be representative of a population of interest. Thus, for example, no data would be obtained for the fifth or the ninety-fifth percentile of potential equipment users, or for any other percentile. However, to the extent that the optical characteristics of the model eye were an approximation to the average optical characteristics of some known population of users, the data would be useful to equipment designers or specifiers.

Such a study was conducted by Spiro (1961) for a collimated display, i.e., a display with an image at optical infinity. The display used an eyepiece with a field of view of 85 degrees and a 4mm exit pupil. Eyepieces with such a huge field of view are rare. Spiro determined, with data from live human observers, that eye rotation angles larger than 30 deg., although possible, were uncomfortable. He also determined how much light cut off (vignetting) could be present without any dimming of the display image being noticed by his observers. He found that up to 55% vignetting at the edge of the field of view was not noticed by any of his observers. His graph for the onset of vignetting of the exit pupil and for total display extinction, as a function of eye rotation angle, being specific to an eye entrance pupil diameter of 4mm and a 13.5mm exit pupil, can not be applied to equipment with other exit pupil diameters or other pupil sizes. Using only one exit pupil size, one eye pupil size, and one angular field of view, however did involve considerable data collection. A major problem with the Spiro report is that it did not specify the experimental procedures and test conditions.

Still another method for obtaining the eye rotation angle at which vignetting begins and the angle when it becomes total is to perform calculations based on a simplified model of the eye. For example, it may be assumed that the eye is a rigid perfect sphere and that it has a mechanical center of rotation about a fixed point within the eye located at a specified distance, such as 10mm, behind the entrance pupil of the eye. Actually, the eyeball is not a perfect sphere, is not very rigid, and its behavior during rotation is quite complex. The above assumptions, then, are only approximations to reality. Calculations based on such assumptions, obviously, supply data of limited accuracy. However, when design or selection decisions must be made, results of such calculations are much more useful than the results from pure guessing.

APERTURE STOPS AND ENTRANCE AND EXIT PUPILS

Many optical display devices, such as binoculars, telescopes, microscopes, periscopes, and helmet-mounted displays, have exit pupils located behind the eyepiece, i.e., on the observer's side of the eyepiece, and present observers with a virtual display image, sometimes called an aerial image. Such optical display equipment usually consists of a lens or a curved mirror that is the first optical component to receive and admit light from an object or a scene. Being closest to the object, this first component is called an objective. It projects a real image which is magnified by a second component for presentation to an observer. The second component, since it is closest to the eye of the observer, is called an eyepiece or ocular. A relay lens, usually called an erector, may be located between the objective and the eyepiece so that the system can present to an observer an image that is correctly oriented relative to the scene or to the object. A relay lens can also supply some magnification of the primary real image projected by the objective.

In any optical system, there is an aperture or opening that limits the amount of light that can be collected by the system. This aperture, the system aperture stop, is called a stop because it stops or cuts out light outside of it. It is often within or very close to the objective. The system entrance pupil is the virtual image of the aperture stop as viewed from the object side of the instrument, i.e., from the front or light entrance side. The aperture stop, then, determines or limits the effective aperture of the objective. To obtain more light at the periphery of the field of view, the objective may have a diameter appreciably larger than the diameter of the entrance pupil. This added size does not increase display luminance at the display center, but it does cause display luminance to decrease at a slower rate as the angle between the display center and the line of regard increases.

To the eyepiece, the entrance pupil is a uniformly-luminous disc-shaped object. More properly, it is an aperture with a sharply-defined edge through which light is emerging. The eyepiece into which the display user looks projects or brings to a focus a real image of the display entrance pupil or opening at a location on the observer side of the eyepiece. This real image, in the form of a featureless disc of light, is the exit pupil of the optical display device. In some older textbooks the exit pupil was referred to as the Ramsden disc, and sometimes it was called the eye ring to emphasize its nature.

The light from the eyepiece converges to the exit pupil or eye ring, then diverges past it. The exit pupil is thus the waist or narrowest part of the beam of light emerging from the eyepiece. Obviously, the light from the eyepiece is most concentrated at the exit pupil. Since the exit pupil is the image of the display entrance pupil, a larger exit pupil will require a larger entrance pupil, hence a larger objective. Equations for the size and distance behind the eyepiece of the exit pupil are derived in an earlier tutorial on optical displays by Self (1992).

As noted above, to obtain a larger exit pupil for an optical instrument requires replacing the objective with an objective having a larger clear or effective diameter, i.e., having a larger entrance pupil. As the real image of the aperture stop, the exit pupil has the same shape as the aperture stop. Since most optical devices have round objectives, most instrument aperture stops are constructed to be round, resulting in exit pupils that are round or disc shaped. For some objectives, such as most telescope objectives, the aperture stop is at the objective, and exit pupil shape is the same as entrance pupil shape. An entrance pupil with a central obstruction, such as is present in most reflecting telescopes and in catadioptric (containing both reflective and refractive components) telescopes and telephoto camera lenses, provide a hollow exit pupil, a disc of light with an area of no light in the center.

This hole or dark area has the same shape as the central obstruction. Only when such devices are used with eyepieces that provide low power and large exit pupils is the hole or dark area large enough relative to the entrance pupil of the eye to be noticeable and bothersome due to display luminance variation with eye rotation or with sideways head motion. Even then, of course, the user doesn't see the hole, since the image of the display entrance exit pupil is at the eye entrance pupil and thus can not be focussed by the eye to form an image of the exit pupil on the observer`s retina.

As noted earlier, a relay lens may be located between the objective and the eyepiece. When the relay lens is not of adequate size, its effective aperture limits the amount of light reaching the eyepiece. In this case, the instrument's exit pupil is the real image of the aperture stop that determines the effective aperture of the relay lens. This is the case in many optical display systems. In such systems, the exit pupil is smaller and closer to the eyepiece.

OPTICAL CHARACTERISTICS OF THE EYE

As an optical device, the human eye differs in several ways from man-made optical devices. In most manufactured optics, the optical elements or components are "centered": the optical axis of the system is a straight line passing through the centers of curvature of the surfaces of the components. In contrast to this, in the human eye the cornea and the crystalline (or focussing) lens do not share a common axis. In the terminology of optical engineers, they are "decentered." In addition, the optical and the visual axis (line of regard or visual fixation axis) are at an angle of 4-5 degrees, sometimes even as much as 7 degrees, to each other. A typical value is 5 degrees.

Optical axes are, as noted above, defined by the geometry of the optical elements or components. For the human eye, as an approximation, the optical axis may be defined as a line perpendicular to the cornea at the cornea's most forward part (the vertex or anterior part of the cornea) and centered on the entrance pupil. The optical axis is usually directed outward and slightly downward from the visual axis. The visual axis is defined by the optics and the part of the retina of the eye where vision is most acute, i.e., where visual resolution of fine image details is greatest. The visual axis may be defined as being, approximately, the line connecting the visual fixation point and the center of the eye's entrance pupil. The optical constants of the human eye are treated by Hopkins (1962) and by Westheimer (1968). A good reference on the anatomy of the eye is found in Brown (1966).

ROTATIONAL CHARACTERISTICS OF THE EYE

The human eye is sometimes, for simplicity, thought of as a rigid sphere rotating in a rigid socket about a fixed mechanical center of rotation, a sighting center. However, the eye is not a perfect sphere: it is slightly flattened in back. Also, the space between the eyeball and the bones of the skull contain pads or cushions of fatty connective tissue. The location of the mechanical center of rotation of the eye varies slightly as the eye moves in its socket. Fry and Hill (1963) found that, with eyeball rotation, the distance from the vertex of the cornea to the mechanical or physical center of rotation of the eye varies with the direction of rotation. This distance was about 2mm longer for vertical rotation than for horizontal rotation. This is a difference of about 20%. Also, the vertical and horizontal axes of rotation did not intersect.

From the above discussion, it is clear that eye movement during eye rotation is quite complex. Alpern (1969) reviewed several studies done prior to 1969 on the center of rotation of the human eye and concluded that the idea of a single fixed center of rotation is an approximation convenient for first-order description of eye movements. A detailed, but somewhat dated, treatment of eye rotation and the center of rotation of the eye is found in Southall (1937). He presents an interesting history, prior to 1937, of concern about and attempts to measure the characteristics of the center of rotation of the eye.

PUPIL POSITION AND RETINAL ILLUMINANCE

To directly view different parts of a displayed image, i.e., to fixate them with the line of sight or line of regard, the eye, the head, or a combination of the two, must move. In addition, the entrance pupil of the eye must intercept or take in some light from the eyepiece. To simultaneously view the entire displayed image, and to admit into the eye as much light as possible, the entrance pupil of the eye should be positioned to coincide with the exit pupil of the display device. The eye pupil should be entirely within the exit pupil of the instrument.

Eye entrance pupils are circular discs, so that when the diameter of the exit pupil is less than the diameter of the eye entrance pupil, the amount of light admitted to the eye is proportional to the exit pupil area, i.e, is proportional to the square of exit pupil diameter. Hence, the amount of light admitted through the eye entrance pupil into the eye increases rapidly with increasing diameter of the exit pupil. The increase in amount of admitted light ceases when the exit pupil light fills the eye entrance pupil, since light that falls outside of the eye entrance pupil is not admitted. It is vignetted or cut off.

When the entrance pupil of the eye is partly out of (to the side of) the exit pupil, the light energy in the retinal image is proportional to the area of the entrance pupil that is in the exit pupil. When eye rotation moves the eye's entrance pupil away from the center of the device exit pupil, the plane of the eye entrance pupil becomes tilted with respect to the optical axis of the eyepiece and to the plane of the exit pupil. This tilting causes a reduction in admitted light, because tilting reduces the projected or effective area of the light-collecting eye entrance pupil.

It is often said that the eye entrance pupil must be in the display exit pupil, or that at least part of the eye pupil must be in the exit pupil, for the display to be seen. This often-quoted statement is not true. As long as even a part of the eye entrance pupil is in the beam of light from the eyepiece, even if none of the eye entrance pupil is at or in the exit pupil, at least part of the display image will be visible. Display users are unlikely to have their eye entrance pupil exactly at the exit pupil: their pupils are usually slightly in front of or behind the exit pupil, and not precisely centered on the exit pupil. Even if requested to place their eye entrance pupil precisely at the exit pupil, they can't do so. Unless the pupil location error is large, observers will be unaware of any loss in the luminance of the displayed image due to eye placement error. In the absence of a sharply-defined adjacent comparison or standard area, humans are rather insensitive as luminance monitors or sensors. When the eye entrance pupil is not at the display exit pupil, retinal illuminance will be proportional to the area of the entrance pupil that is in the beam of light from the eyepiece.

Neglecting light losses from scattering and absorption that are present in all optical devices, and are trivial in quality instruments, when a device exit pupil is as large as or larger than the eye's entrance pupil, and the pupils coincide, retinal illuminance with and without the device are equal. Of course, image angular subtense and linear size on the retina with an optical display device may be very different from their sizes when observing with the unaided eye.

Some optical display devices contain an active component, an electro-optical light amplifier. For devices that contain such an active component, retinal illuminance may be vastly greater than that provided by the unaided eye or by devices containing only passive components. The above and following discussion of entrance pupils, exit pupils, and retinal illuminance is concerned with passive optical devices. However, most of the principles may be readily applied to active optical devices.

THE EFFECT OF EYE ROTATION ON EYE ENTRANCE PUPIL LOCATION AND ON VIGNETTING OF THE DISPLAY EXIT PUPIL

As noted earlier, rotation of the eye about its mechanical center of rotation moves the entrance pupil of the eye. The optical situation is shown in Figure 2. Assume that the eye is a rigid sphere with a stationary center of rotation at a fixed distance r from the center of the eye's entrance pupil. The cited value of r in the literature is usually 10mm. This is an approximate average value. As noted earlier, the above assumptions are only approximations to real eyes. Assume that, initially, the pupil center and the line of visual regard or line of visual fixation are both on the eyepiece optical axis i.e., are on a line passing through the center of the display exit pupil. Based on these simplifying assumptions, equations may be readily derived to calculate the approximate position of the eye entrance pupil following an angular eye rotation. The eye rotation angle at which vignetting of the exit pupil begins and the angle at which vignetting is total (100% vignetting) are also easily derived. Some equations of interest to people who use, choose, or specify exit-pupil containing optical displays are derived in the following paragraphs.

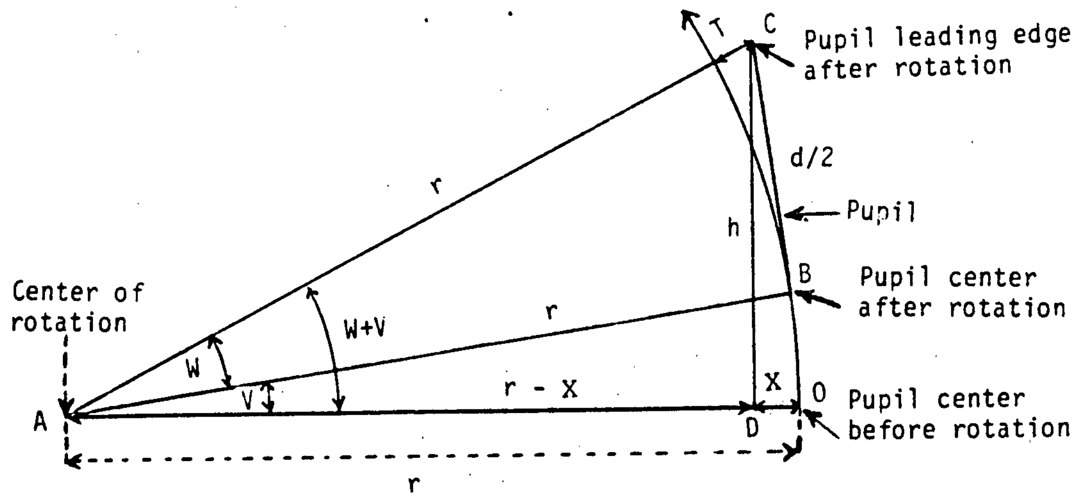
Part (A) of Figure 2 shows the angular subtense at the center of rotation of the eye entrance pupil before eye rotation. In the figure, in right triangle AOC, the radius d/2 of the eye's entrance pupil subtends an angle W. From the figure, $\tan W = (d/2)/r$, from which

Eqn. (1) $W = \arctan(d/2r)$ The angle subtended at the eye's rotation center by the edge of the eye entrance pupil.

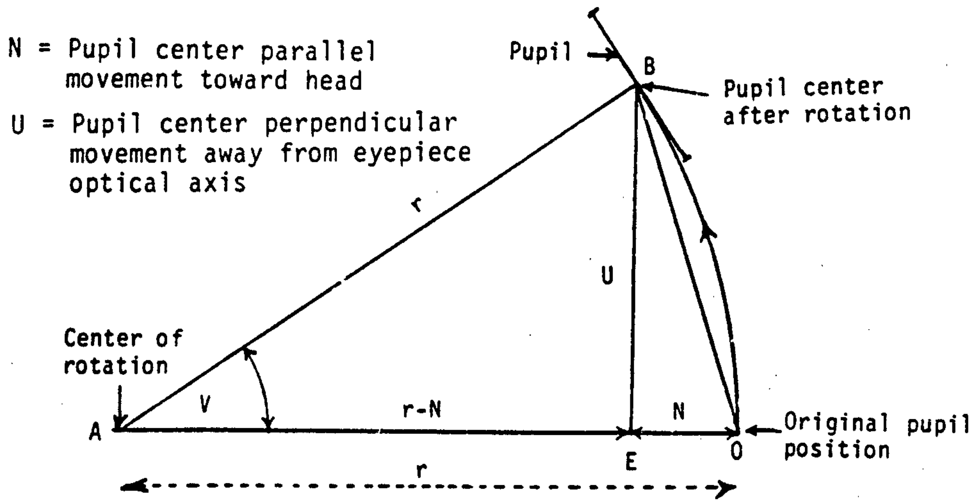
Part (B) of Figure 2 shows the position of the pupil center following an eye rotation angle V. Due to the eye rotation, the center of the entrance pupil has moved up a perpendicular distance U and subtends, at the rotation center, an angle V. From the figure, from right triangle AEB, $\sin V = U/r$, from which

Eqn. (2) $U = r \sin V$ The perpendicular distance from the eyepiece optical axis to the eye entrance pupil center after an eye rotation angle V.

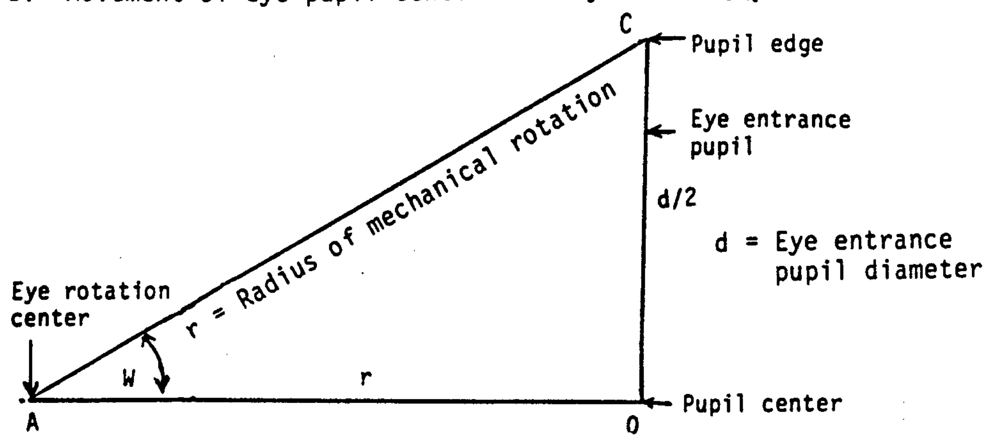
Also, from part B of Figure 2, where N is the parallel distance that the pupil center moves back from a rotation angle V, from right triangle AEB, $\cos V = (r - N)/r$, from which



C. Movement and location of leading edge of eye pupil with eye rotation.



B. Movement of eye pupil center with eye rotation.



A. Angular subtense at the rotation center of the eye entrance pupil edge before eye rotation.

Fig. 2. Eye entrance pupil edge and center locations before and after eye rotation.

Eqn. (3) $N = r - r\cos V$ The parallel distance that the eye pupil center moves back toward the head from an eye rotation angle V.

In part C of Figure 2, let h be the perpendicular distance from the eye pupil leading edge, the top edge in the figure, to the optical axis of the eyepiece after a rotation angle V. From the figure, from right triangle ADC,

$\sin(W + V) = h/(r + T)$. From right triangle ABC,

$$(r + T)^2 = r^2 + (d/2)^2$$
$$= r^2 + d^2/4, \text{ from which}$$
$$(r + T) = \sqrt{r^2 + d^2/4}. \text{ Substituting,}$$

$$\sin(W + V) = h/(r + T)$$
$$= h/\sqrt{r^2 + d^2/4}, \text{ from which}$$

Eqn. (4) $h = \sqrt{r^2 + d^2/4}\,\sin(W + V)$ Perpendicular height above the eyepiece optical axis of the eye pupil leading edge after an eye rotation angle V.

The perpendicular distance delta h that the pupil leading edge has moved up from its initial position above the optical axis of the eyepiece is its final or post-rotation height minus its original height. Thus, delta $h = h - d/2$. Substituting for h the value given above by Eqn. (4),

Eqn. (5) delta $h = \sqrt{r^2 + d^2/4}\,\sin(W + V) - d/2$ Increase in eye pupil leading edge height from an eye rotation angle V.

The rotation angle V at which the leading edge (the top edge in the figure) is a perpendicular distance h from the optical axis of the eyepiece is derivable from Eqn. (4),

$h = \sqrt{r^2 + d^2/4}\,\sin(W + V)$, from which

$\sin(W + V) = h/\sqrt{r^2 + d^2/4}$. Taking the inverse or arc sine of both sides of this equation,

$W + V = \text{ArcSin}\,(h/\sqrt{r^2 + d^2/4})$, from which

$V = \text{ArcSin}(h/\sqrt{r^2 + d^2/4}) - W$. From Eqn. (1),

$W = \text{ArcTan}\,(d/2r)$. Replacing W with this,

Eqn. (6) $V = \text{ArcSin}\,(h/\sqrt{r^2 + d^2/4}) - \text{ArcTan}\,(d/2r)$ The eye rotation angle at which the pupil leading edge is a perpendicular distance h from the optical axis of the eyepiece.

The angle at which vignetting of the exit pupil begins is defined as the eye rotation angle V at which the leading edge of the entrance pupil of the eye is a perpendicular distance $h = e/2$ from the eyepiece optical axis. Eye entrance pupil edge and display exit pupil edge are equally distant from the eyepiece optical axis. With further eye rotation, the entrance pupil of the eye starts moving out of the display exit pupil. Substituting e/2 for h in Eqn. (6), Eqn. (7) $V = \text{ArcSin}\,(e/2\sqrt{r^2 + d^2/4}) - \text{ArcTan}\,(d/2r)$ The eye rotation angle at which the eye pupil leading edge is a height $e/2$ above the eyepiece optical axis. Vignetting begins.

The situation for which the following edge (in the figure, the bottom edge) of the eye entrance pupil is a perpendicular distance H from the eyepiece optical axis is shown in Figure 3. In the figure, from right triangle ABD, $\text{Sin}\,(V - W) = H/(r + T)$. In right triangle ACD, $(r + T)^2 = r^2 + (d/2)^2$, from which $(r + T) = \sqrt{r^2 + d^2/4}$. Substituting for $(r + T)$, $\text{Sin}\,(V - W) = H/(r + T) = H/\sqrt{r^2 + d^2/4}$. Taking the inverse or arc sine of both sides,

$V - W = \text{ArcSin}\,(H/\sqrt{r^2 + d^2/4})$, from which

$V = \text{ArcSin}\,(H/\sqrt{r^2 + d^2/4}) + W$.

From right triangle ACD, $\text{Tan}\,W = (d/2)/r = d/2r$, from which $W = \text{Arc Tan}\,(d/2r)$, as given by Eqn. (1). Substituting this for W,

Eqn. (8) $V = \text{ArcSin}\,(H/\sqrt{r^2 + d^2/4}) + \text{ArcTan}\,(d/2r)$ The eye rotation angle at which the following edge of the eye entrance pupil is a perpendicular distance H from the optical axis of the eyepiece.

In the above equation, note that, except for the algebraic sign of ArcTan $(d/2r)$, equations 6 and 8 are identical.

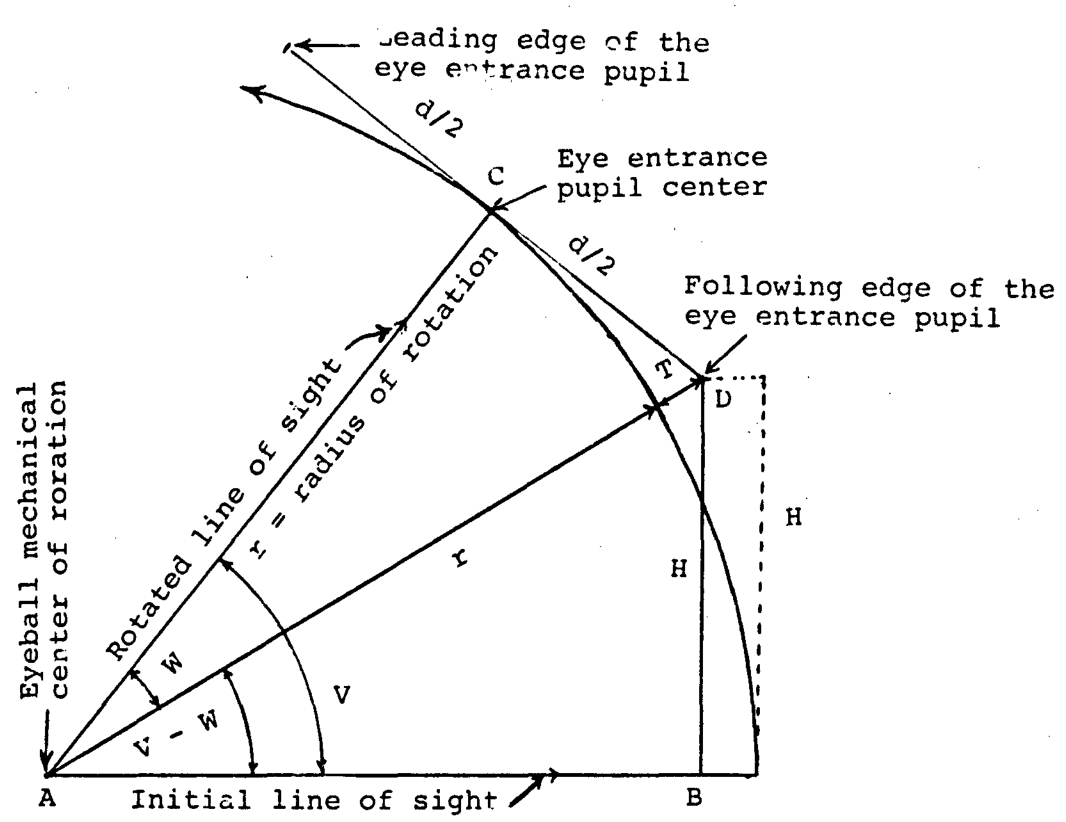


Fig. 3. Optical geometry when eye rotation has moved the eye entrance pupil following edge up to a perpendicular distance H from the optical axis of the eyepiece.

When $H = e/2$, the following edge (the bottom edge in the picture) of the eye entrance pupil is a distance from the optical axis of the eyepiece that is equal to that of the top edge of the exit pupil, and it is assumed that vignetting of the exit pupil is total: the eye entrance pupil is entirely out of the exit pupil. Due to the convergence of the beam of light from the eyepiece to the exit pupil and divergence past the exit pupil, i.e., due to the truncated double cone nature of the light beam emerging from the eyepiece, when $H = e/2$, not all light from the eyepiece is lost. However, here it is assumed that vignetting is total when $H = e/2$. With this approximation, and substituting $e/2$ for H in Eqn. (8), it becomes

Eqn. (9) $v = \mathrm{ArcSin}\ (e/2\sqrt{r^2 + d^2/4}) + \mathrm{ArcTan}\ (d/2r)$ The eye rotation angle at which the following edge of the eye entrance pupil is a perpendicular distance $e/2$ from the eyepiece optical axis. Vignetting of the exit pupil is assumed to be total.

Note that, except for the algebraic sign of the ArcTan term, Eqn. (9) is identical to Eqn. (7).

In Figure 2, part C, X is the parallel distance that the leading edge of the eye entrance pupil has moved back toward the head from an eye rotation angle V. In right triangle ADC of the figure, $\mathrm{Tan}\ (W + V) = h/(r - X)$, from which

$X = r - h/\mathrm{Tan}\ (W + V)$. Since $\mathrm{Tan}(W + V) = \mathrm{Sin}(W + V)/\mathrm{Cos}(W + V)$,

$= r - h/[\mathrm{Sin}\ (W + V)/\mathrm{Cos}\ (W + V)]$

$= r - h\mathrm{Cos}\ (W + V)/\mathrm{Sin}\ (W + V)$. Replacing h with, from Eqn. (4),

$h = \sqrt{r^2 + d^2/4}\ \mathrm{Sin}\ (W + V)$, the equation becomes

$X = r - [\sqrt{r^2 + d^2/4}\ \mathrm{Sin}\ (W + V)]\mathrm{Cos}\ (W + V)/\mathrm{Sin}\ (W + V) =$

Eqn. (10) $X = r - \sqrt{r^2 + d^2/4}\ \mathrm{Cos}\ (W + V)$ The parallel distance that the leading edge of the eye entrance pupil has moved back toward the observer's head from an eye rotation angle V.

In the above equation derivations, Eqn. (7) gives the rotation angle V at which vignetting of the instrument exit pupil by the entrance pupil of the eye begins, i.e., the eye entrance pupil begins to move out of the exit pupil, and Eqn. (9) gives the eye rotation angle at which vignetting is total, i.e., the eye entrance pupil is entirely out of the instrument exit pupil. In deriving these two equations, as noted earlier, it was assumed that vignetting begins when the leading edge of the eye's entrance pupil reaches a perpendicular distance $e/2$ from the optical axis of the eyepiece, i.e., is as far from the device optical axis as is the edge of the exit pupil. Similarly, it was assumed that, when the following edge was this far from the axis, vignetting was total.

Farrell and Booth (1984), presumably deriving from a sketch that was a somewhat simplified representation of the rotational behavior of the eye, supplied an equation for the eye rotation angle at which vignetting begins and an equation for the angle at which vignetting was total. Unfortunately, they did not supply derivations for the equations. In Figure 4, the two equations are derived, with an exit pupil diameter e and an eye entrance pupil diameter D. The 20 in the denominators of the equations is 2r, with r, the radius of mechanical rotation of the eye, assumed to be 10mm. In parts A and B of Figure 4, the equations that they provided are derived, and are as follows:

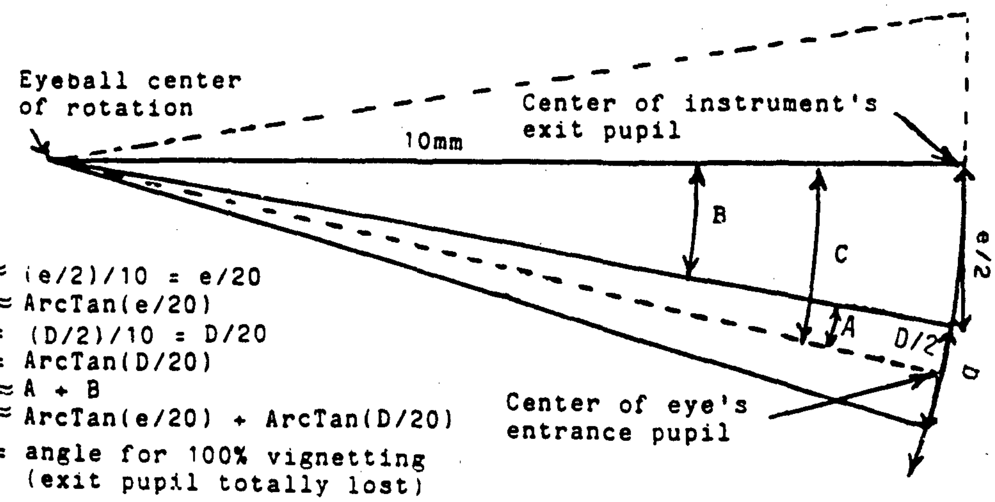
Eqn. (11) $C = \text{ArcTan}\,(e/20) - \text{ArcTan}\,(D/20)$ The eye rotation angle at which vignetting begins. (Farrell and Booth).

Eqn. (12) $F = \text{ArcTan}\,(e/20) + \text{ArcTan}\,(D/20)$ The eye rotation angle at which vignetting is total (Farrell and Booth).

It is instructive to derive equations for the beginning of vignetting of the exit pupil and for total vignetting using a different approach from the one used in deriving equations 7 and 9. Although the resulting equations are quite different, for the same values of entrance and exit pupil diameters, they yield the same values for the vignetting angles. The same assumptions are used as before, but a different sketch of the situation is used.

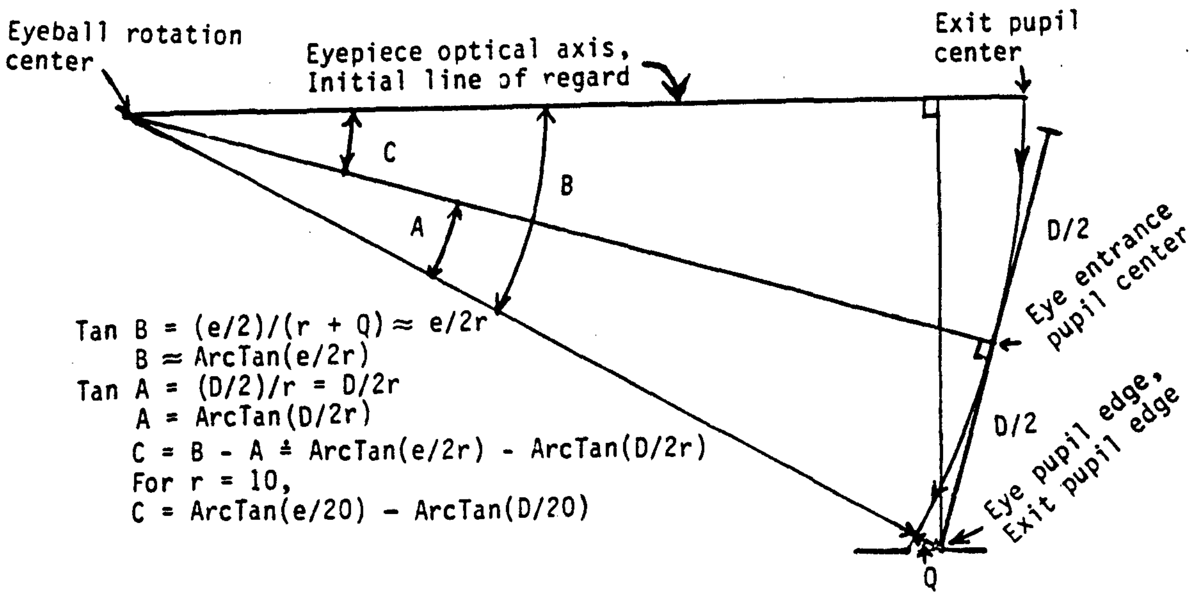
Assume that the line of sight of the eye is initially along the optical axis of the eyepiece. Also assume that total vignetting occurs when the line of sight is at an angle of rotation for which the following or bottom edge of the eye entrance pupil is a perpendicular distance e/2 from the eyepiece optical axis. This distance is the distance of the top edge of the display exit pupil from the optical axis of the eyepiece. These assumptions are the same ones that were made earlier.

In right triangle OAC of Figure 5, $\text{Sin } V = (e/2 + Y)/r$, from which $Y = r\text{Sin } V - e/2$. In right triangle CBD, $\text{Cos } V = Y/(d/2)$, from which $Y = (d/2)\text{Cos } V$. Equating the two expressions for Y, $r\text{Sin } V - e/2 = (d/2)\text{Cos } V$. Multiplying both sides by 2, $2r\text{Sin } V - e = d\text{Cos } V$. Replacing Cos V with $\text{Cos } V = \sqrt{1 - \text{Sin}^2 V}$, $2r\text{Sin } V - e = d\sqrt{1 - \text{Sin}^2 V}$. Squaring both sides, $4r^2\text{Sin}^2 V - 4er\text{Sin } V + e^2 = d^2 - d^2\text{Sin}^2 V$.
Rearranging terms,
$(4r^2 + d^2)\text{Sin}^2 V - 4er\text{Sin } V + (e^2 - d^2) = 0$ The solution to this quadratic equation in sine V, which has the form, where $X = \text{Sin } V$, $AX^2 + BX + C = 0$, is $X = [-B + \sqrt{B^2 - 4AC}]/2A$.
Substituting the corresponding terms from the above equation in Sin V,
$\text{Sin } V = [4er + \sqrt{16e^2r^2 - 4(4r^2 + d^2)(e^2 - d^2)}]/2(4r^2 + d^2)$.
This readily simplifies to
$\text{Sin } V = [2er + d\sqrt{4r^2 + d2 - e^2}]/(4r^2 + d^2)$.



$$\text{Tan } B \approx (e/2)/10 = e/20$$
$$B \approx \text{ArcTan}(e/20)$$
$$\text{Tan } A = (D/2)/10 = D/20$$
$$A = \text{ArcTan}(D/20)$$
$$C \approx A + B$$
$$C \approx \text{ArcTan}(e/20) + \text{ArcTan}(D/20)$$
= angle for 100% vignetting (exit pupil totally lost)

B. Vignetting is total.



$$\text{Tan } B = (e/2)/(r + Q) \approx e/2r$$
$$B \approx \text{ArcTan}(e/2r)$$
$$\text{Tan } A = (D/2)/r = D/2r$$
$$A = \text{ArcTan}(D/2r)$$
$$C = B - A \doteq \text{ArcTan}(e/2r) - \text{ArcTan}(D/2r)$$
For $r = 10$,
$$C = \text{ArcTan}(e/20) - \text{ArcTan}(D/20)$$

A. Vignetting begins.

Fig. 4. Derivation of the equations given by Farrell and Booth (1984) for the eye rotation angles for the beginning of vignetting and for total vignetting.

Taking the inverse or arc sine of both sides of the equation,

Eqn. (13) $V = \text{ArcSin}\ [2er + d\sqrt{4r^2 + d^2 - e^2}/(4r^2 + d^2)]$
The eye rotation angle for total vignetting.

In solving the quadratic equation, the square root term has a $\pm$ algebraic sign. The plus sign is used because, as will be shown later, except for the algebraic sign of the square root term, the equations for the beginning of vignetting and for total vignetting are identical, and the eye rotation angle for total vignetting is clearly larger than the angle at which vignetting begins.

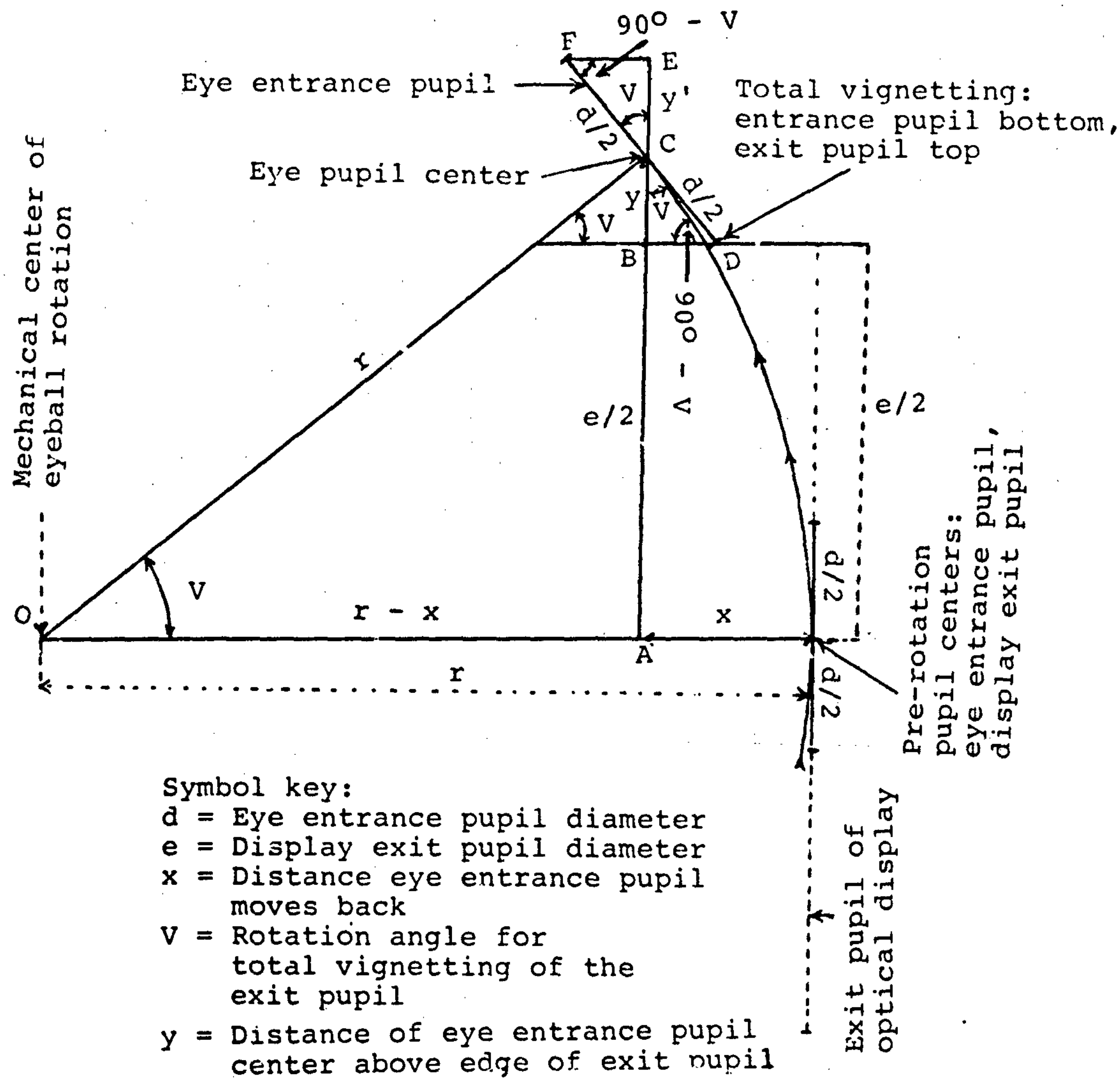


Fig. 5. Total vignetting of the display exit pupil. The figure is for a display exit pupil of 10mm and an eye entrance pupil of 3mm, providing complete vignetting at a rotation angle of 38.2 deg.

Still another equation that yields the same numerical values for the eye rotation angle at which total vignetting occurs is obtained by solving, in a different way, the equation $2r\sin V - e = d\cos V$ obtained in developing Eqn. (13). Rearranging terms, $2r\sin V = e + d\cos V$. Replacing sin V with $\sin V = \sqrt{1 - \cos^2 V}$,
$2r\sqrt{1 - \cos^2 V} = e + d\cos V$. Squaring both sides of the equation,
$4r^2 - 4r^2\cos^2 V = e^2 + 2ed\cos V + d^2\cos^2 V$. Collecting terms,
$(4r^2 + d^2)\cos^2 V + 2ed\cos V + (e^2 - 4r^2) = 0$.
Solving this quadratic equation in cosine V for Cos V,
$\cos V = [-2ed + \sqrt{4e^2d^2 - 4(4r^2 + d^2)(e^2 - 4r^2)}]/2(4r^2 + d^2)$.
This readily reduces to
$\cos V = [-ed + 2r\sqrt{4r^2 + d^2 - e^2}]/(4r^2 + d^2)$.
Taking the inverse or arc cosine of both sides of the equation,

Eqn. (14) $V = \mathrm{ArcCos}\,[(-ed + 2r\sqrt{4r^2 + d^2 - e^2})/(4r^2 + d^2)]$
The eye rotation angle for total vignetting.

In the above equation, a plus sign is used for the square root term, since the angle is not a negative angle.

To derive an equation different from Eqn. (7), yet one providing the same numerical values for the angles at which exit pupil vignetting begins, Figure 6 will be used. Assume as before, that vignetting begins when the leading (or top) edge of the eye entrance pupil is a distance e/2 from the eyepiece optical axis, i.e., is as far from the axis as is the edge of the instrument exit pupil.

From Figure 6, from right triangle OAC, $\sin V = h/r$, from which $h = r\sin V$.
From right triangle CED, $\cos V = y/(d/2)$, from which
$Y = (d/2)\cos V$.
Also, from the figure, $e/2 = h + Y$. Replacing both h and y from above,
$e/2 = h + y$
$= r\sin V + (d/2)\cos V$. Multiplying both sides by 2. and rearranging terms,
$2r\sin V - e = - d\cos V$. Replacing Cos V with $\cos V = \sqrt{1 - \sin^2 V}$,
$2r\sin V - e = - d\sqrt{1 - \sin^2 V}$. Squaring both sided of the equation, $4r^2\sin^2 V - 4er\sin V + e^2 = d^2 - d^2\sin^2 V$.
Collecting terms,
$(4r^2 + d^2)\sin^2 - 4er\sin V + (e^2 + d^2) = 0$. This equation is identical to the one obtained earlier in deriving Eqn. (13), hence will not be solved again. Refer to the earlier solution. For the start of vignetting, the minus sign will be used, instead of the plus sign, for the square root term in the equation, since the rotation angle is smaller for the start of vignetting than for total vignetting. The equation is then

Eqn. (15) $V = \mathrm{ArcSin}\,[(2er - d\sqrt{4r^2 + D^2 - e^2})/(4r^2 + d^2)]$
The eye rotation angle at which vignetting begins.

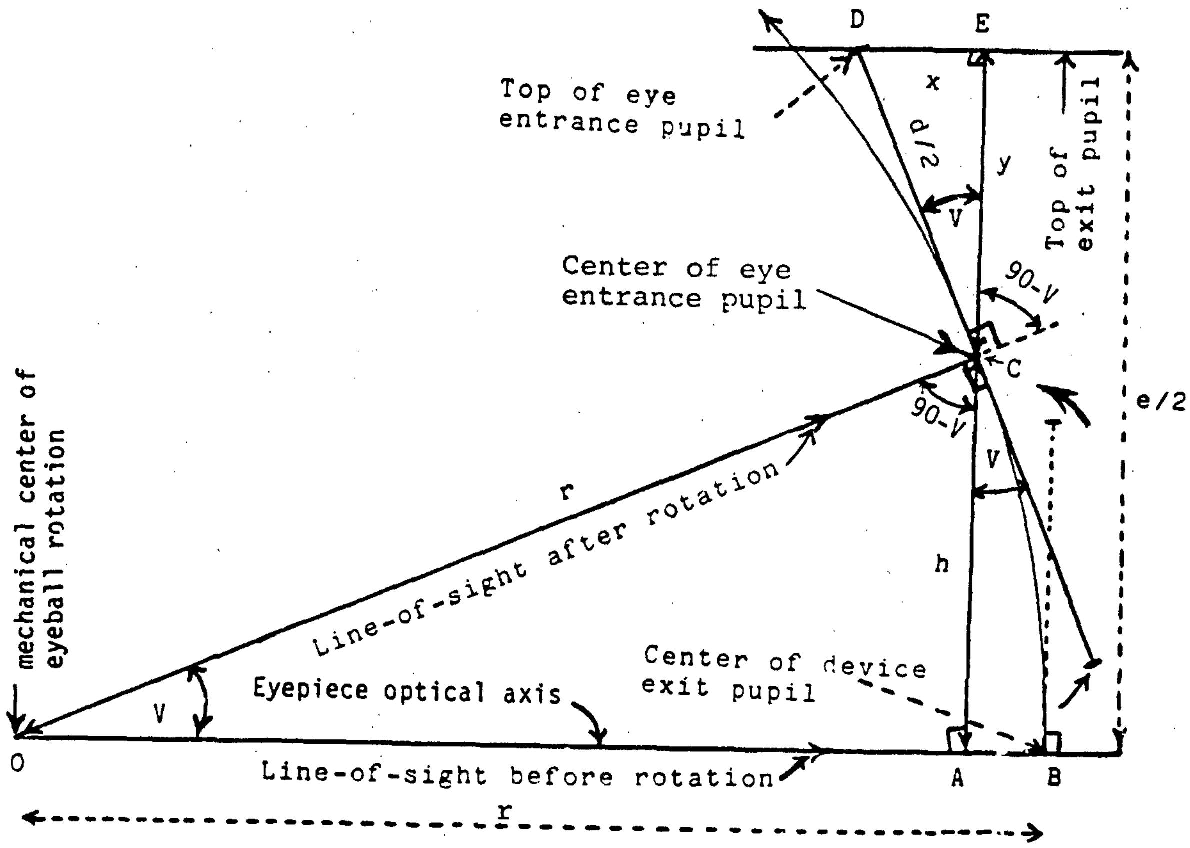


Fig. 6. Optical geometry at the start of exit pupil vignetting.

Instead of an ArcSine solution, an ArcCosine solution may be obtained by rearranging the earlier-obtained equation,
$2r\text{Sin } V - e = - d\text{Cos } V$, as
$2r\text{Sin } V = e - d\text{Cos } V$, and replacing Sin V with $\sqrt{1 - \text{Cos}^2 V}$, to obtain
$2r\sqrt{1 - \text{Cos}^2 V} = e - d\text{Cos } V$. Squaring both sides of the equation and collecting terms,
$(4r^2 + d^2)\text{Cos}^2 V - 2ed\text{Cos } V + (e^2 - 4r^2) = 0$
Using the formula for solving a quadratic equation, in this case a quadratic equation in Cos V,
$\text{Cos } V = [2ed + \sqrt{4e^2d^2 - 4(4r^2 + d^2)(e^2 - 4r^2)}]/2(4r^2 + d^2)$.
This readily reduces to
$\text{Cos } V = [ed + 2r\sqrt{4r^2 + d^2 - e^2}]/(4r^2 + d^2)$.
Taking the inverse cosine or arc cosine of both sides of this equation,

Eqn. (16) $V = \text{ArcCos } [(ed + 2r\sqrt{4r^2 + d^2 - e^2})/(4r^2 + d^2)]$
The eye rotation angle at which vignetting begins.

In solving the above quadratic equation, the plus sign of the square root term was used, because the minus sign would yield a negative value for the angle.

In Figure 5 for total vignetting, the eye entrance pupil is a perpendicular distance $Z' = e/2 + y$ from the optical axis of the eyepiece. In the figure, in right triangle BDC, $\text{Cos } V = y/(d/2)$, from which $y = (d/2)\text{Cos } V$. Replacing y with this value,
$Z' = e/2 + y =$

Eqn. (17) $Z' = e/2 + (d/2)\text{Cos } V$ Exit pupil center distance from eyepiece optical axis at total vignetting.

Figure 5 can be used to derive an equation for the height of the leading or top edge of the eye entrance pupil at total vignetting. In Figure 5, the eye pupil leading edge is a perpendicular height $Z = e/2 + y + y'$ above the optical axis of the eyepiece, and a distance $y + y'$ above a height of $e/2$. Since, right triangles CBD and CEF have a corresponding side, d/2, included between equal corresponding angles V and 90 - V, they are congruent triangles, and $y' = y$. Replacing y' with y,

$Z = e/2 + y + y'$
$= e/2 + y + y$
$= e/2 + 2y$. From triangle CBD, $\text{Cos } V = y/(d/2)$, from which $y = (d/2)\text{Cos } V$. Replacing y with $(d/2)\text{Cos } V$,
$Z = e/2 + 2(d/2)\text{Cos } V =$

Eqn. (18) $Z = e/2 + d\text{Cos } V$ Distance of leading edge of eye entrance pupil from the eyepiece optical axis at the angle v for total vignetting.

COMPARING NUMERICAL VALUES FOR VIGNETTING

Eye rotation angles for the beginning of vignetting and for total vignetting as a function of exit pupil diameter are graphed in Figures 7 and 8, respectively, for several eye entrance pupil diameters. These graphs are based on equations 7 and 9 respectively. Figures 9 and 10 are the corresponding graphs for equations 11 and 12, the Farrell and Booth equations.

It is instructive to compare the numerical values provided by these two sets of vignetting equations. For the two sets of equations, vignetting angles were calculated for a range of values of display exit pupil sizes and eye entrance pupil sizes. The numerical values are presented in Table 1. To facilitate the numerical comparison, differences are presented as percentages for the corresponding vignetting angles. Percent difference is defined as $\%D = 100(V - C)/V$, where V values are from equations 7 and 9, and C, or Farrell and Booth (F&B) values, are from equations 11 and 12.

From examination of Figures 7-10 and the numerical values and comparisons afforded by Table 1, it is apparent that:

(1). Vignetting of the display exit pupil begins at smaller eye rotation angles for larger eye entrance pupils, but total vignetting occurs at larger rotation angles. This is what one would expect.

(2). With larger display exit pupils, both the beginning of vignetting and the occurrence of total vignetting are at larger eye rotation angles, also as expected.

(3). For small exit pupils (7mm or less), the difference between V and F&B angles are not large: less than 5% for beginning of vignetting, and less than 9% for total vignetting.

(4). For large exit pupils (11mm or more), the V angles exceed the F&B angles by 20% to 32%. These are large differences.

(5). From the graphs for the V equations, note that the slopes of the curves for the various eye entrance pupil diameters increase with increased exit pupil diameter. However, note that the slopes for the F&B curves decrease with increase in instrument exit pupil diameter.

As mentioned earlier, the truncated double cone shape of the beam of light from the eyepiece of an aerial image (or virtual image) optical display is not taken into account by either the V equations derived in the present paper, or the Farrell and Booth equations. The angle at which vignetting of the light beam from the eyepiece begins is slightly larger than the angle indicated by the equation for the beginning of vignetting, and total loss of the display image takes place at a slightly larger angle than that indicated by the total vignetting equation, particularly for very wide angle displays. This implies that numerical values used for design or selection of display equipment, whether from the V equations or the Farrell and Booth equations, are on the safe or conservative side.

As noted earlier, the numerical values from the two sets of equations do not differ much for displays with small exit pupils. However, the differences are large for large exit pupils. It is judged that, for making either selection or design decisions, the V equations derived for the present paper would be preferable to the equations listed by Farrell and Booth. It must be kept in mind, as has been mentioned earlier, that, due to the complexity of the rotational movement of the eyeball and the simplifying assumptions upon which the equations were based, the equations provide only approximations to display user reality.

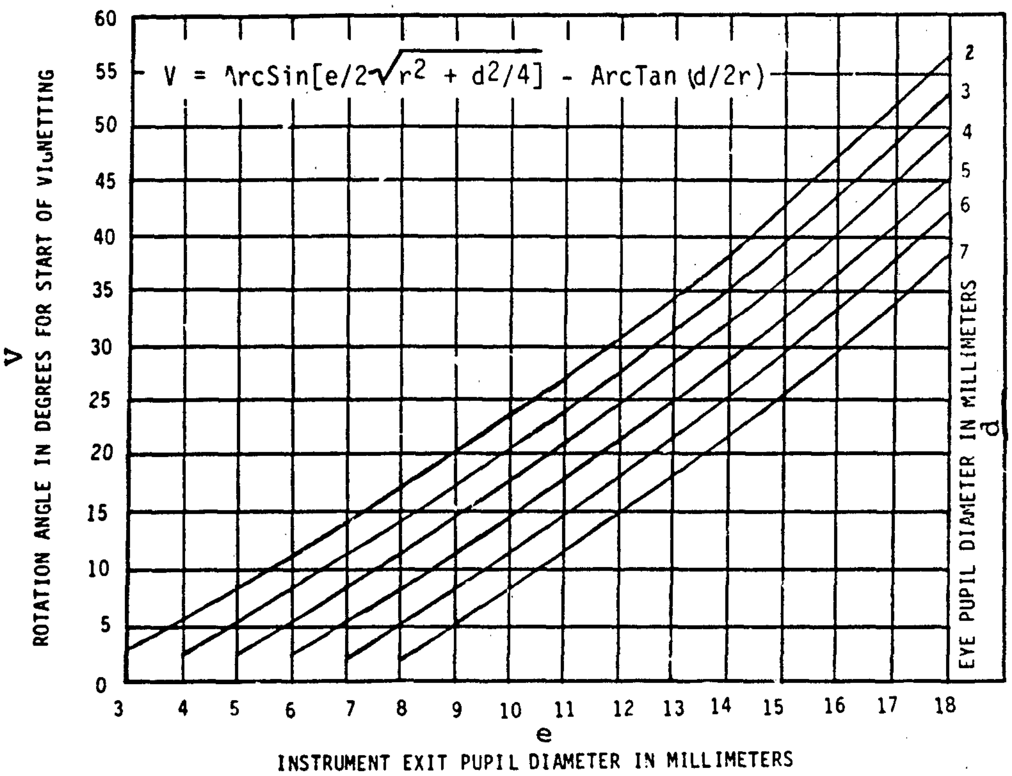


Fig. 7. The eye rotation angle at which vignetting begins as a function of the diameter of the display exit pupil.

TABLE 1

COMPARISON OF NUMERICAL VALUES FROM VIGNETTING EQUATIONS

PUPILS*		Vignetting Begins		Percent	Total Vignetting		Percent
e	d	V	F&B	Difference	V	F&B	Difference
15	3	39.35	28.34	28.0	56.41	45.40	19.5
11	3	24.42	20.28	16.9	41.48	37.34	10.0
7	4	8.76	7.98	8.9	31.38	30.60	2.5
3	2	2.87	2.82	1.7	14.29	14.24	.35
5	3	5.78	5.51	4.7	22.78	22.57	.92
7	3	11.72	10.76	8.2	28.78	27.82	3.7
11	5	18.21	14.77	18.9	46.28	42.85	7.4
15	7	25.77	17.58	31.8	64.35	56.16	12.7

% Difference = 100(V - F&B)/V

*Pupils: e = Instrument exit pupil, d = eye entrance pupil.

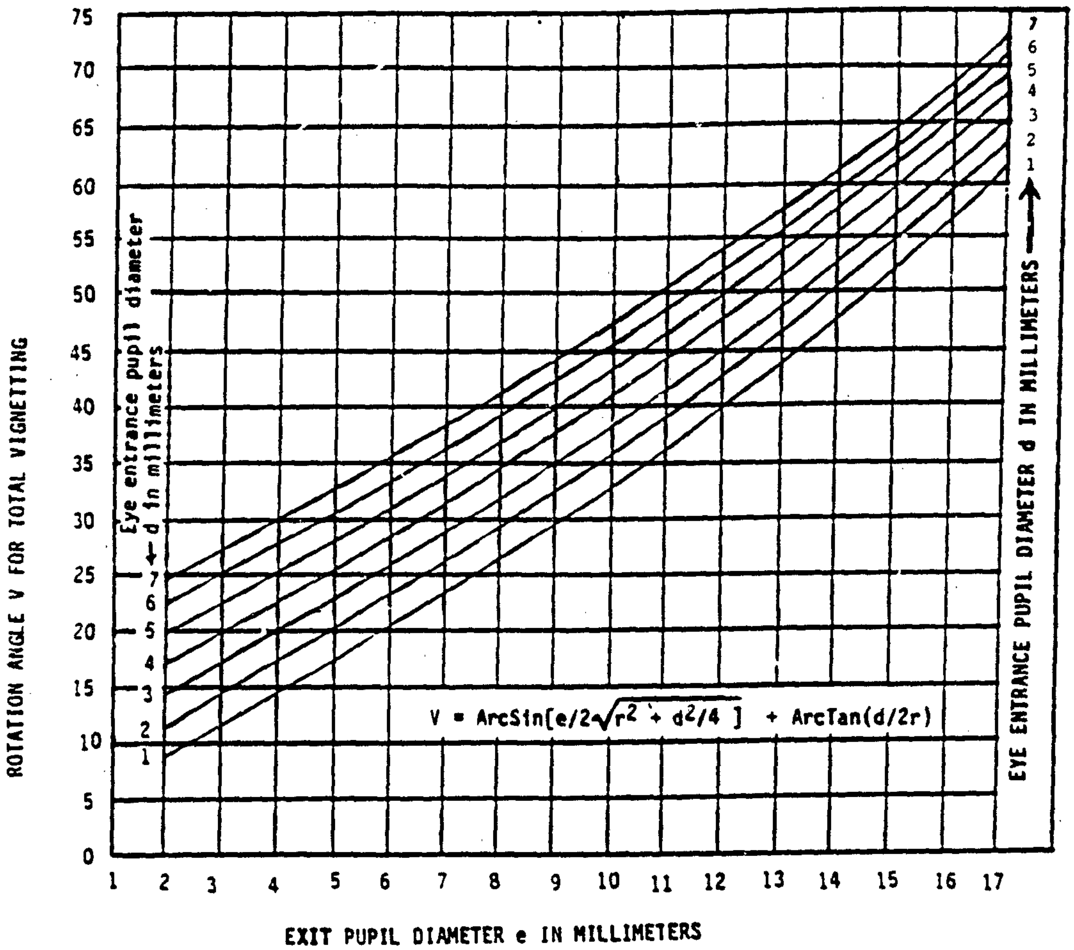


Fig. 8. The eye rotation angle for the beginning of total vignetting of the exit pupil as a function of the diameter of the exit pupil.

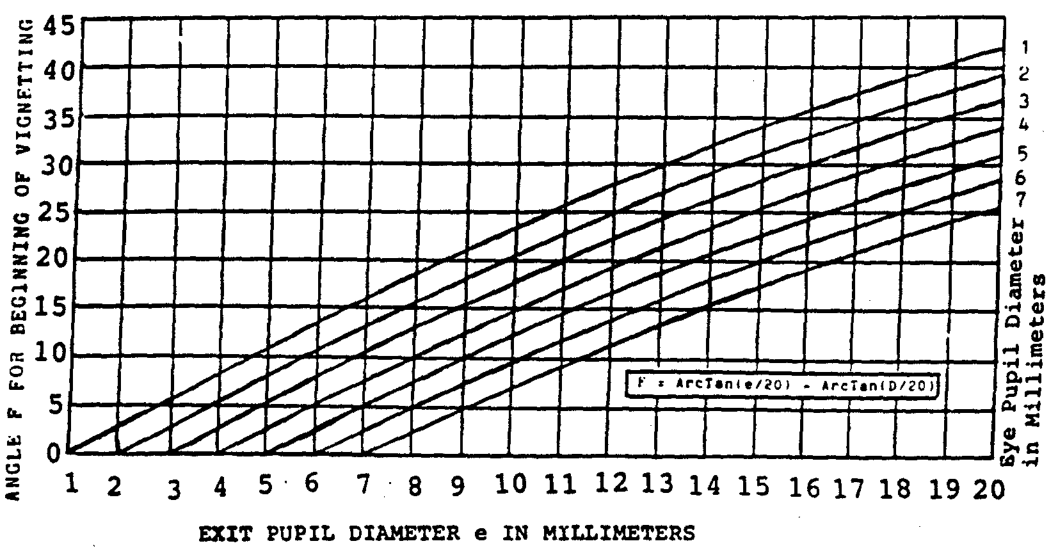


Fig. 9. The eye rotation angle at which partial vignetting begins as a function of exit pupil diameter. From an equation of Farrell and Booth (1984).

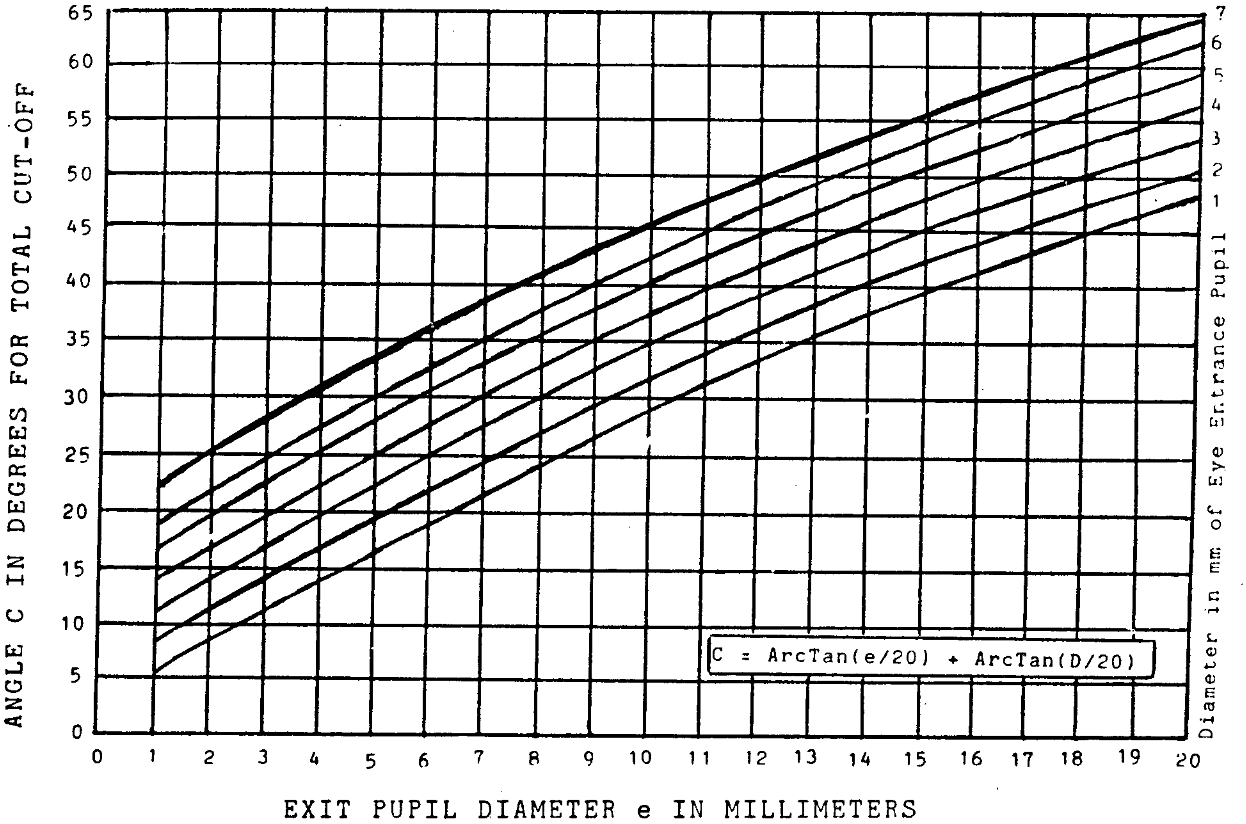


Fig. 1C. The eye rotation angle for total vignetting of the display exit pupil as a function of exit pupil diameter. From an equation of Farrell and Booth (1984).

APPENDIX

COMPUTATIONAL EXAMPLES USING PUPIL POSITION AND VIGNETTING EQUATIONS

As an example of use of the numbered equations derived in this report, let an optical display system have an exit pupil diameter $e = 7mm$, an eye entrance pupil diameter $d = 4mm$, and a mechanical radius of rotation $r = 10mm$.

The angular subtense of the pupil leading edge at the eye rotation center:

From Eqn. (1), $W = \text{ArcTan}\,(d/2r) = \text{ArcTan}\,(4/2x10) = \text{ArcTan}\ .200 = 11.310$ deg.

The angle at which vignetting begins:

From Eqn. (7),

$$V = \text{ArcSin}\,(e/2\sqrt{r^2 + d^2/4}) - \text{ArcTan}\,(d/2r)$$
$$= \text{ArcSin}\,(7/2\sqrt{10^2 + 4^2/4}) - \text{ArcTan}\,(4/2x10)$$
$$= 20.072 - 11.310$$
$$= 8.762\text{deg.}$$

The angle for total vignetting:

From Eqn. (9),

$$V = \text{ArcSin}\,(e/2\sqrt{r^2 + d^2/4}) + \text{ArcTan}\,(d/2r)$$
$$= \text{ArcSin}\,(7/2\sqrt{10^2 + 4^2/4}) + \text{ArcTan}(4/2x10)$$
$$= 20.072 + 11.3099$$
$$= 31.382\text{ deg.}$$

The distance of the eye entrance pupil center from the eyepiece optical axis:

At the beginning of vignetting:

From Eqn. (2), $U = r\text{Sin}\ V = 10\text{Sin}\ 8.762 = 1.523mm$.

At total vignetting:

From Eqn. (2), $U = r\text{Sin}\ V = 10\text{Sin}\ 31.382 = 5.207mm$

The distance of the leading edge of the eye entrance pupil from the optical axis of the eyepiece:

At the beginning of vignetting:

$h = e/2 = 7/2 = 3.500mm$. Also, as a check,

From Eqn. (4),

$$h = \sqrt{r^2 + d^2/4}\ \text{Sin}\ (W + V)$$
$$= \sqrt{10^2 + 4^2/4}\ \text{Sin}\ (11.310 + 8.762)$$
$$= 10.198\text{Sin}\ 20.072 = 3.500mm.$$

Note that this checks with the above value of $e/2$.

For total vignetting:

From Eqn. (4),

$$h = \sqrt{r^2 + d^2/4}\ \text{Sin}\ (W + V)$$
$$= \sqrt{10^2 + 4^2/4}\ \text{Sin}\ (11.318 + 31.382)$$
$$= 10.198\text{Sin}\ 42.692$$
$$= 6.915mm.$$

As a check, using Eqn. (17),

$$Z = e/2 + d\text{Cos}\ V = 7/2 + 4\text{Cos}\ 31.382$$
$$= 3.500 + 3.415 = 6.915,$$

as above.

The distance increase of the eye entrance pupil leading edge from the eyepiece optical axis from eye rotation:

From Eqn. (5), delta h $= [\sqrt{r^2 + d^2/4}\ \mathrm{Sin}\ (W + V)] - d/2$

$= h - d/2$ (use h values calculated above)

At the start of vignetting:

delta h $= h - d/2 = 3.500 - 4/2 = 1.500$mm.

At total vignetting:

delta h $= h - d/2 = 6.916 - 4/2 = 4.916$mm.

The distance that the center of the eye entrance pupil moves toward the observer's head from eye rotation:

At the beginning of vignetting:

From Eqn. (3), $N = r - r\mathrm{Cos}\ V$

$= 10 - 10\mathrm{Cos}\ 8.762$

$= 10 - 9.882$

$= .117$mm

At total vignetting:

From Eqn. (3), $N = r - r\mathrm{Cos}\ V$

$= 10 - 10\mathrm{Cos}31.382$

$= 10 - 8.567$

$= 1.463$mm

The distance that the leading edge of the eye entrance pupil moves toward the observer's head from eye rotation:

From Eqn. (10), $X = r - \sqrt{r^2 + d^2/4}\ \mathrm{Cos}\ (W + V)$

$= r - h/\mathrm{Tan}(W + V)$ (See derivation, and use h values calculated above)

At the beginning of vignetting:

$X = 10 - 3.500/\mathrm{Tan}\ (11.310 + 8.762)$

$= 10 - 3.500/\mathrm{Tan}\ 20.072$

$= 10 - 9.5787$

$= .421$mm.

At total vignetting:

$X = r - h/\mathrm{Tan}\ (W + V)$

$= 10 - 6.916/\mathrm{Tan}\ (11.310 + 31.382)$

$= 10 - 6.916/\mathrm{Tan}\ 42.692$

$= 10 - 7.497$

$= 2.503$mm.

To summarize the computations in the above example, for an exit pupil diameter of 7mm and any eye entrance pupil diameter of 4mm, the pupil leading edge, before eye rotation, subtends, at the mechanical center of rotation, an angle of 11.32 deg. Vignetting begins when the eye has rotated through an angle V = 8.76deg: the eye pupil edge is at the exit pupil edge. The pupil leading or top edge is at a height above the eyepiece optical axis of 3.5mm. i.e., half of the exit pupil diameter. The pupil edge has increased its distance from the eyepiece optical axis by delta h = 1.50mm, and has moved back toward the observer's head by X = .42mm. The center of the eye entrance pupil has moved up by U = 1.52mm, and moved toward the observer's head by N = .12mm.

With still more eye rotation, at an angle with the eyepiece optical axis of 31.4deg., the following or bottom edge of the eye entrance pupil has reached a distance of e/2 = 3.5mm from the optical axis, the distance of the exit pupil edge from the optical axis, and vignetting is total or 100%. The top or leading edge of the eye pupil is a distance of 6.9mm from the eyepiece optical axis, having moved up from its initial position by delta h = 4.92mm, and back toward the observer's head a distance X = 2.50mm. The center of the observer's eye entrance pupil has moved away from the eyepiece optical axis a distance U = 5.21mm and toward the observer's head a distance N = 1.46mm.

For a second example of the use of the numbered equations derived earlier, assume, as before, an exit pupil diameter of e = 7mm, an eye entrance pupil diameter of d = 4mm, and an eye rotation radius of r = 10mm. For this example, the eye rotation angle V for total vignetting will be calculated using four different equations: 9, 13, 14, and 12.

By Eqn. (9),

$$\begin{aligned} V &= \text{ArcSin}\,(e/2\sqrt{r^2 + d^2/4}) + \text{ArcTan}(d/2r) \\ &= \text{ArcSin}\,(7/2\sqrt{10^2 + 4^2/4}) + \text{ArcTan}(4/2\text{x}10) \\ &= \text{ArcSin}\ .343203 + \text{ArcTan}\ .200000 \\ &= 20.0722\text{deg.} + 11.3099\text{deg.} \\ &= 31.38\text{deg.} \end{aligned}$$

By Eqn. (13),

$$\begin{aligned} V &= \text{ArcSin}[(2er + d\sqrt{4r^2 + d^2 - e^2})/(4r^2 + d^2] \\ &= \text{ArcSin}[(2\text{x}7\text{x}10 + 4\sqrt{4\text{x}10^2 + 4^2 - 7^2})/ \\ &\quad (4\text{x}10^2 + 4^2)] \\ &= \text{ArcSin}(216.6290/416) \\ &= \text{ArcSin}(.520742) \\ &= 31.38\text{deg.} \end{aligned}$$

By Eqn. (14),

$$\begin{aligned} V &= \text{ArcCos}[(-ed + 2r\sqrt{4r^2 + d^2 - e^2})/(4r^2 + d^2) \\ &= \text{ArcCos}[(-7\text{x}4 + 2\text{x}10\sqrt{4\text{x}10^2 + 4^2 - 7^2})/ \\ &\quad (4\text{x}10^2 + 4^2)] \\ &= \text{ArcCos}[(-28 + 383.1449)/416] \\ &= \text{ArcCos}\ .853714 \\ &= 31.38\text{deg.} \end{aligned}$$

By Eqn. (12) of Farrell and Booth,

$$\begin{aligned} C &= \text{ArcTan}(e/20) + \text{ArcTan}(d/20) \\ &= \text{ArcTan}(7/20) + \text{ArcTan}(4/20) \\ &= 19.2901\text{deg.} + 11.3099\text{deg.} \\ &= 30.38\text{deg.} \end{aligned}$$

This is slightly less than the 31.38deg. calculated above.

Cautionary note: The numerical values in the examples are sometimes carried out to several decimal places. However, the assumptions used in deriving the equations were approximations, so that numerical values from the equations are also approximations. Numerical answers should be rounded off to two significant figures and regarded as approximations.

As an exercise, the reader may wish to calculate the eye rotation angle for the beginning of vignetting with equations 7, 15, and 16. If the values of e, d and r are 7, 4, and 10, as in the above example, the correct answer for all three equations is 8.76deg which should be rounded off to 8.8 deg. By eqn. (12) of Farrell and Booth, the angle is 7.98deg, which should be rounded off to 8.0 deg.

GLOSSARY

Aperture stop. The opening or aperture in an optical system that limits the amount of light that can be collected by, i.e., get through, the optical system.

Entrance pupil. The virtual image of the aperture stop formed by all of the lenses preceding the aperture stop. The virtual image of the aperture stop as seen from object space, i.e., as seen from the front or light intake end of the optical system. For example, the entrance pupil of your eye is the image of your eye pupil formed by the cornea: it is what someone looking at your eye would see. Due to magnification by the cornea, the eye entrance pupil is larger than the actual pupil.

Exit pupil. The real image of the aperture stop formed by all of the lenses preceding the aperture stop. The real image of the aperture stop as seen from the optical system's light output end, the back or eyepiece end. It is the waist or narrowest part of the truncated double cone of light emerging from the eyepiece.

Eyepiece. The lens or group of lenses next to the observer's eye that forms an enlarged virtual display image of the primary image projected by the objective (or by a relay lens, if one is present, between the objective and the eyepiece).

Objective. The lens or lens group nearest the object that gathers light from the object or viewed field and projects a real image for examination through the eyepiece.

Optics. 1. The science and technology of light. 2. Devices, such as lenses, mirrors, prisms, windows, filters, etc. that influence light by bending. reflecting, focussing, filtering transmitting, polarizing, etc.

Real image. An image that may be recordered by light-sensitive photographic film placed at the optical location of the image, or may be seen on a reflecting screen placed there. The light forming the image is at the optical location of the image.

Vignetting. Cutting out, cutting off, or blocking the passage of light.

Virtual image. An image whose energy content is not located at the image's optical location. An image seen in a mirror or viewed in an eyepiece is a virtual image.

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