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STRONGLY WELL-COVERED GRAPHS

by

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Abstract

A graph is well-covered if every maximal independent set is a maximum independent set. A strongly well-covered graph G has the additional property that G-e is also well-covered for every line e in G. Hence, the strongly well-covered graphs are a subclass of the well-covered graphs. We characterize strongly well-covered graphs with independence number two and determine a parity condition for strongly well-covered graphs with independence number three. More generally, we show that a strongly well-covered graph (with more than four points) is 3-connected and has minimum degree at least four.

STRONGLY WELL-COVERED GRAPHS

INTRODUCTION

A set of points in a graph is independent if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the independence number of G and is denoted by $\alpha(G)$. A set of independent points which attains the maximum size is referred to as a maximum independent set. A set S of independent points in a graph is maximal (with respect to set inclusion) if the addition to S of any other point in the graph destroys the independence. In general, a maximal independent set in a graph is not necessarily maximum.

In a 1970 paper, Plummer [13] introduced the notion of considering graphs in which every maximal independent set is also maximum; he called a graph having this property a well-covered graph. The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. Campbell [2] characterized all cubic well-covered graphs with connectivity at most two, and Campbell and Plummer [3] proved that there are only four 3-connected cubic planar well-covered graphs. Royle and Ellingham [15] have recently completed the picture for cubic well-covered graphs by determining all 3-connected cubic well-covered graphs.

For a well-covered graph with no isolated points, the independence number is at most one-half the size of the graph. Well-covered graphs whose independence number is exactly one-half the size of the graph are called very well-covered graphs. The class of very well-covered graphs was characterized by Staples [16] and includes all well-covered trees and all well-covered bipartite graphs. Independently, Ravindra [14] characterized bipartite well-covered graphs and Favaron [6] characterized the very well-covered graphs. Recently, Dean and Zito [4] characterized the very well-covered graphs as a subset of a more general (than well-covered) class of graphs.

A set S of points in a graph dominates a set V of points if every point in V-S is adjacent to at least one point of S. Finbow and Hartnell [7] and Finbow, Hartnell, and Nowakowski [8] studied well-covered graphs relative to the concept of dominating sets. Finbow, Hartnell, and Nowakowski have also obtained a characterization of well-covered graphs with girth at least five [9].

A well-covered graph is 1-well-covered if and only if the deletion of any point from the graph leaves a graph which is also well-covered (Staples introduced the term 1-well-covered in [16] and [17]). For the analogous line property, we say G is strongly well-covered if and only if G is well-covered and G-e is also well-covered for all lines e in G. Note that if G is not connected, then G is 1-well-covered if and only if all components of G are 1-well-covered. Similarly, if G is not connected, then G is strongly well-covered if and only if all components of G are strongly well-covered. See [10] and [11] for some results on 1-well-covered graphs with girth four.

The class of well-covered graphs contains all complete graphs and all complete bipartite graphs of the form $K_{n,n}$. The only cycles which are well-covered are C_3, C_4, C_5, and C_7. We note that all complete graphs (except K_1) are also 1-well-covered, but no complete bipartite graphs (except $K_{1,1}$) are 1-well-covered. The cycles C_3 and C_5 are the only 1-well-covered cycles. Also note that the only complete graphs which are strongly well-covered are K_1 and K_2, the only complete bipartite graphs which are strongly well-covered are $K_{1,1}$ and $K_{2,2}$, and C_4 is the only strongly well-covered cycle.

In [12], we construct infinite families of strongly well-covered graphs with arbitrarily large (even) independence number. The construction involves the lexicographic product of graphs.

PRELIMINARY RESULTS

Unless otherwise stated, we assume that all graphs are connected. For notation and terminology not defined here, refer to [1].

A line in a graph G is a critical line if its removal increases the independence number. A line-critical graph is a graph with only critical lines. In the following lemma, we show that the deletion of a critical line from a well-covered graph leaves a graph which is no longer well-covered.

Lemma 1. If $G \neq K_2$ is well-covered and e is a critical line in G, then G-e is not well-covered.

Proof. Let e = uv. Since $G \neq K_2$, then (without loss of generality) there exists some point $a \sim u$, $a \neq v$. Since G is well-covered, there exists maximum independent set J in G such that $a \in J$. In the graph G-e, the set J is maximal independent. Thus, G-e has a maximal independent set of size $\alpha(G)$. Since e is a critical line, $\alpha(G\text{-}e) = \alpha(G) + 1$. Hence, the graph G-e is not well-covered. []

Note that as a consequence of Lemma 1, we have the statement that a strongly well-covered graph (other than K_2) has no critical lines. Thus, if $G \neq K_2$ is strongly well-covered, then $\alpha(G\text{-}e) = \alpha(G)$ for all lines e in G.

If x is a point in a graph G, then the closed neighborhood of x is given by N[x] and consists of x and all its neighbors. The next two lemmas will be very helpful in eliminating candidate graphs as we develop the structure of strongly well-covered graphs.

Lemma 2. Suppose G is well-covered. Also suppose that S is an independent set and x is a point in G such that

(i) $x \notin S$ and $x \sim v$ for exactly one v in S, and

(ii) S dominates N[x].

Then G-e is not well-covered, where e = vx.

Proof. Since G is well-covered and S is independent, then there exists maximum independent set $J \supseteq S$ in G. Since v is in S and $x \sim v$, then $x \notin J$. Since S dominates N[x] and $J \supseteq S$, then $N(x) \cap J = \{v\}$. Thus, in the graph G-vx, the set $J \cup \{x\}$ is independent. Hence, vx is a critical line in G. By Lemma 1, the graph G-vx is not well-covered. []

Lemma 3. Suppose G is strongly well-covered with $\alpha(G) \geq 2$. Then every point in G must have at least two nonadjacent neighbors.

Proof. Assume to the contrary that v is a point in G such that every pair of neighbors of v is adjacent. Let $w \sim v$. Then $\{w\} = S$ satisfies the conditions in Lemma 2; hence, G-vw is not well-covered, contradicting the assumption that G is strongly well-covered. []

Let G_v denote the graph obtained from G by deleting the point v and all its neighbors; that is, G_v is the graph induced by G-N[v]. Similarly if u and v are points in G, let G_{uv} be the graph induced by $G-(N[u] \cup N[v])$. The following very useful necessary condition for a graph to be well-covered is proved in [3].

Theorem 4. If a graph G is well-covered and is not complete, then G_v is well-covered for all v in G. Moreover, $\alpha(G_v) = \alpha(G) - 1$.

We obtain a similar necessary condition for a graph to be strongly well-covered in Theorem 6. First we prove the following lemma.

Lemma 5. Suppose e = uv is a line in a well-covered graph G such that G-e is not well-covered. Then either (i) e is a critical line and there exists a maximum independent set I containing {u,v} in G-e, or (ii) e is not a critical line and there exists a maximal independent set J containing {u,v} in G-e such that $|J| < \alpha(G)$.

Proof. Suppose e = uv is a critical line in G. Hence, there exists a maximum independent set I of size $\alpha(G) + 1$ in G-e. Suppose $I \cap \{u,v\} \neq \{u,v\}$. Thus, I is independent in G, a contradiction since $|I| > \alpha(G)$. Therefore, I contains {u,v}.

Suppose e is not a critical line in G. Thus, $\alpha(G\text{-}e) = \alpha(G)$. Consider the independent set {u,v} in the graph G-e. Since G-e is not well-covered (by assumption), then there exists a maximal independent set J in G-e such that $|J| < \alpha(G\text{-}e)$. If $J \cap \{u,v\} \neq \{u,v\}$, then J is maximal independent in G. Since $\alpha(G) = \alpha(G\text{-}e) > |J|$ and G is well-covered, we obtain a contradiction. Thus, J contains {u,v}. []

Theorem 6. If G is strongly well-covered and G is not complete, then G_v is strongly well-covered for all points v in G.

Proof. By Theorem 4, the graph G_v is well-covered and $\alpha(G_v) = \alpha(G) - 1$, for all points v in G. So we need only show that G_v-e is well-covered for all lines e in G_v, for all points v.

Assume to the contrary that there exists v such that G_v-e is not well-covered for some line e in G_v. Let e = uw. By Lemma 5, since G_v is well-covered and G_v-e is not well-covered, then either (i) e is a critical line for G_v, or (ii) if e is not a critical line for G_v, then there exists a maximal independent set $J \supseteq \{u,w\}$ in G_v-e such that $|J| < \alpha(G_v\text{-}e) = \alpha(G_v)$.

Suppose e is a critical line in G_v. Then there exists maximum independent set J in G_v-e such that $|J| = \alpha(G_v) + 1 = \alpha(G)$. But then $J \cup \{v\}$ is independent in G-e, a contradiction since G has no critical lines.

So e is not a critical line in G_v. Thus, there exists a maximal independent set $J \supseteq \{u,w\}$ in G_v-e such that $|J| < \alpha(G_v)$. Then $J \cup \{v\}$ is maximal independent in G-e. Thus, $|J \cup \{v\}| < \alpha(G_v) + 1 = \alpha(G)$; since $\alpha(G\text{-}e) = \alpha(G)$, we contradict the assumption that G-e is well-covered. []

If $G \neq K_2$ is well-covered and e = uv is a line in G, consider maximal independent sets in the graph G-e. Suppose J is a maximal independent set in G-e which does not contain at least one endpoint of e (that is, $J \cap \{u,v\} \neq \{u,v\}$). Then it follows that J is a maximal independent set in G. Since G is well-covered, then $|J| = \alpha(G)$. Thus, every maximal independent set in G-e which does not contain at least one endpoint of e has size $\alpha(G)$. Consequently, to show that G-e is well-covered it suffices to show that every maximal independent set in the graph G-e which contains both endpoints of e has size $\alpha(G)$.

Staples [16] studied well-covered graphs with the property that for all points v in G, the graph G-v is not well-covered . She called these graphs well-covered point-critical. We find a significant connection between such well-covered graphs and strongly well-covered graphs. The following two theorems from Staples [16] will be helpful.

Theorem 7. Suppose G is well-covered and $\alpha(G) = 2$. Then for all points v in G the graph G-v is not well-covered if and only if $\deg(v) = |V(G)| - 2$ for all points v.

Theorem 8. If G is well-covered and has no critical lines, then for all points v in G the graph G-v is not well-covered .

First, we show in Theorem 9 that strongly well-covered is a *sufficient* condition for G to have the property that for all points v the graph G-v is not well-covered. As a consequence, K_2 is the only strongly well-covered graph which is also 1-well-covered.

Theorem 9. If G ($G \neq K_1$ or K_2) is strongly well-covered, then for all points v in G the graph G-v is not well-covered .

Proof. By Lemma 1, if G-e is well-covered, then e is not a critical line. Since G-e is well-covered for all lines e in G, then G contains no critical lines. By Theorem 8, a well-covered graph with no critical lines has the property that for all points v in G the graph G-v is not well-covered . []

The converse of Theorem 9 is not true. Consider the graph G in Figure 1; it can be shown that G is not strongly well-covered yet has the property that for all points v the graph G-v is not well-covered . G is not strongly well-covered because $\alpha(G) = 3$ and G-e has a maximal independent set of size two (namely, the endpoints of e).

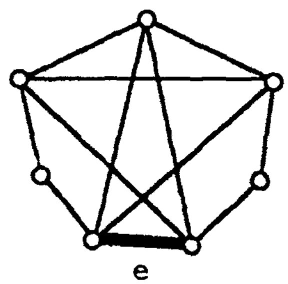

Figure 1

STRUCTURAL RESULTS

First, we completely characterize the strongly well-covered graphs with independence number two. This characterization is quite helpful in building strongly well-covered graphs with independence number larger that two (see [12]).

Theorem 10. Suppose G is well-covered with $\alpha(G) = 2$. Then G is strongly well-covered if and only if G is (|V(G)| - 2) - regular.

Proof. ($\Rightarrow$) Suppose G is strongly well-covered. By Theorem 9, the graph G-v is not well-covered for all points v in G. By Theorem 7, $\deg(v) = |V(G)| - 2$ for all points v in G.

($\Leftarrow$) Suppose G is ($|V(G)| - 2$) - regular. Let e = uv be a line in G. Consider the graph G-e. Since $\deg(v) = |V(G)| - 2$, then $|V(G) - N[v]| = 1$. Let w be the point not in N[v]. Since $\deg(w) = |V(G)| - 2$ and w is not adjacent to v, it follows that w ~ u. Thus, {u,v} is maximal independent in G-e. So every maximal independent set in G-e containing {u,v} has size $\alpha(G)$. Hence, every maximal independent set in G-e has size two, and so we see that G-e is well-covered. Since e is arbitrary, then G-e is well-covered for all lines e in G. Hence, G is strongly well-covered. []

We show in the following theorem that if G is strongly well-covered and v is a point in G, then G_v cannot contain a K_2-component (a component which is a line).

Theorem 11. Suppose G is a connected strongly well-covered graph with $\alpha(G) \geq 2$. If v is a point in G, then G_v cannot contain a K_2-component.

Proof. Assume to the contrary that there exists a point v in G such that G_v contains a K_2-component. Let the K_2-component be e = uw. Let S be a maximum independent set in G_v such that $u \in S$. Then $S \cup \{w\}$ is independent in the graph G_v-e, and so $S \cup \{v,w\}$ is independent in the graph G-e. Now by Theorem 4, we have $|S| = \alpha(G_v) = \alpha(G) - 1$ and hence $|S \cup \{v,w\}| = \alpha(G) + 1$. Thus, e is a critical line for G, a contradiction since G is strongly well-covered. []

Now we are prepared to consider strongly well-covered graphs with independence number three. We show in Theorem 13 that G_v must be connected, for every v in G, if $\alpha(G) = 3$ and G is strongly well-covered. This will be important for an inductive argument given in the proof of Theorem 15. The following iemma is useful in proving Theorem 13.

Lemma 12. Suppose G is strongly well-covered and $\alpha(G) = 3$. If v is a point in G, then G_v cannot have two isolated points.

Proof. Assume to the contrary that there is a point v in G such that G_v has two isolated points. Let a and b be isolated points in G_v. Thus, $V(G) = \{v\} \cup \{a,b\} \cup N(v)$, since $\alpha(G) = 3$. Let $A = N(a) \cap N(v)$ and $B = N(b) \cap N(v)$.

Suppose $A \cap B \neq \varnothing$. Let $w \in A \cap B$. Then $\{v,w\}$ is maximal independent in the graph G-vw, a contradiction since G-vw is well-covered and $\alpha(G\text{-}vw) = \alpha(G) = 3$ ($\alpha(G) = \alpha(G\text{-}vw)$ since a strongly well-covered graph contains no critical lines).

So $A \cap B = \varnothing$. Since $\alpha(G) = 3$, then $\alpha(G_u) = 2$, for all points u in G. By Theorems 6 and 10, it follows that G_a and G_b are each regular strongly well-covered graphs (note that G_a is not complete since v and b are in $V(G_a)$ and v is not adjacent to b; symmetrically, G_b is not complete). Since $a \in G_b$, it follows that $N(v) = A \cup B$. From Lemma 3, the point a must have two nonadjacent neighbors in G, say m and n, and b must have two nonadjacent neighbors in G, say s and t. See Figure 2.

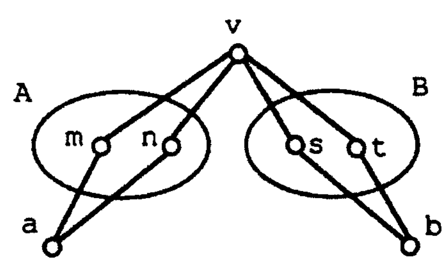

Figure 2

Consider the graph G_m. By Theorem 6, graph G_m is strongly well-covered with $\alpha(G_m) = 2$. By Theorem 11, graph G_m cannot have a K_2-component. Consequently, either G_m consists of the isolated points b and n, or G_m is connected and by Theorem 10 is ($|G_m| - 2$) - regular.

Suppose G_m is connected and ($|G_m|$ - 2) - regular. By Theorem 11, the graph G_m cannot have a K_2-component. So there exist two nonadjacent neighbors, x and y, of b in G_m; that is, $x \sim b$, $y \sim b$, and neither x nor y is adjacent to m. But then $\{x,y,a,m\}$ is independent in the graph G-ma, a contradiction since G is strongly well-covered and therefore contains no critical lines.

So G_m consists of the isolated points b and n. Thus, $x \in B$ implies $x \sim m$. Similarly, by looking at the graph G_s, we conclude that $y \in A$ implies $y \sim s$. Since $V(G) = \{v\} \cup \{a,b\} \cup N(v)$ and $N(v) = A \cup B$, it follows that $\{m,s\}$ is maximal independent in the graph G-ms. This is a contradiction since α(G-ms) = 3 and G-ms is well-covered.

Thus, if v is a point in G, then G_v cannot have two isolated points. []

Theorem 13. If G is strongly well-covered and $\alpha(G) = 3$, then G_v must be connected for all points v in G. Moreover, G_v is ($|G_v|$ - 2) - regular.

<u>Proof</u>. Since $\alpha(G) = 3$, then $\alpha(G_v) = 2$ for any point v in G. By Theorem 11, the graph G_v cannot have a K_2-component. By Lemma 12, graph G_v cannot have two singleton components. By Theorem 6, graph G_v is also strongly well-covered. Since K_1 and K_2 are the only complete graphs which are strongly well-covered, it follows that G_v can have neither isolated points nor any components with independence number one.

Thus, G_v is connected. Since $\alpha(G_v) = 2$, then G_v is ($|G_v|$ - 2) - regular by Theorem 10. []

Theorem 13 gives us enough structural knowledge to obtain in Corollary 14 a parity condition on all point degrees in strongly well-covered graphs with independence number three.

Corollary 14. Suppose G is strongly well-covered and $\alpha(G) = 3$.

(i) If $|V(G)|$ is even, then deg(v) is odd for all v in G.

(ii) If $|V(G)|$ is odd, then deg(v) is even for all v in G.

<u>Proof</u>. For any point v in G, we have $\alpha(G_v) = 2$. From Theorem 13, it follows that G_v is ($|V(G_v)| - 2$) - regular. Hence, $|V(G_v)|$ must be even. Since $|V(G)| = |V(G_v)| + \deg(v) + 1$, then $|V(G_v)| = |V(G)| - \deg(v) - 1$. Thus, $|V(G)|$ and deg(v) must have the opposite parity. []

See Figure 3 for a strongly well-covered graph with independence number three and odd point degrees. The graph given in Figure 4 is strongly well-covered with independence number three and every point has even degree.

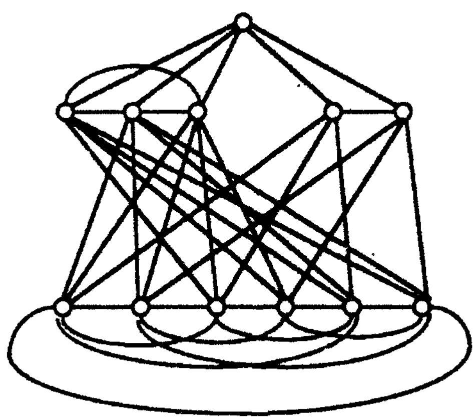

Figure 3

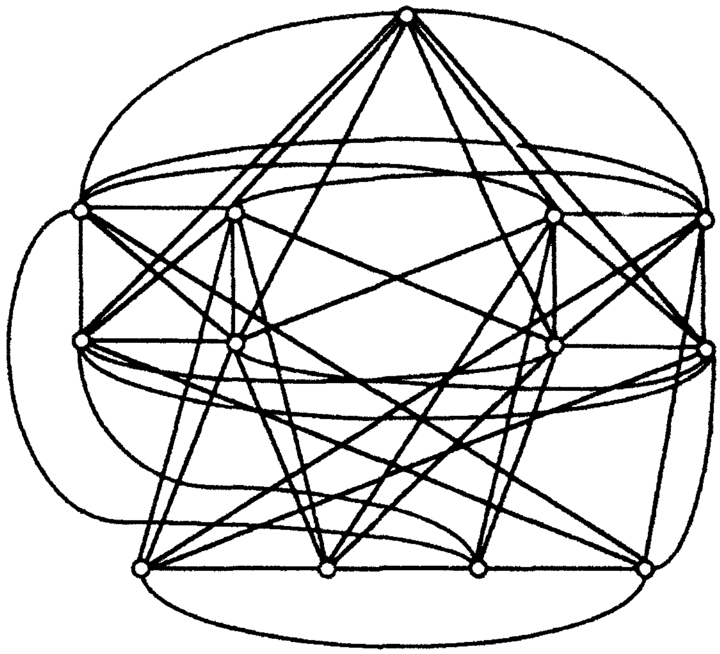

Figure 4

Next we turn to a more general discussion of strongly well-covered graphs. If G is strongly well-covered, then it is possible for G_v to contain an isolated point a, for some point v in G. However, we show in Theorem 15 that if the point a is isolated in the graph G_v, then the points a and v must have the same set of neighbors.

Theorem 15. Suppose G is connected and strongly well-covered and v is a point in G such that G_v has an isolated point a. Then $N_G(a) = N_G(v)$.

Proof. (By induction on $\alpha(G)$.) By Theorem 10, the statement is true for $\alpha(G) = 2$. By Theorem 13, the statement is true (vacuously) for $\alpha(G) = 3$.

Assume the inductive hypothesis: If G is strongly well-covered, $\alpha(G) \leq n-1$ ($n \geq 4$) and v is a point in G such that G_v has an isolated point a, then $N_G(a) = N_G(v)$.

Next, suppose G is a counterexample to the statement with $\alpha(G) = n$, $n \geq 4$. Thus, there exists a point v in G such that G_v has an isolated point a and $N_G(a) \neq N_G(v)$. Clearly,

$N_G(v) \supset N_G(a)$. Let $W = N_G(v) - N_G(a)$, and note that $W \neq \emptyset$. Let $H = G_v\text{-}a$. Since G is well-covered, then so is H and $\alpha(H) = \alpha(G) - 2$. Since $\alpha(G) \geq 4$, then $H \neq \emptyset$.

Suppose x is a point in H. If x is not adjacent to y for some $y \in N_G(a)$ and x is not adjacent to z for some $z \in W$, then v and a are in the same component of the graph G_x. Also, $z \sim v$ in G_x and z is not adjacent to a. By Theorem 6, the graph G_x is strongly well-covered with $\alpha(G_x) = \alpha(G) - 1 = n - 1$. Then v is a point in G_x such that the graph G_{xv} has isolated point a. Since $z \sim v$ in G_x and z is not adjacent to a, then $N_{G_x}(v) \neq N_{G_x}(a)$. But this contradicts the inductive assumption.

Thus, if $x \in H$ then $x \sim y$ for all $y \in N_G(a)$ or $x \sim z$ for all $z \in W$. Let $S = \{x \in H\text{: } x \sim z \text{ for all } z \in W\}$ and $T = \{x \in H\text{: } x \sim y \text{ for all } y \in N_G(a)\}$.

Suppose $y \in N_G(a)$. Since G is strongly well-covered, then G-vy is well-covered. Hence, there exists maximum independent set $J \supset \{v,y\}$ in G-vy and $|J| = \alpha(G)$. Let $J' = J - \{v,y\}$. So $|J'| = \alpha(G) - 2$. Since $\alpha(G) \geq 4$, then $J' \neq \emptyset$. Now $y \sim x$ for all $x \in T$ and $J \supset \{v,y\}$ together imply that J' is contained in S-T. Thus, $S\text{-}T \neq \emptyset$. Note that J' is a maximum independent set in H.

Suppose $x \in S$. Then xz is a line in G, where $z \in W$. Since G is strongly well-covered, then G-xz is well-covered. So there exists maximum independent set $I \supset \{x,z,a\}$ in the graph G-xz. Now, $I' = I - \{a,z\}$ is in H since $\{x\}$ dominates W. Since all points in S, except x, are adjacent to z in the graph G-xz, then I'-x must be in T-S. Since $|I'-x| = \alpha(G) - 3$ and $\alpha(G) \geq 4$, then it follows that $T\text{-}S \neq \emptyset$.

So let $b \in T\text{-}S$. Consider J' from above. Since J' is a maximum independent set in H and J' is contained in S-T, then $b \sim u$ for some $u \in J'$. See Figure 5.

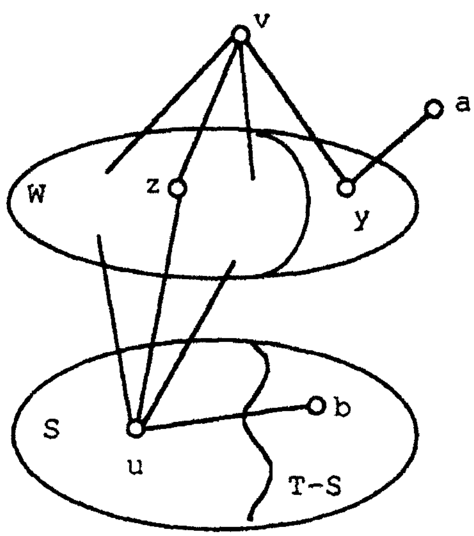

Figure 5

Consider the graph G_a. By Theorem 6, graph G_a is strongly well-covered with $\alpha(G_a) = \alpha(G) - 1$. Since $u \in S$, then $u \sim z$ for all $z \in W$; also, v and u are in the same component of G_a. Hence, v becomes isolated in the graph G_{au}. Since $b \sim u$ in G_a and b is not adjacent to v in G_a, then $N_{G_a}(v) \neq N_{G_a}(u)$. Since $\alpha(G_a) = n - 1$, this contradicts the inductive assumption that $N_{G_a}(u) = N_{G_a}(v)$.

Thus, if G is strongly well-covered with $\alpha(G) = n$ (for $n \geq 4$) and v is a point in G such that G_v has an isolated point a, then $N_G(a) = N_G(v)$. The desired result follows by induction. []

In general, if G is well-covered then G_v can contain up to $\alpha(G) - 1$ isolated points. For example, $K_{n,n}$ is well-covered, $\alpha(K_{n,n}) = n$, and $K_{n,n}$-N[v] contains n-1 isolated points for any point v in $K_{n,n}$. However, for *strongly* well-covered graphs we show in the following corollary that the number of isolated points in G_v is severely restricted.

Corollary 16. If G is connected and strongly well-covered, then G_v has at most one isolated point for any point v in G.

Proof. Assume to the contrary that v is a point in G such that G_v has two isolated points, a and b. By Theorem 15, we have N(a) = N(v) = N(b). Let $w \in N(v)$. By Theorem 6, the graph G_v is strongly well-covered. Moreover, since G is well-covered $\alpha(G_v\text{-a-b}) = \alpha(G) - 3$. Since G is strongly well-covered, there exists maximum independent set J in the graph G-vw which contains $\{v,w\}$. Since $w \sim a$ and $w \sim b$, then $J-\{v,w\}$ is contained in G_v-a-b. This is a contradiction since $|J| = \alpha(G)$ and $\alpha(G_v\text{-a-b}) = \alpha(G) - 3$. Thus, G_v cannot have two isolated points. []

We now have the means to establish an upper bound for the degree of a point in a strongly well-covered graph. It is interesting to compare the bound in the following theorem with the Hajnal type upper bound for a 1-well-covered graph given by Staples in [17].

Theorem 17. Suppose G is connected and strongly well-covered. Then $\deg(v) \leq |V(G)| - 2\alpha(G) + 2$, for all points v in G.

Proof. By Corollary 16, the graph G_v can have at most one isolated point, for all points v in G.

Suppose G_v has no isolated points. Note that for a well-covered graph H with no isolated points, $|V(H)| \geq 2\alpha(H)$. Since G_v is well-covered, then $|V(G_v)| \geq 2\alpha(G_v) = 2\alpha(G) - 2$. Thus, $|V(G)| \geq 1 + \deg(v) + 2\alpha(G) - 2$. So $\deg(v) \leq |V(G)| - 2\alpha(G) + 1$.

Suppose G_v has a single isolated point a. Then G_v-a has no isolated points and is well-covered. So $|V(G_v\text{-a})| \geq 2\alpha(G_v\text{-a}) = 2(\alpha(G) - 2) = 2\alpha(G) - 4$. Hence, $|V(G)| \geq \deg(v) + |\{a,v\}| + 2\alpha(G) - 4 = \deg(v) + 2\alpha(G) - 2$. It follows that $\deg(v) \leq |V(G)| - 2\alpha(G) + 2$. []

The upper bound in Theorem 17 is sharp. Each of the graphs G and H in Figure 6 is strongly well-covered (see [12] for a verification of this) and has at least one point whose degree attains the upper bound. In particular, $|V(G)| = 16$, $\alpha(G) = 6$ and $\Delta(G) = 6$. For H, $|V(H)| = 22$, $\alpha(H) = 8$ and $\Delta(H) = 8$.

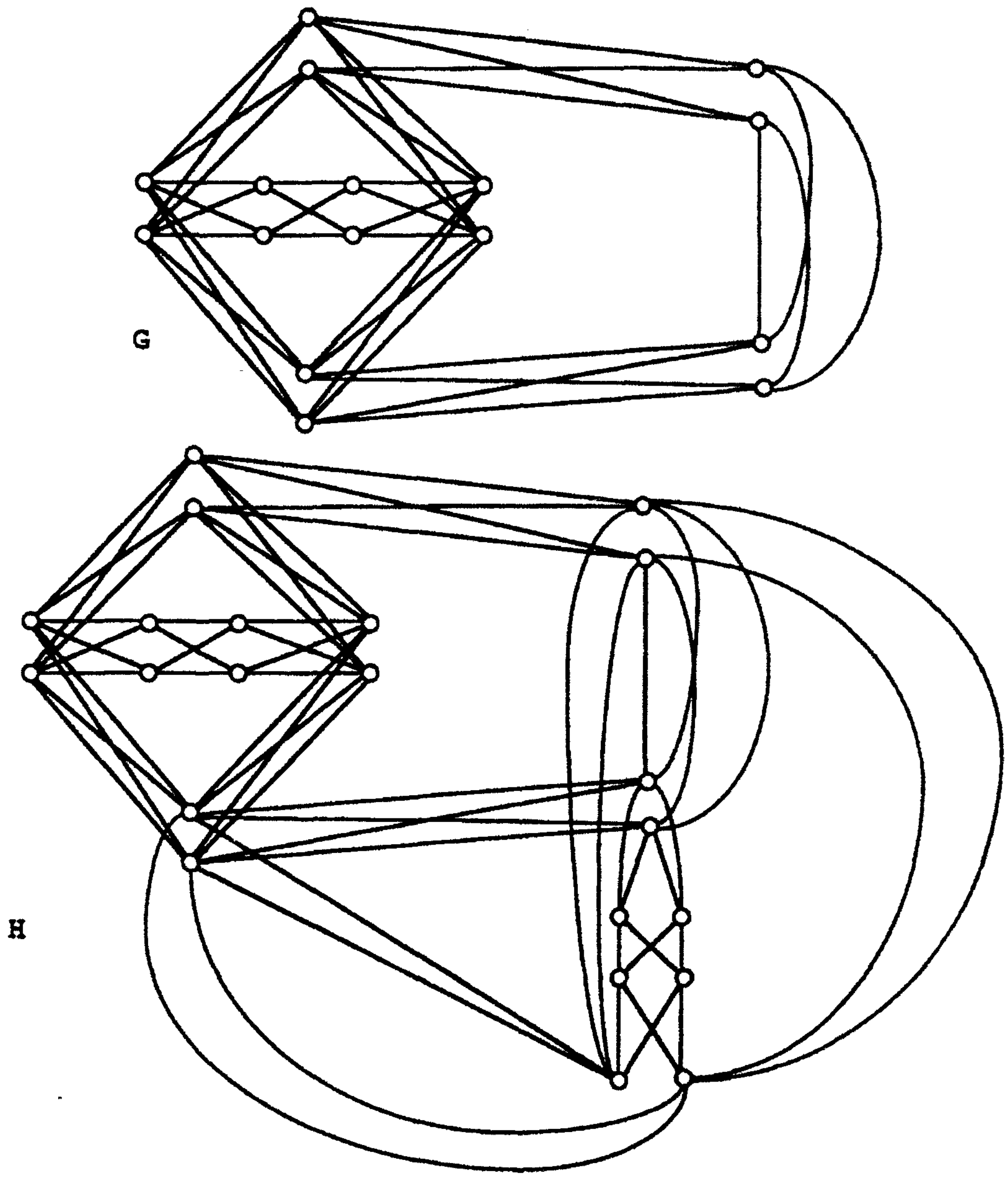

Figure 6

We now turn to developing a *lower* bound on the *minimum* degree δ for a strongly well-covered graph. Assume that G is a strongly well-covered graph, $G \neq K_1$ or K_2. We show next that if $\delta = 2$, then G must be the 4-cycle.

Theorem 18. If G is strongly well-covered with a point of degree two, then G is the 4-cycle.

Proof. Let deg(v) = 2, with N(v) = {a,b}. If there exists $w \sim a$ such that w is not adjacent to b, then {w,b} is independent and dominates N[v], w is not adjacent to v and $b \sim v$. By Lemma 2, the graph G-bv is not well-covered. This contradicts the strongly well-covered assumption.

So $N(b) \supseteq N(a)$. By symmetry, it follows that N(a) = N(b). Let $x \in N(a)$-v. Then v is isolated in the graph G_x. By Theorem 15, we have N(x) = N(v) = {a,b}. Suppose, in addition, there exists $y \sim a$ such that $y \notin \{x,v\}$. Then v is isolated in G_y, and so again by Theorem 15 we have N(y) = N(v) = N(x). But then v and x are isolated in G_y, contradicting Corollary 16.

Hence, G must be the 4-cycle. []

A well-covered graph can have points of degree one, two or three. However, we show in the following theorem that each point in a strongly well-covered graph on more than four points has at least four neighbors.

Theorem 19. If G is strongly well-covered, $G \notin \{K_1, K_2, C_4\}$, then $\delta \geq 4$.

Proof. From Lemma 1, it follows that G cannot have an endpoint. Therefore, from Theorem 18 we see that $\delta \geq 3$. Suppose deg(v) = 3, with N(v) = {a,b,c}.

Case 1. Assume that v lies on a triangle, say triangle vab. If $a \sim c$, then {a} dominates N[v] and $a \sim v$. By Lemma 2, the graph G-av is not well-covered, contradicting the strongly well-covered assumption for G. So a is not adjacent to c and, by symmetry, b is not adjacent to c.

By Lemma 1, c is not an endpoint. So let $w \sim c$, $w \neq v$. If w is not adjacent to a, then {a,w} dominates N[v], $a \sim v$ and w is not adjacent to v . This leads to a contradiction via Lemma 2.

So $w \sim a$ and, by symmetry, $w \sim b$. Thus, $N(a) \supseteq N(c)$. By Theorem 15, it follows that $N(a) = N(c)$. But $b \in N(a)$ and $b \notin N(c)$, a contradiction.

Case 2. So v cannot lie on a triangle; that is, $N(v)$ is independent.

Case 2.1. Suppose $N(a) \cap N(b) \neq \{v\}$. Let $w \in N(a) \cap N(b)$, $w \neq v$. If w is not adjacent to c, then $\{w,c\}$ dominates $N[v]$, $c \sim v$ and w is not adjacent to v. This leads to a contradiction via Lemma 2. So $w \sim c$. Thus, v is an isolated point in the graph G_w and so by Theorem 15 we have $N(w) = N(v)$. Since G_v cannot isolate two points (by Corollary 16), it follows that $N(a) \cap N(b) = N(b) \cap N(c) = N(a) \cap N(c) = \{v,w\}$. Since $\delta \geq 3$, then each of a, b and c has a third neighbor, say $x \sim a$, $y \sim b$ and $z \sim c$, and x, y and z are distinct. See Figure 7.

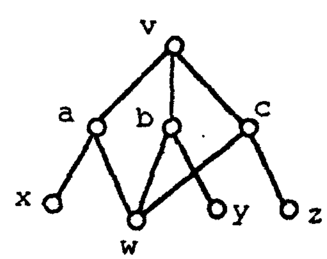

Figure 7

If x is not adjacent to y, then $\{x,y,c\}$ is independent and dominates $N[v]$, neither x nor y is adjacent to v and $c \sim v$. We obtain a contradiction via Lemma 2. So $x \sim y$. By symmetry, $x \sim z$ and $y \sim z$. In fact, if $t \sim c$, $t \notin \{v,w\}$, then $x \sim t$.

Since G is strongly well-covered, then G-xy is well-covered. So there exists maximum independent set $J \supseteq \{x,y,c\}$ in G-xy. But then $(J \cup \{w,v\}) - c$ is independent in G-xy, with $|(J \cup \{w,v\}) - c| = \alpha(G) + 2 - 1 = \alpha(G) + 1$. This is a contradiction since $\alpha(G\text{-}xy) = \alpha(G)$.

Case 2.2. So $N(a) \cap N(b) = \{v\}$. By symmetry, $N(a) \cap N(c) = N(b) \cap N(c) = \{v\}$. Let $x \in N(a)\text{-}v$. If there exists $y \sim b$, $y \neq v$, such that x is not adjacent to y, then

$\{x,y,c\}$ dominates $N[v]$ and is independent. We obtain a contradiction via Lemma 2. So x dominates all neighbors of b, except v. Hence, $\{x,v\}$ is independent and dominates $N[b]$, $v \sim b$ and x is not adjacent to b. This is a contradiction via Lemma 2.

Thus, G cannot have a point of degree 3. So $\delta \geq 4$. []

From the characterization given by Finbow, Hartnell, and Nowakowski in [9], if H is a well-covered graph with girth ≥ 5, then $\delta(H) \leq 3$. Thus, the lower bound on δ for strongly well-covered graphs leads immediately to the following two corollaries.

Corollary 20. There are no cubic strongly well-covered graphs.

Corollary 21. If G is strongly well-covered ($G \neq K_1$ or K_2), then girth(G) ≤ 4.

The last structural result we prove is that a strongly well-covered graph on more than four points is 3-connected. For that purpose, we state as a lemma the following result found by Staples [16].

Lemma 22. If G is well-covered and for all points v in G the graph G-v is not well-covered, then G is 2-connected

Theorem 23. Suppose G is strongly well-covered, $G \notin \{K_1, K_2, C_4\}$. Then G is 3-connected.

Proof. (Induction on $\alpha(G)$.) For $\alpha(G) = 2$, graph G is 3-connected as a result of Theorem 10. This serves as a basis for induction. We assume the inductive hypothesis: If G is strongly well-covered with $\alpha(G) = n$, for $n \geq 2$, then G is 3-connected.

Consider a strongly well-covered graph G with $\alpha(G) = n + 1 \geq 3$. From Theorem 9 and Lemma 22, it follows that G is 2-connected. Suppose $\{u,v\}$ is a cutset for G. We consider two cases.

Case 1. Suppose $N[u] \cup N[v] = V(G)$. If u is not adjacent to v, then $\{u,v\}$ is maximal independent in G. Since G is well-covered and $\alpha(G) \geq 3$, this is a contradiction. So $u \sim v$. Then $\{u,v\}$ is maximal independent in the graph G-uv. Since G is strongly well-covered with $\alpha(G) \geq 3$, then G-uv is well-covered and $\alpha(G\text{-}uv) \geq 3$. Hence, we have a contradiction.

Case 2. So we assume $N[u] \cup N[v] \neq V(G)$. Suppose x is a point in G such that $x \notin N[u] \cup N[v]$. Let G' be the component of the graph $G\text{-}\{u,v\}$ which contains x. Consider the graph G_x. By Theorem 6, graph G_x is strongly well-covered. Let $U_1 = N(u) \cap G'$ and $V_1 = N(v) \cap G'$.

Let H be the component of G_x containing $\{u,v\}$. Then H is strongly well-covered with $\alpha(H) \leq n$. Since $\delta(G) \geq 4$, then H is not a 4-cycle. Thus, by the inductive assumption it follows that H is 3-connected. Therefore, we claim $x \sim a$ for all $a \in U_1$ and $x \sim b$ for all $b \in V_1$. For suppose not; say w is in $U_1 \cup V_1$ and w is not adjacent to x. Then w is in V(H) and is separated from $G\text{-}\{u,v\}\text{-}V(G')$ by $\{u,v\}$. Thus, H is at most 2-connected, a contradiction. Also, since H is 3-connected it follows that $G\text{-}\{u,v\}$ has only two components. Let G" be the other component of $G\text{-}\{u,v\}$. So H is the subgraph of G induced by $V(G'') \cup \{u,v\}$. Let $U_2 = N(u) \cap G''$ and $V_2 = N(v) \cap G''$.

Case 2.1. Suppose $\{u,v\}$ does not dominate V(H). Then there exists some $y \in V(G'')$ such that $y \notin V_2 \cup U_2$. Consider the graph G_y. As argued above for the graph G_x, we have $y \sim a$ for all $a \in U_2$ and $y \sim b$ for all $b \in V_2$.

Consider the graph G_{xy}. Since x and y are in different components of $G\text{-}\{u,v\}$, then $\{x,y\}$ is independent. Since $\alpha(G) \geq 3$, then G_{xy} is not empty. So by Theorem 6, the graph G_{xy} is strongly well-covered. If $u \sim v$, then the line uv is a component of G_{xy}. By Theorem 11, we obtain a contradiction. So u is not adjacent to v. Note that u and v are not isolated points in G_x. However, they are isolated in G_{xy}. Since y is a point in the strongly well-covered graph G_x, then by Corollary 16 at most one of u and v can be isolated in G_{xy}. Hence, we have a contradiction.

Case 2.2. Thus, we assume {u,v} dominates V(H). If u is not adjacent to v, then {u,v} is maximal independent in H. On the other hand, if $u \sim v$, then {u,v} is maximal independent in the graph H-uv. Since H is strongly well-covered, it follows that $\alpha(H) = 2$. By Theorem 10, graph H is $(|V(H)| - 2)$ - regular. Since $\delta(G) \geq 4$, then $|V(H)| \geq 6$.

Case 2.2.1. Suppose $u \sim v$. Since H is well-covered, then there exists a point $y \in V(G'')$ such that u is not adjacent to y. Since H is $(|V(H) - 2)$ - regular, then the graph H_u is just the isolated point y. Since $u \sim v$ and {u,v} is a cutset for G, then y is isolated in the graph G_u. Hence, $N_G(u) = N_G(y)$ by Theorem 15. But this is a contradiction since G is 2-connected and {u,v} separates y from G'.

Case 2.2.2. So u is not adjacent to v. Since H is $(|V(H)| - 2)$ - regular, it follows that $u \sim y$ and $v \sim y$ for all $y \in V(G'')$. Let $t \in U_1$. If t is not adjacent to v, then either v is a cutpoint for the strongly well-covered graph G_t (contradicting the fact that G_t is 2-connected as a consequence of Theorem 9 and Lemma 22), or G_t contains as a component the subgraph of G induced by $V(G'') \cup \{v\}$ (a contradiction since $v \sim y$ for all $y \in V(G'')$). Thus, $t \sim v$. Hence, $t \in U_1$ implies $t \in V_1$. By symmetry, $t \in V_1$ implies $t \in U_1$. Thus, $U_1 = V_1$. It follows that $N_G(u) = N_G(v)$.

Let $t \in U_1$. Suppose x is a point in $G'-U_1$. From earlier, $x \sim a$ for all $a \in U_1$. In particular, $x \sim t$. Thus, t dominates $G'-U_1$.

Consider the graph G-tu. Since G is strongly well-covered and $\alpha(G) \geq 3$, then G-tu is well-covered and $\alpha(G\text{-tu}) \geq 3$. On the other hand, in the graph G-tu, the set {u,t} is maximal independent since t dominates $(G'-U_1) \cup \{v\}$ and u dominates $(U_1\text{-}t) \cup G''$. Thus, we obtain a contradiction.

Therefore, G cannot have a 2-cutset. Thus, G is 3-connected. The result follows by induction on $\alpha(G)$. []

We conjecture that Theorem 23 can be improved to say 4-connected.

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