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LOCKING EFFECTS FOR THE REISSNER-MINDLIN PLATE MODEL

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Abstract: We analyze the robustness of various standard finite element schemes for the Reissner-Mindlin plate and obtain asymptotic convergence estimates that are uniform in terms of the thickness d. We identify h version schemes that show *locking*, i.e. for which the asymptotic convergence rate deteriorates as $d \to 0$ and also show that the p version is free of locking. In order to isolate locking effects from boundary layer effects (which also arise as $d \to 0$), our analysis is carried out for the periodic case, which is free of boundary layers.

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1. INTRODUCTION

It is well-known that the numerical approximation of the Reissner-Mindlin (RM) plate model by certain finite element schemes deteriorates when the thickness d of the plate is close to zero. This occurs because of two phenomena, the existence of *boundary layers* and the presence of *locking*.

Boundary layers arise as components of the exact solution of the RM plate model and (to some extent) model the boundary layers present in the exact solution of the corresponding 3-d problem. In a series of papers (see [1] for example), Arnold and Falk have characterized in detail various aspects of the boundary layer for the exact RM solution, for different boundary conditions. In particular, in [1] it is shown that the layers are strongest for soft simply supported and free plates and weakest for soft clamped conditions. One effect of the presence of these boundary layers is to weaken the a priori regularity of the solution. Since the singular behavior occurs only near the boundary, an effective strategy to overcome any consequent deterioration of numerical schemes is to locally refine the mesh (usually with a low order scheme). See, for instance [2], where this has been done in the context of the h-p version.

The second phenomenon mentioned above, numerical locking, also occurs when the thickness is close to zero, but for a different reason. it is well-known that as the thickness $d \to 0$, the RM solution tends to the Kirchoff plate solution. In the limiting case, the exact solution will therefore satisfy *Kirchoff's hypothesis*. The problem of locking occurs because as $d \to 0$, the finite element solution is also forced to satisfy this hypothesis. Consequently, the number of conforming trial functions (which satisfy Kirchoff's hypothesis) can get severely restricted, resulting in a degradation of the approximation properties of the trial space.

One method to avoid locking is to construct a finite element space which possesses optimal approximation properties even when restricted by Kirchoff's constraint. This leads to a *robust* standard finite element method, i.e. one whose performance is not sensitive to the thickness. An alternate strategy is to use a mixed method, which has the effect of enforcing the constraint in a weaker sense, thereby restricting the finite element space less. Such methods have been developed for various contexts where locking occurs — see, for example, [9] (particularly Chapter 7 for the R-M plate) and the references therein. In this paper, we will concentrate exclusively on the first strategy, which can be used without reformulating the usual variational form. As a consequence, our results will be immediately applicable to several finite elements used in the context of various commercial codes.

The problem of locking is quite different in terms of origin and numerical treatment from that of boundary layers. Consequently, it is more instructive to analyze these problems separately. In this paper, we will be interested only in the problem of locking. To isolate this phenomenon and divorce it from the effect of boundary layers, we will be considering the case of *periodic* boundary conditions, which we will choose so that the solution is smooth. (This models the situation in the interior, in the case boundary layers are present.) For this RM model problem we will characterize the locking and robustness properties of various finite element schemes, using the general theory of locking developed by us in [4]. A key condition from that work, the so-called "condition (α)", will be shown to be satisfied in this case, thereby reducing the question of locking to one of approximability alone. A similar technique was used by us in [5] to analyze *Poisson ratio locking* which occurs in elasticity problems when the Poisson ratio is close to 1/2.

2. THE MODEL PROBLEM AND ITS REGULARITY

We consider as our domain the 2-d midsection $\Omega = (-\pi,\pi)$ of a square isotropic plate with the plate occupying the region $\bar{\Omega} \times [-d,d]$. On Ω we consider the RM plate model for $u_d = (\vec{\phi}_d,\omega_d)$,

$$C_d u_d = -d^2 \frac{D}{2} \{(1-\nu)\, \Delta\vec{\phi}_d + (1+\nu)\, \nabla\nabla\cdot\vec{\phi}_d\} - \lambda(\nabla\omega_d - \vec{\phi}_d) = 0 \tag{2.1}$$

$$-\lambda d^{-2}\nabla\cdot(\nabla\omega_d - \vec{\phi}_d) = g \tag{2.2}$$

which gives the bending of the plate in equilibrium. We assume the periodic boundary conditions,

$$\omega_d(x,\pi) = \omega_d(x,-\pi), \quad \omega_d(\pi,y) = \omega_d(-\pi,y), \quad |x|,|y| \le \pi \tag{2.3a}$$

$$\vec{\phi}_d(x,\pi) = \vec{\phi}_d(x,-\pi), \quad \vec{\phi}_d(\pi,y) = \vec{\phi}_d(-\pi,y), \quad |x|,\ |y| \le \pi. \tag{2.3b}$$

Here, $\vec{\phi}_d$ gives the rotation of fibers normal to the midplane, ω_d measures the transverse displacement of the midplane, and $D = \frac{E}{12(1-\nu^2)}$ is the flexural rigidity scaled by d^3. Also, $\lambda = \frac{\kappa E}{2(1+\nu)}$ with $\kappa > 0$, E and $0 \le \nu < 0.5$ being the shear-correction factor, Young's modulus and Poisson ratio, respectively. Moreover, g is the scaled loading function, i.e. the transverse load density per unit area divided by d^3. (We assume, essentially, that the loading function for a plate of thickness $2d$ is gd^3, where g is independent of d.)

If we allow d to tend to zero in (2.1), we formally obtain Kirchoff's hypothesis,

$$C_0 u_0 := \vec{\phi}_0 - \text{grad}\ \omega_0 = 0 \tag{2.4}$$

The limiting solution $u_0 = (\vec{\phi}_0,\omega_0)$ satisfies (2.4) together with

$$D\Delta^2\omega_0 = g \tag{2.5}$$

and the periodic boundary conditions (2.3) (see e.g. [3]). This, of course, is the classical biharmonic equation of plate bending, which (unlike the RM model) is independent of d (and is sometimes used to model the actual 3-d plate).

For any domain $R \subset \mathbb{R}^n$, we will denote by $H^s(R)$ the usual Sobolev space with s ($= 0,1,2,\dots$) square integrable derivatives. For our domain Ω, $H^s_{per}(\Omega)$ will denote the corresponding space of functions with s periodic derivatives, periodic in both x and y. Using the method of real interpolation [7], the above spaces may be defined for all real s. We will use $|\cdot|_s$ $\|\cdot\|_s$ to denote the seminorm and norm respectively, in both the periodic and non-periodic case. We will also use $C^{(s)}_{per}(\Omega)$, which will denote the space of functions with s periodic continuous derivatives.

Any $u(x) = u(x_1,x_2)$ in $H^s_{per}(\Omega)$ can be expanded as a Fourier series,

$$(2.6) \qquad u(x) = \sum_{k\in\mathbb{Z}^2} u^k e^{ik\cdot x}, \quad k = (k_1,k_2), \quad x = (x_1,x_2)$$

where

$$u^k = \langle u, e^{-ik\cdot x}\rangle, \quad k \in \mathbb{Z}^2$$

with $\langle\cdot,\cdot\rangle$ denoting the usual $L_2(\Omega)$ inner product. Then we have the following equivalences:

$$(2.7) \qquad |u|_s^2 \approx \sum_{k\in\mathbb{Z}^2} |k|^{2s}|u^k|^2, \; \|u\|_s^2 \approx \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^s|u^k|^2 ,$$

which hold for all real $s \ge 0$. We may also use (2.7) to define the norms $\|u\|_s$, $s < 0$ and the corresponding spaces $H^s_{per}(\Omega)$ which are the completions under these norms of $H^0_{per}(\Omega)$.

We now cast (2.1)-(2.3) into the following weak form. Find, for

$d \in (0,1]$, $u_d = (\vec{\phi}_d, \omega_d) \in V = [H^1_{per}(\Omega)]^3$ such that for all $v = (\vec{\theta}, \xi) \in V$,

$$(2.8) \qquad a_d(u_d, v) = b(u_d, v) + \frac{\lambda}{d^2} \langle C_0 u_d, C_0 v\rangle = F(v).$$

Here, C_0 is as defined in (2.4),

$$b(u_d, v) = \frac{D}{2} \iint_\Omega \{(1-\nu)\nabla\vec{\phi}_d \cdot \nabla\vec{\theta} + (1+\nu)(\nabla\cdot\vec{\phi}_d)(\nabla\cdot\vec{\theta})\} dx_1 dx_2$$

and

$$F(v) = \langle g, \xi\rangle = \iint_\Omega g\xi \, dx_1 dx_2$$

with the last integral being understood as the pairing of $H^1_{per}(\Omega)$ and $H^{-1}_{per}(\Omega)$ if $g \in H^{-1}_{per}(\Omega)$. (For existence and uniqueness for this problem, see Theorem 2.1 ahead.) We will denote the problem (2.8) by P_d and assume, without loss of generality, that it is equivalent to (2.1)-(2.3).

Let us define the energy norm corresponding to P_d by

$$\|u\|^2_{E,d} = a_d(u,u), \quad u \in V.$$

Also, for $u \in H^k_{per}(\Omega) \times H^k_{per}(\Omega) \times H^\ell_{per}(\Omega) = H^{k,\ell}_{per}(\Omega)$, let

$$\|u\|^2_{k,\ell} = \|(\vec{\phi},\omega)\|^2_{k,\ell} = \|\phi_1\|^2_k + \|\phi_2\|^2_k + \|\omega\|^2_\ell ,$$

where the norm in ω is understood to be modulo constants. Then it may be shown that

$$(2.9) \qquad A_1\|u\|_{1,1} \le \|u\|_{E,d} \le A_2 d^{-1}\|u\|_{1,1}$$

where A_1 and A_2 are constants independent of d. It is readily observed that for d bounded away from 0, the two norms are equivalent.

Let us now define, for $k \ge 1$, $0 \le d \le 1$, the spaces $H_{k,d} \subset H^{k+1,k}_{per}(\Omega)$ given by

$$(2.10) \qquad H_{k,d} = \{u \in H^{k+1,k}_{per}(\Omega), \; C_d u = 0\}.$$

For any normed linear space H, we will denote the ball of radius $B > 0$ by

$$H^B = \{u \in H, \ \|u\|_H \le B\}.$$

We will use the notation

$$H_k^B = H_{per}^{k+1,k,B}(\Omega) = \{u \in H_{per}^{k+1,k}(\Omega), \quad \|u\|_{k+1,k} \le B\}$$

and

$$H_{k,d}^B = H_k^B \cap H_{k,d}.$$

We note that (2.10) may equivalently be characterized as

$$H_{k,d} = \{u = (\vec{\phi},\omega), \ \vec{\phi} \in H_{per}^{k+1}(\Omega), \quad C_d u = 0\}$$

since using (2.1), it is easily seen that $\vec{\phi} \in H_{per}^{k+1}(\Omega) \Rightarrow \omega \in H_{per}^{k}(\Omega)$. Similarly, in the definition of $H_{k,d}^B$, we may replace $\|u\|_{k+1,k}$ by $\|\vec{\phi}\|_{k+1}$. We note also that for $u \in H_{k,d}^B$, we have, using (2.1),

$$\|C_0 u\|_{k-1} \le Kd^2 B \tag{2.11}$$

for K a constant independent of u, d.

Let us look more closely at the limiting sets $H_{k,0}$ and $H_{k,0}^B$. Since for these $C_0 u = 0$, we see that $\vec{\phi} \in H_{per}^{k+1}(\Omega)$ implies that $\nabla\omega = \vec{\phi} \in H_{per}^{k+1}(\Omega)$, so that $\omega \in H_{per}^{k+2}(\Omega)$. Hence, we see that in this limiting case, the regularity of ω is increased by two derivatives and we have the equivalent characterization,

$$H_{k,0} = \{u \in H_{per}^{k+1,k+2}(\Omega), \quad C_0 u = 0\}$$

$$H_{k,0}^B = \{u \in H_{k,0}, \ \|u\|_{k+1,k+2} \le B\}$$

The choice of the above is motivated by the following theorem, which gives an a priori estimate in these weighted norms.

Theorem 2.1. Let $g \in H^{s-2}_{per}(\Omega)$, $s \geq 1$, satisfy the compatibility condition

$$\langle g,1\rangle = 0. \tag{2.12}$$

Then there exists a unique sequence of solutions $\{u_d\} = \{\vec{\phi}_d,\omega_d\}$ (ω_d unique up to a constant) to (2.1)-(2.3) for $d \in (0,1]$ and (2.3)-(2.5) for $d = 0$, such that $u_d \in H^B_{s,d}$, where $B = C\|g\|_{s-2}$, with C a constant independent of g and d.

Proof. Suppose g, ϕ_{d1}, ϕ_{d2} and ω_d are represented in terms of their respective Fourier series, as in (2.6). Then (2.1)-(2.2) may be written as ($d > 0$)

$$\sum_{k\in\mathbb{Z}^2} e^{ik\cdot x}\left[\begin{pmatrix} \frac{D}{2}\{2k_1^2+(1-\nu)k_2^2\}+\lambda d^{-2} & \frac{D}{2}(1+\nu)k_1k_2 & -i\lambda d^{-2}k_1 \\ \frac{D}{2}\,(1+\nu)k_1k_2 & \frac{D}{2}\{2k_2^2+(1-\nu)k_1^2\}+\lambda d^{2} & -i\lambda d^{-2}k_2 \\ i\lambda d^{-2}k_1 & i\lambda d^{-2}k_2 & \lambda d^{-2}(k_1^2+k_2^2) \end{pmatrix}\begin{pmatrix}\phi_{d1}^k \\ \phi_{d2}^k \\ \omega_d^k\end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ g^k\end{pmatrix}\right] = 0$$

From this it may be easily verified that the solution u_d of (2..1)-(2.3) is

$$u_d = \begin{pmatrix}\phi_{d1} \\ \phi_{d2} \\ \omega_d\end{pmatrix} = \sum_{k\in\mathbb{Z}_0^2} e^{ik\cdot x}\begin{pmatrix} \dfrac{ik_1}{D|k|^4} \\ \dfrac{ik_2}{D|k|^4} \\ \dfrac{1}{D|k|^4}+\dfrac{d^2}{\lambda|k|^2}\end{pmatrix} g^k \tag{2.13}$$

where $\mathbb{Z}_0 = \mathbb{Z}\setminus\{0\}$. Here, we have used the fact that $g^0 = 0$, due to (2.12). Similarly, the solution of (2.3)-(2.5) is given by

(2.14)
$$\omega_0 = \sum_{k\in\mathbb{Z}_0^2} e^{ik\cdot z} \frac{g^k}{D|k|^4}$$

with $\vec{\phi}_0 = \nabla\omega_0$. Note that in (2.13)-(2.14), ω_d is unique up to a constant.

From (2.13), it is easily seen that for $d > 0$,

$$\|u_d\|_{s+1,s} \le C\|g\|_{s-2}$$

proving that $u_d \in H^B_{s,d}$. The case $d = 0$ follows similarly from (2.14). □

We now prove the following theorem, which gives us the co-called "condition (α)" which is central to the locking theory developed in the next section. Essentially, this condition establishes the rate at which the solution of (2.1)-(2.3) tends to the solution of (2.3)-(2.5) in the $H^{s+1,s}_{per}(\Omega)$ norm when d is small.

__Theorem 2.2__. For any $u_d = (\vec{\phi}_d,\omega_d) \in H^B_{s,d}$, there exists $u_0 = (\vec{\phi}_0,\omega_0) \in H^{cB}_{s,0}$ such that

$$\vec{\phi}_d = \vec{\phi}_0$$

$$\|\omega_d - \omega_0\|_s \le Kd^2B$$

where c,K are constants independent of u_d, d and B.

Proof. Given $u_d = (\vec{\phi}_d,\omega_d) \in H^B_{s,d}$, let us define g (possibly depending on d) by (2.2). Then by (2.2),

$$\|g\|_{s-2} \le Kd^{-2}\|C_0u_d\|_{s-1}$$

$$\le KB$$

where we have used (2.11).

Now $\vec{\phi}_d,\omega_d$ may be expressed in terms of g by (2.13). Also, let ω_0 be given by (2.14) and define $\vec{\phi}_0 = \nabla\omega_0$. Then it is easily seen that

$$\vec{\phi}_d = \vec{\phi}_0,$$

and that $(\vec{\phi}_0,\omega_0) \in H^{cB}_{s,0}$, where c is independent of u_d, d and B. Substituting $\vec{\phi}_d = \vec{\phi}_0 = \nabla\omega_0$ in (2.1), we have

$$\nabla(\omega_d-\omega_0) = -\lambda^{-1}d^2 \frac{D}{2} \{(1-\nu)\Delta\vec{\phi}_d + (1+\nu)\ \nabla\nabla\cdot\vec{\phi}_d\}$$

from which it follows that

$$\|\omega_d - \omega_0\|_s \le Kd^2\|\vec{\phi}_d\|_{s+1} \le Kd^2B. \qquad \square$$

The above theorem shows that for d small, functions in $H^B_{s,d}$ are close to functions in $H^B_{s,0}$. In this connection, we will also need the following result.

<u>Theorem</u> <u>2.3</u>. Given $u_0 \in H^B_{s,0}$, $s \ge 1$, there exists a constant C independent of u_0,d such that for any $d \in (0,1]$, there is a $u_d \in H_{s,d}$ satisfying

(2.15) $$\|u_d\|_{E,d} + \|u_d\|_{s+1,s} \le C\|u_0\|_{s+1,s+2}$$

and

(2.16) $$\|u_d - u_0\|_{1,1} \to 0 \text{ as } d \to 0.$$

Proof. First, using (2.11), it is easily seen that for $s \ge 1$,

(2.17) $$\|u_d\|_{E,d} \le C\|u_d\|_{s+1,s}.$$

Next, let $u_0 = (\text{grad } \omega_0,\omega_0) \in H^B_{s,0}$. Then, since $\omega_0 \in H^{s+2}_{per}(\Omega)$, we obtain the decomposition (ω_0 is defined modulo constants)

$$\omega_0 = \sum_{k\in\mathbb{Z}_0^2} e^{ik\cdot x}\,\omega^k, \quad \sum_{k\in\mathbb{Z}_0^2} (|k|^2)^{s+2}|\omega^k|^2 < \infty.$$

Hence, defining $g^k = D\omega^k|k|^4$, we see that

(2.18) $$g \in H^{s-2}_{per}(\Omega), \quad \langle g,1\rangle = 0, \quad \|g\|_{s-2} \le \|\omega_0\|_{s+2} \le C\|u_0\|_{s+1,s+2} .$$

Using Theorem 2.1 together with (2.17)-(2.18) allows us to construct u_d satifying (2.15). Also, the argument of Theorem 2.2 shows that (2.16) holds.□

3. LOCKING AND ROBUSTNESS

Suppose now that we are interested in approximating (2.8). We assume that we are given a sequence $\{V^N\}$ of finite-dimensional subspaces of $V = [H^1_{per}(\Omega)]^3$ (N denoting the dimension, $N \in \mathcal{N}$). Then we can define the sequence of finite element solutions $u^N_d \in V^N$ by

(3.1) $$a_d(u^N_d, v) = a_d(u_d, v) \qquad \forall\, v \in V^N.$$

The sequence $\{V^N\}$ thus defines an <u>extension procedure</u> $\mathcal{F}$, i.e. a rule by which we can increase the dimension N with the idea of increasing accuracy. (3.1) immediately gives

(3.2) $$\|u_d - u^N_d\|_{E,d} \le \inf_{v\in V} \|u_d - v\|_{E,d} .$$

As shown in Theorem 2.1, depending upon the regularity of the data g, the exact solutions of our problem will belong to the sets $H_{k,d} \subset H^{k+1,k}_{per}(\Omega)$, $k \ge 1$, introduced in Section 2. We will assume that the sequence $\{V^N\}$ is F_0-admissible, i.e. it leads to a certain fixed rate $F_0(N)$ of convergence when functions in $H^{k+1,k}_{per}(\Omega)$ are approximated, in the following sense:

(3.3) $$A_1F_0(N) \le \sup_{u\in H^B_k} \inf_{v\in V^N} \|u-v\|_{1,1} \le A_2F_0(N).$$

Here, $F_0(N) \to 0$ as $N\to\infty$ and A_1, A_2 depend upon B but are independent of N. Moreover, we assume that there exists $d_0 \in (0,1)$ such that for

$d_0 \le d \le 1$,

$$(3.4) \qquad A_1(d_0)F_0(N) \le \sup_{u\in H^B_{k,d}} \inf_{v\in V^N} \|u-v\|_{1,1} \le A_2(d_0)F_0(N).$$

(Note that the lower bound in (3.3) follows from the one in (3.4) while the upper bound in (3.4) follows from the one in (3.3)).

Using (2.9), (3.2) and (3.4), we then obtain the following estimate which holds uniformly for all $d_0 \le d \le 1$,

$$(3.5) \qquad \bar{A}_1(d_0)F_0(N) \le \sup_{u_d\in H^B_{k,d}} E_d(u_d-u_d^N) \le \bar{A}_2(d_0)F_0(N)$$

where

$$(3.6) \qquad E_d(v) = \|v\|_{1,1} \ \ \textit{or} \ \ \|v\|_{E,d} .$$

Whether or not $\bar{A}_1$, $\bar{A}_2$ are bounded as $d_0 \to 0$ will depend on the extension procedure being used.

A procedure $\mathcal{F}$ for which (3.5) holds uniformly for *all* $0 < d \le 1$ will be called <u>free from locking</u> for the sets $H_{k,d}$ with respect to the E_d measure. A more precise definition, adapted from the general treatment of locking in [4], is given below.

Let $L(d,N)$, the <u>locking ratio</u> corresponding to $d \in (0,1]$, $N \in \mathcal{N}$, with respect to the spaces $H_{k,d} \subset H^{k+1,k}_{per}(\Omega)$ and error measures $\{E_d\}$ (as in (3.6)) for the problems (3.1), be defined by

$$L(d,N) = \sup_{u_d\in H^B_{k,d}} E_d(u_d-u_d^N)(F_0(N))^{-1}.$$

Then we make the following definitions.

<u>Definition 3.1</u>. The extension procedure $\mathcal{F}$ is <u>free from locking</u> for the family of problems (3.1), $d \in (0,1]$ with respect to the solution sets $H_{k,d} \subset H^{k+1,k}_{per}(\Omega)$ and error measures E_d if and only if

$$\limsup_{N\to\infty} \left[\sup_{d\in(0,1]} L(d,N) \right] = M < \infty.$$

$\mathcal{F}$ shows locking of order $f(N)$ if and only if

$$0 < \limsup_{N\to\infty} \left[\sup_{d} L(d,N) \frac{1}{f(N)} \right] = K < \infty$$

where $f(N) \to \infty$ as $N \to \infty$. It shows locking of <u>at least</u> (respectively <u>at most</u>) order $f(N)$ if $K > 0$ (respectively $K < \infty$).

<u>Definition 3.2</u>. The extension procedure $\mathcal{F}$ is <u>robust</u> for the family of problems (3.1), $d \in (0,1]$ with respect to the solution sets $H_{k,d} \subset H^{k+1,k}_{per}(\Omega)$ and error measures E_d if and only if

$$\operatorname*{Lim\,sup}_{N\to\infty} \sup_{d} \sup_{u_d\in H^B_{k,d}} E_d(u_d-u_d^N) = 0.$$

It is robust with uniform order $g(N)$ if and only if

$$\sup_{d} \sup_{u_d\in H^B_{k,d}} E_d(u_d-u_d^N) \le g(N)$$

where $g(N) \to 0$ as $N \to \infty$.

<u>Remark</u>. In Sections 4, 5 we will frequently use the form $g(N) = N^{-r}$ to characterize the robustness order r. If $r = 0$, then convergence will not be guaranteed.

Definitions 3.1, 3.2 are related by the following theorem, from [4].

<u>Theorem 3.1</u>. $\mathcal{F}$ is free from locking if and only if it is robust with uniform order $F_0(N)$. Moreover, suppose $f(N)$ is such that

$$f(N)F_0(N) = g(N) \to 0 \quad \text{as} \quad N \to \infty.$$

Then $\mathcal{F}$ shows locking of order $f(N)$ if and only if it is robust with maximum uniform order $g(N)$.

It is easily seen that $\mathcal{F}$ is non-robust if and only if it shows locking of order $(F_0(N))^{-1}$.

Let us briefly explain the above ideas. We are assuming that our exact solution has a certain regularity (i.e. it is in $H^{k+1,k}_{per}(\Omega)$). Our extension procedure $\mathcal{F}$ has associated with it a rate of best approximation $F_0(N)$, which gives the best approximation that we could expect to achieve (Eqn. (3.3)), for the most unfavorable exact solution in $H^{k+1,k}_{per}(\Omega)$. Using the finite element method gives this rate for $d \geq d_0$ (Eqn. (3.4)), but does not necessarily give this rate (uniformly) as d approaches 0. The locking ratio compares the accuracy actually achieved by the finite element method (for the least favorable exact solution in $H_{k,d}$) to the best accuracy possible (i.e. to $F_0(N)$). If the achieved accuracy is asymptotically the same as the rate $F_0(N)$ in (3.3), (3.4) uniformly for all $d \in (0,1]$, then we say that the extension procedure $\mathcal{F} = \{V^N\}$ is free of locking for all $u_d \in H^B_{k,d}$, $d \in (0,1]$. If the achieved accuracy is asymptotically not the same, then the robustness $g(N)$ gives the best rate of convergence that can be achieved independent of the parameter d. In this case, $f(N) = g(N)(F_0(N))^{-1}$ characterizes the asymptotic strength of the locking.

In [4], we have formulated a useful condition, called condition (α), under which the question of locking reduces to one of approximability alone. For this condition to hold, we must first be given a sequence of solution spaces $H_{k,d}$ and a limit space $H_{k,0}$ such that Theorem 2.3 holds. Then condition (α) may be stated as: Given $u_d \in H^B_{k,d}$, there exists a $u_0 \in H^{cB}_{k,0}$ (for some c independent of u_d, d, B; u_0 depending on u_d) such that

$$\|u_d - u_0\|_{k+1,k} \leq KdB \tag{3.7}$$

with K a constant independent of B, d and u_d. This condition therefore characterizes the distance of solutions u_d to functions u_0 satisfying (2.4), as $d \to 0$. We have shown in [4] that if the "remainder" $u_d - u_0$ is small in the sense of (3.7), then we need only consider the approximation of functions u_0 in the limit space $H^B_{k,0}$ to answer questions about locking and robustness.

Our choice of periodic boundary conditions for the plate problem is motivated primarily by the fact that condition (α) is satisfied. To prove (3.7) for our problem, we simply choose u_0 (for given u_d) as in Theorem 2.2. Then we get (3.7); in fact, we get a power of d^2 (instead of just d, as needed). As a result, Theorem 2.4 from [4] will hold for our problem. This theorem states that locking and robustness rates are the same no matter which error measure in (3.6) is used. It is stated below.

<u>Theorem</u> <u>3.2</u>. Consider the family of problems (3.1), $d \in (0,1]$ with the solution sets $H_{k,d} \subset H^{k+1,k}_{per}(\Omega)$, $k \geq 1$. Then the extension procedure $\mathcal{F}$ is free from locking with respect to the $V = H^{1,1}_{per}(\Omega)$ norm if and only if it is free with respect to the energy norm. It shows locking of order $f(N)$ in the V norm if and only if it shows locking of order $f(N)$ in the energy norm.

We will now only refer to the locking of $\mathcal{F}$, without specifying an error measure. The following theorem reduces the question of locking to one of approximability alone.

<u>Theorem</u> <u>3.3</u>. Consider the problems (3.1) with solution sets $H_{k,d}$, $k \geq 1$. Let $V^N = Y^N \times Z^N$ where $Y^N \subset \left[H^1_{per}(\Omega)\right]^2$, $Z^N \subset H^1_{per}(\Omega)$ and define $W^N \subset Z^N$ by $W^N = \{\omega \in Z^N, \text{ grad } \omega \in Y^N\}$. Then the extension procedure

$\mathcal{F} = \{V^N\}$ is robust with uniform order $\max\{F_0(N), g(N)\}$ where $g(N)$ is given by

(3.8) $$g(N) = \sup_{\omega \in H^{k+2,B}_{per}} \inf_{z \in W^N} \|\omega - z\|_2 .$$

Also, with $F_0(N)$ as in (3.3), $\mathcal{F}$ is free from locking if and only if

(3.9) $$g(N) \le CF_0(N).$$

It shows locking of order $f(N)$ if and only if

(3.10) $$C_1F_0(N)f(N) \le g(N) \le C_2F_0(N)f(N).$$

Proof. Let us define

(3.11) $$\tilde{g}(N) = \sup_{u \in H^B_{k,0}} \inf_{\substack{v \in V^N \\ C_0 v = 0}} \|u - v\|_{1,1} .$$

Since condition (α) is satisfied, by Theorem 2.2(B) of [4], $\mathcal{F}$ is robust with uniform order $\max\{F_0(N), \tilde{g}(N)\}$. The argument is as follows. For $u_d \in H^B_{k,d}$, we may find (by condition (α)) a $u_0 \in H^B_{k,0}$ such that (3.7) holds. Then we have

$$\|u_d - u_d^N\|_{1,1} \le \|u_d - u_d^N\|_{E,d} \le \inf_{v \in V^N} \|u_d - v\|_{E,d}$$

$$\le \inf_{\substack{v_1+v_2 \in V^N \\ C_0 v_1 = 0}} \left\{\|u_0 - v_1\|_{E,d} + \|(u_d - u_0) - v_2\|_{E,d}\right\}$$

$$\le \tilde{g}(N) + d^{-1} \inf_{v_2 \in V^N} \|(u_d - u_0) - v_2\|_{1,1}$$

$$\le \tilde{g}(N) + d^{-1}[dF_0(N)]$$

using (3.3) and (3.7) (the latter giving $(u_d - u_0) \in H_k^{dB}$). This proves the robustness order. Next, using the fact that Theorem 2.3 and condition (α) hold, we see by Theorem 2.2(B) of [4], that (3.9) and (3.10) will hold with $g(N)$ replaced by $\tilde{g}(N)$.

To show $g(N)$ and $\tilde{g}(N)$ are equivalent, consider a $v = (\vec{\psi}, z) \in V^N$ such that $C_0 v = 0$. Then

$$C_0 v = 0 \Leftrightarrow z \in W^N, \quad \vec{\psi} = \text{grad } z.$$

Hence, for $u = (\vec{\phi}, \omega) \in H_{k,0}^B$, we have

$$\tilde{g}(N) = \sup_{u \in H_{k,0}^B} \inf_{\substack{v \in V^N \\ C_0 v = 0}} \{\|\vec{\phi} - \vec{\psi}\|_1 + \|\omega - z\|_1\}$$

$$= \sup_{\omega \in H_{per}^{k+2,B}(\Omega)} \inf_{z \in W^N} \{\|\text{grad } \omega - \text{grad } z\|_1 + \|\omega - z\|_1\}$$

from which it is easily seen that $g(N)$ and $\tilde{g}(N)$ are equivalent. The theorem follows. □

Let us now define the subspaces $V^N = Y^N \times Z^N$. We will consider both triangular and rectangular meshes. Our results will be for the <u>uniform</u> triangular and rectangular meshes M_1^h, M_2^h shown for the case of our square domain Ω in Figure 3.1.

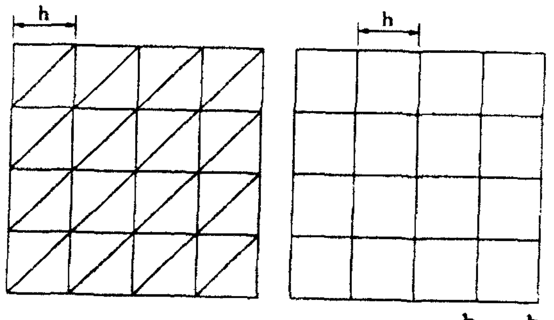

Fig. 3.1. Uniform meshes M_1^h, M_2^h.

Let S be a triangle or parallelogram. Then we define

$\mathcal{P}_p^1(S)$ = polynomials on S of <u>total</u> degree $\le p$

$\mathcal{P}_p^2(S)$ = polynomials on S of degree $\le p$ <u>in each variable</u>

$\mathcal{P}_p^3(S)$ = $\mathcal{P}_p^1(S) \oplus \left\{x_1^p x_2,\ x_1 x_2^p\right\}$, the serendipity elements

($\mathcal{P}_p^2(S)$, $\mathcal{P}_p^3(S)$ are defined only for S parallelogram).

For any mesh M^h, we now define for $i = 1,2,3$

$$\mathcal{P}_{p,-1}^{i,h} = \mathcal{P}_{p,-1}^{i,h}(M^h) = \left\{v \in L_2(\Omega),\ v|_{\Omega_j^h} \in \mathcal{P}_p^i(\Omega_j^h)\right\}$$

where Ω_j^h are the elements of M^h. For $k > 0$, we define

$$\mathcal{P}_{p,k}^{i,h} = \mathcal{P}_{p,k}^{i,h}(M^h) = \mathcal{P}_{p,-1}^{i,h} \cap C_{per}^{(k)}(\Omega)$$

where $\mathcal{P}_{p,k}^{2,h}$ and $\mathcal{P}_{p,k}^{3,h}$ are defined for rectangular meshes only. In the succeeding sections, we will consider the locking effects of the spaces $V^N = V_{p,q}^{i,h}$ defined by taking $Y^N = Y_p^{i,h} = [\mathcal{P}_{p,0}^{i,h}]^2$ and $Z^N = Z_q^{i,h} = \mathcal{P}_{q,0}^{i,h}$. Note that for the space $\mathcal{P}_{p,0}^{i,h}$, $N = O(h^{-2}p^2)$.

4. THE h VERSION

In this section, we consider the robustness of the finite element spaces $V^N = V_{p,q}^{i,h}$ when p, q are held fixed and h is decreased to attain accuracy. Let us first estimate $F_0(N)$ in (3.3), (3.4).

<u>Lemma 4.1</u>. Consider an h version sequence of spaces $\{V^N\} = \left\{V_{p,q}^{i,h}\right\}$ defined by $V_{p,q}^{i,h} = Y_p^{i,h} \times Z_q^{i,h} = \left[\mathcal{P}_{p,0}^{i,h}\right]^2 \times \mathcal{P}_{q,0}^{i,h}$ on an appropriate triangular or parallelogram quasiuniform mesh, where $p,q \ge 1$ are fixed and h varies.

Then for any $d_0 > 0$, (3.4) is satisfied with

(4.1) $$F_0(N) = CN^{-\min(p,q,k-1)/2}$$

with C independent of N but depending on p,q,k. Moreover, (3.3) is also satisfied with the above F_0.

Proof. We first note that for $\ell > 1$, $r \geq 1$, with $\beta = \min(r,\ell-1)/2$,

(4.2) $$C_1N^{-\beta} \leq \sup_{z\in H^{\ell,B}_{per}(\Omega)} \inf_{v\in\mathcal{P}^{1,h}_{r,0}} \|z - v\|_1 \leq C_2N^{-\beta} .$$

(4.2) has been shown in Theorem 4.1 of [5] (for example) for the non-periodic case, from which the periodic case is easily deduced. Using (4.2), it is easily seen that the upper estimate in (3.3) (and hence (3.4)) holds with F_0 given by (4.1). To establish the lower estimate in (3.4) (and hence (3.3)), we note that given a periodic ω_d (or conversely a periodic $\tilde{\vec{\phi}}$), we can find a corresponding periodic function $\vec{\phi}$ (or a periodic $\tilde{\omega}_d$) satisfying (2.1), such that $u_d = (\vec{\phi}_d,\omega_d)$ (or $\tilde{u}_d = (\vec{\phi}_d,\tilde{\omega}_d)$) lies in $H^B_{k,d}$ (for appropriate k). Hence, (4.2) can be used to establish the required lower bound. □

We may now analyze the locking and robustness properties of various types of elements by calculating $g(N)$ defined by (3.8) and comparing it with $F_0(N)$ given by (4.1). We first consider <u>triangular</u> elements.

Suppose that $V^N = V^{1,h}_{p,q}$ (as in Lemma 4.1) is defined on mesh M^h_1. Then we see that

(4.3) $$\begin{aligned} W^N &= \mathcal{P}^{1,h}_{q,1} && q \leq p \\ &= \mathcal{P}^{1,h}_{p+1,1} && q \geq p+1. \end{aligned}$$

Let us define

$$r(q) = \max(2,q) \quad \text{for } 1 \leq q \leq 4$$

$$= q + 1 \qquad \text{for} \quad q \geq 5.$$

Then the following lemma follows from Theorem 5.1 and Lemma 5.2 of [5], where it has been established for the non-periodic case.

Lemma 4.2. Let M_1^h be the uniform triangular mesh of Figure 3.1. Then for $\omega \in H_{per}^{k+2}(\Omega)$, $k \geq 0$, $q \geq 1$,

$$\inf_{z \in \mathcal{P}_{q,1}^{1,h}} \|\omega - z\|_2 \leq Ch^{\min(k, r(q)-2)} \|\omega\|_{k+2}.$$

Moreover, there exists a function $Q \in C_{per}^{(\infty)}(\Omega)$ satisfying

$$\inf_{z \in \mathcal{P}_{q,1}^{1,h}} \|Q - z\|_2 \geq Ch^{r(q)-2}.$$

We can now prove the following theorem.

Theorem 4.1. Let the extension procedure $\mathcal{F}$ consist of the h version on the uniform triangular mesh M_1^h, with spaces $\{V^N\} = \left\{V_{p,q}^{1,h}\right\} = \left\{\left[\mathcal{P}_{p,0}^{1,h}\right]^2 \times \mathcal{P}_{q,0}^{1,h}\right\}$. Let the solution sets be $H_{k,d}$. Then for the p and q shown below, $\mathcal{F}$ is robust with uniform order N^{-r} when $k \geq 2r+1$ and shows locking of order N^{ℓ} when $k \geq p + 1$.

Degree p	Degree q	Robustness order $= O(N^{-r})$ r	Locking order $= O(N^{\ell})$ ℓ
1	$q \geq 1$	0	1/2
$2 \leq p \leq 4$	$q = p$ $q \geq p + 1$	$(p-2)/2$ $(p-1)/2$	1 1/2
$p \geq 5$	$q = p$ $q \geq p + 1$	$(p-1)/2$ $p/2$	1/2 0

(For $p > q$, the same results as for the case $p = q$ hold.)

Remark. The above theorem shows that with the customary choice $p = q$, convergence is not guaranteed for $p = 1$ or 2 (i.e. r, the robustness order is 0) and is only guaranteed (with reduced order) if $p \geq 3$. In fact, locking cannot be avoided whenever we take $p = q$. It is $O(N)$ for $p = 2,3,4$ and $O(N^{1/2})$ for $p \geq 5$ (for $p = 1$, it is technically $O(N^{1/2})$ as well, since the maximum possible rate in this case is $O(N^{-1/2})$). To avoid locking, p has to be taken to be 5 or larger with q being chosen to be $p + 1$. (Note that taking $q > p + 1$ will not increase the robustness rate.)

Proof of Theorem 4.1. We illustrate the proof for $p = 2$, for the two cases $q = 2$, $q \geq 3$. By Lemma 4.1, we see that for both cases, for $k \geq 1$,

$$F_0(N) = CN^{-\min(2,k-1)/2} \tag{4.4}$$

and for $k \geq 3$ (i.e. $k \geq p + 1$), we have the best rate that we can expect, i.e. $O(N^{-1})$, We now calculate the robustness rate actually achieved, given by $\max(F_0(N), g(N))$ where $g(N)$ is defined by (3.8).

For $q = 2$, we see that using (4.3), we have

$$g(N) = \sup_{\omega \in H_{per}^{k+2,B}(\Omega)} \inf_{z \in \mathcal{P}_{2,1}^{1,h}} \|\omega - z\|_2 . \tag{4.5}$$

By Lemma 4.2, for any $k \geq 0$, we see that

$$C_1 \leq g(N) \leq C$$

and hence, for $k \geq 1$, the robustness rate is $\max(F_0(N), g(N)) \approx C$. Hence, this method is <u>not</u> robust. By (4.4), for $k \geq 3$, $F_0(N) = CN^{-1}$, so that by Theorem 3.3, the locking is $O(N^1)$.

For $q \geq 3$, we get (4.5) again, except z is now in $\mathcal{P}_{3,1}^{1,h}$. By Lemma 4.2, for $k \geq 1$,

$$C_1 h \le g(N) \le C_2 h.$$

i.e. $g(N) \approx CN^{-1/2}$. Using (4.4), we see that for $k \ge 2$, $F_0(N) \le CN^{-1/2}$, so that the robustness rate is $\max(F_0(N), g(N)) = CN^{-1/2}$ for $k \ge 2$. Also, for $k \ge 3$, $F_0(N) = CN^{-1}$ and by Theorem 3.3, for $k \ge 3$, the locking is $O(N^{1/2})$. □

Remark. For $p \ge 5$, the above results hold even for the case that the quasiuniform version of M_1^h is used (see [5]).

Let us now consider the uniform rectangular mesh M_2^h. Suppose first that the extension $\mathcal{F}$ is based on $\mathcal{P}_p^2(S)$ elements, with $V^N = V_{p,q}^{2,h} = \left[\mathcal{P}_{p,0}^{2,h}\right]^2 \times \mathcal{P}_{q,0}^{2,h}$. Then it may be observed that

$$W^N = \mathcal{P}_{q,1}^{2,h} \qquad q \le p \tag{4.6}$$

$$= \mathcal{P}_{p+1,1}^{1,h} \cup \mathcal{P}_{p,1}^{2,h} \qquad q \ge p+1.$$

The following is an analog of Lemma 4.2 for this case. The non-periodic version of this result is established in [8], [5].

Lemma 4.3. Let M_2^h be the uniform rectangular mesh of Figure 3.1. Then for $\omega \in H_{per}^{k+2}(\Omega)$, $k \ge 0$, $q \ge 1$,

$$\inf_{z \in \mathcal{P}_{q,1}^{2,h}} \|\omega - z\|_2 \le Ch^{\min(k,q-1)} \|\omega\|_{k+2}. \tag{4.7}$$

Moreover, there exists a function $Q \in C_{per}^{(\infty)}(\Omega)$ satisfying

$$\inf_{z \in \mathcal{P}_{q+1,1}^{1,h} \cup \mathcal{P}_{q,1}^{2,h}} \|Q - z\|_2 \ge Ch^{q-1}. \tag{4.8}$$

Obviously, the two bounds (4.7) and (4.8) hold for both the spaces

$\mathcal{P}^{1,h}_{q+1,1} \cup \mathcal{P}^{2,h}_{q,1}$ and $\mathcal{P}^{2,h}_{q,1}$. Now using Theorem 3.3, Lemma 4.1, (4.6) and Lemma 4.3, we obtain the following theorem, whose proof is similar to that of Theorem 4.1.

Theorem 4.2. Let the extension procedure $\mathcal{F}$ consist of the h version on the uniform rectangular mesh M^h_2, with spaces $\{V^N\} = \left\{V^{2,h}_{p,q}\right\} = \left\{\left[\mathcal{P}^{2,h}_{p,0}\right]^2 \times \mathcal{P}^{2,h}_{q,0}\right\}$. Let the solution sets be $H_{k,d}$. Then for the p and q shown below, $\mathcal{F}$ is robust with uniform order N^{-r} when $k \geq 2r + 1$ and shows locking of order N^{ℓ} when $k \geq p+1$.

Degree p	Degree q	Robustness order $= O(N^{-r})$ r	Locking order $= O(N^{\ell})$ ℓ
1	$q \geq 1$	0	1/2
$p \geq 2$	$q \geq p$	$(p-1)/2$	1/2

We now consider an extension procedure on M^h based on $\mathcal{P}^3(S)$ type elements. Accordingly, suppose $V^N = V^{3,h}_{p,q}$. Then it can be shown that

$$
\begin{aligned}
(4.7) \qquad W^N &= \mathcal{P}^{3,h}_{q,1} && q \leq p \\
&= \mathcal{P}^{1,h}_{p+1,1} && p \neq 2, \quad q \geq p+1, \text{ or } (p,q) = (2,3). \\
&= \mathcal{P}^{1,h}_{3,1} \cup \mathcal{P}^{2,h}_{2,1} && p = 2, \quad q \geq 4.
\end{aligned}
$$

In this case, we have the following lemma.

Lemma 4.4. Let M^h_2 be the uniform rectangular mesh of Figure 3.1 Then for $\omega \in H^{k+2}_{per}(\Omega)$, $k \geq 0$, $q \geq 1$

$$(4.8) \qquad \inf_{z\in\mathcal{P}^{1,h}_{q,1}} \|\omega - z\|_2 \le Ch^{\min(k,m(q))}\|\omega\|_{k+2} ,$$

where $m(q) = \max(0,q-3)$. Moreover, there exists a function $Q \in C^{(\infty)}_{per}(\Omega)$ satisfying

$$(4.9) \qquad \inf_{z\in\mathcal{P}^{3,h}_{q,1}} \|Q - z\|_2 \ge Ch^{m(q)} .$$

Proof. (4.8) follows from Theorems 1, 2 of [8], as shown in [5]. Next, we note that in Lemma 5.2 of [5], (4.9) was established for $\mathcal{P}^{1,h}_{q,1}$ instead of $\mathcal{P}^{3,h}_{q,1}$, for the non-periodic case. Essentially the same proof can be used to prove (4.9) here as well. □

Note that the bounds in (4.8), (4.9) will hold for $\mathcal{P}_{q,1}$ and $\mathcal{P}_{q,1}$, respectively, as well, Using (4.7) and Lemma 4.4, we may once again establish the following theorem, analogously to Theorem 4.1.

<u>Theorem 4.3</u>. Let the extension procedure $\mathcal{F}$ consist of the h version on the uniform rectangular mesh M^h_2, with spaces $\{V^N\} = \left\{V^{3,h}_{p,q}\right\} = \left\{\left[\mathcal{P}^{3,h}_{p,0}\right]^2 \times \mathcal{P}^{3,h}_{q,0}\right\}$. Let the solution sets be $H_{k,d}$. Then for the p and q shown below, $\mathcal{F}$ is robust with uniform order N^{-r} when $k \ge 2r + 1$ and shows locking of order N^{ℓ} when $k \ge p + 1$.

Degree p	Degree q	Robustness order $= O(N^{-r})$ r	Locking order $= O(N^{\ell})$ ℓ
1	$q \geq 1$	0	1/2
2	$q = 2,3$ $q \geq 4$	0 1/2	1 1/2
$p \geq 3$	$q = p$ $q \geq p+1$	$(p-3)/2$ $(p-2)/2$	3/2 1

Remark. Theorems 4.2 and 4.3 show that locking cannot be avoided when rectangular elements are used, no matter what choices of p and q are made. Both $\mathcal{P}_p^2$ and $\mathcal{P}_p^3$ elements are robust only when $p \geq 2$ (for $\mathcal{P}_2^2$, we only get robustness if, moreover, $q \geq 4$). For $p \geq 3$, $\mathcal{P}_p^3$ elements show twice to three times the locking rate as $\mathcal{P}_p^2$ elements, depending on the choice of q.

5. THE p AND h-p VERSIONS

Let us now consider a p version extension procedure $\mathcal{F}$, with $\{V^N\} = \left\{V_{p,q}^{1,h}\right\}$, where h is kept fixed and $p,q \to \infty$. Also, let us consider an h-p version over a quasiuniform family of meshes $\{M^h\}$, where both h and p,q are changed for accuracy. Then we have the following estimate for F_0 (Theorem 4.2 of [5]).

Lemma 5.1. Let $\{V^N\} = \left\{V_{p,q}^{1,h}\right\}$ be a sequence of p version spaces on a fixed mesh M^h, with $p,q \to \infty$. Then (3.3) and (3.4) (for any fixed $d_0 > 0$) are satisfied with

$$(5.1) \qquad F_0(N) = CN^{-(k-1)/2}$$

with C independent of N but depending on h, k. Moreover, as $p,q \to \infty$, (5.1) also holds if the h-p version over a quasiuniform family of meshes $\{M^h\}$ is used. In this case, we have the following more refined estimate:

$$F_0(N) = Ch^{\min(r,k-1)}r^{-(k-1)}, \tag{5.2}$$

where $r = \min(p,q)$.

Note that in the above, $\{M^h\}$ does not have to be a family of uniform meshes, but can be a quasiuniform family of meshes.

We now show that there is no locking when the p version is used for our model problem.

<u>Theorem 5.1</u>. Let the extension procedure $\mathcal{F}$ consist of the p version using a mesh consisting of triangles or parallelograms (which can be arbitrary). Then with solution sets $H_{k,d}$, $k \geq 1$, $\mathcal{F}$ is free of locking and is robust with uniform order $N^{-(k-1)/2}$ as $p,q \to \infty$.

Proof. Using the results of [11], [10], we can show that for the p version with $C^{(1)}_{per}$ continuous triangular or parallelogram straight-sided elements, for $\omega \in H^{k+2}_{per}(\Omega)$, $k \geq 1$, (h fixed),

$$\inf_{z \in \mathcal{P}^{1,h}_{r,1}} \|\omega - z\|_2 \leq Cr^{-k}\|\omega\|_{k+2} .$$

Now for any $i = 1,2,3$, with $V^N = V^{1,h}_{p,q} = \left[\mathcal{P}^{1,h}_{p,0}\right]^2 \times \mathcal{P}^{1,h}_{q,0}$, we have $\mathcal{P}^{1,h}_{r,1} \subset W^N$ for $r = \min(p,q)$. Since $r = O(N^{1/2})$, we have, with $g(N)$ defined by (3.8),

$$g(N) = CN^{-k/2},$$

so that the method is robust with order $\max(g(N),F_0(N)) = CN^{-(k-1)/2}$ by Lemma 5.1. The theorem follows, using Theorem 3.3. □

Let us remark that in the above proof, it is observed that $g(N)$ is of a smaller order than $F_0(N)$. This implies that for the limiting case (where we have the biharmonic problem), the p version actually shows an <u>increase</u> in the rate of convergence (by one order of p), rather than a decrease due to locking. The reason is that the solution to the biharmonic is of higher regularity than that of the plate problem, and the asymptotic rate of convergence of the p version <u>only</u> depends upon the regularity of the solution.

For the h-p version, we may show the following theorem for triangular meshes.

<u>Theorem 5.2</u>. Let the extension procedure $\mathcal{F}$ consist of the h-p version, using quasiuniform meshes consisting of triangles. Let $V^N = V^{1,h}_{p,q}$ with $p \geq 5$, $q \geq p + 1$. Then with solution sets $H_{k,d}$, $k \geq 1$, $\mathcal{F}$ is free of locking and is robust with uniform order $N^{-(k-1)/2}$ (or, more precisely, $h^{k-1}p^{-(k-1)}$).

Proof. The essential idea is to use the following estimate for the h-p version with $C^{(1)}_{per}$ continuous triangular elements for $\omega \in H^{k+2}_{per}(\Omega)$, $k \geq 1$

$$\inf_{z \in \mathcal{P}^{1,h}_{r,1}} \|\omega - z\|_2 \leq Ch^k r^{-k} \|\omega\|_{k+2} \tag{5.2}$$

provided $r \geq 5$, $r \geq k + 1$. An analog of (5.2) has been established in [6] for the case of $C^{(0)}$ elements — a similar technique, combining (5.1) with a scaling argument, works for the $C^{(1)}$ case as well. For $V^N = V^{1,h}_{p,q}$ with $q \geq p + 1$, $p \geq 5$, we have $\mathcal{P}^{1,h}_{p,1} \subset W^N$. Hence, using (5.2) and Theorem 3.3, we have

$$g(N) = Ch^k p^{-k} = CN^{-k/2}$$

in (3.8). The theorem follows easily. □

We see from the above that when the h-p version is used, the separate robustness rates of the h and p versions are combined. Theorem 5.2 therefore combines the results of Theorems 4.1 and 5.1, and says that if the p version is combined with a locking-free h version, then the resulting h-p extension procedure is also free from locking.

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- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

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