NUWC-NPT Technical Report 10,275
21 January 1993

(12)

Exact Detection Performance of Normalizer with Multiple-Pulse Frequency-Shift-Keyed Signals in a Partially-Correlated Fading Medium With Generalized Noncentral Chi-Squared Statistics

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PREFACE

This research was conducted under NUWC Project Number A70272, Subproject Number RR00N00, **Selected Statistical Problems in Acoustic Signal Processing**, Principal Investigator Dr. Albert H. Nuttall (Code 302). This technical report was prepared with funds provided by the NUWC In-House Independent Research Program, sponsored by the Office of Naval Research.

The technical reviewer for this report was Keith Peters (Code 3314).

REVIEWED AND APPROVED: 21 JANUARY 1993

D. Mayer for
Donald W. Counsellor
Director, Surface Antisubmarine Warfare

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

1. AGENCY USE ONLY (*Leave blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	21 January 1993	Progress

4. TITLE AND SUBTITLE Exact Detection Performance of Normalizer with Multiple-Pulse FSK Signals in a Partially-Correlated Fading Medium with Generalized Noncentral Chi-Squared Statistics

5. FUNDING NUMBERS
PE 61152N

6. AUTHOR(S)

Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Undersea Warfare Center Detachment
New London, Connecticut 06320

8. PERFORMING ORGANIZATION REPORT NUMBER
NUWC-NPT
TR 10275

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Chief of Naval Research
Office of Naval Research
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release;
distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

The detection and false alarm probabilities, for a normalizer operating in a partially-correlated fading medium with an additive unknown noise level, are derived exactly in the form of series expansions which can be quickly and accurately computed. This new work is an extension of [1], where the characteristic function of the decision variable, for a system operating in a known background noise level, was derived exactly and evaluated for a variety of cases. These new results also replace the approximate analysis of the same normalizer system, that was conducted in [2], with an exact analysis here.

The specific situation considered here is as follows. The transmitted signal consists of K pulses separated in time, frequency space so as to be nonoverlapping. In passing through the medium to the receiver, each signal pulse is subjected to fading. In particular, pulse pairs which are closely spaced in

14. SUBJECT TERMS

detection performance	normalizer
multiple pulses	frequency-shift-keyed
partial-correlation	fading medium

15. NUMBER OF PAGES
56

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102

13. ABSTRACT (continued)

time, frequency space can fade in a highly dependent fashion, while those pairs more widely separated can have relatively independent fading behavior; that is, the transmitted frequency-shift-keyed signal pulses undergo partially-correlated fading of a very general character that can contain both deterministic components as well as random components. Thus, for example, the amplitude-fading statistics are not limited to be Rayleigh.

Additive zero-mean Gaussian noise of unknown level, which is stationary over the total signal transmission time and which has a flat spectrum over the total signal bandwidth, is present at the input to the receiver, in addition to the fluctuating signal pulses (when present).

Receiver processing consists of matched filtering of each of the K time-delayed, frequency-shifted pulse locations, followed by squared-envelope detection, sampling, summation, and comparison of this decision variable with a variable threshold, for statement on signal presence or absence. The variable threshold is obtained by estimating the noise power in L time, frequency bins in which signal is known to be absent. The additional uncertainty, caused by this lack of knowledge of the additive noise level, requires additional input signal-to-noise ratio to maintain the same performance as for the fixed-threshold (known noise level) system. The exact quantitative evaluation of this degradation is the subject of this study.

14. SUBJECT TERMS (continued)

noncentral
detection probability
additive noise
characteristic function
matched filtering
variable threshold
chi-squared statistics
false alarm probability
unknown noise level
time, frequency space
squared-envelope detection
noise level estimation

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

K	number of signal pulses, figure 1
L	number of noise-only bins or samples, figure 1
t	time, figure 1
f	frequency, figure 1
γ	signal processor output, sum of K signal bins, (7)
ξ	characteristic function argument, (7), (A-1)
f_γ	characteristic function of γ, (7), (A-3)
Q_{mk}	auxiliary constant, (8)
e_{mk}	auxiliary constant, (8)
z	transformation variable, (11), (A-1)
β	scale factor in transformation, (11)
f_p	series coefficients, (12), (15), (A-7)
F	auxiliary constant, (13)
a_p	auxiliary constants, (14)
p_γ	probability density function of γ, (17)
u	fixed threshold, (18)
E_γ	exceedance distribution function of γ, (18)
$Pr(a)$	probability of event a, (20)
$\underline{E}(\nu,u)$	exceedance distribution function, (21)
v	scale factor, (22); threshold for normalizer, (25)
γ_o	sum of L noise-only bin outputs, (22)
p_{γ_o}	probability density function of γ_o, (24)
$E_n(v)$	exceedance distribution function of normalizer, (25)
overbar	ensemble average, (1), (25), (C-1)
$T(m,n,x)$	auxiliary summation, (26)

U(k,n,x)	auxiliary function, (26)
${}_1F_0$	confluent hypergeometric function, (32)
$P_F(v)$	false alarm probability of normalizer output, (48)
E_1	average random received signal energy in one pulse
N_o	received noise one-sided spectral level
SNR	signal-to-noise ratio E_1/N_o in dB, figure 2
$\underline{C}(\nu,x)$	cumulative distribution function, (B-1)
$\underline{E}(\nu,x)$	exceedance distribution function, (B-14)
$\underline{E}(n,x)$	exceedance distribution function, (B-17)
μ_n	n-th moment of ratio γ/γ_o, (C-1)
Var	variance, (C-5)
A(p)	auxiliary array, (D-1)
F(p)	auxiliary array, (D-2)
Be, Bl	auxiliary arrays, (D-3)
Cp, Bp	auxiliary arrays, (D-4)
Dp	scaled power, (D-5)

EXACT DETECTION PERFORMANCE OF NORMALIZER WITH MULTIPLE-PULSE FREQUENCY-SHIFT-KEYED SIGNALS IN A PARTIALLY-CORRELATED FADING MEDIUM WITH GENERALIZED NONCENTRAL CHI-SQUARED STATISTICS

INTRODUCTION

In a recent study [1], the detection performance capability of a multiple-pulse, frequency-shift-keyed system, with received signals subject to partially-correlated fading, was analyzed exactly in terms of the characteristic function of the system output decision variable. It was assumed there that the additive background noise level was known, so that a fixed threshold could be set for an arbitrarily specified false alarm probability. Then, the detection probability was evaluated exactly, as a function of the threshold level, the received signal-to-noise ratio, the number K of signal pulses, and the fading statistics.

Here, we extend these earlier results to cover the case where, instead, the background noise level is unknown and must be estimated on the basis of a finite number L of noise-only samples. Furthermore, the current analysis will also be exact, yielding new series expansions for the detection and false alarm probabilities. These new results replace the approximate analysis of the same normalizer system that was conducted in [2]. This normalizer possesses the important capability of constant false alarm rate; that is, a prescribed false alarm probability can be achieved, exactly, in the absence of knowledge of the actual noise background level.

The reader is referred to [1] for additional background, motivation, interpretations, and related references. For the sake of brevity, we will employ the same notation as in [1] and presume that the reader has complete familiarity with the earlier material and developments.

PROBLEM DEFINITION

BASIC SIGNAL PROCESSING

The problem of interest will be couched in a time, frequency setting, in which the individual transmitted signal pulses occupy disjoint nonoverlapping cells; however, the extension to other variables, such as space or angle, should be obvious.

A set of K signal pulses is transmitted at known time, frequency locations, as indicated in figure 1. These pulses may be abutting in time and/or frequency, or they may be widely separated in time, frequency space. The individual signal pulses need not have unity time-bandwidth product, although many practical applications would utilize that format.

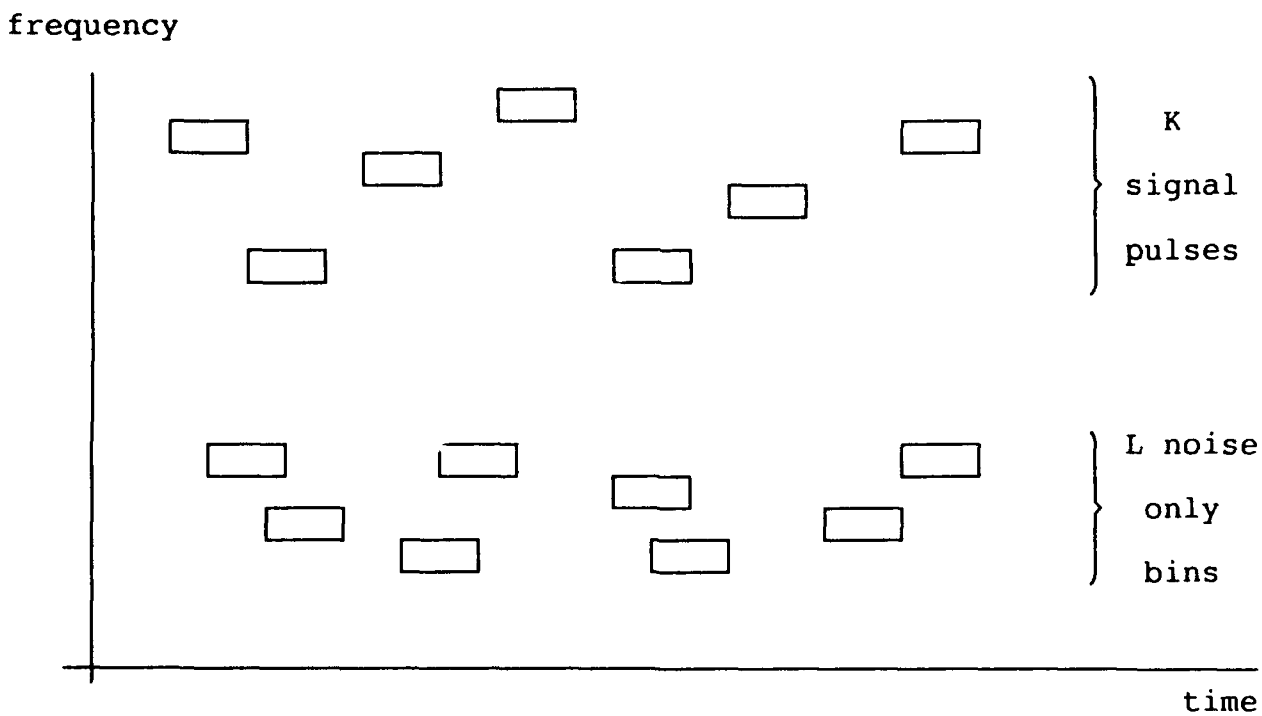

Figure 1. Time, Frequency Occupancy Diagram

At the receiver, the noisy received waveform is processed through K matched filters, which utilize knowledge of the particular time, frequency occupancy pattern of the signal pulses. Their squared envelope outputs are then appropriately sampled and summed for comparison with a scaled estimate of the background level obtained from the reference collection of L noise-only bins. The scale factor applied to the noise-level estimate yields the threshold value, which is selected to realize the desired false alarm probability. If the threshold is exceeded by the sum of K squared envelopes, a signal is declared present in that pattern of K pulses.

Depending on the time, frequency separation of the signal pulses in figure 1, the received signal strengths of the individual pulses, when signal is present, may fade considerably and with statistical dependence between them. The exact amount and effect of the fading depends on the distribution of the fading and on the K×K covariance matrix between the fading amplitudes applied to each signal pulse.

It is presumed, as in [1], that during the time extent of a single pulse, the fading of that particular pulse is essentially constant, resulting in a constant amplitude scaling and phase shift applied to each received pulse. The phase shifts of distinct pulses can have arbitrary statistics and dependencies; the particular phase shifts are irrelevant in the method of signal processing adopted here and in [1].

The choice of L, the number of noise-only pulses, must be based on a compromise. Larger L results in a more stable estimate of the background noise level, in which case the performance of the normalizer processor for figure 1 approaches that for the fixed-threshold processor in [1]. However, L cannot be chosen so large as to encounter nonstationary and/or nonwhite noise fields over the time,frequency extent depicted in figure 1.

For signal pulses consisting of bursts of pure tones, the receiver implementation of figure 1 can be accomplished by means of fast Fourier transforms. This case would utilize signal bursts and noise slots that have approximately unity time-bandwidth products, and would result in minimum consumption of time, frequency space in figure 1. However, the results presented below are not limited to this special case.

REVIEW OF PARAMETERS

Very heavy reliance will be made upon the methods and results presented for the fixed-threshold system analyzed in [1], and which the reader must be familiar with. However, it is very worthwhile at this point to review and summarize the most important and critical parameters that governed performance in [1], and which are relevant to the normalizer analysis here, as well. These parameters are listed below; their first appearance and detailed explanation can be found in [1; pages iv - viii] in the list of symbols there.

K	number of transmitted signal pulses; see figure 1 above
M	number of components in fading model; see [1; (45) - (47)]
E_{1m}	average random received signal energy in the m-th component of a single pulse, $1 \le m \le M$; see [1; (52)]
N_o	one-sided spectral level of received white noise (unknown)
$\underline{C}^{(m)}$	K×K normalized covariance matrix of signal amplitude fading; see [1; (127)]
$\underline{\Lambda}^{(m)}$	K×K eigenvalue matrix of $\underline{C}^{(m)}$
$\underline{\lambda}_k^{(m)}$	k-th eigenvalue of matrix $\underline{\Lambda}^{(m)}$, $1 \le k \le K$
D_{km}	deterministic received signal energy in the m-th component of the k-th pulse; see [1; (50)]
$V_k^{(m)}$	k-th eigenvector of $\underline{C}^{(m)}$
$v_{kj}^{(m)}$	j-th component of vector $V_k^{(m)}$, $1 \le j \le K$
$\underline{\varepsilon}_k^{(m)}$	auxiliary constants; see [1; (137)]
ψ_m	fractional strengths of random components; see [1; (191)]

It is also worthwhile at this point to expound on the covariances of the fading model, since these will be used at a later stage of the development. The covariance function of the alternating component $g_m(t,f)$ in [1; (45) & (47)] is

$$R_{mn}(\tau,\nu) = \overline{g_m(t,f)\ g_n(t-\tau,f-\nu)} \ , \tag{1}$$

while the covariance of power-scaling variate $q_k = q(t_k,f_k)$ is, from [1; (43), (44), (54), (55)]

$$\tilde{R}_{kj} = \mathrm{Cov}(q_k,q_j) = \mathrm{Cov}\Big(q(t_k,f_k),\ q(t_j,f_j)\Big) \ . \tag{2}$$

The covariance coefficient between q_k and q_j is

$$\rho_{kj} = \frac{\tilde{R}_{kj}}{\left(\tilde{R}_{kk}\ \tilde{R}_{jj}\right)^{\frac{1}{2}}} = \left[\frac{R_{11}(t_k-t_j,f_k-f_j)}{R_{11}(0,0)}\right]^2 \ , \tag{3}$$

from [1; (56), (66) or (69)]. We also define the normalized covariance function

$$\mathrm{Cov}(\tau,\nu) = \frac{R_{11}(\tau,\nu)}{R_{11}(0,0)} \tag{4}$$

as in [1; (194)], from which there follows

$$\rho_{kj} = \mathrm{Cov}^2\Big(t_k-t_j,f_k-f_j\Big) \tag{5}$$

as in [1; (56), (66) or (69)]. Finally, we define a special case of (4), for the case of equally time-spaced signal pulses without any frequency shifts, according to

$$\mathrm{Cov}_1 = \mathrm{Cov}(t_{k+1}-t_k,0) = \frac{R_{11}(t_{k+1}-t_k,0)}{R_{11}(0,0)} \quad \text{for all } k \ , \tag{6}$$

since this quantity will be very useful later; this last quantity in (6) was denoted by Cov in [1] and all the figures therein.

EXCEEDANCE DISTRIBUTION FUNCTION OF NORMALIZER OUTPUT

FIXED-THRESHOLD PROCESSOR OUTPUT

Before we begin the analysis of the normalizer, we need to review and manipulate the results for the fixed-threshold case. The general result for the characteristic function of signal processor output γ, in the case of M uncorrelated <u>components</u> in fading model [1; (45)], is given by [1; (136)] as

$$f_\gamma(\xi) = (1 - i2\xi)^{K(\frac{1}{2}M-1)} \left(\prod_{m=1}^{M} \prod_{k=1}^{K} \left(1 - i2\xi Q_{mk}\right) \right)^{-\frac{1}{2}} \times$$

$$\times \exp\left(i\xi \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{1 - i2\xi Q_{mk}} \right) , \tag{7}$$

where

$$Q_{mk} = 1 + \frac{2E_{1m}}{N_o} \underline{\lambda}_k^{(m)} , \quad e_{mk} = \underline{\varepsilon}_k^{(m)2} \quad \text{for } 1 \le m \le M , \ 1 \le k \le K . \tag{8}$$

This situation pertains to ignoring the L noise-only outputs in the lower portion of figure 1.

We will limit consideration here to the special case 6 delineated in [1; page 23]. Then, eigenvalues $\{\underline{\lambda}_k\}$ and eigenvectors $\{V_k\}$ are not functions of component number m, and

$$Q_{mk} = 1 + \frac{2E_1}{N_o} \psi_m \underline{\lambda}_k \tag{9}$$

from (8) and [1; (191)]. Also, from [1; (157)],

$$\underline{\xi}_k^{(m)} = \sum_{j=1}^{K} v_{kj} \left(\frac{2D_{jm}}{N_o}\right)^{1/2} . \tag{10}$$

We now want to perform a transformation of variable ξ in characteristic function (7). In particular, consider the use of the slightly more general transformation (relative to that used in [1; (D-3)]), where scale factor β is under our choice:

$$z = \frac{1}{1 - i\xi\beta} \quad (\beta > 0) . \tag{11}$$

Substitution into (7), and expansion in a power series in z (according to appendix A), yields the characteristic function of the fixed-threshold processor output γ in the (modified but exact) form

$$f_\gamma(\xi) = F\, z^K \sum_{p=0}^{\infty} f_p\, z^p = F \sum_{p=0}^{\infty} f_p\, (1 - i\xi\beta)^{-K-p} , \tag{12}$$

where the constants are found according to

$$F = \left(\frac{\beta}{2}\right)^K \left(\prod_{m=1}^{M} \prod_{k=1}^{K} Q_{mk}\right)^{-1/2} , \qquad a_o = -\frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \frac{e_{mk}}{Q_{mk}} , \tag{13}$$

$$p\, a_p = \frac{1}{2} \sum_{m=1}^{M} \sum_{k=1}^{K} \left(1 - \frac{\beta}{2Q_{mk}}\right)^{p-1} \left[1 - \frac{\beta}{2Q_{mk}}\left(1 - p\,\frac{e_{mk}}{Q_{mk}}\right)\right] -$$

$$- K \left(\frac{M}{2} - 1\right) \left(1 - \frac{\beta}{2}\right)^p \quad \text{for } p \ge 1 , \tag{14}$$

$$f_o = \exp(a_o) , \qquad f_p = \frac{1}{p} \sum_{n=1}^{p} n\, a_n\, f_{p-n} \quad \text{for } p \ge 1 . \tag{15}$$

For $\beta = 2$, (14) reduces to [1; (185) or (D-10)]. However, we want to choose β here so that the magnitudes of the coefficients $\{F\ f_p\}$ in (12) decay as rapidly as possible with p. On the other hand, some of the $\{f_p\}$ can then be negative.

Now, suppose we have found the characteristic function of any random variable γ in the series form (12), where K is integer. Then, the origin value of the characteristic function is

$$f_\gamma(0) = F \sum_{p=0}^{\infty} f_p = 1 \ , \tag{16}$$

while the corresponding probability density function of γ is

$$p_\gamma(x) = F \sum_{p=0}^{\infty} f_p \frac{x^{K+p-1}}{\beta^{K+p}} \frac{\exp(-x/\beta)}{(K+p-1)!} \quad \text{for } x > 0 \ . \tag{17}$$

The exceedance distribution function of γ is then

$$E_\gamma(u) = \int_u^\infty dx\ p_\gamma(x) = F \sum_{p=0}^{\infty} f_p \int_u^\infty dx \frac{x^{K+p-1}}{\beta^{K+p}} \frac{\exp(-x/\beta)}{(K+p-1)!} =$$

$$= F \sum_{p=0}^{\infty} f_p\ \underline{E}\left(K+p, \frac{u}{\beta}\right) = \tag{18}$$

$$= 1 - F \sum_{p=0}^{\infty} f_p \left[1 - \underline{E}\left(K+p, \frac{u}{\beta}\right)\right] \quad \text{for } u > 0 \ , \tag{19}$$

where we used (16). Physically, this exceedance distribution function is the probability that fixed-threshold processor output γ is greater than a constant value u; that is,

$$E_{\gamma}(u) = \Pr(\gamma > u) \; . \qquad (20)$$

Here, in (18), we used the following definition of the exceedance distribution function $\underline{E}$, which can be expressed in terms of incomplete gamma functions:

$$\underline{E}(\nu,u) = \int_u^\infty dx \, \frac{x^{\nu-1} \exp(-x)}{\Gamma(\nu)} = \frac{\Gamma(\nu,u)}{\Gamma(\nu)} \quad \text{for } u \ge 0 \; , \quad \nu > 0 \; . \qquad (21)$$

More details on recursions for these functions, when ν is equal to an integer or a half-integer, are given in appendix B.

NORMALIZER OUTPUT

Now, let fixed threshold u in (20) be replaced by a <u>variable</u> threshold according to

$$u = v\,\gamma_o\ , \tag{22}$$

where v is a positive scaling constant, and γ_o is an independent positive random variable given by the sum of the L independent noise-only bin outputs in figure 1. This latter quantity is proportional to an estimate of the noise background power level in which the normalizer system is operating. The characteristic function of γ_o can be found from [1; (40)], by replacing K by L and by setting signal strength $\tilde{E}$ to zero:

$$f_{\gamma_o}(\xi) = (1 - i2\xi)^{-L}\ . \tag{23}$$

The corresponding probability density function of γ_o is

$$p_{\gamma_o}(x) = \frac{x^{L-1}\exp(-x/2)}{2^L\,(L-1)!} \quad \text{for } x > 0\ . \tag{24}$$

We are now interested in the exceedance distribution function $E_n(v)$ of the normalizer output, namely ratio γ/γ_o:

$$E_n(v) \equiv \Pr\left(\frac{\gamma}{\gamma_o} > v\right) = \Pr(\gamma > v\,\gamma_o) = \overline{E_\gamma(v\,\gamma_o)}^{\gamma_o} =$$

$$= \int dx\; p_{\gamma_o}(x)\, E_\gamma(v\,x) = \int_0^\infty dx\, \frac{x^{L-1}\exp(-x/2)}{2^L\,(L-1)!}\, F \sum_{p=0}^{\infty} f_p\, \underline{E}\left(K+p, \frac{v}{\beta}x\right) =$$

$$= F \sum_{p=0}^{\infty} f_p \int_0^{\infty} dx \; \frac{x^{L-1} \; \exp(-x/2)}{2^L \; (L-1)!} \; \exp\left(- \frac{v}{\beta} x\right) \sum_{k=0}^{K-1+p} \frac{1}{k!} \left(\frac{v}{\beta} x\right)^k =$$

$$= F \sum_{p=0}^{\infty} f_p \sum_{k=0}^{K-1+p} \frac{1}{k!} \left(\frac{v}{\beta}\right)^k \int_0^{\infty} dx \; \frac{x^{L-1+k}}{2^L \; (L-1)!} \exp\left(-x\left(\frac{1}{2} + \frac{v}{\beta}\right)\right) =$$

$$= F \left(\frac{\beta}{\beta + 2v}\right)^L \sum_{p=0}^{\infty} f_p \sum_{k=0}^{K-1+p} \binom{L-1+k}{k} \left(\frac{2v}{\beta + 2v}\right)^k =$$

$$= F \left(\frac{\beta}{\beta + 2v}\right)^L \sum_{p=0}^{\infty} f_p \; T\left(K-1+p, L-1, \frac{2v}{\beta + 2v}\right) , \tag{25}$$

where we define the auxiliary functions

$$T(m,n,x) \equiv \sum_{k=0}^{m} \binom{n+k}{k} x^k = \sum_{k=0}^{m} (n+1)_k \frac{x^k}{k!} \equiv \sum_{k=0}^{m} U(k,n,x) . \tag{26}$$

As a partial check on (25), we have origin value

$$E_n(0) = F \sum_{p=0}^{\infty} f_p \; T(K-1+p, L-1, 0) = F \sum_{p=0}^{\infty} f_p = 1 , \tag{27}$$

using (16).

For rapid evaluation of functions T and U in (26), we have recursions

$$T(m,n,x) = T(m-1,n,x) + U(m,n,x) \quad \text{for } m \geq 1 , \tag{28}$$

$$U(m,n,x) = U(m-1,n,x) \frac{n+m}{m} x \quad \text{for } m \geq 1 , \tag{29}$$

along with starting values

$$T(0,n,x) = 1 , \quad U(0,n,x) = 1 . \tag{30}$$

A general result that will find frequent application is

$$(1-x)^{n+1}\ T(m,n,x) = 1 - x^{m+1}\ T(n,m,1-x) \ . \tag{31}$$

As a special case, we also have [3; 15.1.8]

$$T(\infty,n,x) = \sum_{k=0}^{\infty} (n+1)_k \frac{x^k}{k!} = {}_1F_0(n+1;x) = (1-x)^{-n-1} \ . \tag{32}$$

By means of this last relation, we have

$$T\left(\infty, L-1, \frac{2v}{\beta + 2v}\right) = \left(1 - \frac{2v}{\beta + 2v}\right)^{-L} = \left(\frac{\beta + 2v}{\beta}\right)^{L} , \tag{33}$$

which enables the normalizer exceedance distribution function in (25) to be expressed in the alternative form

$$E_n(v) = F \left(\frac{\beta}{\beta+2v}\right)^{L} \sum_{p=0}^{\infty} f_p \left[T\left(K-1+p, L-1, \frac{2v}{\beta+2v}\right) - \left(\frac{\beta+2v}{\beta}\right)^{L} + \left(\frac{\beta+2v}{\beta}\right)^{L}\right] =$$

$$= 1 - F \sum_{p=0}^{\infty} f_p \left[1 - \left(\frac{\beta}{\beta+2v}\right)^{L} T\left(K-1+p, L-1, \frac{2v}{\beta+2v}\right)\right] , \tag{34}$$

where we used (16). This latter form is recommended over (25) because <u>both</u> f_p and the bracketed term, [ ], in (34) go to zero as $p \to \infty$. The approach of f_p to zero follows from (16). It is also very important to notice that the bracketed term in (34) is <u>not</u> a function of the signal-to-noise ratio.

By means of (31), (34) can also be expressed as

$$E_n(v) = 1 - F \sum_{p=0}^{\infty} f_p \left(\frac{2v}{\beta+2v}\right)^{K+p} T\left(L-1, K-1+p, \frac{\beta}{\beta+2v}\right) \ . \tag{35}$$

Although both f_p and the succeeding power in (35) do tend to zero as $p \to \infty$, function T does not; of course, the product of the power term and T do tend to zero. Nevertheless, (34) will be more useful, since a simple recursion on p exists. (It has been numerically verified that (25), (34), and (35) are all independent of the particular value of the positive scaling constant β in transformation (11); we are now free to choose the value of β that results in the most rapid convergence of the various series.) Results (25), (34), and (35) are alternative expressions for the detection probability $P_D(v)$ of the normalizer output ratio γ/γ_o, with threshold value v; that is, $P_D(v) = \Pr(\gamma/\gamma_o > v)$ for fixed v.

The numerical evaluation of form (34) requires a difference in the bracketed term and thereby suffers a loss of accuracy. A more useful numerical alternative, using a sum of positive terms, is furnished by

$$E_n(v) = 1 - F \left(\frac{\beta}{\beta + 2v}\right)^L \sum_{p=0}^{\infty} f_p \sum_{k=K+p}^{\infty} \frac{(L)_k}{k!} \left(\frac{2v}{\beta + 2v}\right)^k . \qquad (36)$$

This result follows upon use of (26) and (33). The inner sum on k in (36) obviously goes to zero as p increases, as does coefficient f_p. Recursions for the inner sum on k make this a rapid accurate alternative to (34); result (36) is programmed and used in this investigation.

CHECK ON EXCEEDANCE DISTRIBUTION (25)

Result (25) gave the normalizer exceedance distribution function

$$E_n(v) = \Pr(\gamma > v\ \gamma_o) = \Pr\left(\frac{\gamma}{\gamma_o} > v\right) . \qquad (37)$$

If we now let the normalizer fixed threshold v be given by

$$v = \frac{u}{2L} , \qquad (38)$$

then we find

$$E_n(v) = E_n\left(\frac{u}{2L}\right) = \Pr\left(\gamma > u\ \frac{\gamma_o}{2L}\right) . \qquad (39)$$

But as $L \to \infty$, random variable $\gamma_o/(2L)$ tends to the value 1, since the mean of γ_o is 2L and its standard deviation is $2\sqrt{L}$; see (24), for which the k-th cumulant is $L\ (k-1)!\ 2^k$. Thus, we should have

$$\lim_{L\to\infty} E_n\left(\frac{u}{2L}\right) = \Pr(\gamma > u) = E_\gamma(u) , \qquad (40)$$

where the latter exceedance distribution was given by (18).

In order to confirm this result, substitute $v = u/(2L)$ into (25) to obtain

$$E_n\left(\frac{u}{2L}\right) = F\ \left(\frac{\beta}{\beta + u/L}\right)^L \sum_{p=0}^{\infty} f_p\ T\left(K-1+p, L-1, \frac{u/L}{\beta + u/L}\right) . \qquad (41)$$

But

$$\left(\frac{\beta}{\beta + u/L}\right)^L = \left(\frac{1}{1 + u/(\beta L)}\right)^L \sim \exp\left(-\frac{u}{\beta}\right) \quad \text{as } L \to \infty , \qquad (42)$$

while

$$T\left(m, L-1, \frac{u/L}{\beta + u/L}\right) = \sum_{k=0}^{m} \binom{L-1+k}{k} \left(\frac{u/L}{\beta + u/L}\right)^k =$$

$$= \sum_{k=0}^{m} \frac{(L)_k}{k!} \frac{(u/\beta)^k}{(L + u/\beta)^k} \sim \sum_{k=0}^{m} \frac{(u/\beta)^k}{k!} \quad \text{as } L \to \infty . \tag{43}$$

Therefore,

$$\lim_{L\to\infty} E_n\left(\frac{u}{2L}\right) = F \exp\left(-\frac{u}{\beta}\right) \sum_{p=0}^{\infty} f_p \sum_{k=0}^{K-1+p} \frac{(u/\beta)^k}{k!} = F \sum_{p=0}^{\infty} f_p \, \underline{E}\left(K+p, \frac{u}{\beta}\right) \tag{44}$$

in agreement with (18). Here, we used (B-17).

FALSE ALARM PROBABILITY

If signal is absent from the processor input, the earlier results for the normalizer output exceedance distribution function should reduce to the false alarm probability. The moments of normalizer output γ/γ_o are presented in appendix C, when signal is absent. Also, in this case, we have $Q_{mk} = 0$, $e_{mk} = 0$, giving, from (13) - (14),

$$F = \left(\frac{\beta}{2}\right)^K , \qquad a_o = 0 , \qquad p\, a_p = K \left(1 - \frac{\beta}{2}\right)^p \quad \text{for } p \geq 1 . \tag{45}$$

There follows (see (A-5) - (A-6))

$$\sum_{p=0}^{\infty} f_p\, z^p = \exp\left(\sum_{p=0}^{\infty} a_p\, z^p\right) = \exp\left(K \sum_{p=1}^{\infty} \frac{1}{p} \left(1 - \frac{\beta}{2}\right)^p z^p\right) =$$

$$= \exp\left(-K \ln\left[1 - \left(1 - \frac{\beta}{2}\right) z\right]\right) = \left[1 - \left(1 - \frac{\beta}{2}\right) z\right]^{-K} =$$

$$= \sum_{p=0}^{\infty} \binom{-K}{p} \left(1 - \frac{\beta}{2}\right)^p (-z)^p = \sum_{p=0}^{\infty} \frac{(K)_p}{p!} \left(1 - \frac{\beta}{2}\right)^p z^p , \tag{46}$$

from which we find coefficients

$$f_p = \frac{(K)_p}{p!} \left(1 - \frac{\beta}{2}\right)^p \quad \text{for } p \geq 0 . \tag{47}$$

Then (25) reduces to the normalizer false alarm probability

$$P_F(v) = \left(\frac{\beta}{2}\right)^K \left(\frac{\beta}{\beta + 2v}\right)^L \sum_{p=0}^{\infty} \frac{(K)_p}{p!} \left(1 - \frac{\beta}{2}\right)^p T\left(K-1+p, L-1, \frac{2v}{\beta + 2v}\right) . \tag{48}$$

Furthermore, since this result must be independent of β, we set $\beta = 2$ to get

$$P_F(v) = \frac{1}{(1 + v)^L}\, T\left(K-1, L-1, \frac{v}{1 + v}\right) =$$

$$= \frac{1}{(1 + v)^L} \sum_{k=0}^{K-1} \frac{(L)_k}{k!} \left(\frac{v}{1 + v}\right)^k = \tag{49}$$

$$= 1 - \left(\frac{v}{1 + v}\right)^K \sum_{j=0}^{L-1} \frac{(K)_j}{j!} \frac{1}{(1 + v)^j}\,, \tag{50}$$

where we used (31). This result, (50), agrees with [2; (22)]. The form in (49) utilizes a sum of all-positive terms and thereby retains significance for very small P_F.

GRAPHICAL RESULTS

The scenario to be considered here is the same as that used in [1], namely, a sequence of K signal pulses, equally-spaced in time but with no frequency shifts. Also, the covariance between the signal amplitude fading coefficients will be taken to be exponential in time separation. In this manner, we can make a direct comparison of the results for the fixed-threshold processor in [1] with the normalizer here, and numerically determine the degradation associated with utilizing too small a value for L, the number of noise-only bins. As in [1; page 68], the KM deterministic signal-to-noise ratios $\{D_{km}/N_o\}$ are all taken to be zero, and the M strengths $\{r^{(m)}\}$ are all 1, for the numerical examples investigated here.

Due to the multitude of parameters in this investigation, it is expedient to hold all but one of them fixed, and to let one parameter vary, thereby determining the effect of that parameter by itself. The first series of plots in figures 2, 3, 4, 5, 6 correspond to varying K, the number of signal pulses in figure 1, over the range K = 2, 3, 4, 5, 6. The curve for K = 1 has already been given in [2; figure 32]; that analysis was, in fact, exact for K = 1, M = 2.

The abscissa in all the plots is L, the number of noise-only bins that are used to estimate the background noise level, and which varies over the range from 1 to 100 in all cases. The asymptotic values of required signal-to-noise ratio at L = ∞ are available in [1]. The ordinate in the following plots is the input signal-to-noise ratio (SNR) E_1/N_o in dB, required in order to realize the specified performance levels in terms of detection and false alarm probabilities, where E_1 is the average random received signal energy in one pulse, and N_o is the additive noise power density level. The curves are parameterized in terms of the false alarm probability, which is set to values 10^{-2}, 10^{-4}, 10^{-6}, and 10^{-8}. The particular parameter values that apply to each plot appear in the title for each figure. Parameter Cov_1 was defined and explained in (1) - (6); basically, it is the covariance between adjacent signal pulses.

For most rapid convergence of the series for the exceedance distribution function of the normalizer output, the value of scaling β in transformation (11) should be changed for each point computed on these plots. Some samples for the best β values are tabulated in appendix D, along with the particular program used for computing the results in all the figures here.

The first result in figure 2, for K = 2, replaces the approximate result presented in [2; figure 33]. It will be seen that approximately 1 dB less is required, according to the exact results of figure 2 here; thus, the approximate analysis in [2] was somewhat pessimistic in terms of the input signal-to-noise ratio required to realize a specified performance level. This same conclusion was also reached upon a comparison of the exact analysis of the fixed-threshold processor in [1] with the corresponding approximate analysis in [4].

As the number of pulses, K, increases from 2 to 6 in figures 2 through 6, the required value of per-pulse SNR, E_1/N_o, decreases monotonically. However, the total SNR, summed over all K pulses, increases with K; this same effect was observed and investigated quantitatively in [4], although by an approximate analysis.

The next series of plots, in figures 7 - 11, correspond to varying covariance coefficient Cov_1 in (6) over the range 0 to 1 in steps of $(n/4)^{1/2}$, for n = 0, 1, 2, 3, 4. The remaining parameters are held fixed at values $P_D = .5$, K = 4, M = 2. As expected, the required input signal-to-noise ratio increases monotonically with Cov_1. The increases, as n makes the four changes from 0 to 4, are in the ranges .15 to .21 dB, .21 to .28 dB, .31 to .35 dB, and .36 to .45 dB, respectively. The total increases, as Cov_1 varies from 0 (figure 7) to 1 (figure 11) are in the range 1.1 to 1.2 dB. Thus, at least for this example, the cost of the fading covariance coefficient approaching unity is moderate.

The final series of plots, in figures 12 - 14, depict the variation of required input signal-to-noise ratio with M, the number of components in the fading model [1; (45) - (47)]. (The plot in figure 12 is identical to figure 4 above.) As M changes from 2 to 3, with other parameters fixed, the required input SNR decreases by approximately 1 dB, while the change of M from 3 to 4 allows for an additional decrease of .5 dB in required input SNR. Generally, larger M means a fading channel model subject to less severe deep fades; see [1; figures 2 - 8].

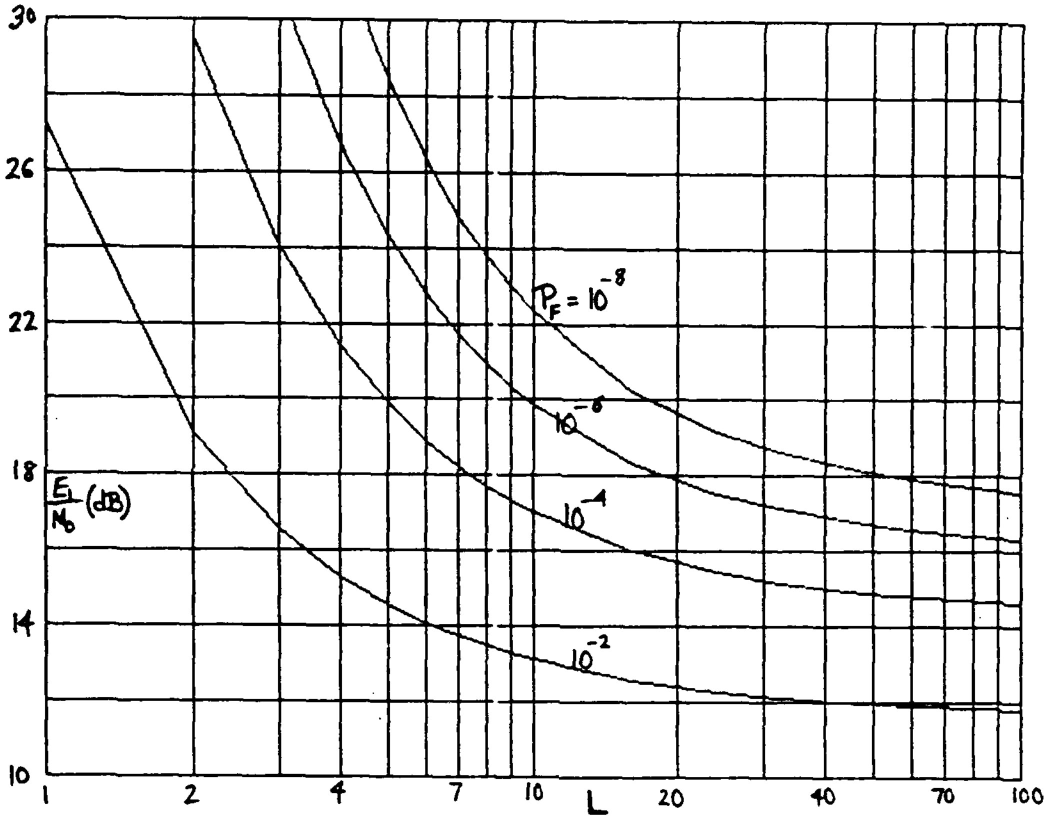

Figure 2. Required SNR for P_D = .9, K = 2, M = 2, Cov_1 = $\sqrt{.5}$

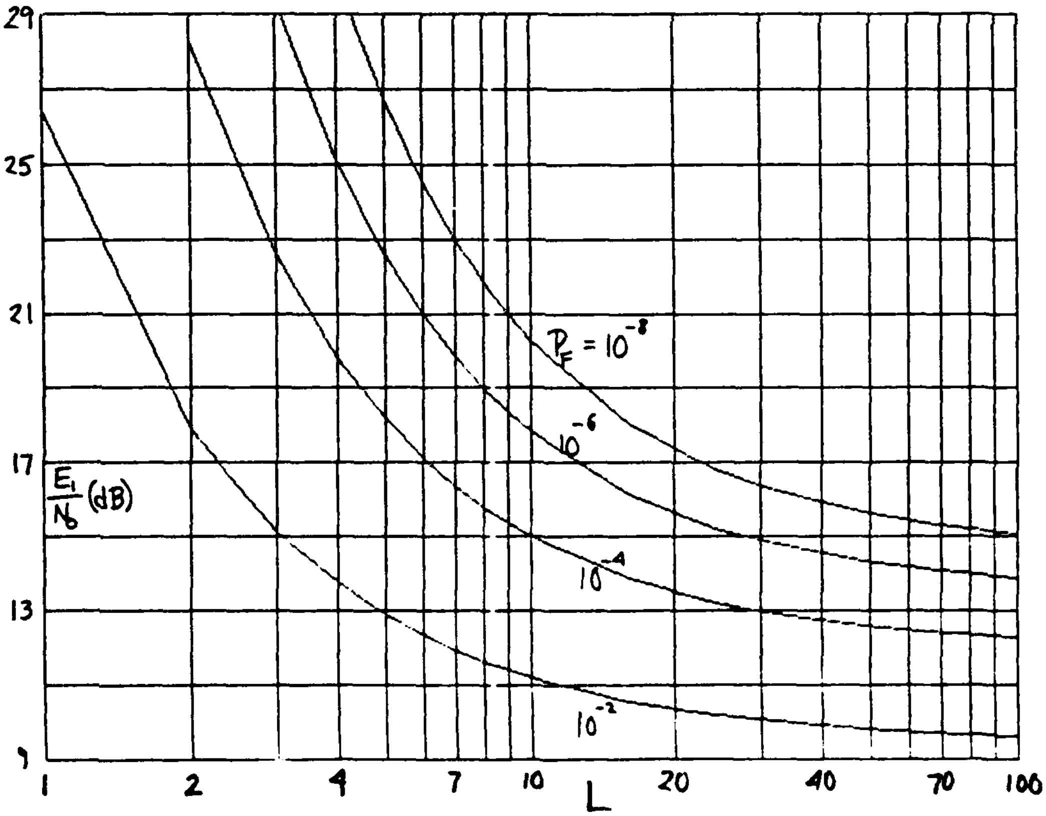

Figure 3. Required SNR for P_D = .9, K = 3, M = 2, Cov_1 = $\sqrt{.5}$

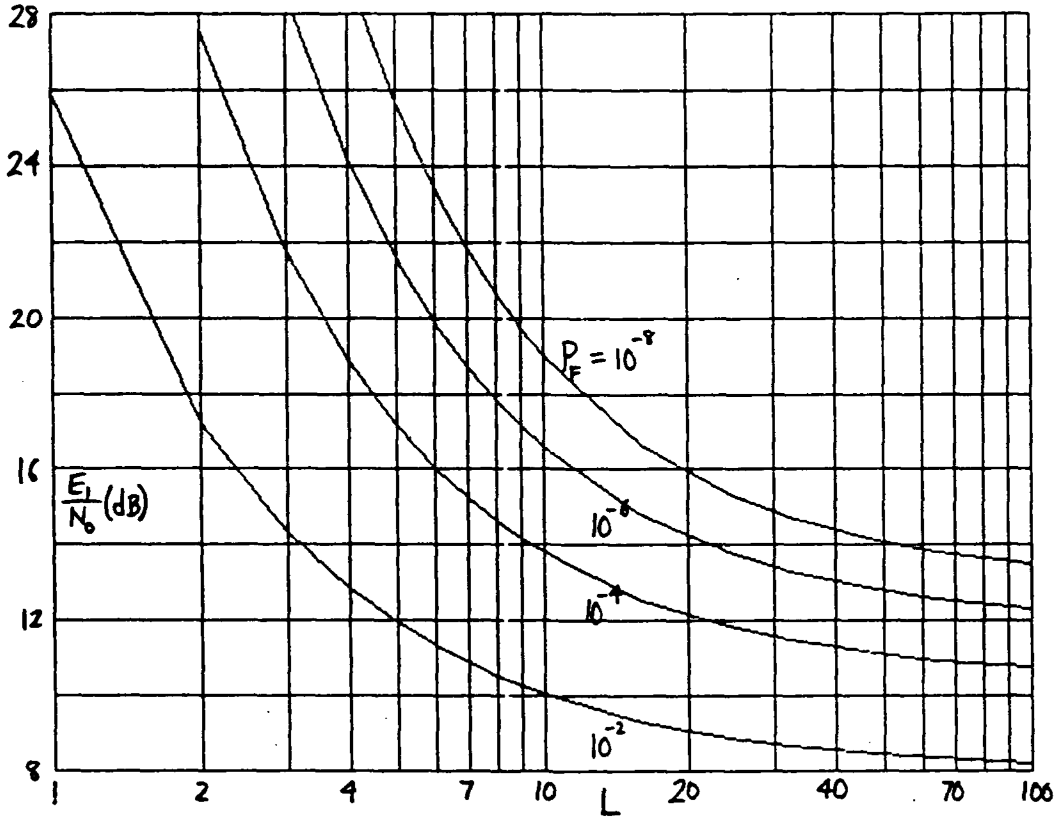

Figure 4. Required SNR for P_D = .9, K = 4, M = 2, Cov_1 = √.5

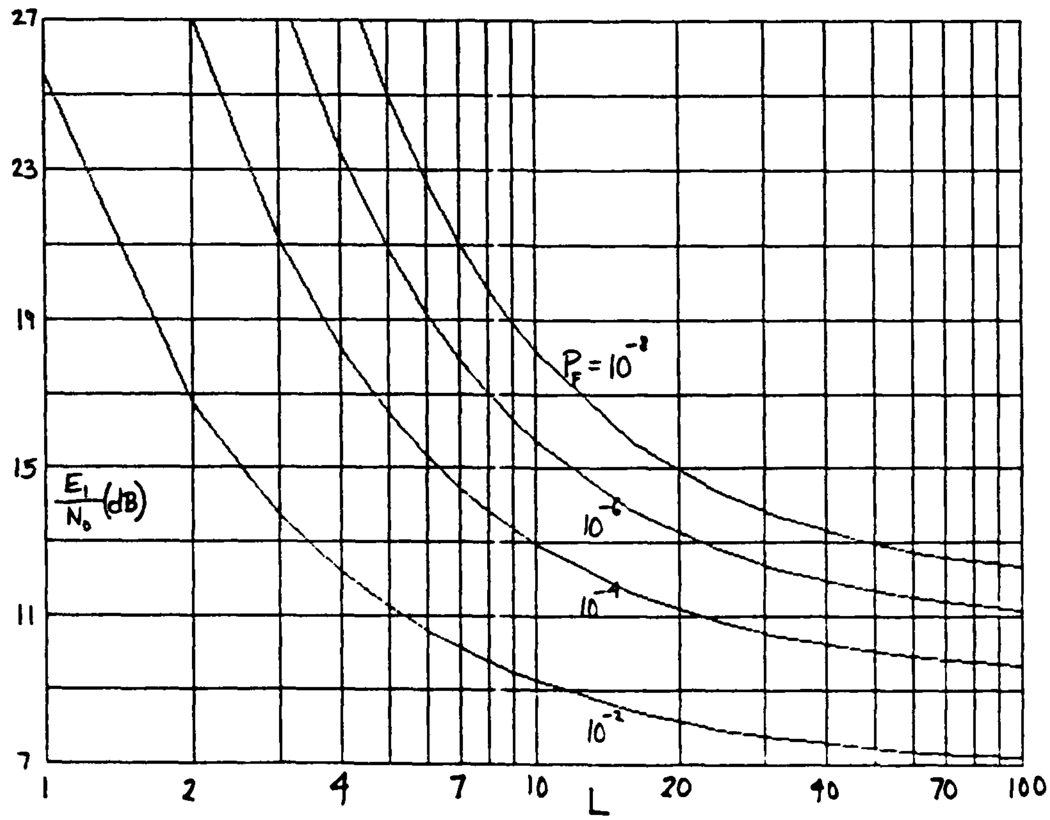

Figure 5. Required SNR for P_D = .9, K = 5, M = 2, Cov_1 = √.5

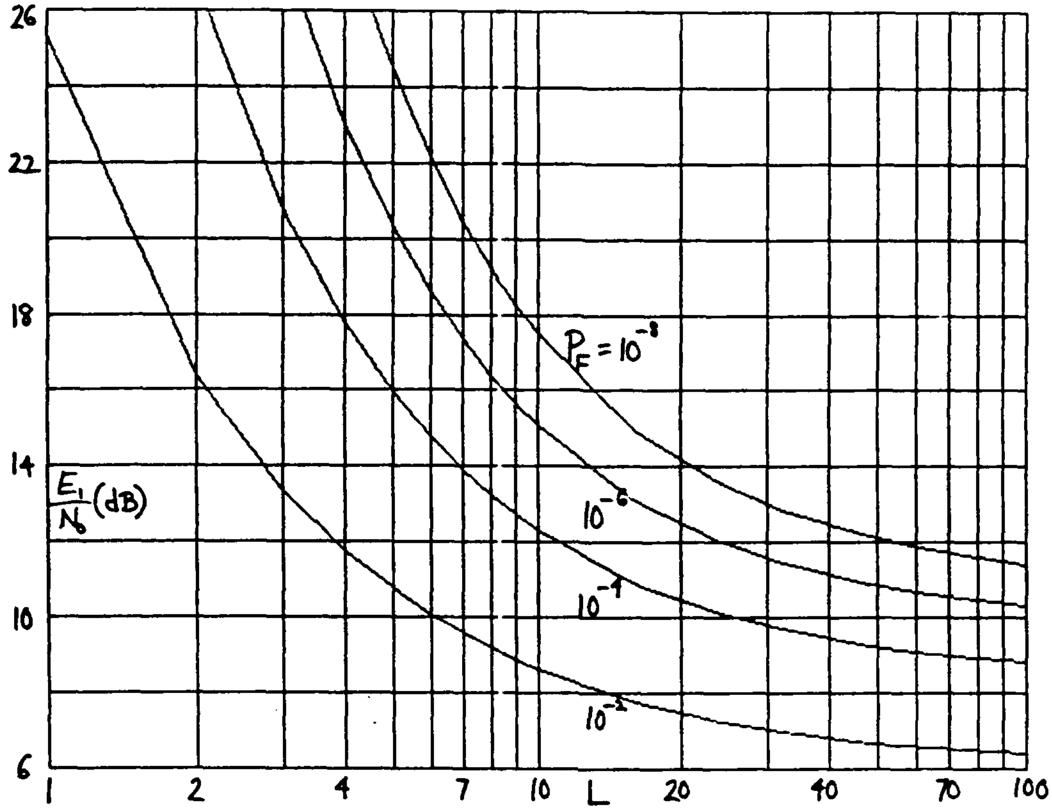

Figure 6. Required SNR for P_D = .9, K = 6, M = 2, Cov_1 = $\sqrt{.5}$

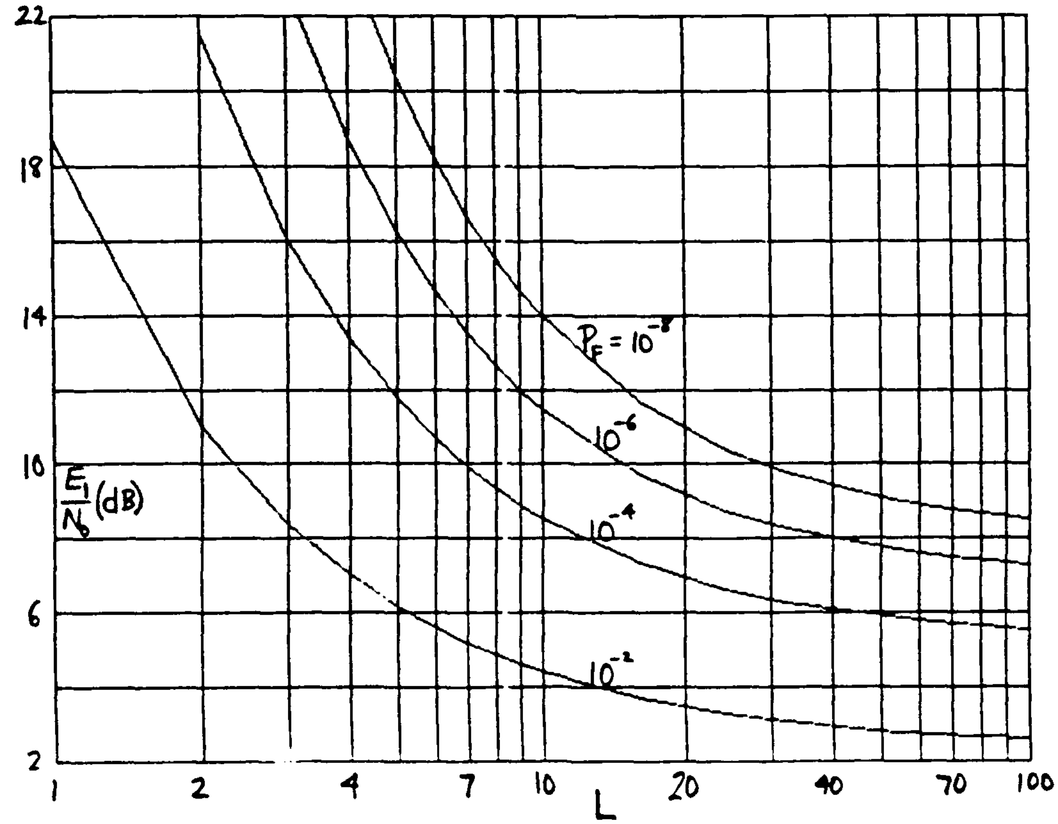

Figure 7. Required SNR for P_D = .5, K = 4, M = 2, Cov_1 = 0

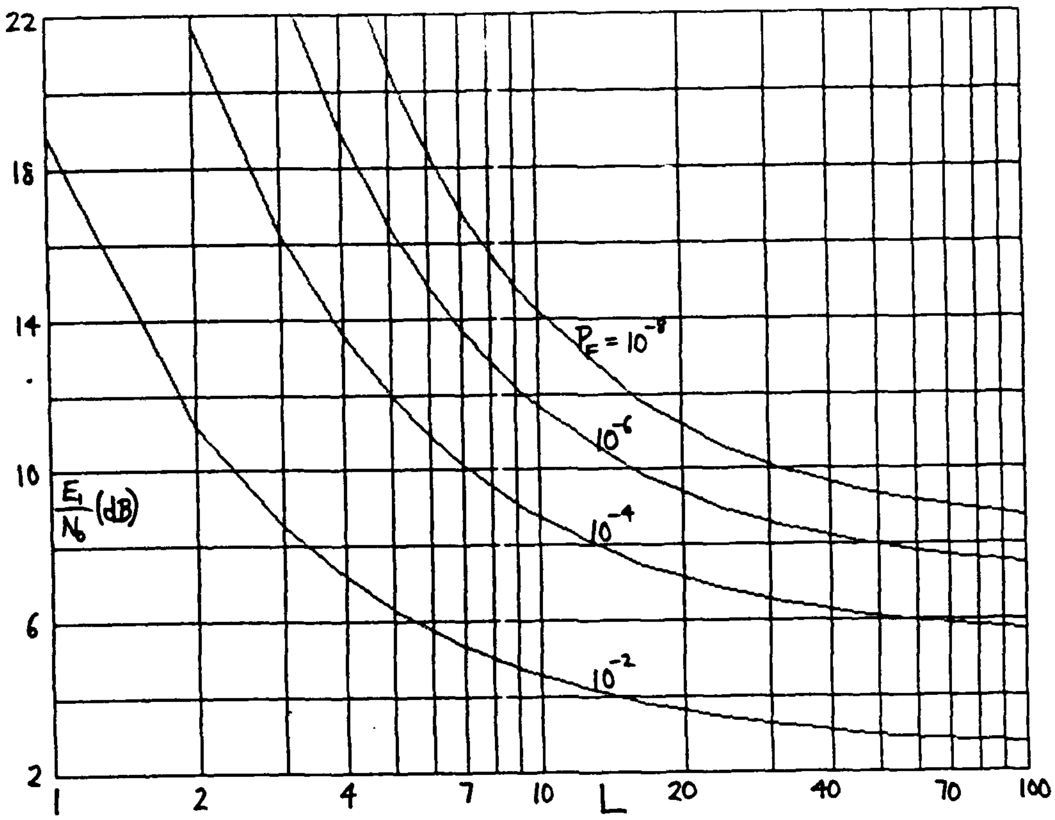

Figure 8. Required SNR for P_D = .5, K = 4, M = 2, Cov_1 = $\sqrt{.25}$

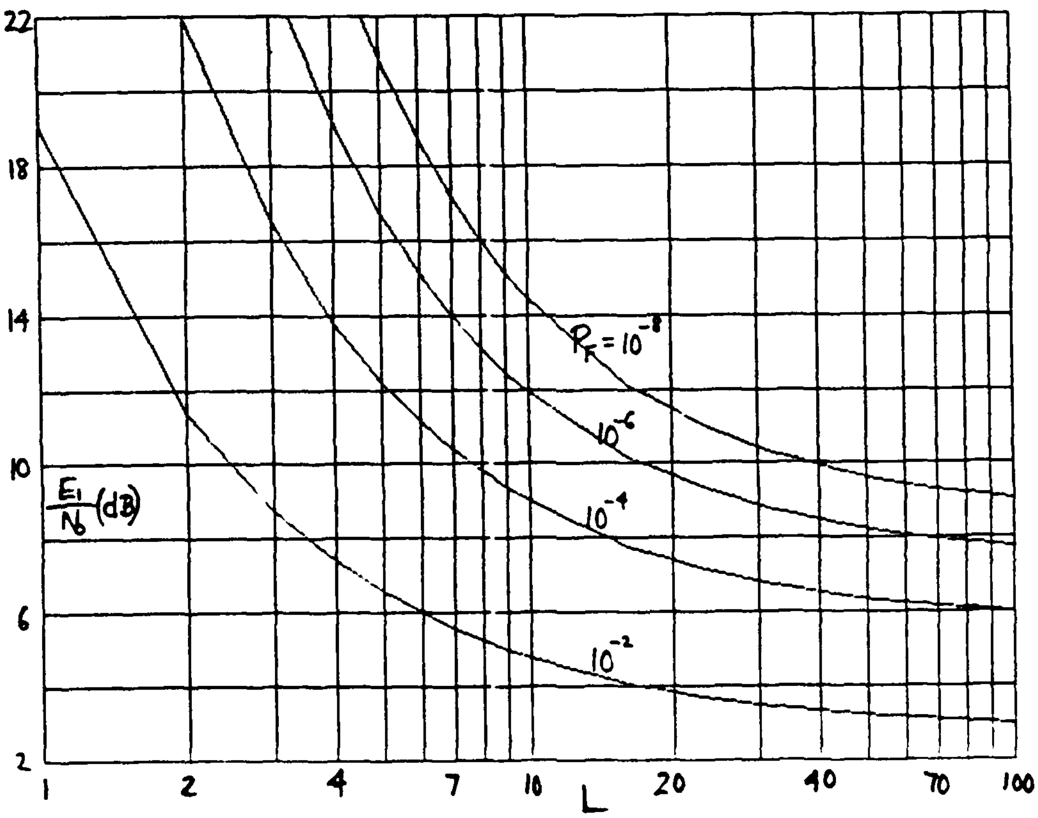

Figure 9. Required SNR for P_D = .5, K = 4, M = 2, Cov_1 = $\sqrt{.5}$

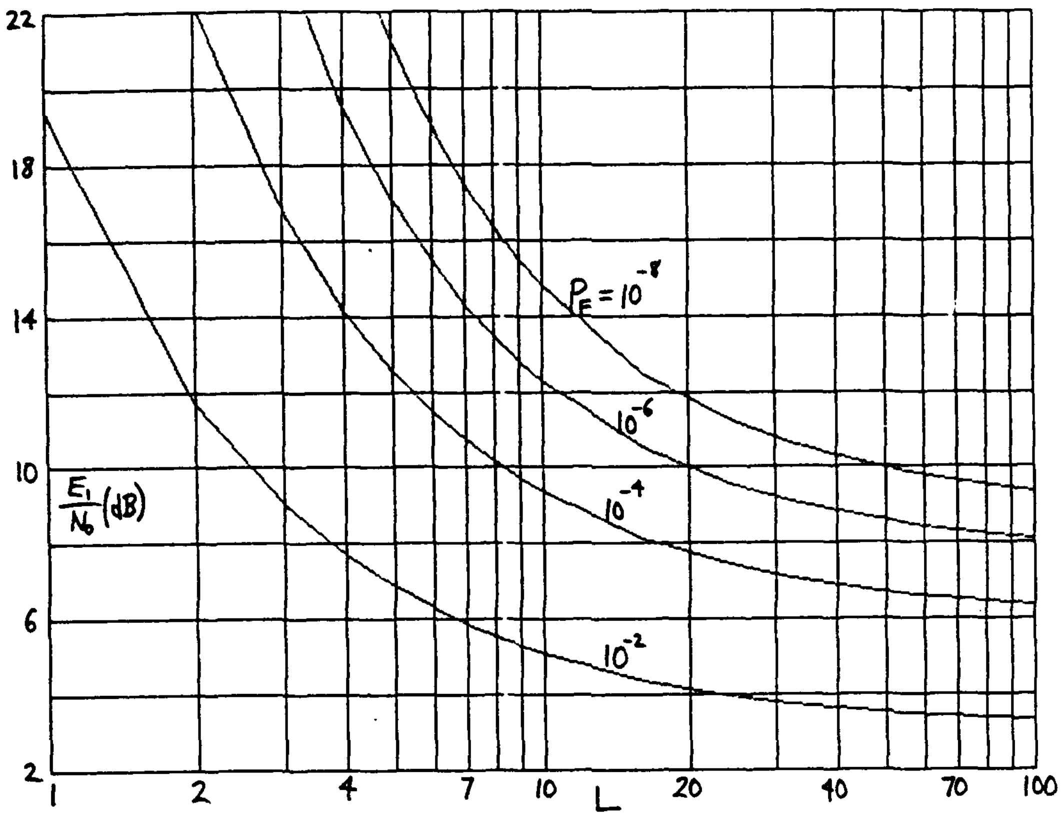

Figure 10. Required SNR for P_D = .5, K = 4, M = 2, Cov_1 = $\sqrt{.75}$

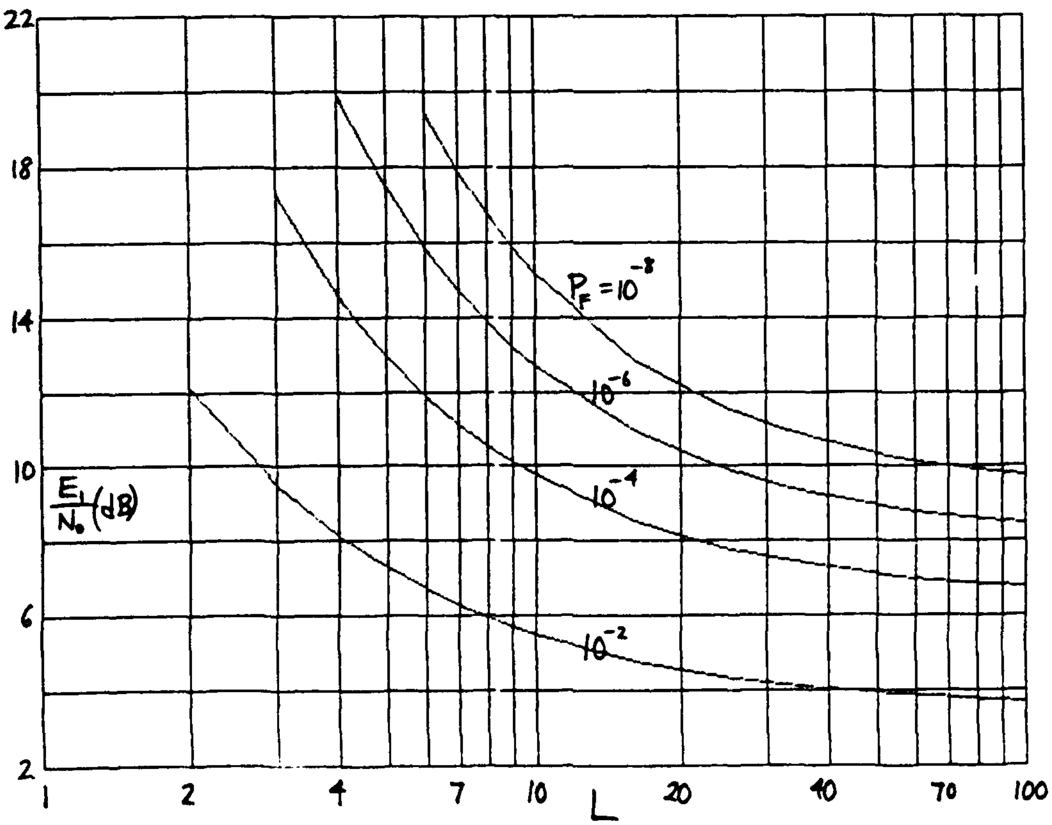

Figure 11. Required SNR for P_D = .5, K = 4, M = 2, Cov_1 = 1

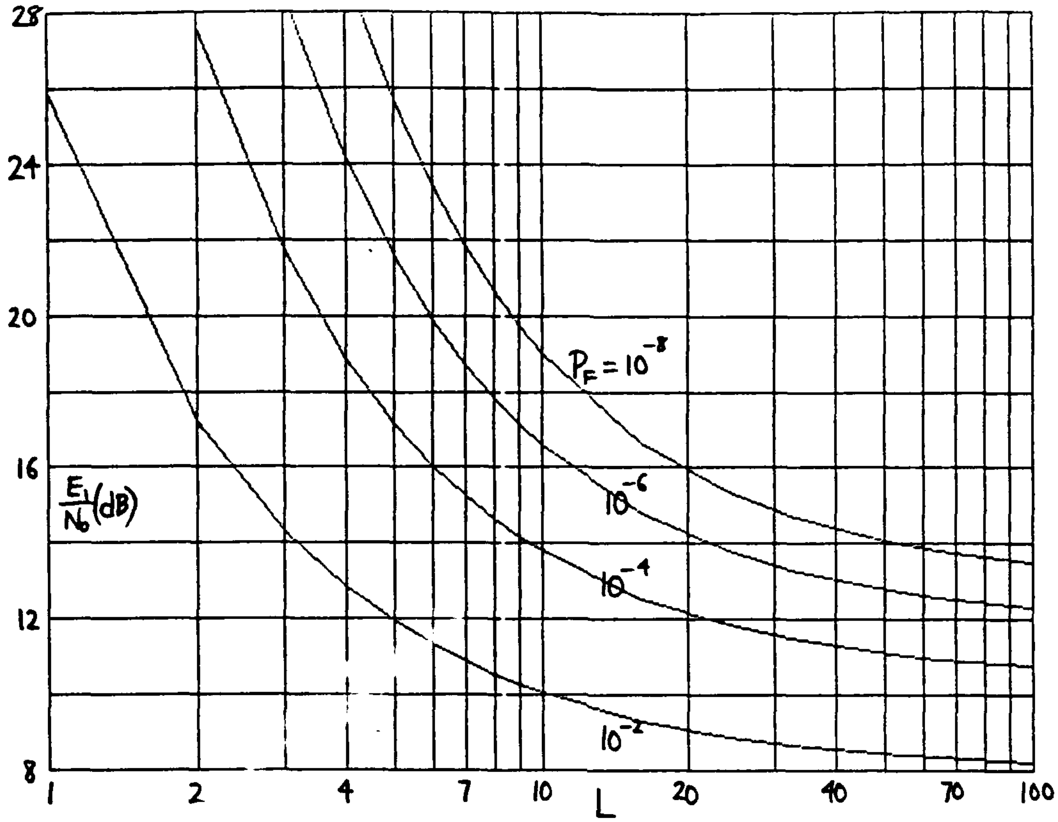

Figure 12. Required SNR for $P_D = .9$, $K = 4$, $M = 2$, $Cov_1 = \sqrt{.5}$

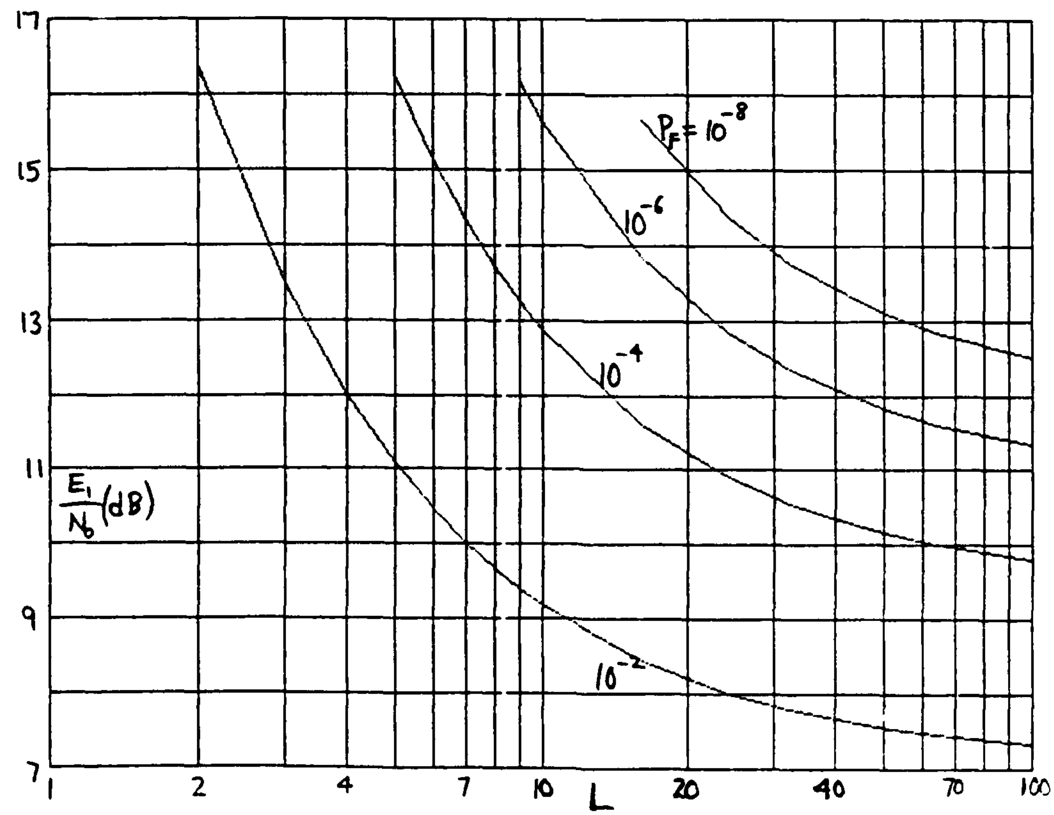

Figure 13. Required SNR for $P_D = .9$, $K = 4$, $M = 3$, $Cov_1 = \sqrt{.5}$

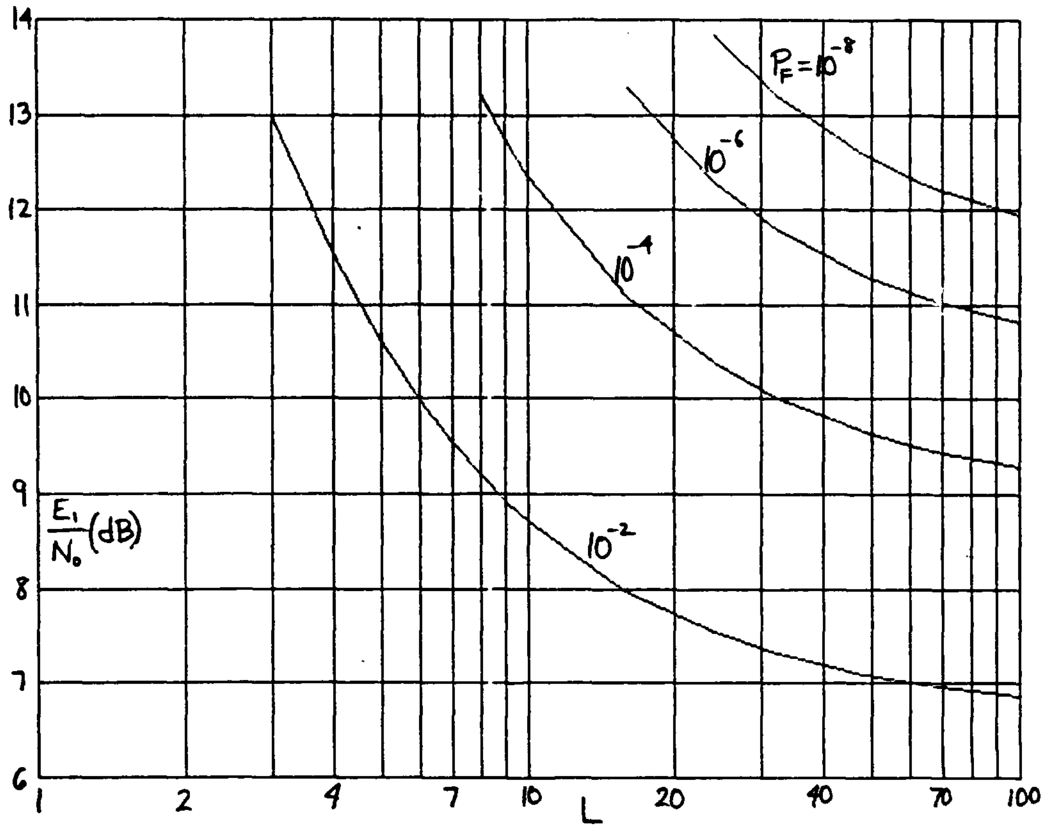

Figure 14. Required SNR for P_D = .9, K = 4, M = 4, $Cov_1 = \sqrt{.5}$

SUMMARY

The false alarm and detection probabilities, for a normalizer operating in a fading medium, have been derived exactly in the form of series expansions which are capable of efficient recursive numerical evaluation. These results allow for multiple signal pulses with an arbitrary covariance matrix between the amplitude fadings applied to each signal pulse. Lack of knowledge of the additive background noise level requires the use of a group of noise-only bins, from which an estimate of the noise level is made and used to establish a threshold for the normalizer, in order to realize a specified false alarm probability. The degradation associated with this noisy threshold has been evaluated quantitatively; a program for its evaluation in a specific environment is furnished in appendix D. Some examples of its use have been presented in the figures herein.

The expansion of the characteristic function in a series, as developed here in appendix A, will be extended in [6] to include a more general form of transformation than (11). This should allow for series with more rapid convergence properties than those derived here in (34) - (36), for example. In particular, extensions of the Hermite series and the generalized Laguerre series used in [5] will be developed.

APPENDIX A. SERIES EXPANSION OF CHARACTERISTIC FUNCTION (7)

The transformation of variable in (11) is given by

$$z = \frac{1}{1 - i\xi\beta}, \quad -i\xi = \frac{1 - z}{\beta z}, \tag{A-1}$$

which yields

$$1 - i2\xi = \frac{1 - z(1 - \tfrac{1}{2}\beta)}{\tfrac{1}{2}\beta z}, \quad 1 - i2\xi Q = \frac{1 - z(1 - \tfrac{1}{2}\beta/Q)}{\tfrac{1}{2}\beta z/Q}. \tag{A-2}$$

When these expressions are substituted into characteristic function (7) and simplified, there follows

$$f_\gamma(\xi) = F\, z^K \frac{[1 - z(1 - \tfrac{1}{2}\beta)]^{K(\frac{1}{2}M-1)}}{\left(\prod_{m=1}^{M}\prod_{k=1}^{K}\left[1 - z\left(1 - \tfrac{1}{2}\beta/Q_{mk}\right)\right]\right)^{\frac{1}{2}}} \times$$

$$\times \exp\left(-(1 - z)\sum_{m=1}^{M}\sum_{k=1}^{K}\frac{\tfrac{1}{2}e_{mk}/Q_{mk}}{1 - z\left(1 - \tfrac{1}{2}\beta/Q_{mk}\right)}\right), \tag{A-3}$$

where F is given by (13).

Except for the factor $F\, z^K$, we now take the natural logarithm of the right-hand side of (A-3) and expand in a power series in z. After considerable simplification and grouping of similar terms together, we find that this logarithm is given by

$$\sum_{p=0}^{\infty} a_p\, z^p, \tag{A-4}$$

where the coefficients $\{a_p\}$ are given in (13) and (14). Recalling the factor $F\, z^K$ in (A-3), we obtain the characteristic function of the fixed-threshold processor output γ in the form

$$f_\gamma(\xi) = F\ z^K \exp\left(\sum_{p=0}^{\infty} a_p\ z^p\right) . \tag{A-5}$$

We now employ [5; page 93] to obtain the desired series in powers of z according to

$$f_\gamma(\xi) = F\ z^K \sum_{p=0}^{\infty} f_p\ z^p , \tag{A-6}$$

where z is given by (A-1), and

$$f_o = \exp(a_o) , \quad f_p = \frac{1}{p} \sum_{n=1}^{p} n\ a_n\ f_{p-n} \quad \text{for } p \ge 1 . \tag{A-7}$$

Expression (A-6) is the desired result. Numerous recursions can be developed from these equations that enable rapid efficient numerical evaluation of characteristic function f_γ. More will be said on this topic in the program for evaluation of the detection probability for the normalizer output; see appendix D.

APPENDIX B. INCOMPLETE GAMMA RELATIONS

Define the function

$$\underline{C}(\nu,x) = \int_0^x dy\, \frac{y^{\nu-1}\exp(-y)}{\Gamma(\nu)} \quad \text{for } x \ge 0 \;, \quad \nu > 0 \;. \tag{B-1}$$

(This is $P(\nu,x)$ in [3; 6.5.1].) Then, from [3; 6.5.2 and 6.5.3],

$$\underline{C}(\nu,x) = \frac{\gamma(\nu,x)}{\Gamma(\nu)} = 1 - \frac{\Gamma(\nu,x)}{\Gamma(\nu)} \;, \tag{B-2}$$

in terms of incomplete gamma functions. The function $\underline{C}(\nu,x)$ is a cumulative distribution function; that is,

$$\underline{C}(\nu,\infty) = 1 \;. \tag{B-3}$$

It is also useful to notice that

$$\underline{C}(\infty,x) = 0 \;, \tag{B-4}$$

since the unit area of the integrand of (B-1) moves progressively farther out in the positive direction as ν increases. This can be seen from the fact that the density (integrand) in (B-1) has mean ν and standard deviation $\sqrt{\nu}$.

Also, define function

$$p(\nu,x) = \frac{x^{\nu-1}\exp(-x)}{\Gamma(\nu)} \quad \text{for } x \ge 0 \;, \quad \nu > 0 \;. \tag{B-5}$$

Then function (B-1) satisfies recurrence

$$\underline{C}(\nu,x) = \int_0^x dy\ p(\nu,y) = \underline{C}(\nu-1,x) - p(\nu,x) \quad \text{for } \nu > 1\ , \tag{B-6}$$

while

$$p(\nu,x) = p(\nu-1,x)\ \frac{x}{\nu-1} \quad \text{for } \nu > 1\ . \tag{B-7}$$

Convenient starting values for (B-6) and (B-7) are

$$\underline{C}(\tfrac{1}{2},x) = 2\ \Phi\left((2x)^{\frac{1}{2}}\right) - 1\ , \qquad \underline{C}(1,x) = 1 - \exp(-x)\ , \tag{B-8}$$

$$p(\tfrac{1}{2},x) = (\pi x)^{-\frac{1}{2}} \exp(-x)\ , \qquad p(1,x) = \exp(-x)\ . \tag{B-9}$$

As an example, with $\nu = n$,

$$p(n,x) = p(n-1,x)\ \frac{x}{n-1} \quad \text{for } n = 2,3,4,\ldots \tag{B-10}$$

$$\underline{C}(n,x) = \underline{C}(n-1,x) - p(n,x) \quad \text{for } n = 2,3,4,\ldots \tag{B-11}$$

along with the starting values for $p(1,x)$ and $\underline{C}(1,x)$ given in (B-9) and (B-8), respectively.

As a second example, with $\nu = n + \frac{1}{2}$,

$$p(n+\tfrac{1}{2},x) = p(n-\tfrac{1}{2},x)\ \frac{x}{n-\frac{1}{2}} \quad \text{for } n = 1,2,3,\ldots \tag{B-12}$$

$$\underline{C}(n+\tfrac{1}{2},x) = \underline{C}(n-\tfrac{1}{2},x) - p(n+\tfrac{1}{2},x) \quad \text{for } n = 1,2,3,\ldots \tag{B-13}$$

along with the starting values for $p(\frac{1}{2},x)$ and $\underline{C}(\frac{1}{2},x)$ given in (B-9) and (B-8), respectively.

We also define exceedance distribution function

$$\underline{E}(\nu,x) = 1 - \underline{C}(\nu,x) = \int_x^\infty dy\, \frac{y^{\nu-1}\exp(-y)}{\Gamma(\nu)} =$$

$$= \Gamma(\nu,x)/\Gamma(\nu) \quad \text{for } x \ge 0 \ , \quad \nu > 0 \ . \tag{B-14}$$

Then

$$\underline{E}(\nu,x) = \underline{E}(\nu-1,x) + p(\nu,x) \quad \text{for } \nu > 1 \ , \tag{B-15}$$

with recursion (B-7) for $p(\nu,x)$, and starting values

$$\underline{E}(\tfrac{1}{2},x) = 2\,\Phi\left(-(2x)^{\frac{1}{2}}\right) \ , \qquad \underline{E}(1,x) = \exp(-x) \ . \tag{B-16}$$

We also have closed form

$$\underline{E}(n,x) = \exp(-x)\sum_{k=0}^{n-1} x^k/k! \quad \text{for } n \ge 1 \ . \tag{B-17}$$

APPENDIX C. MOMENTS OF NORMALIZER OUTPUT FOR SIGNAL ABSENT

For ease in determining the value of threshold v to use in (49), in order to realize specified false alarm probability P_F, it is helpful to know the moments of the normalizer output γ/γ_o when signal is absent. Here, we will determine the n-th moment of ratio γ/γ_o:

$$\mu_n \equiv \overline{(\gamma/\gamma_o)^n} = \overline{\gamma^n}\ \overline{\gamma_o^{-n}} , \tag{C-1}$$

where we have used the statistical independence of γ and γ_o. By reference to (24), we find

$$\overline{\gamma_o^{-n}} = \int_0^\infty dx\ x^{-n} \frac{x^{L-1} \exp(-x/2)}{2^L\ (L-1)!} = \frac{(L-1-n)!}{2^n\ (L-1)!} \quad \text{for } L \geq n+1 . \tag{C-2}$$

In a similar manner, for signal absent (replacing L by K),

$$\overline{\gamma^n} = \int_0^\infty dx\ x^n \frac{x^{K-1} \exp(-x/2)}{2^K\ (K-1)!} = \frac{(K-1+n)!\ 2^n}{(K-1)!} . \tag{C-3}$$

The first two moments of ratio γ/γ_o are then given by

$$\mu_1 = \frac{K}{L-1} \quad \text{for } L \geq 2 , \qquad \mu_2 = \frac{(K+1)\ K}{(L-1)(L-2)} \quad \text{for } L \geq 3 . \tag{C-4}$$

The variance of the normalizer output is then given by

$$\mathrm{Var}(\gamma/\gamma_o) = \mu_2 - \mu_1^2 = \frac{(L+K-1)\ K}{(L-1)^2\ (L-2)} \quad \text{for } L \geq 3 , \tag{C-5}$$

when signal is absent.

APPENDIX D. PROGRAM FOR NORMALIZER SIGNAL-TO-NOISE RATIO

The program in this appendix computes the false alarm probability by means of (49) and the detection probability by means of (36). Since exceedance distribution function value (36) is independent of β, a guess at β is required, preferably one that leads to a rapidly convergent series. Here, this choice has been made by trial and error and by observing and noting values for β that were advantageous for "nearby" parameter values. For the purpose of ease of programming and for interpretation of the recursions in the listing below, we have utilized the following notation:

$$A(p) = p\, a_p \quad \text{for } p \geq 1 \,, \tag{D-1}$$

$$F(p) = F\, f_p \quad \text{for } p \geq 0 \,, \tag{D-2}$$

$$Be(m,k) = \frac{\beta}{2Q_{mk}} \frac{e_{mk}}{Q_{mk}} \,, \qquad Bl(m,k) = 1 - \frac{\beta}{2Q_{mk}} \,, \tag{D-3}$$

$$Cp(m,k) = Bl(m,k) + p\, Be(m,k) \,, \qquad Bp(m,k) = Bl(m,k)^{p-1} \,, \tag{D-4}$$

$$Dp = K\left(\frac{M}{2} - 1\right)\left(1 - \frac{\beta}{2}\right)^{p} \,. \tag{D-5}$$

The program below allows for arbitrary values of the deterministic signal-to-noise ratios $\{D_{km}/N_o\}$; however, it was utilized here only for the special case of all $\{D_{km}/N_o\}$ equal to zero. Also, the program is written such that $\{D_{km}/N_o\}$ are held fixed while E_1/N_o is varied to meet the specified detection probability P_D.

```
10  ! NORMALIZER, NUWC-NPT TR 10275, 21 JANUARY 1993
20    Pf=.01                  !  FALSE ALARM PROBABILITY
30    Pd=.9                   !  DETECTION PROBABILITY
40    K=2                     !  NUMBER OF SIGNAL PULSES
50    L=10                    !  NUMBER OF NOISE-ONLY BINS
60    M=2                     !  NUMBER OF FADING COMPONENTS
70    DATA 1.,1.              !  POWER RATIOS r(m)   FOR 1 <= m <= M
80    DATA 0.,0.,0.,0.        !  Dkm/No FOR 1 <= k <= K, 1 <= m <= M
90    DATA 1.,2.              !  TIMES tk (SEC)      FOR 1 <= k <= K
100   DATA 0.,0.              !  FREQUENCIES fk (HZ) FOR 1 <= k <= K
110     E1no0=20.             !  E1/No STARTING VALUE (GUESS)
120     E1no1=E1no0*.1        !  E1/No INCREMENT
130     Ef=1.E-15             !  TOLERANCE ON Pf
140     Ed=1.E-10             !  TOLERANCE ON Pd
150     P=500                 !  MAXIMUM NUMBER OF TERMS IN SUM (36)
160   OPTION BASE 1
170   COM DOUBLE K,L,M,P      !  INTEGERS (NOT DOUBLE PRECISION)
180   DOUBLE Ms,Ks,Js,L1      !  INTEGERS
190   DIM Rs(5),Dn(10,5),Ts(10),Fs(10),Psi(5),Cbar(10,10)
200   DIM U(10,10),V(10,10),Eig(10),Sq(10,5)
210   COM V,B2,Prod(5,10),Es(5,10),E(0:500),B1(5,10)
220   COM Be(5,10),Bp(5,10),Cp(5,10),F(0:500),A(500)
230   REDIM Rs(M),Dn(K,M),Ts(K),Fs(K),Psi(M),Cbar(K,K)
240   REDIM U(K,K),V(K,K),Eig(K),Sq(K,M)
250   REDIM Prod(M,K),Es(M,K),E(0:P),B1(M,K)
260   REDIM Be(M,K),Bp(M,K),Cp(M,K),F(0:P),A(P)
270   PRINT
280   PRINT "Pf =";Pf;"     K =";K;"     L =";L;"     M =";M
290   READ Rs(*),Dn(*),Ts(*),Fs(*)    !  Dn(*) WAS FILLED IN THE ORDER:
300   S=0.            !  Dn(1,1),Dn(1,2),Dn(1,3),...,Dn(k,m),...,Dn(K,M)
310   FOR Ms=1 TO M
320   S=S+Rs(Ms)
330   NEXT Ms
340   FOR Ms=1 TO M
350   Psi(Ms)=Rs(Ms)/S        !  [1; (191)]
360   NEXT Ms
370   FOR Ks=1 TO K
380   FOR Js=1 TO K
390   Cov=FNCov(Ts(Ks)-Ts(Js),Fs(Ks)-Fs(Js))
400   Cbar(Ks,Js)=Cov         !  NORMALIZED COVARIANCE MATRIX [1; (127)]
410   NEXT Js
420   NEXT Ks
430   MAT U=Cbar
440   CALL Svd(K,K,U(*),V(*),Eig(*))  !  OUTPUTS: U(*),V(*),Eig(*)
450   PRINT "EIGENVALUES:"
460   PRINT Eig(*)
470   FOR Ms=1 TO M
480   T=2.*Psi(Ms)
490   FOR Ks=1 TO K
500   Prod(Ms,Ks)=T*Eig(Ks)
510   Sq(Ks,Ms)=SQR(2.*Dn(Ks,Ms))
520   NEXT Ks
530   NEXT Ms
540   FOR Ks=1 TO K
550   FOR Ms=1 TO M
560   S=0.
```

```
570     FOR Js=1 TO K
580     S=S+V(Js,Ks)*Sq(Js,Ms)
590     NEXT Js
600     Es(Ms,Ks)=S*S
610     NEXT Ms
620     NEXT Ks
630       IF L<3 THEN 680
640       Mean=K/(L-1)                       !  (C-4)
650       Var=(L+K-1)*K/((L-1)^2*(L-2))  !  (C-5)
660       V0=Mean+SQR(Var)*3. !  THRESHOLD STARTING VALUE (GUESS)
670       GOTO 690
680       V0=200               !  THRESHOLD STARTING VALUE (GUESS)
690       V1=V0*.1             !  THRESHOLD INCREMENT
700     CALL Inversfunction1(-Pf,Ef,V0,V1,V)
710       Beta=23.             !  SCALING FACTOR (GUESS)
720     PRINT "Threshold V =";V;"        Beta =";Beta
730     B2=Beta/2.
740     X=V/(B2+V)
750     Tk=(B2/(B2+V))^L
760     L1=L-1
770     FOR Ks=1 TO K-1
780     Tk=Tk*X*(Ks+L1)/Ks
790     NEXT Ks
800     FOR Ks=K TO K+P
810     Tk=Tk*X*(Ks+L1)/Ks
820     E(Ks-K)=Tk
830     NEXT Ks
840     S=Tk
850 A:  FOR Ks=K+P+1 TO 2000
860     Tk=Tk*X*(Ks+L1)/Ks
870     S=S+Tk
880       IF Tk<=S*1.E-15 THEN 920  !  ERROR TOLERANCE
890     NEXT Ks
900     PRINT "2000 TERMS ARE INSUFFICIENT IN LINE A:"
910     PAUSE
920     E(P)=S
930     FOR Ks=P-1 TO 0 STEP -1
940     E(Ks)=S=S+E(Ks)
950     NEXT Ks
960     CALL Inversfunction2(Pd,Ed,E1no0,E1no1,E1no)
970     Db=10.*LGT(E1no)       !  SNR E1/No in dB
980     Y1=MIN(F(*))
990     Y2=MAX(F(*))
1000    PRINT "dB =";Db;"     Y1 = ";Y1;"     Y2 =";Y2
1010    GINIT
1020    PLOTTER IS "GRAPHICS"
1030    GRAPHICS ON
1040      WINDOW 0.,50.,Y1,Y2
1050    LINE TYPE 3
1060      GRID 10.,.1
1070    LINE TYPE 1
1080    FOR Js=0 TO P
1090    PLOT Js,F(Js)          !  EXPANSION COEFFICIENTS  F f(sub p)
1100    NEXT Js
1110    PENUP
1120    PAUSE
1130    END
1140    !
```

```
1150     DEF FNCov(Tau,Nu)        !  NORMALIZED COVARIANCE:
1160       Rho=.5                 !  R11(Tau,Nu)/R11(0,0); SEE (4)
1170     A=-.5*LOG(Rho)
1180       B=0.
1190     Cov=EXP(-A*ABS(Tau)-B*ABS(Nu))  !  EXPONENTIAL
1200     RETURN Cov
1210     FNEND
1220     !
1230     DEF FNPf(V)              !  PROBABILITY OF FALSE ALARM
1240     COM DOUBLE K,L           !  INTEGERS
1250     DOUBLE Ks,L1             !  INTEGERS
1260     IF V<=0. THEN RETURN 1.
1270     V1=V/(1.+V)
1280     Pf=T=EXP(-L*LOG(1.+V))
1290     L1=L-1
1300     FOR Ks=1 TO K-1
1310     T=T*V1*(L1+Ks)/Ks
1320     Pf=Pf+T
1330     NEXT Ks
1340     RETURN -Pf               !  - TO YIELD INCREASING FUNCTION OF V
1350     FNEND
1360     !
1370     DEF FNPd(E1no)           !  PROBABILITY OF DETECTION
1380     COM DOUBLE K,L,M,P       !  INTEGERS
1390     COM V,B2,Prod(*),Es(*),E(*),B1(*),Be(*),Bp(*),Cp(*),F(*),A(*)
1400     DOUBLE Ms,Ks,Ps          !  INTEGERS
1410       Tol=1.E-8              !  RELATIVE ERROR OF SUM
1420     F=1.
1430     S=0.
1440     FOR Ms=1 TO M
1450     FOR Ks=1 TO K
1460     Q=1.+E1no*Prod(Ms,Ks)
1470     Eq=Es(Ms,Ks)/Q
1480     F=F*Q
1490     S=S+Eq
1500     Bq=B2/Q
1510     B1(Ms,Ks)=1.-Bq
1520     Be(Ms,Ks)=Bq*Eq
1530     Bp(Ms,Ks)=1.
1540     Cp(Ms,Ks)=1.-Bq
1550     NEXT Ks
1560     NEXT Ms
1570     F(0)=EXP(K*LOG(B2)-.5*S)/SQR(F)
1580     Pd=1.-F(0)*E(0)
1590     B21=1.-B2
1600     Dp=K*(M/2.-1.)
1610     T1=1.
1620 B:  FOR Ps=1 TO P
1630     Dp=Dp*B21
1640     S=0.
1650     FOR Ms=1 TO M
1660     FOR Ks=1 TO K
1670     IF Ps=1 THEN 1690
1680     Bp(Ms,Ks)=T1=Bp(Ms,Ks)*B1(Ms,Ks)
1690     Cp(Ms,Ks)=T2=Cp(Ms,Ks)+Be(Ms,Ks)
1700     S=S+T1*T2
1710     NEXT Ks
1720     NEXT Ms
```

```
1730    A(Ps)=.5*S-Dp
1740    S=0.
1750    FOR Ms=1 TO Ps
1760    S=S+A(Ms)*F(Ps-Ms)
1770    NEXT Ms
1780    F(Ps)=F=S/Ps            !  APPENDIX A
1781    Do=Del
1790    Del=F*E(Ps)
1800    Pd=Pd-Del
1801    IF Ps=1 THEN 1820
1810    IF ABS(Do)+ABS(Del)<=Pd*Tol THEN 1850
1820    NEXT Ps
1830    PRINT P;"TERMS ARE INSUFFICIENT IN LINE B:"
1840    PAUSE
1850    FOR Ms=Ps+1 TO P
1860    F(Ms)=A(Ms)=0.
1870    NEXT Ms
1880    PRINT "Pd =";Pd;"       Ps =";Ps,"E1/No =";E1no
1890    RETURN Pd
1900    FNEND
1910    !
1920    SUB Inversfunction1(Desired,Error,X1,Del,X2)
1930    X2=X1+Del
1940    F1=FNPf(X1)             !  FALSE ALARM PROBABILITY
1950    F2=FNPf(X2)
1960    IF F2>=Desired THEN 2010
1970    X1=X2
1980    X2=X2+Del
1990    F1=F2
2000    GOTO 1950
2010    IF F1<=Desired THEN 2070
2020    X2=X1
2030    X1=X1-Del
2040    F2=F1
2050    F1=FNPf(X1)
2060    GOTO 2010
2070    Xa=X1
2080    Xb=X2
2090    IF (F2-Desired)<(Desired-F1) THEN 2160
2100    T=X1
2110    X1=X2
2120    X2=T
2130    T=F1
2140    F1=F2
2150    F2=T
2160    IF ABS(F2-Desired)<Error THEN 2260
2170    IF F2=F1 THEN 2260
2180    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
2190    T=MAX(T,Xa)
2200    T=MIN(T,Xb)
2210    X1=X2
2220    X2=T
2230    F1=F2
2240    F2=FNPf(X2)
2250    GOTO 2160
2260    SUBEND
2270    !
```

```
2280    SUB Inversfunction2(Desired,Error,X1,Del,X2)
2290    X2=X1+Del
2300    F1=FNPd(X1)             !  DETECTION PROBABILITY
2310    F2=FNPd(X2)
2320    IF F2>=Desired THEN 2370
2330    X1=X2
2340    X2=X2+Del
2350    F1=F2
2360    GOTO 2310
2370    IF F1<=Desired THEN 2430
2380    X2=X1
2390    X1=X1-Del
2400    F2=F1
2410    F1=FNPd(X1)
2420    GOTO 2370
2430    Xa=X1
2440    Xb=X2
2450    IF (F2-Desired)<(Desired-F1) THEN 2520
2460    T=X1
2470    X1=X2
2480    X2=T
2490    T=F1
2500    F1=F2
2510    F2=T
2520    IF ABS(F2-Desired)<Error THEN 2620
2530    IF F2=F1 THEN 2620
2540    T=(X1*(F2-Desired)-X2*(F1-Desired))/(F2-F1)
2550    T=MAX(T,Xa)
2560    T=MIN(T,Xb)
2570    X1=X2
2580    X2=T
2590    F1=F2
2600    F2=FNPd(X2)
2610    GOTO 2520
2620    SUBEND
2630    !
2640    SUB Svd(DOUBLE M,N,REAL A(*),V(*),W(*))
2650  ! THIS SUBROUTINE COMPUTES THE SINGULAR VALUE DECOMPOSITION
2660  ! OF AN ARBITRARY REAL MxN MATRIX A:  A = U W Vt, M >= N.
2670  ! U IS MxN, V IS NxN, W IS NxN:  W = DIAG(D(n)).
2680    ALLOCATE Rv1(1:N)                !  NUMERICAL RECIPES, PAGES 60-64
2690    IF M>=N THEN 2720                !  A(*) IS OVER-WRITTEN
2700    PRINT "M<N IS DISALLOWED"
2710    PAUSE
2720    DOUBLE I,J,K,L,Its,Nm,Jj         !  INTEGERS (NOT DOUBLE PRECISION)
2730    G=Scale=Anorm=0.
2740    FOR I=1 TO N
2750    L=I+1
2760    Rv1(I)=Scale*G
2770    G=S=Scale=0.
2780    IF I>M THEN 3060
2790    FOR K=I TO M
2800    Scale=Scale+ABS(A(K,I))
2810    NEXT K
2820    IF Scale=0. THEN 3060
2830    FOR K=I TO M
2840    Aa=A(K,I)=A(K,I)/Scale
2850    S=S+Aa*Aa
2860    NEXT K
```

```
2870    F=A(I,I)
2880    G=-SQR(S)
2890    IF F<0. THEN G=-G
2900    H=F*G-S
2910    A(I,I)=F-G
2920    IF I=N THEN 3030
2930    FOR J=L TO N
2940    S=0.
2950    FOR K=I TO M
2960    S=S+A(K,I)*A(K,J)
2970    NEXT K
2980    F=S/H
2990    FOR K=I TO M
3000    A(K,J)=A(K,J)+F*A(K,I)
3010    NEXT K
3020    NEXT J
3030    FOR K=I TO M
3040    A(K,I)=A(K,I)*Scale
3050    NEXT K
3060    W(I)=Scale*G
3070    G=S=Scale=0.
3080    IF (I>M) OR (I=N) THEN 3380
3090    FOR K=L TO N
3100    Scale=Scale+ABS(A(I,K))
3110    NEXT K
3120    IF Scale=0. THEN 3380
3130    FOR K=L TO N
3140    Aa=A(I,K)=A(I,K)/Scale
3150    S=S+Aa*Aa
3160    NEXT K
3170    F=A(I,L)
3180    G=-SQR(S)
3190    IF F<0. THEN G=-G
3200    H=F*G-S
3210    A(I,L)=F-G
3220    FOR K=L TO N
3230    Rv1(K)=A(I,K)/H
3240    NEXT K
3250    IF I=M THEN 3350
3260    FOR J=L TO M
3270    S=0.
3280    FOR K=L TO N
3290    S=S+A(J,K)*A(I,K)
3300    NEXT K
3310    FOR K=L TO N
3320    A(J,K)=A(J,K)+S*Rv1(K)
3330    NEXT K
3340    NEXT J
3350    FOR K=L TO N
3360    A(I,K)=A(I,K)*Scale
3370    NEXT K
3380    Anorm=MAX(Anorm,ABS(W(I))+ABS(Rv1(I)))
3390    NEXT I
3400    FOR I=N TO 1 STEP -1
3410    IF I>=N THEN 3580
3420    IF G=0. THEN 3550
```

```
3430    FOR J=L TO N
3440    V(J,I)=A(I,J)/A(I,L)/G
3450    NEXT J
3460    FOR J=L TO N
3470    S=0.
3480    FOR K=L TO N
3490    S=S+A(I,K)*V(K,J)
3500    NEXT K
3510    FOR K=L TO N
3520    V(K,J)=V(K,J)+S*V(K,I)
3530    NEXT K
3540    NEXT J
3550    FOR J=L TO N
3560    V(I,J)=V(J,I)=0.
3570    NEXT J
3580    V(I,I)=1.
3590    G=Rv1(I)
3600    L=I
3610    NEXT I
3620    FOR I=N TO 1 STEP -1
3630    L=I+1
3640    G=W(I)
3650    IF I>=N THEN 3690
3660    FOR J=L TO N
3670    A(I,J)=0.
3680    NEXT J
3690    IF G=0. THEN 3860
3700    G=1./G
3710    IF I=N THEN 3820
3720    FOR J=L TO N
3730    S=0.
3740    FOR K=L TO M
3750    S=S+A(K,I)*A(K,J)
3760    NEXT K
3770    F=S/A(I,I)*G
3780    FOR K=I TO M
3790    A(K,J)=A(K,J)+F*A(K,I)
3800    NEXT K
3810    NEXT J
3820    FOR J=I TO M
3830    A(J,I)=A(J,I)*G
3840    NEXT J
3850    GOTO 3890
3860    FOR J=I TO M
3870    A(J,I)=0.
3880    NEXT J
3890    A(I,I)=A(I,I)+1.
3900    NEXT I
3910    FOR K=N TO 1 STEP -1
3920    FOR Its=1 TO 30
3930    FOR L=K TO 1 STEP -1
3940    Nm=L-1
3950    IF (ABS(Rv1(L))+Anorm)=Anorm THEN 4170
3960    IF (ABS(W(Nm))+Anorm)=Anorm THEN 3980
3970    NEXT L
3980    C=0.
3990    S=1.
```

```
4000    FOR I=L TO K
4010    F=S*Rv1(I)
4020    Rv1(I)=C*Rv1(I)
4030    IF (ABS(F)+Anorm)=Anorm THEN 4170
4040    G=W(I)
4050    H=SQR(F*F+G*G)
4060    W(I)=H
4070    H=1./H
4080    C=G*H
4090    S=-F*H
4100    FOR J=1 TO M
4110    Y=A(J,Nm)
4120    Z=A(J,I)
4130    A(J,Nm)=Y*C+Z*S
4140    A(J,I)=-Y*S+Z*C
4150    NEXT J
4160    NEXT I
4170    Z=W(K)
4180    IF L<>K THEN 4250
4190    IF Z>=0. THEN 4240
4200    W(K)=-Z
4210    FOR J=1 TO N
4220    V(J,K)=-V(J,K)
4230    NEXT J
4240    GOTO 4780
4250    IF Its<30 THEN 4280
4260    PRINT "NO CONVERGENCE IN 30 ITERATIONS"
4270    PAUSE
4280    X=W(L)
4290    Nm=K-1
4300    Y=W(Nm)
4310    G=Rv1(Nm)
4320    H=Rv1(K)
4330    F=((Y-Z)*(Y+Z)+(G-H)*(G+H))/(2.*H*Y)
4340    G=SQR(F*F+1.)
4350    Aa=ABS(G)
4360    IF F<0. THEN Aa=-Aa
4370    F=((X-Z)*(X+Z)+H*((Y/(F+Aa))-H))/X
4380    C=S=1.
4390    FOR J=L TO Nm
4400    I=J+1
4410    G=Rv1(I)
4420    Y=W(I)
4430    H=S*G
4440    G=C*G
4450    Z=SQR(F*F+H*H)
4460    Rv1(J)=Z
4470    C=F/Z
4480    S=H/Z
4490    F=X*C+G*S
4500    G=-X*S+G*C
4510    H=Y*S
4520    Y=Y*C
4530    FOR Jj=1 TO N
4540    X=V(Jj,J)
4550    Z=V(Jj,I)
4560    V(Jj,J)=X*C+Z*S
4570    V(Jj,I)=-X*S+Z*C
4580    NEXT Jj
```

```
4590    Z=SQR(F*F+H*H)
4600    W(J)=Z
4610    IF Z=0. THEN 4650
4620    Z=1./Z
4630    C=F*Z
4640    S=H*Z
4650    F=C*G+S*Y
4660    X=-S*G+C*Y
4670    FOR Jj=1 TO M
4680    Y=A(Jj,J)
4690    Z=A(Jj,I)
4700    A(Jj,J)=Y*C+Z*S
4710    A(Jj,I)=-Y*S+Z*C
4720    NEXT Jj
4730    NEXT J
4740    Rv1(L)=0.
4750    Rv1(K)=F
4760    W(K)=X
4770    NEXT Its
4780    NEXT K
4790    SUBEND
```

The values of threshold v and required input signal-to-noise ratio E_1/N_o, that are yielded by the above program, are listed below, as a function of L, the number of noise-only bins. The values of scaling factor β are chosen to approximately minimize the number of terms, N_p, used in the p-summation in (36).

L	v	E_1/N_o (dB)	β	N_p
1	198.5	27.263	535	14
2	15.98	19.123	85	12
3	6.099	16.566	48	10
4	3.503	15.317	36	10
5	2.398	14.577	31	10
6	1.804	14.086	28	10
7	1.439	13.738	26	10
8	1.193	13.477	25	9
9	1.018	13.274	24	9
10	.8861	13.113	23	9
16	.4962	12.569	21	8
24	.3115	12.267	20	8
32	.2267	12.117	19	8
48	.1467	11.966	18	8
64	.1084	11.891	18	8
100	.0683	11.810	18	8

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