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1. REPORT DATE 1/18/93		3. REPORT TYPE AND DATES COVERED <i>Technical</i>	
4. TITLE AND SUBTITLE Electroelastic Equations for Electroded Thin Plates Subject to Large Driving Voltages		5. FUNDING NUMBERS DAA03-92-G-0123	
6. AUTHOR(S) H.F. Tiersten			
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Rensselaer Polytechnic Institute Dept. of Civil & Env. Engineering, JEC 4049 110 - 8th Street Troy, NY 12180-3590		8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709-2211		10. SPONSORING/MONITORING AGENCY REPORT NUMBER <i>ARO 30378.3-EG-URT</i>	
11. SUPPLEMENTARY NOTES The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.			
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited.		12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) Existing rotationally invariant electroelastic equations are reduced to the case of large electric fields and small strain. These latter equations are specialized to the case of thin plates with completely electroded major surfaces, and it is shown that in this case the charge equation of electrostatics is satisfied trivially to lowest order. It is further shown that for the thin stress-free polarized ferroelectric ceramic plate subject to large electric fields, the resulting equations readily account for experimental data in existence in the literature.			
14. SUBJECT TERMS		15. NUMBER OF PAGES 16	
		16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL

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ELECTROELASTIC EQUATIONS FOR ELECTRODED THIN PLATES SUBJECT TO  
LARGE DRIVING VOLTAGES

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ABSTRACT

Existing rotationally invariant electroelastic equations are reduced to the case of large electric fields and small strain. These latter equations are specialized to the case of thin plates with completely electroded major surfaces, and it is shown that in this case the charge equation of electrostatics is satisfied trivially to lowest order. It is further shown that for the thin stress-free polarized ferroelectric ceramic plate subject to large electric fields, the resulting equations readily account for experimental data in existence in the literature

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## 1. Introduction

In the field of active vibration control thin fully electroded polarized ceramic plates are used as actuators and sensors. The equations of linear piezoelectricity are employed in the analytical description of the polarized ceramic even though large electric driving fields are employed during actuation, for which the material behavior is strongly nonlinear. To account for this inherent nonlinearity it has been suggested<sup>1</sup> that the coefficients of the linear theory be made field dependent, which is inconsistent from a theoretical standpoint. This is the current state of affairs in spite of the fact that a rotationally invariant general nonlinear system of electroelastic equations, which can readily account for the observed behavior, has been in existence in the literature<sup>2,3</sup> for many years. In addition, all the work on piezoelectric actuators uses only the linear piezoelectric constitutive equations and does not discuss the extent to which the stress equations of motion or charge equation of electrostatics are or are not satisfied.

In this paper the general rotationally invariant nonlinear electroelastic equations are presented and specialized to the case of large fields and small strain. These latter three-dimensional equations are then reduced to the case of anisotropic plane stress for the thin electroelastic plate. The equations for the thin plate are further specialized to the case when the major surfaces are completely covered with electrodes, and it is shown that the charge equation of electrostatics is satisfied trivially to lowest order for this configuration. The reduced form of the constitutive equations for the case of anisotropic plane stress when the large surfaces of the thin plate are completely covered with electrodes is obtained from the more general equations. These latter equations may be used in the extensional equations of

motion to obtain a determinate system of equations. The reduced constitutive equations are specialized to the case of the polarized ceramic poled normal to the major surfaces and it is shown that the resulting equations readily account for measurements by Crawley and Anderson<sup>1</sup> of the stress-free PZT ceramic plate subject to large electric fields.

## 2. Electroelastic Equations

The stress equations of motion and charge equation of electrostatics for material points of an electroelastic solid may be written in the respective forms<sup>3</sup>

$$K_{Lj,L} = \delta_{jM} \rho^0 \ddot{u}_M, \quad (2.1)$$

$$\mathcal{Q}_{L,L} = 0, \quad (2.2)$$

where we have employed the conventions that capital and lower case indices, respectively, refer to the reference Cartesian coordinates and present Cartesian coordinates of material points. We also employ the conventions that a comma followed by a capital index denotes partial differentiation with respect to the known reference coordinates, i.e., the independent variables excluding time, a dot over a variable denotes partial differentiation with respect to time and repeated tensor indices are to be summed. The symbols  $\rho^0$ ,  $u_M$ ,  $K_{Lj}$ , and  $\mathcal{Q}_L$ , respectively, denote the reference mass density, the mechanical displacement, the total Piola-Kirchhoff stress tensor (material plus Maxwell electrostatic) and the reference electric displacement vector, which is related to the ordinary electric displacement vector  $D_i$ , by

$$\mathcal{Q}_L = J X_{L,i} D_i. \quad (2.3)$$

In (2.3) we have made use of the fact that the motion of a material point is described by the relation

$$y_i = y_i(X_L, t), \quad (2.4)$$

which is one-to-one and differentiable as often as required, where  $y_i$  and  $X_L$ , respectively, denote the present and reference position of material points,

$$J = \det y_{i,L}, \quad (2.5)$$

and a comma followed by a lower case index denotes partial differentiation with respect to the unknown present coordinates of material points. The symbol  $\delta_{jM}$  in (2.1) is a translation operator, which serves to translate a vector from the present to the reference position and vice-versa and is required for notational consistency and clarity because of the use of capital and lower case indices, respectively, to refer to the reference and present positions of material points. Clearly, the mechanical displacement vector,  $u_M$ , and present and reference positions of material points,  $y_i$  and  $X_M$ , are related by

$$y_i = \delta_{iM}(X_M + u_M). \quad (2.6)$$

The total Piola-Kirchhoff stress tensor  $K_{Lj}$  and reference electric displacement vector  $\mathcal{Q}_L$  may be written in the respective forms<sup>3</sup>

$$K_{Lj} = H_{Lj} + M_{Lj}, \quad (2.7)$$

$$\mathcal{Q}_L = \epsilon_0 \mathcal{E}_L + \mathcal{P}_L, \quad (2.8)$$

where  $\epsilon_0$  is the electric permittivity of free space, MKS units are being employed and

$$H_{Lj} = \rho^0 y_{j,M} \partial \chi / \partial E_{LM}, \quad \mathcal{P}_L = -\rho^0 \partial \chi / \partial W_L, \quad (2.9)$$

$$M_{Lj} = J X_{L,i} T_{ij}^{ES}, \quad \mathcal{E}_L = J X_{L,i} E_i. \quad (2.10)$$

The symmetric Maxwell electrostatic stress tensor  $T_{ij}^{ES}$ , material strain tensor  $E_{LM}$ , and rotationally invariant electric variable  $W_L$  are given by

$$T_{ij}^{ES} = \epsilon_0 \left[ E_i E_j - \frac{1}{2} E_k E_k \delta_{ij} \right] \quad (2.11)$$

$$E_{LM} = \frac{1}{2} (y_{i,L} y_{i,M} - \delta_{LM}), \quad W_L = y_{i,L} E_i. \quad (2.12)$$

The electric field  $E_i$  is related to the quasistatic electric scalar potential  $\phi$  by

$$E_i = -\phi_{,i}, \quad (2.13)$$

and  $\chi = \chi(E_{KL}, W_L)$  is a scalar state function, which for purposes of this work may be written in the form

$$\begin{aligned} \rho^0 \chi = & \frac{1}{2} c_{ABCD} E_{AB} E_{CD} - e_{ABC} W_A E_{BC} - \frac{1}{2} \chi_{AB} W_A W_B \\ & - \frac{1}{2} b_{ABCD} W_A W_B E_{CD} - \frac{1}{6} \chi_{ABC} W_A W_B W_C \\ & - \frac{1}{6} d_{ABCDE} W_A W_B W_C E_{DE} - \frac{1}{24} \chi_{ABCD} W_A W_B W_C W_D, \end{aligned} \quad (2.14)$$

where the material constants  $c_{2ABCD}$ ,  $e_{ABC}$ ,  $\chi_{2AB}$ ,  $b_{ABCD}$ ,  $\chi_{3ABC}$ ,  $d_{3ABCDE}$ , and  $\chi_{4ABCD}$  are called the second order elastic, piezoelectric, second-order electric permeability, electrostrictive, third-order electric permeability, third odd electroelastic, and fourth-order electric permeability, respectively. At this point we note, from (2.12)<sub>2</sub> and (2.13), and the chain rule of differentiation, that the rotationally invariant electric vector  $W_L$  may be written in the simple linear form

$$W_L = -\phi_{,L} \quad (2.15)$$

The system of equations may readily be reduced to four equations in the four dependent variables  $u_M$  and  $\phi$  by straightforward substitution. The four equations consist of (2.1) and (2.2).

The boundary conditions across material surfaces of discontinuity may be written in the form

$$N_L [K_{Lj}] = 0, \quad [u_M] = 0, \quad (2.16)$$

$$N_L [\mathcal{Q}_L] = \mathcal{Q}, \quad [\phi] = 0, \quad (2.17)$$

where we have introduced the conventional notation  $[a_i]$  for  $a_i^+ - a_i^-$ ,  $N_L$  denotes the unit normal directed from the - to the + side of the reference position of the material surface of discontinuity, and  $\mathcal{Q}$  is the electric surface charge density per unit reference area that can exist at a surface separating an insulator from a conductor. At a material surface separating two insulators  $\mathcal{Q}$  vanishes. The boundary conditions may readily be expressed in terms of the four dependent variables  $u_M$  and  $\phi$  on each side of the material surface of discontinuity by straightforward substitution<sup>4</sup>.

### 3. Small Strain Quadratic Field

In this section we reduce the general rotationally invariant electroelastic equations presented in Section 2 to the special case of infinitesimal strain and quadratic electric fields. In doing this we ignore the terms containing the  $d_{3ABCDE}$  and  $\chi_{3ABCD}$  in (2.14) in the

systematic treatment of this and the next section. However, we note that in the last section of this paper we present the extended results that would have been obtained for the simple case treated there, for which only one term of each of these material tensors occurs.

In making the reduction to infinitesimal strain powers of  $u_M$  and its gradients higher than the first are dropped in all expressions, and even the linear terms themselves are dropped in comparison with any finite quantity, such as the Kronecker delta or 1, in the usual manner. In this way we obtain

$$y_{i,K} \rightarrow \delta_{iK}, \quad X_{L,i} \rightarrow \delta_{Li} \quad J \rightarrow 1, \quad (3.1)$$

$$E_{LM} \rightarrow S_{LM} = \frac{1}{2}(u_{L,M} + u_{M,L}). \quad (3.2)$$

Then, introducing (3.1) and (3.2) in (2.9) and (2.10) and employing (2.11) and (2.15), we obtain

$$H_{Lj} = \delta_{jM} \left[ c_{LMAB} S_{AB} + e_{ALM} \phi_{,A} - \frac{1}{2} b_{ABLM} \phi_{,A} \phi_{,B} \right], \quad (3.3)$$

$$\varphi_L = e_{LBC} S_{BC} - \chi_{AL} \phi_{,A} + \frac{1}{2} \chi_{ABL} \phi_{,A} \phi_{,B}, \quad (3.4)$$

$$M_{Lj} = \varepsilon_0 \delta_{jM} \left[ \phi_{,L} \phi_{,M} - \frac{1}{2} \phi_{,K} \phi_{,K} \delta_{LM} \right], \quad (3.5)$$

$$\xi_L = -\phi_{,L}. \quad (3.6)$$

The substitution of (3.3) and (3.5) into (2.7) enables us to write

$$K_{Lj} = \delta_{jM} T_{LM}, \quad (3.7)$$

where

$$T_{LM} = c_{LMAB} S_{AB} + e_{ALM} \phi_{,A} - \frac{1}{2} \hat{b}_{ABLM} \phi_{,A} \phi_{,B}, \quad (3.8)$$

and

$$\hat{b}_{ABLM} = b_{ABLM} - 2\varepsilon_0 \left[ \delta_{LA} \delta_{MB} - \frac{1}{2} \delta_{AB} \delta_{LM} \right]. \quad (3.9)$$

The further substitution of (3.4) and (3.6) into (2.8) yields

$$\mathcal{Q}_L = e_{LBC} S_{BC} - \epsilon_{LA} \phi_{,A} + \frac{1}{2} \chi_{ABL} \phi_{,A} \phi_{,B}, \quad (3.10)$$

where the tensor of dielectric constants  $\epsilon_{LA}$  is given by

$$\epsilon_{LA} = \epsilon_0 \delta_{LA} + \chi_{LA}. \quad (3.11)$$

The substitution of (3.7) into (2.1) yields the stress equations of motion in the form

$$T_{LM,L} = \rho^0 \ddot{u}_M, \quad (3.12)$$

and we still have the charge equation of electrostatics (2.2), which we rewrite here for completeness

$$\mathcal{Q}_{L,L} = 0. \quad (3.13)$$

At this point we have the stress equations of motion (3.12) and charge equation of electrostatics (3.13), both referred to the known reference coordinates. These are the four field equations for the electroelastic continuum. The associated constitutive equations, which are linear in the infinitesimal strain and up to quadratic in the electric field are given by (3.8) and (3.10). The linear portion of these constitutive equations is identical with those of linear piezoelectricity<sup>5</sup>, and we note that the effective electrostrictive tensor  $\hat{b}_{ABLM}$  in (3.8) contains the influence of the Maxwell electrostatic stress tensor as shown by (3.9) with (3.7) and (3.5). The substitution of (3.8) and (3.10) with (3.2) into (3.12) and (3.13) yields four differential equations in the four dependent variables  $u_M$  and  $\phi$ .

#### 4. Extensional Equations for the Thin Plate with Fully Electroded Major Surfaces

A diagram of the thin plate with fully electroded major surfaces is shown in Figure 1 along with the associated coordinate system. Since we have restricted the description to infinitesimal strain in Section 3, we dispense with the no longer necessary convention that capital indices refer to the reference coordinates and lower case indices refer to the present coordinates of material points, and use lower case indices exclusively to refer to the reference coordinates of material points, to which the entire description is referred. In addition, for

convenience and because it is conventional we introduce the usual compressed notation for tensor indices for stress and strain in the constitutive equations, in which  $p, q, r, s$  take the values 1, 2, 3, 4, 5, 6 as  $ij$  or  $kl$  take the values 11, 22, 33, 23 or 32, 31 or 13, 12 or 12. Accordingly, we rewrite the constitutive equations (3.8) and (3.10) in the compressed notation thus

$$T_p = c_{pq} S_q - e_{kp} E_k - \frac{1}{2} \hat{b}_{k\ell p} E_k E_\ell, \quad (4.1)$$

$$D_\ell = e_{\ell q} S_q + \epsilon_{\ell k} E_k + \frac{1}{2} \chi_{kj\ell} E_k E_j, \quad (4.2)$$

where we have taken the liberty of writing  $D_\ell$  for  $\mathcal{D}_\ell$  and  $E_\ell$  for  $\mathcal{E}_\ell$  since the distinction between the quantities is lost when the assumption of infinitesimal strain is employed.

Since the tractions  $T_{3m}$  on the electroded major surfaces vanish and the electroelastic plate is very thin, we may make the assumptions of plane-stress and take  $T_{3m} = 0$  throughout the plate. Similarly, since the electric fields vanish in the conducting electrodes and the electroelastic plate is thin, we may take  $E_a = 0$  throughout the plate, where we have introduced the convention that  $a, b, c, d$ , take the values 1 and 2, but not 3. As in the case of anisotropic plane stress, when so many of the stress components vanish it is advantageous to obtain the strain-stress relations. To this end we operate on (4.1) with  $c_{rp}^{-1} = s_{rp}$  to obtain

$$S_q = s_{qp} T_p + d_{kq} E_k + \frac{1}{2} \beta_{jkq} E_j E_k, \quad (4.3)$$

where

$$d_{kp} = e_{kp} s_{pq}, \quad \beta_{k\ell q} = \hat{b}_{k\ell p} s_{pq}, \quad (4.4)$$

and we note that the  $d_{kq}$  are a well-known set of previously defined<sup>6</sup> piezoelectric constants, which are useful for plane-stress type calculations, whereas the plane-stress type nonlinear electroelastic constants  $\beta_{k\ell}$  are new. The substitution of (4.3) into (4.2) yields

$$D_\ell = d_{\ell p} T_p + \epsilon_{\ell k}^T E_k + \frac{1}{2} \chi_{\ell jk}^T E_j E_k, \quad (4.5)$$

where

$$\varepsilon_{\ell k}^T = \varepsilon_{\ell k} + e_{\ell q} d_{kq}, \quad \chi_{\ell jk}^T = \chi_{\ell jk} + e_{\ell q} \beta_{jkq}, \quad (4.6)$$

and we note that the  $\varepsilon_{\ell k}^T$  are a well-known set of previously defined<sup>6</sup> dielectric constants, while the nonlinear electroelastic susceptibility constants  $\chi_{\ell qk}^T$  are new.

In the present notation the charge equation of electrostatics takes the form

$$D_{\ell, \ell} = 0. \quad (4.7)$$

Since for the thin plate shown in Figure 1 the  $D_k$  vary slowly with respect to  $X_a$ , we first rewrite (4.7) in the form

$$D_{3,3} + D_{a,a} = 0, \quad (4.8)$$

and then note that  $D_{a,a}$  is of smaller order than  $D_{3,3}$ . Hence, to lowest order we have

$$D_{3,3} = 0, \quad (4.9)$$

which yields

$$D_3 = D_3(X_a, t). \quad (4.10)$$

Furthermore, since

$$E_a = 0, \quad (4.11)$$

and for a state of plane-stress to lowest order  $T_{k\ell}$  and  $S_{k\ell}$  are independent of  $X_3$ , we have shown that

$$E_3 = -V/t, \quad (4.12)$$

where  $V$  is the driving voltage across the electrodes and  $t$  is the thickness of the plate. Thus we have shown that (4.11) and (4.12) assure that (4.7) is satisfied to lowest order for the nonlinear constitutive equations (4.5) or (4.2). Consequently, with (4.11) and (4.12), Eq.(4.7) may be ignored from here on because it has been satisfied to lowest order for the nonlinear constitutive equations (4.2) or (4.5).

Since we have (4.11) and by virtue of the validity of plane stress for the thin plate, we also have

$$T_{3m} = 0, \quad (4.13)$$

the constitutive equations (4.3) and (4.5) may be written in the reduced form

$$S_q = s_{qv} T_v + d_{3q} E_3 + \frac{1}{2} \beta_{33q} E_3^2, \quad (4.14)$$

$$D_\ell = d_{\ell v} T_v + \epsilon_{\ell 3}^T E_3 + \frac{1}{2} \chi_{33\ell} E_3^2, \quad (4.15)$$

where we have introduced the convention that  $v$  takes the numbers 1,2,6 but not 3,4,5. For the symmetry of the polarized ceramic poled in the  $X_3$ -direction, the constitutive equations (4.14) and (4.15) when written out in full take the reduced form

$$\begin{aligned} S_1 &= s_{11} T_1 + s_{12} T_2 + d_{31} E_3 + \beta_{31} E_3^2, \\ S_2 &= s_{12} T_1 + s_{11} T_2 + d_{31} E_3 + \frac{1}{2} \beta_{31} E_3^2, \\ S_3 &= s_{13} T_1 + s_{13} T_2 + d_{33} E_3 + \frac{1}{2} \beta_{33} E_3^2, \\ S_4 &= 0, \quad S_5 = 0, \quad S_6 = s_{66} T_6, \quad D_1 = 0, \quad D_2 = 0, \\ D_3 &= d_{31} (T_1 + T_2) + \epsilon_{33}^T E_3 + \frac{1}{2} \chi_{33}^T E_3^2. \end{aligned} \quad (4.16)$$

At this point we note that since the  $T_{3m}$  have been taken to vanish for the thin plate,  $\ddot{u}_3$  must be negligible on account of (3.12), which is always the case for the low frequency extension of thin plates<sup>7</sup>. As a consequence, from (3.12) the stress equations of motion for extension of the thin plate take the form

$$T_{ab,a} = \rho \ddot{u}_b, \quad (4.17)$$

in our notation. Since for the low frequency extension of thin plates Eqs.(4.17) remain to be satisfied, we must solve the constitutive equations (4.16) for  $T_{ab}$  in terms of  $S_{ab}$  and  $E_3$ . The result is

$$\begin{aligned} T_1 &= c_{11}^p S_1 + c_{12}^p S_2 - e_{31}^p E_3 - \frac{1}{2} b_{31}^p E_3^2, \\ T_2 &= c_{12}^p S_1 + c_{11}^p S_2 - e_{31}^p E_3 - \frac{1}{2} b_{31}^p E_3^2, \\ T_6 &= c_{66}^p S_6, \quad D_3 = e_{31}^p S_1 + e_{31}^p S_2 + \epsilon_{33}^p E_3 + \frac{1}{2} \chi_{33}^p E_3^2, \end{aligned} \quad (4.18)$$

where the plate constants with the superscript  $p$  are given by

$$\begin{aligned}
c_{11}^p &= \frac{s_{11}}{\Delta}, & c_{12}^p &= -\frac{s_{12}}{\Delta}, & \Delta &= s_{11}^2 - s_{12}^2, \\
e_{31}^p &= \frac{d_{31}}{s_{11} + s_{12}}, & \hat{b}_{31}^p &= \frac{\beta_{31}}{s_{11} + s_{12}}, & \epsilon_{33}^p &= \epsilon_{33}^T - \frac{2d_{31}^2}{s_{11} + s_{12}}, \\
\chi_{33}^p &= \chi_{33}^T - 2 \frac{d_{31}\beta_{31}}{s_{11} + s_{12}},
\end{aligned} \tag{4.19}$$

and we note that although the constants  $c_{ab}^p$ ,  $e_{31}^p$  and  $\epsilon_{33}^p$  are well known the expressions for the nonlinear constants  $\hat{b}_{31}^p$  and  $\chi_{33}^p$  are new. The substitution of (4.18) with (3.2) into (4.17) yields two differential equations in the two dependent variables  $u_A$ . When the major surfaces of the polarized ceramic plate are completely covered with conducting electrodes, the differential equations are independent of the voltage  $V$ , which appears only in certain of the boundary or continuity conditions at the edges.

### 5. Stress-Free Plate Subject to Large Electric Fields

When the thin ferroelectric ceramic plate polarized normal to the major surfaces, which are fully electroded and subject to a large static voltage, is stress-free, the stress equations of equilibrium (4.17) with  $\bar{u}_b = 0$  are satisfied trivially and the constitutive equations for the in-plane strains, (4.16)<sub>1,2</sub>, take the relevant form

$$S_1 = S_2 = d_{31}E_3 + \frac{1}{2}\beta_{31}E_3^2. \tag{5.1}$$

In Section 3 we ignored the material tensors  $d_{3ABCDE}$  and  $\chi_{4ABCD}$  for the systematic treatment presented in Sections 3 and 4, but we noted that we would present the extended result for the relevant constitutive equations that would have been obtained for the simple case treated had they been included. For the stress-free plate the constitutive equations that would have been obtained in place of (5.1) are given by

$$S_1 = S_2 = d_{31}E_3 + \frac{1}{2}\beta_{31}E_3^2 \left[ + \frac{1}{6}\gamma_{331}E_3^3 \right], \tag{5.2}$$

in which we have enclosed the term that was not obtained in the systematic treatment in square brackets.

In Figure 2 the discrete solid boxes denote the data obtained by Crawley and Anderson<sup>1</sup> and the straight dotted line tangent to the data near the origin is the prediction of linear piezoelectricity for the known value of  $d_{31}$  in nm/V. The curved solid line that departs from the data for values of strain larger than  $10^{-4}$  was obtained using the constitutive equation given in (5.1) with the known value of  $d_{31}$  and employing a least squares fit to the measured data for values of microstrain below 100 to evaluate  $\beta_{31}$  as  $.7949 \text{ pm}^2/\text{V}^2$ . The curved solid line that goes through all the data points was obtained using the constitutive equation given in (5.2) with the known value of  $d_{31}$  and employing a least squares fit to all the measured data to evaluate  $\beta_{31}$  and  $\gamma_{331}$ , which in this fit have the values  $\beta_{31} = .8054 \text{ pm}^2/\text{V}^2$  and  $\gamma_{331} = -.7754 \text{ fm}^3/\text{V}^3$ . Thus we have shown that the general nonlinear electroelastic description, which existed long before the measurements were made, can readily be reduced to the form that describes the thin electroelastic polarized ceramic plate subject to large electric fields and accurately accounts for the measurements.

#### Acknowledgements

The author wishes to thank Dr. Y.S. Zhou for performing the calculations for the curves presented in Figure 2.

This work was supported in part by the Army Research Office under Grant No. DAAL03-92-G-0123.

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2. J.C. Baumhauer and H.F. Tiersten, "Nonlinear Electroelastic Equations for Small Fields Superposed on a Bias," *J. Acoust. Soc. Am.*, **54**, 1017 (1973).
3. H.F. Tiersten, "Nonlinear Electroelastic Equations Cubic in the Small Field Variables," *J. Acoust. Soc. Am.*, **57**, 660 (1975).
4. If one side of the interface is a conducting electrode attached to a circuit,  $\phi$  is constant in the electrode and only the three mechanical displacement components  $u_M$  occur on the electrode side of the other boundary conditions in (2.16) and (2.17).
5. IEEE Standard on Piezoelectricity, IEEE Std. 176-1978, Institute of Electrical and Electronics Engineers, New York (1978), Eqs.(17) and (18).
6. Ref.5, Eqs.(43).
7. In this standard approximation the free  $u_3$ -displacement takes place quasi-statically through the strains  $S_{3m}$  by virtue of (4.13). These equations for the extensional motion of thin plates hold as long as the frequency is well below the lowest thickness resonance of the thin plate, which is certainly the case here.

## FIGURE CAPTIONS

1. Thin Plate with Fully Electroded Major Surfaces
2. Electric Field vs Strain for Stress-Free Thin Plate of PZT G-1195 Polarized Ceramic.  
(Experimental data after Crawley and Anderson<sup>1</sup>.)

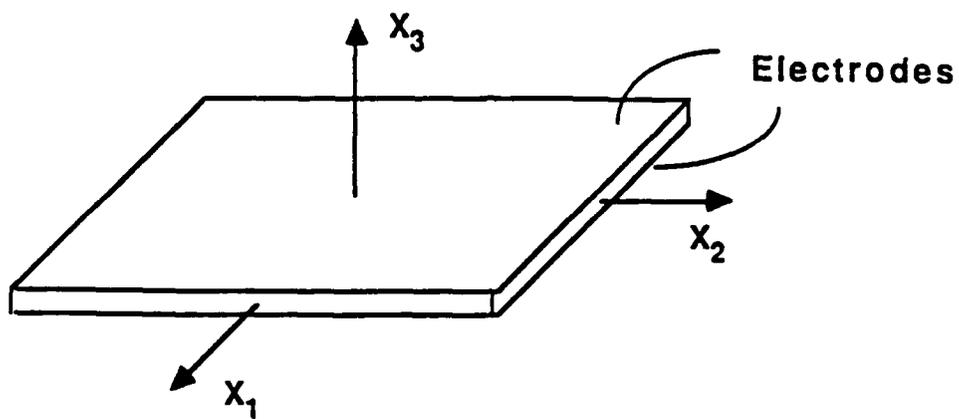


Figure 1

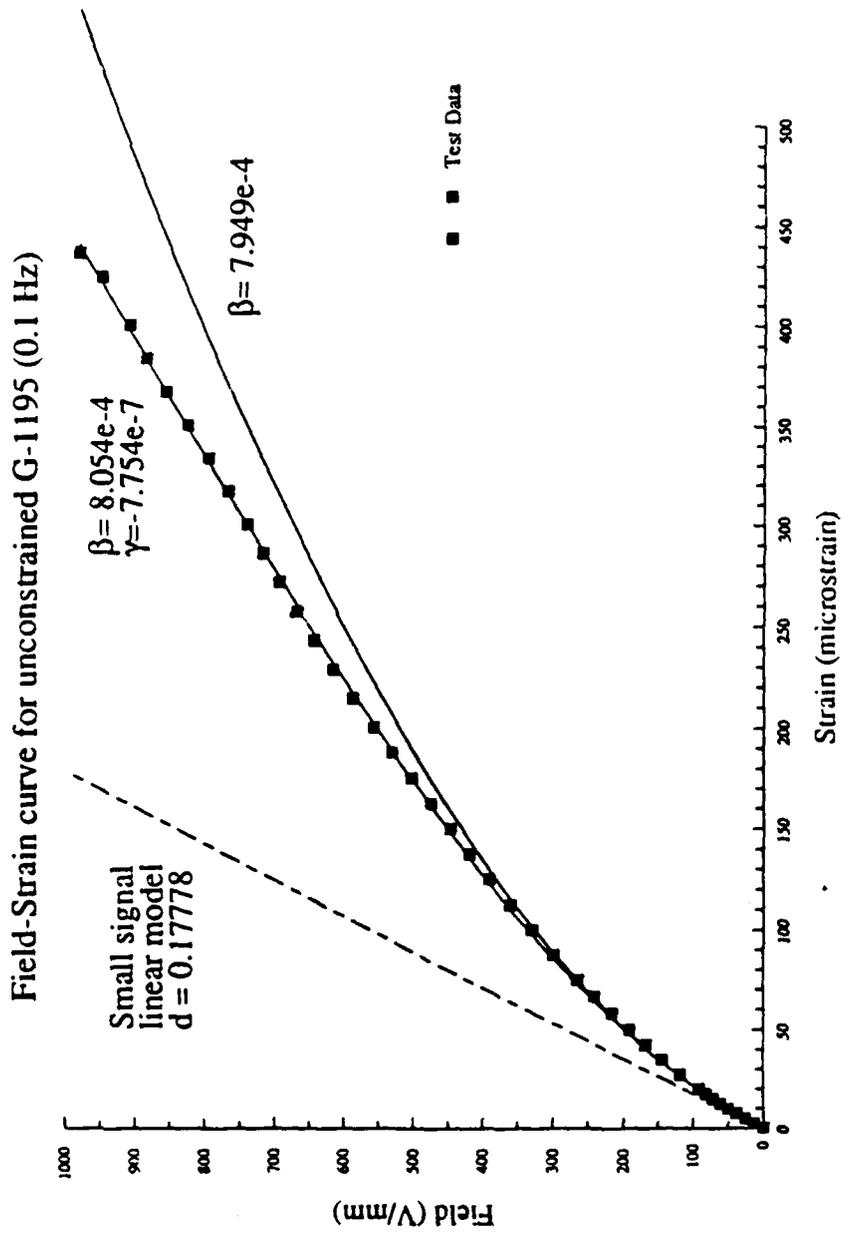


Figure 2