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We have constructed explicitly the most general class of finite dimensional filters which inlcude both Kalman-Bucy filters and Benes filters as special cases. We also proved that if the state space dimenison is less than three, then generically all finite dimenisonal filters must be those constructed by us from the Lie algebraic point of view. Without making any assumption on the drift term of the filtering system, we can write down the asymptotic solution to the famous Kolmogorov equation which is a fundamental equation in Applied Science. Moreover we have an explicit algorithm to construct the convergent solution from this formal asymptotic solution.



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# PDE, DIFFERENTIAL GEOMETRIC AND ALGEBRAIC METHODS IN NONLINEAR FILTERING

## Final Report

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**A. Statement of problem studied.**

The purpose of this paper is to report some of our recent results on nonlinear filtering which appeared only in preprint form [Ya]. We have constructed a new class of filters explicitly which includes both Kalman-Bucy filters and Benes filters as special cases.

The filtering problem considered here is based on the following signal observation model:

$$(1)\qquad \begin{cases} dx(t) = f(x(t))dt + g(x(t))dv(t) & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0 \end{cases}$$

in which $x$, $v$, $y$ and $w$, are, respectively, $\mathbf{R}^n$, $\mathbf{R}^p$, $\mathbf{R}^m$ and $\mathbf{R}^m$ valued processes, and $v$ and $w$ have components which are independent, standard Brownian process. We further assume that $n = p$, $f$, $h$ are $C^\infty$ smooth, and that $g$ is an orthogonal matrix. We will refer to $x(t)$ as the state of the system at time $t$ and $y(t)$ as the observation at time $t$.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s); 0 \le s \le t\}$. It is well-known that $\rho(t,x)$ is given by normalizing a function, $\sigma(t,x)$, which satisfies the following Duncan-Mortensen-Zakai equation:

$$(2)\qquad d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy(t), \sigma(0,x) = \sigma_0$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \dots, m$, $L_i$ is the zero degree differential operator of multiplication by $h_i$. $\sigma_0$ is the probability density of the initial point, $x_0$.

Equation (2) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to defining a new unnormalized density

$$\xi(t,x) = \exp\Big(-\sum_{i=1}^{m} h_i(x)y_i(t)\Big)\sigma(t,x).$$

It is easy to show that $\xi(t,x)$ satisfies the following time varying partial differential equation

$$(3)\qquad \begin{aligned} \frac{d\xi}{dt}(t,x) &= L_0\xi(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\big[[L_o, L_i], L_j\big]\xi(t,x) \\ \xi(0,x) &= \sigma_0 \end{aligned}$$

where $[L_0, L_i]$ denotes the Lie bracket of $L_0$ and $L_i$.

**Definition.** The estimation algebra $E$ of a filtering problem (1), is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$, or $E = \langle L_0, L_1, \ldots, L_m \rangle_{L.A.}$

If in addition there exists a potential function $\psi$ such that $f_i = \frac{\partial \psi}{\partial x_i}$ for all $1 \leq i \leq n$, then the estimation algebra is called exact.

The estimation algebra is said to be with maximal rank if $x_i + c_i$ is in $E$ for all $1 \leq i \leq n$ where $c_i$ is a constant.

The problem is to solve explicitly equation (3). In particular, we want to construct all possible finite dimensional filters via Wei-Norman approach. This includes solving the Brockett problem on classification of finite dimensional estimation algebras.

## B. Summary of the most important results.

Recently Tan, Wong and the present author [T-W-Y 1,2] have examined the properties of finite dimensional estimation algebras and the Wei-Norman approach in detail. There a class of filtering systems having the property that the drift-term $f$ of the state evolution equation is a gradient vector field was considered. In [Wo 2], the concept of $\Omega$ is introduced, which is defined as the $n \times n$ matrix whose $(i,j)$-entry is $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$. In view of Poincare lemma, $f$ is a gradient vector field if and only if $\Omega = 0$. In [Ya 1, 3], we consider a more general class of filtering systems having the property that $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ are constants for all $i$, $j$ i.e. $\Omega$ is a skew symmetric constant matrix.

**Theorem 1.** $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = c_{ij}$ *are constants for all $i$ and $j$ if and only if* $(f_1, \ldots, f_n) = (\ell_1, \ldots, \ell_n) + (\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n})$ *where $\ell_1, \ldots, \ell_n$ are polynomials of degree one and $\varphi$ is a $C^\infty$ function.*

Notice that in the statement of Theorem 1, if $\varphi \equiv 0$ on $\mathbf{R}^n$, then we are in the situation of the Kalman-Buchy filtering system; while if $(\ell_1, \ldots, \ell_n) \equiv 0$, then we have the Bene's filtering system as special case.

Define

$$D_i \equiv \frac{\partial}{\partial x_i} - f_i$$

and

$$\eta = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 + \sum_{i=1}^m h_i^2.$$

**Theorem 2.** Let E be a finite dimensional estimation algebra of (1) satisfying $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \equiv c_{ij}$ where $c_{ij}$ are constant for all $1 \leq i, j \leq n$. then $h_1, \ldots h_m$ are polynomials of degree at most one.

**Theorem 3.** Let $F(x_1, \ldots, x_n)$ be a $C^\infty$ function on $\mathbf{R}^n$. Suppose that there exists a path $C : \mathbf{R} \to \mathbf{R}^n$ and $\delta > 0$ such that $\lim_{t\to\infty} ||C(t)|| = \infty$ and $\lim_{t\to\infty} \sup_{B_\delta(C(t))} F = -\infty$, where $B_\delta(C(t)) = \{x \in \mathbf{R}^n : ||x - C(t)|| < \delta\}$. then there are no $C^\infty$ functions $f_1, f_2, \ldots, f_n$ on $\mathbf{R}^n$ satisfying the equation

$$\sum_{i=1}^n \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^n f_i^2 = F.$$

**Corollary.** Let $F(x_1, \dots, x_n)$ be a polynomial on $\mathbf{R}^n$. Suppose that there exists a polynomial path $C : \mathbf{R} \to \mathbf{R}^n$ such that $\lim_{t\to\infty} ||C(t)|| = \infty$ and $\lim_{t\to\infty} F \circ C(t) = -\infty$. Then there is no $C^\infty$ functions $f_1, f_2, \dots, f_n$ on $\mathbf{R}^n$ satisfying the equation

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F.$$

**Definition.** Let $\mathbf{E}$ be an estimation algebra of (1) satisfying $\frac{\partial f_i}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = c_{ij}$ where $c_{ij}$ are constants for all $1 \le i, j \le n$. If $\mathbf{E}$ is finite dimensional, then the matrix

$$H = [\nabla h_1, \nabla h_2, \dots, \nabla h_m]$$

is a constant matrix in view of Theorem 2. $H$ is called the observation matrix of (1).

The following result provides a simple characterization of when the dimension of an estimation algebra is finite dimensional.

**Theorem 4.** Let $\mathbf{E}$ be an estimation algebra of (1) satisfying $\frac{\partial f_i}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = c_{ij}$ where $c_{ij}$ are constants for all $1 \le i, j \le n$.

(i) if $\eta$ is a polynomial of degree at most two, then $\mathbf{E}$ is finite dimensional and has a basis consisting of $E_0 = L_0$, differential operators $E_1, \dots, E_p$ (for some $p$) of the form

$$\sum_{j=1}^{n} \alpha_{ij} D_j + \beta_i$$

where $\alpha_{ij}$'s are constants and $\beta_i$'s are affine in $x$, and zero degree differential operators $E_{p+1}, \dots, E_q, 1$ (form some $q > p$) where $E_i$'s are affine in $x$ for $p+1 \le i \le q$. Moreover the quadratic part of $\eta - \sum_{i=1}^{m} h_i^2$ is positive semi-definite.

(ii) Conversely if $\mathbf{E}$ is finite dimensional, then $h_1, \dots, h_m$ are affine in $x$ i.e. the observation matrix is a constant matrix. Furthermore if the observation matrix has rank $n$ (in particular $m \ge n$), then $\eta$ is a polynomial of degree at most two.

**Theorem 5.** Let $\mathbf{E}$ be an estimation algebra of (1) satisfying $\frac{\partial f_i}{\partial x_j} - \frac{\partial f_i}{\partial x_j} = c_{ij}$ where $c_{ij}$ are constants for all $1 \le i, j \le n$. Suppose $m \ge n$ and the observation matrix is a constant matrix with full rank. If $\mathbf{E}$ is finite dimensional, then it is of dimension $2n+2$ with basis given by $1, x_1, \dots, x_n, D_1, \dots, D_n$ and $L_0$.

Finally the following theorem describes the finite dimensional filters explicitly.

**Theorem 6.** Let $\mathbf{E}$ be an estimation algebra of (1) satisfying $\frac{\partial f_i}{\partial x_i} - \frac{\partial f_i}{\partial x_j} = c_{ij}$ where $c_{ij}$ are constants for all $1 \le i, j \le n$. Suppose $\mathbf{E}$ is finite dimensional, then $\eta = \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + d$ where $a_{ij}$, $b_i$ and $d$ are constants for all $1 \le i, j \le n$ and the observation matrix is a constant matrix. Suppose further that $m \ge n$ and the observation matrix has full rank. Then the robust Duncan-Mortensen-Zakai equation (3) has a solution for all $t \geq 0$ of the form:

$$\xi(t,x) = e^{T(t)} e^{r_n(t)x_n} \cdots e^{r_1(t)x_1} e^{s_n(t)D_n} \cdots e^{s_1(t)D_1} e^{tL_0} \sigma_0 \tag{4}$$

where $T(t), r_1(t), \ldots, r_n(t), s_1(t), \ldots, s_n(t)$ satisfy the following ordinary differential equation (5), (6) and (7). For $1 \leq i \leq n$

$$\frac{ds_i}{dt}(t) = r_i(t) + \sum_{i=1}^{n} s_j(t)c_{ji} + \sum_{k=1}^{n} h_{ki}y_k(t) \tag{5}$$

where $h_k(t) = \sum_{j=1}^{n} h_{kj}x_j + e_k$, for $1 \leq k \leq m$; $h_{kj}$ and $e_k$ are constants.

For $1 \leq j \leq n$

$$\frac{dr_j}{dt}(t) = \frac{1}{2}\sum_{i=1}^{n} s_i(t)a_{ij} \tag{6}$$

and

$$\begin{aligned} \frac{dT}{dt}(t) = {} & \frac{1}{2}\sum_{i=1}^{n} r_i^2(t) - \frac{1}{2}\sum_{i=1}^{n} s_i^2(t)\left(\sum_{j=1}^{n} c_{ij}^2 - \frac{1}{2}a_{ii}\right) \\ & + \sum_{1\leq i<k\leq n} s_i(t)s_k(t)\left(\sum_{j=1}^{n} c_{ij}c_{jk} + \frac{1}{2}a_{ik}\right) \\ & - \sum_{i,j=1}^{n} s_i(t)s_j(t)c_{ij} + \sum_{i=1}^{n} r_i(t) \\ & - \sum_{j=2}^{n}\sum_{i=1}^{j} s_j(t)c_{ij} + \frac{1}{2}\sum_{i=1}^{n} s_i(t)b_i \\ & + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\sum_{k=1}^{n} h_{ik}h_{ik}. \end{aligned} \tag{7}$$

The proofs of these theorems are presented in [Ya 3].

Let $n$ be the dimension of the state space. It turns out that all finite dimensional estimation algebras with nontrivial observation are automatically exact with maximal rank if $n = 1$. We are able to classify all finite dimensional estimation algebras with maximal rank if the dimension of the state space is at most two. Theorem 8 below is the most important step towards the complete solution of the famous Brockett problem. The major difference between our Theorem 7 and the corresponding theorem in Tam-Wong-Yau [T-W-Y 1, 2] is that we are able to remove the exactness assumption in their Theorem. The novelty of the problem is that there is no assumption on the drift term of the nonlinear filtering system. However, in the course of this proof of Theorem 8, we show that the drift term cannot be very arbitrary if the estimation algebra is finite dimensional. In fact we show that the drift term must be linear vector field plus gradient vector field.

**Theorem 7.** Suppose that the state space of the filtering system (1) is of dimension one. If the estimation algebra $E$ is finite dimensional, then either

(i) $E$ is a real vector space of dimension 4 with basis given by 1, $x$, $D = \frac{\partial}{\partial x} - f$ and $L_0 = \frac{1}{2}(D^2 - \eta)$ or

(ii) $E$ is a real vector space of dimension 2 with basis given by 1 and $L_0 = \frac{1}{2}(D^2 - \eta)$ or

(iii) $E$ is a real vector space of dimension 1 with basis given by $L_0 = \frac{1}{2}(D^2 - \eta)$.

**Theorem 8.** Suppose that the state space of the filtering system (1) is of dimension two. If $E$ is the finite dimensional estimation algebra $E$ with maximal rank, then $E$ is a real Lie algebra of dimension 6 with basis given by 1, $x_1$, $x_2$, $D_1$, $D_2$ and $L_0$.

The complete proof of this theorem is in [Ch-Ya].

It is quite easy for people to think that the Wei-Norman approach is a way to solve P.D.E. (3) by means of O.D.E. This is not exactly the case. As it was shown for example in Theorem 6 that the robust Duncan-Mortensen-Zakai equation (3) has a solution for all $t \geq 0$ of the form

$$\xi(t,x) = e^{T(t)} e^{r_n(t)x_n} \dots e^{r_1(t)x_1} e^{s_n(t)D_n} \dots e^{s_1(t)D_1} e^{tL_0} \sigma_0$$

where $D_i = \frac{\partial}{\partial x_i} = f_i$ and $T(t)$, $r_1(t), \dots, r_n(t)$, $s_1(t), \dots, s_n(t)$ satisfy a certain system of ordinary differential equations. However one still has to write down $e^{tL_0}\sigma_0$ explicitly. This means that one has to solve the so-called Kolmogorov equation. In what follows we shall outline the scheme on how to solve the Kolmogorov equation explicitly. The full details can be found in [Ya-Ya 1,2]. In fact, using the similar technique introduced here, one can actually solve some special class of Duncan-Mortensen-Zakai equation (2).

**Formal solution to Kolmogorov equation.**

**Lemma 9.** Equation (3) is equivalent to the following equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) = \Bigg\{ \frac{1}{2} \sum_{i=1}^{n} \Bigg[ \frac{\partial}{\partial x_i} - \Big( f_i(x) - \sum_{j=1}^{m} y_j(t) \frac{\partial h_j}{\partial x_i}(x) \Big) \Bigg]^2 \\ - \frac{1}{2} \Big( \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) + \sum_{i=1}^{m} h_i^2(x) \Big) \Bigg\} u(t,x) \\ u(0,x) = \sigma_0(x). \end{aligned} \tag{8}$$

**Theorem 10.** The equation (3) has a formal asymptotic solution on $\mathbf{R}^n$ if $h_i(x)$ are constants for all $1 \leq i \leq m$. In fact, the solution is of the following form

(9)

$$u(t,x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi})^n} t^{-n/2} \exp\Big( - \sum_{j=1}^{n} (x_j - \xi_j)^2 / 2t \Big) b(t,x,\xi) \sigma_0(\xi) d\xi_1, \dots, d\xi_n$$

where $b(t,x,\xi)=\sum_{k=0}^{\infty} a_k(x,\xi)t^k$. Here $a_k(x,\xi)$ are described explicitly as follows. Let

$$a(x,\xi)=\int_0^1 \sum_{i=1}^{n}(x_i-\xi_i)f_i(\xi+t(x-\xi))dt. \tag{10}$$

Then

$$a_0(x,\xi)=e^{a(x,\xi)}. \tag{11}$$

Suppose that $a_{k-1}(x,\xi)$ is given. Let

(12)
$$g_k(x,\xi)=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a_{k-1}}{\partial x_i^2}(x,\xi)-\sum_{i=1}^{n} f_i(x)\frac{\partial a_{k-1}}{\partial x_i}(x,\xi)-\frac{1}{2}\left(\sum_{i=1}^{m} h_i^2\right)a_{k-1}(x,\xi)$$
$$-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)a_{k-1}(x,\xi).$$

Then, for $k\geq 1$

$$a_k(x,\xi)=e^{a(x,\xi)}\int_0^1 t^{k-1}e^{-a\left(\xi+t(x-\xi),\xi\right)}g_k(\xi+t(x-\xi),\xi)dt. \tag{13}$$

**Theorem 11.** The equation (3) has a formal solution on $\mathbf{R}^n$ if $h_i(x)$ are constants for all $1\leq i\leq m$. In fact the solution is of the following form

(14)
$$u(t,x)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{(\sqrt{2\pi})}t^{-n/2}\exp\Big(-\frac{1}{2t}\sum_{j=1}^{n}(x_j-y_j)^2+$$
$$\int_0^2\sum_{i=1}^{n}(x_i-y_i)f_i(y+t(x-y))dt\Big)\cdot\big[1+\widetilde{a}_1(x,y)t$$
$$+\ldots+\widetilde{a}_k(x,y)t^k+\ldots\big]\sigma_0(y)dy_1,\ldots,dy_n$$

where $\widetilde{a}_k(x,y)=\int_0^1 t^{k-1}\widetilde{g}_k(y+t(x-y),y)dt$ and

$$\widetilde{g}_k(x,y)=\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\widetilde{a}_{k-1}}{\partial x_i^2}(x,y)+\sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x,y)-f_i(x)\right)\frac{\partial\widetilde{a}_{k-1}}{\partial x_i}(x,y)$$
$$+\left[\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 a}{\partial x_i^2}(x,y)+\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial a}{\partial x_i}(x,y)\right)^2-\sum_{i=1}^{n}f_i(x)\frac{\partial a}{\partial x_i}(x,y)\right.$$
$$\left.-\frac{1}{2}\left(\sum_{i=1}^{m}h_i^2\right)-\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)\right]\widetilde{a}_{k-1}(x,y).$$

**Theorem 12.** For $t \leq \frac{\epsilon}{N}$ so that

$$4Nt^2 + nCt < \frac{1}{8} \text{ and } nC < \frac{1}{8t}$$

are satisfied, the following series converges.

$$\widetilde{\phi}_N(t,x,y) + \sum_{m=0}^{\infty}(-1)^{m+1}\frac{1}{\sqrt{m+2}}\int_{\sum_{i=0}^{m+1}\tau_i=t}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}$$
$$\widetilde{\phi}_N(\tau_{m+1},x,x_{m+1})e_N(\tau_m,x_{m+1},x_m)e_N(\tau_{m-1},x_m,x_{m-1})\cdots e_N(\tau_0,x_1,y)$$

and it represents a kernel $\phi(t,x,y)$ which satisfies

$$\frac{\partial\phi}{\partial t}(t,x,y) = L_x\phi(t,x,y)$$
$$\lim_{t\to 0}\phi(t,x,y) = \delta_x(y).$$

In this way, for $t \leq \frac{\epsilon}{N}$, we have found an explicit kernel for the equation. When time is equal to $T$ which may be large, we can find the kernel up to time $T$ by the formula

$$\phi(T,x,y) = \int_{x_1}\int_{x_2}\cdots\int_{x_K}\phi\left(\frac{T}{K},x,x_1\right)\phi\left(\frac{T}{K},x_1,x_2\right)\cdots\phi\left(\frac{T}{K},x_K,y\right).$$

Here $K$ is the smallest integer greater than $\frac{TN}{\epsilon}$.

**Corollary 13.** The fundamental solution $\phi(t,x,y)$ in Theorem 12 is approximated by

$$\widetilde{\phi}_N(t,x,y) + \sum_{m=0}^{K}(-1)^{m+1}\phi_m(t,x,y)$$

which is readily computable. Here

$$\phi_m(t,x,y) = \int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_m}\int_{x_{m+1}}\int_{x_m}\cdots\int_{x_1}\widetilde{\phi}_N(t-t_1,x,x_{m+1})e_N(t_1-t_2,x_{m+1},x_m)$$
$$e_N(t_2-t_3,x_m,x_{m-1})\cdots e_N(t_m-t_{m+1},x_2,x_1)e_N(t_{m+1},x_1,y)dt_{m+1}\cdots dt_1.$$

The error for such an approximation is given by

$$\sum_{m=K+1}^{\infty}(-1)^{m+1}\phi_m(t,x,y)$$

which can be estimated by

$$(1+\sqrt{t}|x|)^{2N}(1+\sqrt{t}|y|)^{2N}\exp\left[(x-y)\cdot f(x) + \left(4Nt+\frac{1}{4}\right)|y|^2\right]\exp\left(-\frac{|x|^2}{4t}\right)$$
$$\sum_{m=K+1}^{\infty}2N^{6(m+2)N}(\sqrt{2\pi})^{-(m+2)N}2^{4N(m+1)}(\sqrt{4\pi})^{(m+1)n}\frac{t^{-\frac{n}{2}+m+1}}{(m+1)!}$$

which clearly tends to zero rapidly if $t$ is small and $K$ is large.

**C. List of all Publications and Technical Reports.**

[Ch-Ya] W.-L. Chiou and S.S.-T. Yau, Finite dimensional filters with nonlinear drift II: Brockett's problem on classification of finite dimensional estimation algebras (to appear, *SIAM J. Control and Optimization.*)

[D-T-W-Y] R.T. Dong, L.F. Tam, W.S. Wong and S.S.-T. Yau, Structure and Classification Theorems of Finite Dimensional Exact Estimation Algebras, *SIAM J. Control and Optimization,* vol. **29** July 1991, 866-877.

[T-W-Y 1] Luen-Fai Tam, Wing-Shing Wong and Stephen S.-T. Yau, Recent results on finite dimensional exact estimation algebras, Proceedings of the 28th Conference on Decision and Control, Tampa, Florida, Dec. 1988, 2574-2575.

[T-W-Y 2] L.-F. Tam, W.-S. Wong and S.S.-T. Yau, On a necessary and sufficient condition for finite dimensionality of estimation algebras, *SIAM J. Control and Optimization,* vol. **28**, no. 1 (1990), 173-185.

[Wo 1] W.-S. Wong, New classes of finite dimensional nonlinear filters, *Systems Control Letter,* **3** (1983), 155-164.

[Wo 2] W.-S. Wong, On a new class of finite dimensional estimation algebras, *Systems Control Letter,* **9** (1987), 79-83.

[Ya-Ch] Stephen S.-T. Yau and Wen-Lin Chiou, Recent results on classification of finite dimensional estimation algebras: dimension of state space $\leq 2$, Proceedings of the 30th Conference on Decision and Control, Brighton, England, Dec. 1991.

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| | |
|---|---|
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**Report of Inventions.**

New finite dimensional filters and algorithm for solving Kolmogorov equation.