THE SINGLE-GAP HOLLOW SPHERICAL DIPOLE IN NON-CONDUCTING MEDIA

Carl E. Baum

January 1993

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93-01343

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13. ABSTRACT (Maximum 200 words)
This note considers the response of a hollow spherical dipole in non-conducting media. This sensor is a sphere with a slot around the equator which is uniformly resistively loaded. The current through the resistive load across the slot is proportional to the time rate of change of the displacement vector for low frequencies. The response characteristics of such a device at high frequencies are calculated using expansions in spherical wave functions. These calculations include the dependence of the sensor response on both frequency and the direction of wave incidence.

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Foreword

The calculations in this note have a form similar to those in a few previous notes on cylindrical loops. This note extends these types of calculations to spherical geometries where the sensor in this case measures the displacement current density. For convenience the figures are grouped together after the summary and before the appendices. Appendix C was written by Mr. Joe P. Martinez of Dikewood and we would like to thank him and Mr. Larry Berg of AFWL for the numerical calculations and graphs.

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I. Introduction

Among the problems of electromagnetic sensor design for use in non-conducting media there are the general problems of sensor accuracy, directional sensitivity at high frequencies, and maximizing the upper frequency response for a given sensitivity. Here we are concerned with sensors for measuring some electromagnetic field component, or its derivative, with a flat frequency response over the bandwidth of interest so that in measuring pulsed fields there is no distortion of the waveform, within limits like the risetime. As an example, a multi-gap cylindrical loop has a well calculable equivalent area for measuring a component of $\mathbf{B}$, and by increasing the number of gaps the upper frequency response can be raised and the directional sensitivity at high frequencies is reduced (for waves still propagating perpendicular to the loop axis).\footnote{Capt Carl E. Baum, Sensor and Simulation Note 41, The Multi-Gap Cylindrical Loop in Non-Conducting Media, May 1967.}

In this note we consider another kind of sensor which we term a hollow spherical dipole. Since this sensor is based on a sphere we can analyze its performance using vector eigenfunction expansions in spherical coordinates. (This in itself is a good reason for considering a spherical sensor.) The analysis will follow an approach similar to that used in two previous notes concerning cylindrical loops where cylindrical vector eigenfunctions were used.\footnote{Capt Carl E. Baum, Sensor and Simulation Note 30, The Single-Gap Cylindrical Loop in Non-Conducting and Conducting Media, January 1967.}

The basic sensor geometry is shown in figure 1. It consists of a hollow sphere of radius $a$ with a gap of angular width $2\psi_0$ symmetrically cut around the equator of the sphere. The sensor is described in spherical coordinates $(r, \theta, \phi)$ as a conducting surface on $r = a$ for $0 < \theta < \theta_0$ and $\pi - \theta_0 < \theta < \pi$. The gap is described by $r = a$ and $\theta_0 < \theta < \pi - \theta_0$ where $0 < \theta_0 < \pi/2$. We have the relation

$$\psi_0 = \frac{\pi}{2} - \theta_0 \quad (1)$$

There is also the cylindrical coordinate system $(\psi, \phi, z)$ and the sensor geometry is constrained to have axial symmetry (about the $z$ axis) so that its response is independent of $\phi$. The gap is assumed to have resistive loading uniform in $\phi$ to preserve
axial symmetry. This resistive loading might in practice be many cable inputs evenly spaced around the gap; the cables would bring the total current crossing the gap with equal delays to one common point where the signal is desired. Such cable networks are not considered in this note, but they are assumed to be located in positions which do not significantly perturb the sensor geometry.

The basic mode of operation of this type of sensor uses the short circuit current across the loop gap. As will appear in the analysis, for wavelengths much larger than \( a \) the short circuit current is just \( 3\pi a^2 \) times the \( z \) component of the total current density. If the medium conductivity is zero then the total current density is just the displacement current density. (There are no source currents in these calculations.) Thus one might refer to this sensor as a \( D \) sensor or a total current density sensor, depending on the specific application.\(^3\) The sensor has an equivalent area of \( 3\pi a^2 \) which is quite accurate as long as \( \Psi_o \) is small. The actual accuracy of this number for the equivalent area is not considered in this note. The simplifying assumptions allowing the present high-frequency analysis give \( 3\pi a^2 \) as the equivalent area.

In outline this note first considers the expansion of an incident electromagnetic plane wave in terms of spherical vector eigenfunctions. Then this plane wave is imposed on the sphere with a shorted gap in order to calculate the short circuit current as a function of frequency and the angle of wave incidence. Then assuming small \( \Psi_o \) and a quasi static electric field distribution across the gap we calculate the sensor admittances associated with both the volume inside and outside the sensor. Combining the admittances due to the sensor geometry and the assumed cable loading with the short circuit current then gives the sensor response functions for an incident plane wave. From these results things like optimum cable loading can be calculated. For this sensor with the case of interest that the medium conductivity is zero the admittance due to the cable loading is larger compared to the sensor admittance (basically a capacitance) for frequencies below the upper frequency response of the sensor. The sensor response is then normalized by dividing the current output by \( 3\pi a^2 \) times \( D \), the ideal low-frequency response.

\(^3\) Capt Carl E. Baum, Sensor and Simulation Note 38, Parameters for Some Electrically-Small Electromagnetic Sensors, March 1967.
II. Electromagnetic Fields in Spherical Coordinates

Consider that we have a linear, homogeneous, isotropic medium with permittivity $\varepsilon$, permeability $\mu$, and conductivity $\sigma$. We have propagation constants

$$k = \sqrt{-i\omega \mu (\sigma + i\omega \varepsilon)} \quad (2)$$

$$\gamma = \sqrt{s\mu (\sigma + s\varepsilon)}$$

and a wave impedance

$$Z = \sqrt{\frac{s\mu}{\sigma + s\varepsilon}} = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\varepsilon}} \quad (3)$$

where $s$ is the Laplace transform variable which we take as $i\omega$ for the frequency-domain analysis in this note. The radian frequency is $\omega$ and $i$ is the unit imaginary. Our interest lies for the most part in $\sigma = 0$, as is used for the numerical calculations. However we keep $\sigma$ in the analysis for generality.

With time harmonic fields and with $e^{i\omega t}$ suppressed Maxwell's equations have the form

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B} \quad , \quad \nabla \times \mathbf{H} = \mathbf{J} + i\omega \mathbf{D} \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0 \quad , \quad \nabla \cdot \mathbf{D} = \rho$$

together with the constitutive relations and Ohm's law

$$\mathbf{B} = \mu \mathbf{H} \quad , \quad \mathbf{D} = \varepsilon \mathbf{E} \quad , \quad \mathbf{J} = \sigma \mathbf{E} \quad (5)$$

From equations 4 and 5 one obtains vector wave equations of the form

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = \mathbf{0} \quad , \quad \nabla^2 \mathbf{H} + k^2 \mathbf{H} = \mathbf{0} \quad (6)$$

Note that there are assumed to be no source currents or charges ($\rho = 0$) in the medium, but there will be charges (and currents) on the spherical sensor. Thus $\mathbf{E}$ (as well as $\mathbf{H}$) has zero divergence away from boundaries allowing the result in equations 6.

4. All units are rationalized MKSA.
In spherical coordinates the solution of the scalar wave equation can be written as a linear combination of functions of the form\(^5\)

\[
\Xi^{(\ell)}(n, m, \phi) = f^{(\ell)}_n(kr)P_m^o(\cos(\theta)) \begin{bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{bmatrix}
\]

where \(f^{(\ell)}_n(kr)\) is one of the spherical Bessel functions \(j_n(kr), y_n(kr), h_n^{(1)}(kr), h_n^{(2)}(kr)\) for \(\ell = 1, 2, 3, 4\) in that order. The third argument \(\phi\) is listed as \(e\) or \(o\) (meaning even or odd); \(e\) corresponds to using \(\cos(m\phi)\) and \(o\) to using \(\sin(m\phi)\). Unless noted to the contrary the definitions of the special functions correspond to those in a standard reference work.\(^6\) In particular the Legendre functions of the first kind \(P_n^m(\xi)\) of degree \(n\) and order \(m\) on the cut \((-1 < \xi < 1)\) in the complex \(\xi\) plane have the form (ref. 6 eqn. 8.6.6)

\[
P_n^m(\xi) = (-1)^m(1 - \xi^2)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi)
\]

where

\[
P_n(\xi) = P_n^0(\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n
\]

where we only consider \(n\) and \(m\) as non negative integers and \(\xi\) as a real argument with \(-1 < \xi < 1\). Our definition differs from that in some texts on electromagnetic theory\(^5,7,8\) in which the factor of \((-1)^m\) is not included. The form of equation 8 is however consistent with various texts dealing with the special


8. Morse and Feshbach, Methods of Theoretical Physics, McGraw Hill, 1953.
functions of mathematical physics. For as a general complex number not on the cut \((-1 \leq \xi \leq 1)\) the definition differs somewhat. The form in equation 8 agrees with more general definitions for real \(m\) not necessarily an integer. In any case we only use \(-1 \leq \xi \leq 1\) in this note.

Similar to Stratton (ref. 5, section 7.11) define two independent sets of vector wave functions which can be used to construct any divergenceless vector field satisfying the vector wave equation (as in equations 6). The first set of vector wave functions is defined by

\[
\hat{M}^{(\ell)}(n, m, \xi_o) \equiv \nabla \times \left[ r^2(\ell)(n, m, \xi_o) \hat{e}_r \right]
\]

where \(\hat{e}_r\) is the unit vector in the \(r\) direction, and similarly for other unit vectors. These vector wave functions have components as

\[
M_r^{(\ell)}(n, m, \xi_o) = 0
\]

\[
M_\theta^{(\ell)}(n, m, \xi_o) = \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} \bar{z}^{(\ell)}(n, m, \xi_o)
\]

\[
M_\phi^{(\ell)}(n, m, \xi_o) = -\frac{d}{d\theta} \bar{z}^{(\ell)}(n, m, \xi_o)
\]

where the last two components can be expanded as

\[
M_\theta^{(\ell)}(n, m, \xi_o) = f_\ell(kr) \frac{P_n^m(\cos(\theta))}{\sin(\theta)} \begin{vmatrix} -\sin(m\phi) \\ \cos(m\phi) \end{vmatrix}
\]

\[
M_\phi^{(\ell)}(n, m, \xi_o) = -f_\ell(kr) \frac{dP_n^m(\cos(\theta))}{d\theta} \begin{vmatrix} \cos(m\phi) \\ \sin(m\phi) \end{vmatrix}
\]


The second set of vector wave functions is defined by

\[ \mathbf{N}^{(\ell)}(n, m, e) = \frac{1}{k} \nabla \times \mathbf{M}^{(\ell)}(n, m, e) \]  

(13)

which has components

\[ N_r^{(\ell)}(n, m, e) = \frac{n(n + 1)}{kr} \bar{\varepsilon}^{(\ell)}(n, m, e) \]

\[ N_\theta^{(\ell)}(n, m, e) = \frac{1}{kr} \frac{\partial^2}{\partial (kr)^2} \left[ kr \bar{\varepsilon}^{(\ell)}(n, m, e) \right] \]  

(14)

\[ N_\phi^{(\ell)}(n, m, e) = \frac{1}{kr \sin(\theta)} \frac{\partial^2}{\partial (kr)^2} \left[ kr \bar{\varepsilon}^{(\ell)}(n, m, e) \right] \]

which can be expanded as

\[ N_r^{(\ell)}(n, m, e) = n(n + 1) \frac{f_n^{(\ell)}(kr)}{kr} P_n^m(\cos(\theta)) \begin{vmatrix} \cos(m\phi) \\ \sin(m\phi) \end{vmatrix} \]

\[ N_\theta^{(\ell)}(n, m, e) = \frac{\left[ kr f_n^{(\ell)}(kr) \right]'}{kr} \frac{d^m}{d\theta^m} P_n^m(\cos(\theta)) \begin{vmatrix} \cos(m\phi) \\ \sin(m\phi) \end{vmatrix} \]  

(15)

\[ N_\phi^{(\ell)}(n, m, e) = \frac{\left[ kr f_n^{(\ell)}(kr) \right]'}{kr} \frac{P_n^m(\cos(\theta))}{\sin(\theta)^m} \begin{vmatrix} -\sin(m\phi) \\ \cos(m\phi) \end{vmatrix} \]

where a prime is used to indicate differentiation with respect to the argument of the spherical Bessel function being considered.

Similar to equation 13 we also have the relation

\[ \mathbf{M}^{(\ell)}(n, m, e) = \frac{1}{k} \nabla \times \mathbf{N}^{(\ell)}(n, m, e) \] 

(16)

The \( \mathbf{N} \) and \( \mathbf{M} \) functions are mutually orthogonal on a sphere (of constant kr). Also on the unit sphere we have the orthogonality relations that two \( \mathbf{N} \) functions with different \( n \) or \( m \) are orthogonal, or if one is even and the other odd they are orthogonal. The
same results hold for the \( \hat{M} \) functions. For all indices the same we have

\[
\int_0^{2\pi} \int_0^\pi \hat{M}(n, m, \theta, \phi) \cdot \hat{M}(n, m, \theta, \phi) \sin(\theta) d\theta d\phi
\]

\[
= [1 + \delta_{m, o}] 2\pi \frac{n(n + 1)}{2n + 1} \frac{(n + m)!}{(n - m)!} \left[ f_n(kr) \right] ^2
\]

(17)

and

\[
\int_0^{2\pi} \int_0^\pi \hat{N}(n, m, \theta, \phi) \cdot \hat{N}(n, m, \theta, \phi) \sin(\theta) d\theta d\phi
\]

\[
= [1 + \delta_{m, o}] 2\pi \frac{n(n + 1)}{2n + 1} \frac{(n + m)!}{(n - m)!} \cdot \left\{ (n + 1) \left[ f_{n-1}(kr) \right] ^2 + n \left[ f_n(kr) \right] ^2 \right\}
\]

(18)

where \( \delta_{m_1, m_2} \) is the Kronecker delta function defined by

\[
\delta_{m_1, m_2} = \begin{cases} 1 & \text{for } m_1 = m_2 \\ 0 & \text{for } m_1 \neq m_2 \end{cases}
\]

(19)

With the \( \hat{N} \) and \( \hat{M} \) vector wave functions we can now expand an electric field with zero divergence as

\[
\hat{E} = E_0 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ \alpha_{n,m} \hat{M}^{(l)}(n, m, \theta, \phi) + \beta_{n,m} \hat{N}^{(l)}(n, m, \theta, \phi) \right]
\]

(20)

where \( E_0 \) is some constant with dimensions volts/meter and where \( \alpha_{n,m} \) and \( \beta_{n,m} \) are dimensionless constants. Note that we can also sum over \( l \) and over even and odd functions but typically for some particular form of wave only one \( l \) will be needed and with appropriate symmetry only one of \( e \) and \( o \) will be needed for each type of vector wave function. Comparing equations 4 and 5 with the
relations between the \( \hat{M} \) and \( \hat{N} \) functions (equations 13 and 16).

Note that we can find \( \hat{H} \) from \( \hat{E} \) by replacing

\[
\hat{M}(\ell)(n, m, e_o) + \frac{i}{2} \hat{N}(\ell)(n, m, e_o)
\]

\[
\hat{N}(\ell)(n, m, e_o) + \frac{i}{2} \hat{M}(\ell)(n, m, e_o)
\]

giving an \( \hat{H} \) to go with equation 20 as

\[
\hat{H} = i \frac{E}{Z} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \alpha_{n,m} \hat{N}(\ell)(n, m, e_o) + \beta_{n,m} \hat{M}(\ell)(n, m, e_o) \right]
\]

Similarly if we are given an expansion for \( \hat{H} \) we can find \( \hat{E} \) by substituting

\[
\hat{M}(\ell)(n, m, e_o) + -iz\hat{N}(\ell)(n, m, e_o)
\]

\[
\hat{N}(\ell)(n, m, e_o) + -iz\hat{M}(\ell)(n, m, e_o)
\]

With the \( \hat{N} \) and \( \hat{M} \) functions we can expand any electromagnetic field distribution as long as the fields have no divergence which requires \( \rho = 0 \) in equations 4. For completeness note that the wave equation for \( \hat{E} \) found from equations 4 and 5 is

\[
-\nabla \times (\nabla \times \hat{E}) + k^2 \hat{E} = 0
\]

which reduces to

\[
\nabla^2 \hat{E} - \nabla(\nabla \cdot \hat{E}) + k^2 \hat{E} = 0
\]

As discussed by Stratton another set of wave functions are needed which we define as

\[
\tilde{L}(\ell)(n, m, e_o) \equiv \frac{1}{k} \nabla \hat{E}(\ell)(n, m, e_o)
\]

which has components
\[
L_r^{(\ell)}(n, m, e_o) = f_n^{(\ell)}(kr) P_n^m(\cos(\theta)) \begin{bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{bmatrix}
\]

\[
L_\theta^{(\ell)}(n, m, e_o) = \frac{f_n^{(\ell)}(kr)}{kr} \frac{d P_n^m(\cos(\theta))}{d\theta} \begin{bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{bmatrix}
\]

(27)

\[
L_\phi^{(\ell)}(n, m, e_o) = \frac{f_n^{(\ell)}(kr)}{kr} \frac{P_n^m(\cos(\theta))}{\sin(\theta)} m \begin{bmatrix} \cos(m\phi) \\ -\sin(m\phi) \end{bmatrix}
\]

The \( \hat{L} \) and \( \hat{M} \) vector functions are related as

\[
\hat{M}^{(\ell)}(n, m, e_o) = kr \hat{L}^{(\ell)}(n, m, e_o) \times \hat{e}_r
\]

(28)

The \( \hat{L} \) and \( \hat{M} \) functions are mutually orthogonal on a sphere (of constant \( kr \)). However the \( L \) and \( N \) functions are only orthogonal if at least one of the indices (\( \ell, n, m, e_o \)) differs. The \( L \) functions are orthogonal to each other unless the indices are all the same, in which case we have

\[
\int_0^{2\pi} \int_0^\pi \hat{L}^{(\ell)}(n, m, e_o) \cdot \hat{L}^{(\ell)}(n, m, e_o) \sin(\theta)d\theta d\phi
\]

\[
= [1 + \delta_{m,0}] \frac{2\pi}{2n + 1} \frac{(n + m)!}{(n - m)!} \left[ f_n^{(\ell)}(kr) \right]^2
\]

\[
+ \frac{n(n + 1)}{(kr)^2} \left[ f_n^{(\ell)}(kr) \right]^2
\]

(29)

For another reference concerning these spherical vector wave functions see Morse and Feshback. The definitions used here are similar to Morse and Feshback except that we use \( e^{i\omega t} \) whereas Stratton and Morse and Feshback both use \( e^{-i\omega t} \) for the harmonic time dependence.

\textit{In.} Ref. 8, Part II, pp. 1864-1866.
III. Vector Plane Waves in Spherical Coordinates

Now consider vector plane waves in spherical coordinates. Such a wave has the general form

\[ \mathbf{\mathbf{F}} = F_0 \mathbf{\hat{u}} e^{-i \mathbf{k} \cdot \mathbf{r}} \]  

where \( \mathbf{\hat{u}} \) is some unit vector with fixed direction and where the propagation vector is given by

\[ \mathbf{k} = k \mathbf{\hat{e}}_1 \]  

where \( \mathbf{\hat{e}}_1 \) is the direction of propagation of the wave. Figure 2A shows a general plane wave at some position \( \mathbf{r} \) propagating in the \( \mathbf{\hat{e}}_1 \) direction with \( \mathbf{E} \), \( \mathbf{H} \), and \( \mathbf{k} \) all mutually orthogonal vectors. We assume that \( \mathbf{E} \) has a fixed polarization (and thus \( \mathbf{H} \) also has a fixed polarization) for purposes of illustration. This plane wave also has

\[ \mathbf{E} = \mathbf{H} \times \mathbf{\hat{e}}_1 \]  

so that the wave is propagating in the \( +\mathbf{\hat{e}}_1 \) direction.

\( \mathbf{\hat{e}}_1 \) is a fixed direction in space determining the direction of wave propagation. \( \mathbf{\hat{e}}_2 \) and \( \mathbf{\hat{e}}_3 \) are parallel to a plane which is in turn perpendicular to \( \mathbf{\hat{e}}_1 \). Construct two orthogonal unit vectors, \( \mathbf{\hat{e}}_2 \) and \( \mathbf{\hat{e}}_3 \), parallel to this plane as shown in figure 2B. Now since \( \mathbf{\hat{e}}_1 \) is a fixed direction in space it can be described by fixed angles \( \theta_1 \) and \( \phi_1 \) in a spherical coordinate system with respect to the cartesian unit vectors \( (\mathbf{\hat{e}}_x, \mathbf{\hat{e}}_y, \mathbf{\hat{e}}_z) \) which are also fixed directions in space. As shown in figure 2B choose \( \mathbf{\hat{e}}_2 \) to be parallel to the same plane to which both \( \mathbf{\hat{e}}_1 \) and \( \mathbf{\hat{e}}_3 \) are parallel; also make \( \mathbf{\hat{e}}_2 \) perpendicular to \( \mathbf{\hat{e}}_1 \). Finally choose \( \mathbf{\hat{e}}_3 \) perpendicular to both \( \mathbf{\hat{e}}_1 \) and \( \mathbf{\hat{e}}_2 \), thereby making it parallel to the \( x, y \) plane.

These new unit vectors are made to form a right handed system so that we have

\[ \mathbf{\hat{e}}_1 \times \mathbf{\hat{e}}_2 = \mathbf{\hat{e}}_3 \text{,} \quad \mathbf{\hat{e}}_2 \times \mathbf{\hat{e}}_3 = \mathbf{\hat{e}}_1 \text{,} \quad \mathbf{\hat{e}}_3 \times \mathbf{\hat{e}}_1 = \mathbf{\hat{e}}_2 \]  

Note that \( \mathbf{\hat{e}}_2 \) is chosen such that for \( 0 < \theta_1 < \pi/2 \) the polar angle of \( \mathbf{\hat{e}}_2 \) is \( \pi/2 - \theta_1 \). In terms of the cartesian unit vectors we have
\[ \mathbf{e}_1 = \sin(\theta_1)\cos(\phi_1)\mathbf{e}_x + \sin(\theta_1)\sin(\phi_1)\mathbf{e}_y + \cos(\theta_1)\mathbf{e}_z \]
\[ \mathbf{e}_2 = -\cos(\theta_1)\cos(\phi_1)\mathbf{e}_x - \cos(\theta_1)\sin(\phi_1)\mathbf{e}_y + \sin(\theta_1)\mathbf{e}_z \] (34)
\[ \mathbf{e}_3 = \sin(\phi_1)\mathbf{e}_x - \cos(\phi_1)\mathbf{e}_y \]

Cartesian and spherical coordinates are related as
\[ x = r \sin(\theta)\cos(\phi) \]
\[ y = r \sin(\theta)\sin(\phi) \] (35)
\[ z = r \cos(\theta) \]

The cartesian and spherical unit vectors are related as
\[ \mathbf{e}_x = \sin(\theta)\cos(\phi)\mathbf{e}_r + \cos(\theta)\cos(\phi)\mathbf{e}_\theta - \sin(\phi)\mathbf{e}_\phi \]
\[ \mathbf{e}_y = \sin(\theta)\sin(\phi)\mathbf{e}_r + \cos(\theta)\sin(\phi)\mathbf{e}_\theta + \cos(\phi)\mathbf{e}_\phi \] (36)
\[ \mathbf{e}_z = \cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta \]
or
\[ \mathbf{e}_r = \sin(\theta)\cos(\phi)\mathbf{e}_x + \sin(\theta)\sin(\phi)\mathbf{e}_y + \cos(\theta)\mathbf{e}_z \]
\[ \mathbf{e}_\theta = \cos(\theta)\cos(\phi)\mathbf{e}_x + \cos(\theta)\sin(\phi)\mathbf{e}_y - \sin(\theta)\mathbf{e}_z \] (37)
\[ \mathbf{e}_\phi = -\sin(\phi)\mathbf{e}_x + \cos(\phi)\mathbf{e}_y \]

Substitute for the cartesian unit vectors in equations 34 from equations 36 and use some trigonometric identities to give
\[ \dot{\hat{e}}_1 = [\cos(\theta_1)\cos(\theta) + \sin(\theta_1)\sin(\theta)\cos(\phi - \phi_1)]\hat{e}_r + [-\cos(\theta_1)\sin(\theta) + \sin(\theta_1)\cos(\theta)\cos(\phi - \phi_1)]\hat{e}_\theta - \sin(\theta_1)\sin(\phi - \phi_1)\hat{e}_\phi \]

\[ \dot{\hat{e}}_2 = [\sin(\theta_1)\cos(\theta) - \cos(\theta_1)\sin(\theta)\cos(\phi - \phi_1)]\hat{e}_r - [\sin(\theta_1)\sin(\theta) + \cos(\theta_1)\cos(\theta)\cos(\phi - \phi_1)]\hat{e}_\theta + \cos(\theta_1)\sin(\phi - \phi_1)\hat{e}_\phi \]

\[ \dot{\hat{e}}_3 = -\sin(\theta)\sin(\phi - \phi_1)\hat{e}_r - \cos(\theta)\sin(\phi - \phi_1)\hat{e}_\theta - \cos(\phi - \phi_1)\hat{e}_\phi \]

Note that \( \phi \) and \( \phi_1 \) appear in the combination \( \phi - \phi_1 \) in this formulation.

The general unit vector \( \hat{u} \) (with constant direction) used in equation 30 can be considered as a linear combination of the system of orthogonal unit vectors \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \). Now \( \hat{e}_1 \) gives the direction of propagation of the vector plane wave. For our purposes we have an electromagnetic wave which is an assumed TEM plane wave. This plane wave will be used as the incident wave for calculating the sensor response. Calling the electric field \( \hat{E}_{\text{inc}} \) and the magnetic field \( \hat{H}_{\text{inc}} \) then \( \hat{E}_{\text{inc}} \) and \( \hat{H}_{\text{inc}} \) must be perpendicular to \( \hat{e}_1 \); they can then be formed from linear combinations of \( \hat{e}_1 \) and \( \hat{e}_2 \). In figure 3 we illustrate such a plane wave incident on a sphere centered on \( r = 0 \). The slot around the sphere is to be centered on \( \theta = \pi/2 \) so as to make the sensor symmetric with respect to the \( x, y \) plane. By symmetry if \( \hat{E}_{\text{inc}} \) were parallel to \( \hat{e}_2 \) and thus parallel to the \( x, y \) plane it would drive no net current across the gap; note we are going to integrate the current all around the gap to obtain the short-circuit current for the sensor. Thus we are only interested in the polarization of
the incident wave for which \( \hat{\mathbf{E}}_{\text{inc}} \) is parallel to \( \hat{\mathbf{e}}_2 \) and our incident wave is defined by

\[
\hat{\mathbf{E}}_{\text{inc}} = E_0 \hat{\mathbf{e}}_2 e^{-i\mathbf{k} \cdot \mathbf{r}}
\]

\[
\hat{\mathbf{H}}_{\text{inc}} = \frac{E_0}{\mu} \hat{\mathbf{e}}_3 e^{-i\mathbf{k} \cdot \mathbf{r}}
\]

Since one can convert an electric field distribution to the associated magnetic field by the transformations in equations 21, we only need to expand \( \hat{\mathbf{E}}_{\text{inc}} \) in terms of the \( M \) and \( N \) functions; \( \hat{\mathbf{H}}_{\text{inc}} \) is directly obtainable from the expansion of \( \hat{\mathbf{E}}_{\text{inc}} \).

As a further simplification we will later set \( \phi_1 = 0 \) to make \( \mathbf{k} \) parallel to the \( x, z \) plane as shown in figure 3. Since the sensor geometry is independent of \( \phi \) this represents no loss in generality. Consider an \( r_1, \theta_1, \phi_1 \) spherical coordinate system where \( \theta_1 \) and \( \phi_1 \) are the angles fixing the direction of propagation of the incident plane wave as above. The unit vectors for such a coordinate system are

\[
\hat{\mathbf{e}}_{r_1} = \hat{\mathbf{e}}_1, \quad \hat{\mathbf{e}}_{\theta_1} = -\hat{\mathbf{e}}_2, \quad \hat{\mathbf{e}}_{\phi_1} = -\hat{\mathbf{e}}_3
\]

As in Morse and Feshback\(^{12}\) we define vector spherical harmonics using the spherical angles \( \theta_1, \phi_1 \) as

\[
\hat{\mathbf{P}}(n, m, \theta_1, \phi_1) = \hat{\mathbf{e}}_{r_1} P^m_n(\cos(\theta_1)) \begin{bmatrix} \cos(m\phi_1) \\ \sin(m\phi_1) \end{bmatrix}
\]

\[
\hat{\mathbf{B}}(n, m, \theta_1, \phi_1) = \frac{\sqrt{n(n+1)}}{(2n+1)\sin(\theta_1)} \begin{bmatrix} \hat{\mathbf{e}}_{\theta_1} P^m_{n-1}(\cos(\theta_1)) \\ \frac{1}{n(n+1)} P^m_n(\cos(\theta_1)) \end{bmatrix}
\]

\[
- \frac{n+m}{n} P^m_{n-1}(\cos(\theta_1)) \begin{bmatrix} \cos(m\phi_1) \\ \sin(m\phi_1) \end{bmatrix}
\]

\[
+ \hat{\mathbf{e}}_{\phi_1} \frac{m(2n+1)}{n(n+1)} P^m_n(\cos(\theta_1)) \begin{bmatrix} -\sin(m\phi_1) \\ \cos(m\phi_1) \end{bmatrix}
\]

\(^{12}\) Ref. 8, Part II, pp. 1898, 1899.
\[ c(n, m, e) = \frac{\sqrt{n(n+1)}}{(2n+1)\sin(\theta_1)} \left[ \begin{array}{c} e_0 \frac{m(2n+1)}{n(n+1)} p_n^m(\cos(0_1)) \\ \sin(m\phi_1) \end{array} \right] \]

\[ - e_0 \phi_1 \left[ \frac{n-m+1}{n+1} p_{n+1}^m(\cos(0_1)) \right] \]

\[ - \frac{n+m}{n} p_n^{m-1}(\cos(0_1)) \left[ \begin{array}{c} \cos(m\phi_1) \\ \sin(m\phi_1) \end{array} \right] \]

The letters e and o refer to even and odd functions and are associated with the upper and lower functions in braces just as with the vector wave functions previously introduced.

Now we introduce the result of Morse and Feshback\textsuperscript{13} for a dyadic plane wave with i switched to -i giving

\[ (\delta_{\alpha, \beta}) e^{-i\hat{k} \cdot \hat{r}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{-i\hat{k} \cdot \hat{r}} \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{e,o} [2-\delta_{m,o}][(-i)^n(2n+1) \frac{(n-m)!}{(n+m)!}] \left[ i\hat{p}(n,m,e)\hat{\ell}(1)(n,m,e) \right] 

\[ + \frac{1}{\sqrt{n(n+1)}} \left[ \hat{c}(n,m,e)\hat{\ell}(1)(n,m,e) + i\hat{b}(n,m,e)\hat{\ell}(1)(n,m,e) \right] \]

The \( \hat{p}, \hat{b}, \hat{c} \) vector spherical harmonics are functions of \( \theta_1 \) and \( \phi_1 \) while the \( \hat{L}, \hat{M}, \hat{N} \) spherical vector wave functions are functions of \( r, \theta, \) and \( \phi. \) Note the use of dyadic notation where two vectors are written side by side to form a dyadic.\textsuperscript{(\delta_{\alpha, \beta})} is the identity dyadic which is also written in matrix form in equation 42. A vector plane wave is formed by taking the dot product of some constant vector on the left of the terms in equation 42. Using a general unit vector \( \hat{u} \) with fixed direction (as in equation 30) we have a vector plane wave as

\textsuperscript{13} Ref. 8, Part II, p. 1866.
\[ u e^{-i \mathbf{k} \cdot \mathbf{r}} = \mathbf{u} \cdot (\delta_{\alpha, \beta}) e^{-i \mathbf{k} \cdot \mathbf{r}} \]

\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{e, o} [2-\delta_{m, o}] (-i)^n (2n+1) \frac{(n-m)!}{(n+m)!} \left[ i[\mathbf{u} \cdot \mathbf{p}(n, m, e)] \mathbf{L}_n^{(1)}(n, m, e) \right] \]

\[ + \frac{1}{\sqrt{n(n+1)}} \left[ [\mathbf{u} \cdot \mathbf{c}(n, m, e)] \mathbf{M}_n^{(1)}(n, m, e) + i[\mathbf{u} \cdot \mathbf{b}(n, m, e)] \mathbf{N}_n^{(1)}(n, m, e) \right] \]

(43)

The problem of expanding a vector plane wave then reduces to finding the coefficients expressed above as dot products.

Our plane wave of interest (equations 39) can be found by setting \( u \) equal to \( \mathbf{e}_2 \) and then equal to \( \mathbf{e}_3 \) while using the relations for \( \mathbf{e}_2 \) and \( \mathbf{e}_3 \), given in equations 40 and setting \( \phi_1 = 0 \).

Considering first \( \mathbf{e}_2 \) we have

\[ i\mathbf{e}_2 \cdot \mathbf{p}(n, m, e) = -i\mathbf{e}_{\theta_1} \cdot \mathbf{p}(n, m, e) = 0 \]

\[ \frac{1}{\sqrt{n(n+1)}} \mathbf{e}_2 \cdot \mathbf{c}(n, m, e) = \frac{-1}{\sqrt{n(n+1)}} \mathbf{e}_{\theta_1} \cdot \mathbf{c}(n, m, e) \]

\[ = -\frac{1}{\sin(\theta_1)} \frac{m}{n(n+1)} \mathbf{p}_n^m(\cos(\theta_1)) \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} \]

(44)

\[ \frac{i}{\sqrt{n(n+1)}} \mathbf{e}_2 \cdot \mathbf{b}(n, m, e) = \frac{-i}{\sqrt{n(n+1)}} \mathbf{e}_{\theta_1} \cdot \mathbf{b}(n, m, e) \]

\[ = \frac{-i}{\sin(\theta_1)} \frac{1}{2n+1} \left[ \frac{n-m+1}{n+1} \mathbf{p}_n^m(\cos(\theta_1)) - \frac{n+m}{n} \mathbf{p}_{n-1}^m(\cos(\theta_1)) \right] \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} \]

Thus only the second set of coefficients for odd harmonics and the third set for even harmonics can be non zero. Note that \( n = 0 \) is not used because the corresponding \( \mathbf{M} \) and \( \mathbf{N} \) functions are identically zero. The last of equations 44 can be rewritten using identities for the Legendre functions 14 as

The incident electric field from equations 39 can then be written as

\[
\tilde{E}_{\text{inc}} = E_0 \hat{e}_2 e^{-i\mathbf{k} \cdot \mathbf{r}}
\]

\[
\hat{e}_2 e^{-i\mathbf{k} \cdot \mathbf{r}} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[ a_{n,m} \hat{\Phi}^{(1)}(n,m,o) + b_{n,m} \hat{\Phi}^{(1)}(n,m,e) \right]
\]

where

\[
a_{n,m} = [2-\delta_{m,0}] (-i)^{n+2} \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \frac{p^m_n(\cos(\theta_1))}{\sin(\theta_1)}
\]

\[
b_{n,m} = [2-\delta_{m,0}] (-i)^{n+1} \frac{2n+1}{n(n+1)} \frac{(n-m)!}{(n+m)!} \frac{dp^m_n(\cos(\theta_1))}{d\theta_1}
\]

Setting \( \hat{\Phi} = \hat{e}_3 \) we have another set of coefficients as

\[
i \hat{e}_3 \cdot \hat{\Phi}(n,m,o) = -i \hat{e}_3 \cdot \hat{\Phi}(n,m,e) = 0
\]

\[
\frac{1}{\sqrt{n(n+1)}} \hat{e}_3 \cdot \hat{\Phi}(n,m,e) = \frac{-1}{\sqrt{n(n+1)}} \hat{e}_3 \cdot \hat{\Phi}(n,m,e)
\]

\[
= \frac{1}{\sin(\theta_1)} \frac{1}{2n+1} \left[ \frac{n-m+1}{n+1} p^m_{n+1}(\cos(\theta_1)) - \frac{n+m}{n} p^m_{n-1}(\cos(\theta_1)) \right]
\]

\[
= \frac{1}{n(n+1)} \frac{dp^m_n(\cos(\theta_1))}{d\theta_1}
\]
The incident magnetic field from equations 39 can then be written as

\[
\frac{i}{\sqrt{n(n+1)}} \hat{e}_3 \cdot \hat{B}(n,m,o) = \frac{-i}{\sqrt{n(n+1)}} \hat{e}_\phi \cdot \hat{B}(n,m,o)
\]

\[
= \frac{-i}{\sin(\theta)} \frac{m}{n(n+1)} P^m_n(\cos(\theta)) \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

We then have the complete expansion of our incident wave.

Actually, with \( \hat{e}_2 e^{-ik \cdot \hat{r}} \) and \( \hat{e}_3 e^{-ik \cdot \hat{r}} \) expanded any polarization of the incident electric field (perpendicular to \( \hat{k} \)) can be written as a linear combination of these two plane waves. The associated incident magnetic field can be similarly written. Beginning with equations 44 we restricted \( \phi_1 = 0 \). However, if it is desired to have \( \phi_1 \neq 0 \) there are at least two approaches to generalize the present results. First, one can simply leave \( \phi_1 \neq 0 \) in calculating the coefficients as in equations 44 and 48; this will put terms like \( \cos(m\phi_1) \) and \( \sin(m\phi_1) \) in the coefficients. Or second, one can take advantage of the axial symmetry of the spherical coordinate system and consider \( \phi - \phi_1 \) as the azimuthal angle in the present results; in this case one just substitutes \( \phi - \phi_1 \) in place of \( \phi \) in the \( M \) and \( N \) functions in equations 46 and 49. Which of these methods if used is a matter of convenience related to the problem at hand. As stated before we assume the spherical sensor in the present problem to have axial symmetry so that we use \( \phi_1 = 0 \) for convenience.

The expansion of \( \hat{e}_1 e^{-ik \cdot \hat{r}} \) would be another extension of the present results for cases where there were source currents in the incident wave such that the incident electric field had a non zero divergence. This, of course, would be accomplished by substituting \( e_1 \) for \( \hat{u} \) and obtaining a set of coefficients analogous to equations 44 or 48.
IV. Short Circuit Current

Having the incident plane wave, next consider the short circuit current from the sensor. For this calculation short circuit the gap in the sphere so that it looks like a complete perfectly conducting sphere with no gap. Then we consider the currents induced on the sphere by the incident plane wave. The incident plane wave is given by equations 46 and 49 in the previous section. Add to this a scattered wave of the form

\[ E_{sc} = E_0 \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[ c_{n,m} M^{(4)}(n,m,o) + d_{n,m} N^{(4)}(n,m,e) \right] \]

\[ H_{sc} = \frac{E_0}{z} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[ ic_{n,m} M^{(4)}(n,m,o) + id_{n,m} N^{(4)}(n,m,e) \right] \]

The fourth kind of spherical bessel functions are used to give an outward propagating wave; the M and N functions are chosen even or odd to match the \( \phi \) dependence of those appearing in the expansion for the incident wave.

The boundary condition at \( r = a \) is that \( E_\theta = E_\phi = 0 \). From the spherical vector wave functions in equations 12 and 15 this requires

\[ a_{n,m} j_n(ka) + c_{n,m} h_n^{(2)}(ka) = 0 \]

\[ b_{n,m} \frac{[kaj_n(ka)]'}{ka} + d_{n,m} \frac{[kah_n^{(2)}(ka)]'}{ka} = 0 \]

\[ \]

giving

\[ c_{n,m} = - \frac{j_n(ka)}{h_n^{(2)}(ka)} a_{n,m} \]

\[ d_{n,m} = - \frac{[kaj_n(ka)]'}{[kah_n^{(2)}(ka)]'} b_{n,m} \]

Note that the total field is
\[
\dot{E} = \dot{E}_{\text{inc}} + \dot{E}_{\text{sc}}, \quad \dot{H} = \dot{H}_{\text{inc}} + \dot{H}_{\text{sc}}
\]

(53)

The surface current density \( \mathbf{J}_s \) on the perfectly conducting spherical surface on \( r = a \) has \( \theta \) and \( \phi \) components which can be found from the magnetic field just outside the surface as

\[
\mathbf{J}_s \theta = -H_{\phi} \bigg|_{r=a+}
\]

\[
\left. \frac{E_0}{Z} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left( -\mathbf{i} a_{n,m} \frac{[k a j_n(ka)]'}{ka} - \frac{j_n(ka)}{h_n^{(2)}(ka)} \frac{[k a h_n^{(2)}(ka)]'}{ka} \right) \right)
\]

\[
\cdot \frac{P_n^m(\cos(\theta))}{\sin(\theta)} m \cos(m\phi)
\]

\[
+ \mathbf{i} b_{n,m} \left[ j_n(ka) - \frac{[k a j_n(ka)]'}{[k a h_n^{(2)}(ka)]'} h_n^{(2)}(ka) \right] \left\{ \frac{d P_n^m(\cos(\theta))}{d\theta} \cos(m\phi) \right\}
\]

(54)

\[
\mathbf{J}_s \phi = H_{\theta} \bigg|_{r=a+}
\]

\[
\left. \frac{E_0}{Z} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left( \mathbf{i} a_{n,m} \frac{[k a j_n(ka)]'}{ka} - \frac{j_n(ka)}{h_n^{(2)}(ka)} \frac{[k a h_n^{(2)}(ka)]'}{ka} \right) \right)
\]

\[
\cdot \frac{d P_n^m(\cos(\theta))}{d\theta} \sin(m\phi)
\]

\[
- \mathbf{i} b_{n,m} \left[ j_n(ka) - \frac{[k a j_n(ka)]'}{[k a h_n^{(2)}(ka)]'} h_n^{(2)}(ka) \right] \left\{ \frac{P_n^m(\cos(\theta))}{\sin(\theta)} m \sin(m\phi) \right\}
\]
To simplify this we use a Wronskian relation\textsuperscript{15} to obtain

\begin{equation}
\begin{split}
\frac{1}{k^2}[\frac{\partial}{\partial k} h_n^{(2)}(ka) [j_n(ka)]' - j_n(ka)[k h_n^{(2)}(ka)]'] \\
= \frac{i}{k} [k j_n(ka) [k j_n(ka)]' - j_n(ka) [k j_n(ka)]'] \\
= \frac{i}{k^2}
\end{split}
\end{equation}

Using this result in equations 54 gives

\begin{equation}
J_{s\theta} = \frac{E_0}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[ a_{n,m} \frac{1}{(ka)^2 h_n^{(2)}(ka)} \frac{p_n^m(\cos(\theta))}{\sin(\theta)} m \sin(m\phi) \\
+ b_{n,m} \frac{1}{k a [k h_n^{(2)}(ka)]'} \frac{d P_n^m(\cos(\theta))}{d\theta} \cos(m\phi) \right]
\end{equation}

\begin{equation}
J_{s\phi} = \frac{E_0}{2} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left[ -a_{n,m} \frac{1}{(ka)^2 h_n^{(2)}(ka)} \frac{d P_n^m(\cos(\theta))}{d\theta} \sin(m\phi) \\
- b_{n,m} \frac{1}{k a [k h_n^{(2)}(ka)]'} \frac{p_n^m(\cos(\theta))}{\sin(\theta)} m \sin(m\phi) \right]
\end{equation}

With surface current density on the sphere determined we can calculate the total current crossing a circle of constant $\theta$ on the spherical surface; this is

\textsuperscript{15} Ref. 6, eqn. 10.3.4.
\[ I(\theta) = a \sin(\theta) \int_0^{2\pi} J_{\nu}(\theta) d\phi \]

\[ = 2\pi a \sin(\theta) \sum_{n=0}^{\infty} b_{n,0} \frac{1}{ka \left[ \text{kh}_n^{(2)}(ka) \right]} \frac{dp_n^0(\cos(\theta))}{d\theta} \]

where

\[ b_{n,0} = (-i)^{n+1} \frac{2n+1}{n(n+1)} \frac{dp_n^0(\cos(\theta_1))}{d\theta_1} \]

Rewriting the Legendre functions with the relation

\[ \frac{dp_n^0(\cos(\theta))}{d\theta} = p_n^{1}(\cos(\theta)) \]

we then have

\[ I(\theta) = 2\pi a \sin(\theta) \sum_{n=0}^{\infty} (-i)^{n+1} \frac{2n+1}{n(n+1)} \frac{1}{ka \left[ \text{kh}_n^{(2)}(ka) \right]} \]

\[ \cdot p_n^{1}(\cos(\theta_1)) p_n^{1}(\cos(\theta)) \]

For small \( |ka| \) we have for the spherical Hankel functions

\[ h_n^{(2)}(ka) = i(2n - 1)!!(ka)^{-n-1} + O((ka)^{-n}) \]

The double factorial function is defined by

\[ (2n)!! \equiv (2n)(2n-2) \cdots (4)(2) \quad \text{(even)} \]

\[ (2n-1)!! \equiv (2n-1)(2n-3) \cdots (3)(1) \quad \text{(odd)} \]

with the conventions
\[ 0!! = 1, \quad (-1)!! = 1 \]  

(63)

For small \(|ka|\) the \(n = 1\) term in equation 60 dominates and we have as \(ka \to 0\)

\[
I(\theta) = 3\pi a \frac{E_0}{z} (-ika) \sin(\theta_1)[\sin(\theta)]^2 + O((ka)^2)
\]

\[
= -3\pi a^2 E_0 (\sigma + i\omega \epsilon) \sin(\theta_1)[\sin(\theta)]^2 + O((ka)^2)
\]

(64)

Thus for low frequencies the short circuit current is proportional to \((\sigma + i\omega \epsilon) E_0 \sin(\theta_1)\) which is the \(z\) component of the total current density. For the case that \(\sigma = 0\) then the response is proportional to \(i\omega \epsilon E_0 \sin(\theta_1)\) which (in the time domain) is just the \(z\) component of \(B\). Setting \(\theta = \pi/2\) so that the position of the loop gap is on the equator of the sphere we have as \(ka \to 0\)

\[
I(\pi/2) = -A_{eq} E_0 (\sigma + i\omega \epsilon) \sin(\theta_1) + O((ka)^2)
\]

(65)

where the equivalent area of the sensor is

\[ A_{eq} = 3\pi a^2 \]

(66)

or as a vector

\[ \vec{A}_{eq} = 3\pi a^2 \vec{e}_z \]

(67)

so that we can write as \(ka \to 0\)

\[
I(\pi/2) = -(\sigma + i\omega \epsilon) E_{inc} \bigg|_{r=0} \cdot \vec{A}_{eq} + O((ka)^2)
\]

(68)

For convenience we define a short circuit current transfer function as

\[ T(\theta_1) \equiv I(\pi/2) [-A_{eq} E_0 (\sigma + i\omega \epsilon)]^{-1} \]

(69)

so that as \(ka \to 0\) we have \(T = \sin(\theta_1)\) which is the ideal low-frequency angular dependence of the sensor. Then we have
Some special values for the Legendre functions are

\[ P_m^n(0) = \begin{cases} 0 & \text{for } n + m \text{ odd} \\ \frac{n+m}{2} \frac{(n + m - 1)!!}{(n - m)!!} & \text{for } n + m \text{ even} \end{cases} \] (71)

giving

\[ T(\theta_1) = \sum_{n=1}^{\infty} \frac{2}{3} \frac{2n+1}{n(n+1)} \frac{n!!}{(n-1)!!} \frac{i}{(ka)^2 \left( \frac{k a h_n^2}{ka} \right)} P_n^1(\cos(\theta_1))P_n^1(0) \] (72)

where the second index above the summation sign is the increment which is added to the value of \( n \) to obtain the next value of \( n \) for the summation. Here only odd \( n \) result. For convenience define

\[ T_n(\theta_1, \theta) = (-i)^n \frac{2}{3} \frac{2n+1}{n(n+1)} \frac{1}{(ka)^2 \left( \frac{k a h_n^2}{ka} \right)} P_n^1(\cos(\theta_1))P_n^1(\theta) \] (73)

For \( \theta = \pi/2 \) and odd \( n \) this is

\[ T_n(\theta_1, \pi/2) = \frac{2}{3} \frac{2n+1}{n(n+1)} \frac{n!!}{(n-1)!!} \frac{i}{(ka)^2 \left( \frac{k a h_n^2}{ka} \right)} P_n^1(\cos(\theta_1)) \] (74)

For our present case of interest just call this \( T_n \) so that the short circuit current transfer function can be written as

\[ T(\theta_1) = \sum_{n=1}^{\infty} T_n \] (75)

Note that as \( ka \rightarrow 0 \) we have \( T \rightarrow T_1 \rightarrow \sin(\theta_1) \).

The short circuit current transfer function is plotted in figure 4 as a function of $ka$ with $\sigma = 0$. Note that we plot $T/\sin(\theta_1)$ for several values of $\theta_1$. This is so that all the curves tend to one for low frequency and we can observe at what frequency (i.e. what $ka$) they start to spread from one another. For comparison $T_1$ from equation 74 is included. $T_1$ will be used later in calculating some of the sensor response functions. In figure 4 the magnitude and phase of $T/\sin(\theta_1)$ are plotted. For convenience the phase is plotted as $\arg(T) - ka$ which corresponds to multiplying $T$ by $e^{-ika}$. This is the same type of display as used in reference 2. Note that the magnitude of $T/\sin(\theta_1)$ peaks up slightly at around $ka = .7$. Also, between $ka = 1$ and $ka = 2$ the response functions for various values of $\theta_1$ begin to spread apart; some $ka$ roughly between 1 and 2 can then be considered the upper frequency for which the sensor maintains a response proportional to $\sin(\theta_1)$ which can be considered the ideal angular dependence of the sensor. This range of $ka$ is also where the frequency response (for the short circuit current) starts to fall off for higher frequencies.

$T_1/\sin(\theta_1)$ is a useful function to use for some of the later response function calculations because it is the first term in the expansion of $T/\sin(\theta_1)$ and because it has no dependence on $\theta_1$.

The Hankel function for $n = 1$ is just

$$h_1^{(2)}(ka) = \left[ -\frac{1}{ka} + \frac{i}{(ka)^2} \right] e^{-ika} \quad (76)$$

Then $T_1$ is just

$$\frac{T_1}{\sin(\theta_1)} = \frac{-i}{(ka)^2} \left[ kah_1^{(2)}(ka) \right] e^{-ika} = \frac{e^{ika}}{1 + ika - (ka)^2} \quad (77)$$

and we also have

$$\frac{T_1 e^{-ika}}{\sin(\theta_1)} = [1 + ika - (ka)^2]^{-1} \quad (78)$$

$T_1/\sin(\theta_1)$ has a peak magnitude of $2/\sqrt{3} \approx 1.155$ occurring at $ka = 1/\sqrt{2} \approx .707$. 

25
V. Admittances

Now consider the admittances when the sensor is driven at the gap. As illustrated in figure 5 the loop gap is centered on \( \theta = \pi/2 \) and has an angular width of \( 2\psi_0 \). There is a voltage \( V_{gap} \) uniformly distributed around the gap. Associated with \( V_{gap} \) there are three surface current densities which are parallel to \( \theta \). These are \( J_{ext} \), \( J_{int} \), and \( J_c \) and are associated respectively with fields external to the sphere, fields internal to the sphere, and currents into the cables or other transmission lines loading the gap. Taking the conventions for the directions of these currents as indicated in figure 5 we have three admittances to define, namely

\[
Y_{int} = 2\pi a \frac{J_{int}}{V_{gap}}, \quad Y_{ext} = 2\pi a \frac{J_{int}}{V_{gap}}, \quad Y_c = 2\pi a \frac{J_c}{V_{gap}}
\]  

(79)

In normalized form we also define

\[
Y_{int} = Z_{int} Y_{int}, \quad Y_{ext} = Z_{ext} Y_{ext}, \quad Y_c = Z Y_c
\]  

(80)

For the numerical calculations we take \( \sigma = 0 \), as before. Also define a normalized cable conductance as

\[
r_c = \frac{1}{Y_c} = \frac{Z_c}{Z}
\]  

(81)

where \( Z_c \) is the net cable impedance (resistive) loading the gap. Since we have \( \sigma = 0 \) for the numerical calculations then \( r_c \geq 0 \) for such calculations and \( r_c \) is a constant which we can specify parametrically.

A. Boundary Conditions at Gap in Sphere

Here we use approximately the same quasi static approximation for the electric field distribution in the gap as was used in references 1 and 2. Defining

\[
\psi = \frac{\pi}{2} - \theta
\]  

(82)

we have
This field distribution has the proper form of singularity at the edges of the gap, considering the assumption of a perfectly conducting spherical shell of zero thickness. The presence of cable connections across the gap will of course distort this field somewhat near the connections. However, of the various choices for the gap field one might choose the above choice seems to be a reasonable one. Note that we assume $\psi_0 << 1$ for these calculations. Later we change to another field distribution $f_{E}$ which closely approximates $f_{E}$ for small $\psi$. This other distribution will be more convenient for the evaluation of a certain integral.

B. Internal Admittance

To calculate the internal admittance we expand the fields inside the sphere in terms of the spherical vector wave functions. The boundary conditions at $r = a$ are taken as independent of $\phi$ so that $m = 0$ for the wave functions. There are three non-zero field components: $E_r$, $E_\theta$, $H_\phi$. The fields are finite at $r = 0$ so that the spherical Bessel functions of the first kind are used. The fields are then expanded as

$$E_{\text{int}} = E_1 \sum_{n=0}^{\infty} a_n \tilde{N}^{(1)}(n, 0, e)$$

$$H_{\text{int}} = \frac{E_1}{2} \sum_{n=0}^{\infty} i a_n \tilde{N}^{(1)}(n, 0, e)$$

where $E_1$ is some constant with dimensions volts/meter. The non-zero components of the spherical vector wave functions for $m = 0$ from equations 12 and 15 are
\begin{align*}
\mathbf{M}_\phi(n,0,e) &= -f_n^{(\ell)}(kr) \frac{dP_n^0(\cos(\theta))}{d\theta} = -f_n^{(\ell)}(kr)P_n^1(\cos(\theta)) \\
N_r^{(\ell)}(n,0,e) &= n(n + 1) \frac{f_n^{(\ell)}(kr)}{kr} P_n^0(\cos(\theta)) \\
N_\theta^{(\ell)}(n,0,e) &= \left[ \frac{krf_n^{(\ell)}(kr)}{kr} \right]' \frac{dP_n^0(\cos(\theta))}{d\theta} = \left[ \frac{krf_n^{(\ell)}(kr)}{kr} \right]' P_n^1(\cos(\theta)) \quad (86)
\end{align*}

The boundary conditions for \( E_\theta \) on \( r = a \) are given by equations 83 and 84. Writing out \( E_\theta \) on \( r = a \) gives

\[ E_\theta \bigg|_{r=a} = E_1 \sum_{n=0}^{\infty} \frac{[kaj_n(ka)]'}{ka} P_n^1(\cos(\theta)) \quad (87) \]

Multiplying both sides by \( P_n^1(\cos(\theta)) \sin(\theta) \), integrating over \( \theta \) from 0 to \( \pi \), and using the orthogonality of the Legendre functions gives\(^{17}\)

\[ \frac{V_{\text{gap}}}{\alpha_{\psi}} \int_0^\pi f_1(\psi) P_n^1(\cos(\theta)) \sin(\theta) d\theta = E_1 \sum_{n=0}^{\infty} \frac{[kaj_n(ka)]'}{ka} \frac{2n(n + 1)}{2n + 1} \quad (88) \]

For the integral we define a convenient term

\[ \Lambda_n' = \frac{1}{\psi_o} \int_0^\pi f_1(\psi) P_n^1(\cos(\theta)) \sin(\theta) d\theta \]

\[ = \frac{1}{\psi_o} \int_{-\pi/2}^{\pi/2} f_1(\psi) P_n^1(\sin(\psi)) \cos(\psi) d\psi \]

\[ = \frac{1}{\pi\psi_o} \int_{-\psi_o}^{\psi_o} \left[ 1 - \left( \frac{\psi}{\psi_o} \right)^2 \right]^{-1/2} P_n^1(\sin(\psi)) [1 - \sin^2(\psi)]^{1/2} \quad (89) \]

\(^{17}\) Ref. 6, eqn. 8.14.13.
Since we assume $\psi_0 << 1$, then for $|\psi| \leq \psi_0$ we can use $\sin(\psi) = \psi$ to write

$$\Lambda_n' = \frac{1}{\pi \psi_0} \int_{-\psi_0}^{\psi_0} \left[ 1 - \left( \frac{\psi}{\psi_0} \right)^2 \right]^{-1/2} \left[ 1 - \psi^2 \right]^{1/2} \frac{1}{n} P_n(\psi) d\psi \equiv \Lambda_n$$  \hspace{1cm} (90)

Then let $\zeta = \psi/\psi_0$ to give

$$\Lambda_n = \frac{1}{\pi} \int_{-1}^{1} \left[ 1 - \zeta^2 \right]^{-1/2} \left[ 1 - \psi_0^2 \zeta^2 \right]^{1/2} \frac{1}{n} P_n(\psi_0 \zeta) d\zeta$$  \hspace{1cm} (91)

Since the form of $f_E^i$ is only approximate we take $\Lambda_n$ instead of $\Lambda_n'$ to calculate the coefficients. This integral is treated in appendix A. Using the approximation good for small $\psi_0$ we then have for our coefficients

$$\alpha_n = \frac{\sqrt{gap}}{\varepsilon E_1} \frac{2n + 1}{2n(n + 1)} \frac{ka}{[ka_j^e(n)]^r} \Lambda_n$$  \hspace{1cm} (92)

In obtaining the result of equation 91 for $\Lambda_n$ the approximation $\sin(\psi) = \psi$ was used for small $\psi$. This is equivalent to altering slightly the field distribution $f_E$ to one which we call $f_E$. To see this let $\psi_0 \zeta = \sin(\psi)$ in equation 91 giving

$$\Lambda_n = \frac{1}{\pi \psi_0} \int_{-\arcsin(\psi_0)}^{\arcsin(\psi_0)} \left[ 1 - \left( \frac{\sin(\psi)}{\psi_0} \right)^2 \right]^{-1/2} \cos(\psi) P_n^1(\sin(\psi)) \cos(\psi) d\psi$$

$$= \frac{1}{\psi_0} \int_{-\pi/2}^{\pi/2} f_E P_n^1(\sin(\psi)) \cos(\psi) d\psi$$  \hspace{1cm} (93)

where we have defined

$$f_E = \begin{cases} \frac{1}{\pi} \left[ 1 - \left( \frac{\sin(\psi)}{\psi_0} \right)^2 \right]^{-1/2} \cos(\psi) & \text{for } |\psi| < \arcsin(\psi_0) \\ 0 & \text{for } |\psi| \geq \arcsin(\psi_0) \end{cases}$$  \hspace{1cm} (94)
The voltage across the gap is given from equation 83 (except using $f_E$) by the integral

$$V_o = \frac{V_{\text{gap}}}{a\psi_o} \int_{-\pi/2}^{\pi/2} f_E \, d\psi$$  \hspace{1cm} (95)$$

Substituting from equation 94 and letting $\psi_o \zeta = \sin(\psi)$ we have

$$V_o = \frac{V_{\text{gap}}}{\pi} \int_{-1}^{1} \left[ 1 - \zeta^2 \right]^{-1/2} d\zeta = V_{\text{gap}}$$  \hspace{1cm} (96)$$

Thus this field distribution is still exactly consistent with $V_{\text{gap}}$ as the gap voltage. The angular gap half width is $\arcsin(\psi_o)$ for this distribution and is very nearly $\psi_o$ for small $\psi$. We thus take $f_E$ as the field distribution across the gap instead of the simpler form given by $f_E'$.

The surface current density associated with the internal admittance is taken at $\theta = \pi/2$ for convenience so that we have

$$J_{\text{int}} = -H_{\text{int}} \phi \bigg|_{r=a}^{\theta=\pi/2}$$  \hspace{1cm} (97)$$

From equations 85 we then have

$$J_{\text{int}} = \frac{E_1}{\pi} \sum_{n=0}^{\infty} i \alpha_n j_n(ka) P_n^1(0)$$  \hspace{1cm} (98)$$

From equation 71 we then have

$$J_{\text{int}} = \frac{E_1}{\pi} \sum_{n=1}^{\infty} i(-1)^{n+1} \frac{n+1}{(n-1)!} \alpha_n j_n(ka)$$  \hspace{1cm} (99)$$

Note that only odd $n$ contribute to this sum.

From equations 79 and 80 the normalized internal admittance is then
\[
Y_{\text{int}} = \frac{2\pi aZ}{V_{\text{gap}}} J_{\text{int}} = \frac{2\pi aE_1}{V_{\text{gap}}} \sum_{n=1}^{\infty,2} i p_n^1(0) \alpha_n j_n(ka)
\]  

Substituting for \(\alpha_n\) gives

\[
Y_{\text{int}} = \sum_{n=1}^{\infty,2} i \pi \frac{2n + 1}{n(n + 1)} p_n^1(0)\Lambda_n \frac{kaj_n(ka)}{[kaj_n(ka)]'}
\]

From appendix A, equation A21, we have

\[
\Lambda_n = p_n^1(0) F\left(- \frac{n+1}{2}, \frac{n}{2}; 1; \psi_o^2\right)
\]

where \(F\) is a hypergeometric function given by

\[
F\left(- \frac{n+1}{2}, \frac{n}{2}; 1; \psi_o^2\right) = \sum_{q=0}^{\infty} \frac{\left(- \frac{n+1}{2}\right)\left(\frac{n}{2}\right)}{(q!)^2} \psi_o^{2q}
\]

and where \((\alpha)_q\) is a Pochhammer symbol\(^{18}\) defined by

\[
(\alpha)_0 = 1
\]

\[
(\alpha)_q = \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+q-1) \quad \text{for } q = 1, 2, \cdots
\]

Note that the series in equation 103 has only a finite number of terms giving a polynomial function of \(\psi_o\). This is convenient for the numerical calculations. The normalized internal admittance can then be written as

\[
Y_{\text{int}} = \sum_{n=1}^{\infty,2} i \pi \frac{2n + 1}{n(n+1)} \left[p_n^1(0)\right]^2 F\left(- \frac{n+1}{2}, \frac{n}{2}; 1; \psi_o^2\right) \frac{kaj_n(ka)}{[kaj_n(ka)]'}
\]

\[
= \sum_{n=1}^{\infty,2} i \pi \frac{2n + 1}{n(n+1)} \left[\frac{n!!}{(n-1)!!}\right]^2 F\left(- \frac{n+1}{2}, \frac{n}{2}; 1; \psi_o^2\right) \frac{kaj_n(ka)}{[kaj_n(ka)]'}
\]

\(^{18}\) Ref. 6, eqn. 6.1.22.
The asymptotic properties of this series for large \(n\) are treated in appendix B.

The normalized internal admittance is plotted in figure 6 as a function of \(ka\) with \(\sigma = 0\) for several values of \(\psi_0\). Note that \(Y_{\text{int}}\) has only an imaginary part. For convenience \(Y_{\text{int}}/ka\) is plotted in figure 6; \(Y_{\text{int}}/ka\) is an imaginary function for real \(ka\) which we are using. As \(ka \to 0\) \(Y_{\text{int}}/ka\) tends to a constant; the numerically determined coefficients are listed in table 1.

<table>
<thead>
<tr>
<th>(\psi_0)</th>
<th>(Y_{\text{int}}/ka)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.001</td>
<td>14.22</td>
</tr>
<tr>
<td>.01</td>
<td>9.62</td>
</tr>
<tr>
<td>.1</td>
<td>5.10</td>
</tr>
</tbody>
</table>

Table 1. Asymptotic form of \(Y_{\text{int}}\) for small \(ka\)

As one would expect for small \(ka\) this admittance represents a capacitance. As \(ka\) is increased \(Y_{\text{int}}\) has singularities corresponding to the zeros of \([kaj_n(ka)]'\). The first two singularities occur at \(ka = 2.744\) and \(ka = 4.973\).

C. External Admittance

The calculation of the external admittance follows the same development as the internal admittance. The fields outside the sphere are expanded as outward propagating waves in the forms

\[
\hat{E}_{\text{ext}} = E_1 \sum_{n=0}^{\infty} \beta_n \hat{N}_n^{(4)}(n,0,e)
\]

\[
\hat{H}_{\text{ext}} = \frac{E_1}{z} \sum_{n=0}^{\infty} i\beta_n \hat{M}_n^{(4)}(n,0,e)
\]

The only difference on the boundary \(r = a\) between these fields and those of equations 85 is the replacement of \(j_n(ka)\) by \(h_n^{(2)}(ka)\). Analogous to equation 92 we then have the expansion coefficients given by

\[
\beta_n = \frac{V_{\text{gap}}}{\mathbf{a}E_1} \frac{2n + 1}{2n(n + 1)} \frac{ka}{[kah_n^{(2)}(ka)]} \Lambda_n
\]

where only odd \(n\) are of interest.
The surface current density associated with the external admittance is taken at \( \theta = \pi/2 \) giving

\[
J_{s_{\text{ext}}} = H_{\text{ext}} \bigg|_{\theta=\pi/2} \bigg|_{r=a}
\]

\[= - \frac{E_1}{2} \sum_{n=1}^{\infty} i \beta_n h_n^{(2)}(ka) P_n^1(0) \tag{108}\]

From equations 79 and 80 the normalized external admittance is then

\[Y_{\text{ext}} = \frac{2\pi a Z}{V_{\text{gap}}} J_{s_{\text{ext}}} = \frac{2\pi a E_1}{V_{\text{gap}}} \sum_{n=1}^{\infty} (-i) P_n^1(0) \beta_n h_n^{(2)}(ka) \]

\[= \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} \beta_n h_n^{(2)}(ka) \frac{k a h_n^{(2)}(ka)}{k a h_n^{(2)}(ka)} \tag{109}\]

Or, substituting for \( \Lambda_n \) we have

\[Y_{\text{ext}} = \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} \left[ P_n^1(0) \right]^2 F \left( -\frac{n+1}{2}, \frac{n}{2}; 1; \psi_o^2 \right) \frac{k a h_n^{(2)}(ka)}{k a h_n^{(2)}(ka)} \]

\[= \sum_{n=1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} \left[ \frac{n!}{(n-1)! n!} \right]^2 F \left( -\frac{n+1}{2}, \frac{n}{2}; 1; \psi_o^2 \right) \frac{k a h_n^{(2)}(ka)}{k a h_n^{(2)}(ka)} \tag{110}\]

This series is very similar to the one in equation 105 and its asymptotic properties for large \( n \) are treated in appendix B.

The normalized external admittance is plotted in figure 7 as a function of \( ka \) with \( \sigma = 0 \) for several values of \( \psi_o \). In this figure the real and imaginary parts of \( Y_{\text{ext}} \) are plotted, again in the form \( Y_{\text{ext}}/ka \) which tends to a constant for small \( ka \); the numerically determined coefficients are listed in table 2.
Table 2. Asymptotic form of $y_{ext}$ for small $ka$

<table>
<thead>
<tr>
<th>$\psi_o$</th>
<th>$\frac{y_{ext}}{ika}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.001</td>
<td>17.36</td>
</tr>
<tr>
<td>.01</td>
<td>12.75</td>
</tr>
<tr>
<td>.1</td>
<td>8.04</td>
</tr>
</tbody>
</table>

$y_{ext}$ is also a capacitance for small $ka$. Note that as $ka$ is increased $y_{ext}$ does not have singularities as was the case with $y_{int}$. 

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VI. Frequency Response Characteristics

Now that we have the short circuit current and the admittances in suitably normalized forms we combine these results to calculate the frequency response characteristics. First define a response function including only the admittances as

\[ R_y = \frac{Y_c}{Y_{int} + Y_{ext} + Y_c} = \left[ 1 + r_c (y_{int} + y_{ext}) \right]^{-1} \]  

This is plotted as a function of \( ka \) in figures 8 and 9 for two values of \( \psi_o (0.01 \text{ and } 0.1) \) with \( \sigma = 0 \). For each graph several values of \( r_c \) are used. Note for \( r_c > 0 \) that \( R_y \) has zeros at the singularities of \( y_{int} \). As one would expect decreasing \( r_c \) maintains \( R_y \) as a flat response characteristic out to larger values of \( ka \) which represents higher frequency response.

Including the short circuit current transfer function from equation 75 we have the response function

\[ R(\theta) = \frac{\cos(\theta)}{\sin(\theta_c)} \left[ 1 + r_c (y_{int} + y_{ext}) \right]^{-1} \]  

Another convenient, but somewhat artificial response function is given by

\[ R_1 = \frac{\cos(\theta)}{\sin(\theta_c)} \left[ 1 + r_c (y_{int} + y_{ext}) \right]^{-1} \]

This last response function uses only the first term in the expansion for \( T(\theta) \) and is independent of \( \theta_1 \).

In figures 10 and 11 we have \( R_1 \) plotted as a function of \( ka \) for two values of \( \psi_o (0.01 \text{ and } 0.1) \) and several values of \( r_c \). Note that the factor \( T_1/\sin(\theta_1) \) peaks up the frequency response in the vicinity of \( ka = 1 \), partially compensating for the rolloff with frequency associated with \( R_y \).

Based on \( R_1 \) we define an upper frequency response as the minimum value of \( ka \) for which

\[ |R_1| = \frac{1}{\sqrt{2}} \]
This value of $ka$ is plotted as a function of $r_c$ in figure 12 for two values of $\psi_0$. Frequency response is increased by increasing $\psi_0$ and by decreasing $r_c$. It may not be desirable to decrease $r_c$ too far because as $r_c \to 0$ no power is delivered to the output.

In figures 13 and 14 we have plotted $R$ as a function of $ka$ with $\sigma = 0$ for various values of $\theta_1$ with $\psi_0 = .1$ for two specific values of $r_c$. These two values are $r_c = .1327$ and $r_c = .2654$ which correspond to $Z_c = 50 \, \Omega$ and $Z_c = 100 \, \Omega$ respectively if the media inside and outside the sphere are assumed to have the same constitutive parameters as free space so that the wave impedance is

$$Z = Z_0 = 376.7 \, \Omega \quad (115)$$

Using $R_1$ as an average of $R$ to remove the dependence on $\theta_1$, the frequency responses for these two cases as defined by equation 114 are given by $ka = .60$ and $ka = .30$ for $Z_c = 50 \, \Omega$ and $Z_c = 100 \, \Omega$ respectively.
VII. Summary

In this note we have developed frequency and angular response curves for the hollow spherical dipole with a uniformly resistively loaded equatorial slot. For these calculations the media inside and outside the sphere were assumed to have the same permittivity and permeability and zero conductivity. For low frequencies the response of this kind of sensor is proportional to one component of the displacement current density (which is also called the time rate of change of the displacement vector).

As an extension of the present calculations one might consider the case that the external medium was a linear conducting medium. In this case if the resistance due to the signal cable loading \( Z_c \) were small compared to the resistance loading due to the conducting medium, then for low frequencies this sensor would have a response proportional to the total current density (conduction plus displacement) in the external medium.

Perhaps the design considered in this note can be extended to the case of several slots around the sphere at different values of \( \theta \). This would be a multi-gap spherical dipole analogous to the multi-gap cylindrical loop. Another extension of the present calculations would be to loops with spherical geometries. Perhaps some of these and related topics can be considered in future notes.
FIGURE 1. SINGLE-GAP HOLLOW SPHERICAL DIPOLE
A. PLANE WAVE WITH FIXED POLARIZATION

\[
\begin{align*}
\vec{E} & = -\hat{e}_1 e^{-j\omega t} \\
\vec{k} & = \hat{e}_2 \\
\vec{H} & = \hat{e}_3
\end{align*}
\]

\(\hat{e}_1, \hat{e}_2, \hat{e}_3\) are in the same plane.

\(\hat{e}_x, \hat{e}_y, \hat{e}_3\) are in the same plane.

B. UNIT VECTORS FOR PLANE WAVES

FIGURE 2. VECTOR PLANE WAVES IN SPHERICAL COORDINATES
FIGURE 3. PLANE WAVE INCIDENT ON SPHERE: CROSS-SECTION VIEW WITH $\phi_1 = 0$
FIGURE 4. SHORT CIRCUIT CURRENT TRANSFER FUNCTION
y IS POINTING INTO THE PAGE.

FIGURE 5. CURRENTS ON SPHERE FOR ADMITTANCES: CROSS SECTION VIEW
A. $\frac{y_{\text{int}}}{ka}$ WITH $\psi_0$ AS A PARAMETER

$\psi_0 = 0.001$, $0.01$, $0.1$

ka = 2.744, ka = 4.973

B. SCALE OF A EXPANDED

$\frac{y_{\text{int}}}{ka}$

$\psi_0 = 0.001$, $0.01$, $0.1$

ka = 2.744

FIGURE 6. NORMALIZED INTERNAL ADMITTANCE
FIGURE 7. NORMALIZED EXTERNAL ADMITTANCE

A. $\frac{\text{Re}[y_{\text{ext}}]}{ka}$ WITH $\psi_0$ AS A PARAMETER

B. $\frac{\text{Im}[y_{\text{ext}}]}{ka}$ WITH $\psi_0$ AS A PARAMETER
A. MAGNITUDE OF $R_y$ WITH $r_c$ AS A PARAMETER

B. PHASE OF $R_y$ WITH $r_c$ AS A PARAMETER

FIGURE 8. EFFECT OF ADMITTANCES ON RESPONSE: $\psi_0 = .01$
A. MAGNITUDE OF $R_y$ WITH $r_c$ AS A PARAMETER

B. PHASE OF $R_y$ WITH $r_c$ AS A PARAMETER

FIGURE 9. EFFECT OF ADMITTANCES ON RESPONSE: $\psi_0 = 0.1$
FIGURE 10. ARTIFICIAL RESPONSE CHARACTERISTICS: $\psi_0 = .01$
A. MAGNITUDE OF $R_1$ WITH $r_c$ AS A PARAMETER

B. PHASE OF $R_1$ WITH $r_c$ AS A PARAMETER

FIGURE 11. ARTIFICIAL RESPONSE CHARACTERISTICS: $\psi_0 = .1$
FIGURE 12. DEPENDENCE OF FREQUENCY RESPONSE ON CABLE IMPEDANCE
WITH $\psi_0$ AS A PARAMETER: $|R_1| = 1/\sqrt{2}$
A. MAGNITUDE OF R WITH $\theta_1$ AS A PARAMETER

B. PHASE OF R WITH $\theta_1$ AS A PARAMETER

FIGURE 13. RESPONSE CHARACTERISTICS: $\psi_0 = .1$, $r_c = 50 \Omega / Z_0 \approx .1327$
A. MAGNITUDE OF R WITH $\theta_1$ AS A PARAMETER

B. PHASE OF R WITH $\theta_1$ AS A PARAMETER

FIGURE 14. RESPONSE CHARACTERISTICS: $\psi_0 = .1$, $r_c = 100\Omega / Z_0 \approx .2654$
Appendix A: Calculation of $\Lambda_n$

In equation 91 we have a function which is expressed as an integral given by

$$\Lambda_n = \frac{1}{\pi} \int_{-1}^{1} [1 - \zeta^2]^{-1/2} \left[1 - \psi_o^2 \zeta^2 \right]^{1/2} p_{n}^{-1}(\psi_o \zeta) d\zeta \quad (A1)$$

This function appears in the coefficients of the field expansions used to calculate the internal and external admittances of the sensor. The purpose of this appendix is to calculate this integral.

Define

$$\Xi_1 = \int_{-1}^{1} [1 - \zeta^2]^{1/2} \left[1 - \psi_o^2 \zeta^2 \right]^{\mu/2} p_{V}^{p}(\psi_o \zeta) d\zeta \quad (A2)$$

Rearrange this as

$$\Xi_1 = \int_{-1}^{0} [1 - \zeta^2]^{1/2} \left[1 - \psi_o^2 \zeta^2 \right]^{\mu/2} p_{V}^{p}(\psi_o \zeta) d\zeta$$

\[= \int_{0}^{1} [1 - \zeta^2]^{1/2} \left[1 - \psi_o^2 \zeta^2 \right]^{\mu/2} p_{V}^{p}(\psi_o \zeta) d\zeta \quad (A3) \]

\[+ \int_{0}^{1} [1 - \zeta^2]^{1/2} \left[1 - \psi_o^2 \zeta^2 \right]^{\mu/2} p_{V}^{p}(-\psi_o \zeta) d\zeta \]

\[+ \int_{0}^{1} [1 - \zeta^2]^{1/2} \left[1 - \psi_o^2 \zeta^2 \right]^{\mu/2} p_{V}^{p}(\psi_o \zeta) d\zeta \quad (A3) \]

giving

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\[ \Xi_1 = \int_0^1 (1 - \zeta^2)^{\lambda-1} \left[ 1 - \psi_0^2 \zeta^2 \right]^{\mu/2} \left[ \frac{\mathbf{P}_\nu^\mu(\psi_0 \zeta) + \mathbf{P}_\nu^\mu(-\psi_0 \zeta)}{\mathbf{P}_\nu^\mu(\psi_0 \zeta) + \mathbf{P}_\nu^\mu(-\psi_0 \zeta)} \right] d\zeta \quad (A4) \]

Now we have a special case of a hypergeometric function as \[ A, 2A \]

\[ \left[ 1 - \psi_0^2 \zeta^2 \right]^{\frac{1}{2} \left( \frac{1}{2} - \alpha - \beta \right)} \left[ \frac{1}{\pi} \frac{\Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \beta \right) \Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \gamma \right)}{\Gamma \left( \frac{1}{2} + \alpha \right) \Gamma \left( \frac{1}{2} + \beta \right)} \right] \left[ \frac{1}{\pi} \frac{\Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \beta \right) \Gamma \left( \frac{1}{2} \alpha + \frac{1}{2} \gamma \right)}{\Gamma \left( \frac{1}{2} + \alpha \right) \Gamma \left( \frac{1}{2} + \beta \right)} \right] \]

for \( 0 < \psi_0^2 \zeta^2 < 1 \). Let

\[ \mu = \frac{1}{2} - \alpha - \beta , \quad \nu = \alpha - \beta - \frac{1}{2} \quad (A6) \]

giving

\[ \alpha = \frac{\nu - \mu + 1}{2} , \quad \beta = - \frac{\nu + \mu}{2} \quad (A7) \]

and

\[ \left[ 1 - \psi_0^2 \zeta^2 \right]^{\mu/2} \left[ \frac{\mathbf{P}_\nu^\mu(\psi_0 \zeta) + \mathbf{P}_\nu^\mu(-\psi_0 \zeta)}{\mathbf{P}_\nu^\mu(\psi_0 \zeta) + \mathbf{P}_\nu^\mu(-\psi_0 \zeta)} \right] \]

\[ = \frac{\pi^{1/2} \mu + \nu}{\Gamma \left( \frac{1}{2} + \alpha \right) \Gamma \left( \frac{1}{2} + \beta \right)} \left[ \mathbf{P}_\nu^\mu(\psi_0 \zeta) + \mathbf{P}_\nu^\mu(-\psi_0 \zeta) \right] \]

1A. Ref. 6, eqn. 15.4.23.

2A. Ref. 9, p. 53.
This then gives

\[ \Xi_1 = \frac{\pi^{1/2} \mu + 1}{\Gamma\left(\frac{1}{2} - \mu - \nu\right) \Gamma\left(1 - \mu + \nu\right)} \int_0^1 [1 - \zeta^2]^{\lambda - 1} F\left(-\frac{\mu + \nu}{2}, \frac{1 - \mu + \nu}{2}; \frac{1}{2}; \psi_0^2 \zeta^2\right) d\zeta \]  

(A9)

The hypergeometric function is given by

\[ F(a, b; c; z) = \sum_{q=0}^{\infty} \frac{(a)_q (b)_q}{(c)_q} \frac{z^q}{q!} \]  

(A10)

where the Pochhammer symbol\(^3\) is defined by

\[ (a)_0 \equiv 1 \]  

(A11)

\[ (a)_q = a(a + 1)(a + 2) \cdots (a + q - 1) \]

and provided \(a\) is not a negative integer or zero

\[ (a)_q = \frac{\Gamma(a + q)}{\Gamma(a)} \]  

(A12)

Then we have the integral

\[ \Xi_2 \equiv \int_0^1 [1 - \zeta^2]^{\lambda - 1} F\left(-\frac{\mu + \nu}{2}, \frac{1 - \mu + \nu}{2}; \frac{1}{2}; \psi_0^2 \zeta^2\right) d\zeta \]

\[ = \sum_{q=0}^{\infty} \frac{(-\frac{\mu + \nu}{2})^q (\frac{1 - \mu + \nu}{2})^q \psi_0^{2q}}{q!} \int_0^1 [1 - \zeta^2]^{\lambda - 1} \zeta^{2q} d\zeta \]  

(A13)

\(^3\) Ref. 6, eqn. 6.1.22.
This leaves us with the integral (in which we let $\xi = \xi^2$) as

$$
\Xi_3 = \int_0^1 [1 - \xi^2]^{\lambda-1} \xi^2 d\xi
$$

$$
= \frac{1}{2} \int_0^1 [1 - \xi]^{\lambda-1} \xi^{-\frac{1}{2}} d\xi
$$

$$
= \frac{1}{2} B(\lambda, q + \frac{1}{2}) = \frac{1}{2} \frac{\Gamma(\lambda)\Gamma(q + \frac{1}{2})}{\Gamma(\lambda + q + \frac{1}{2})}
$$

(A14)

where $B$ is the well known beta function and we need $\text{Re}[\lambda] > 0$.

Recombining these terms gives

$$
\Xi_2 = \frac{\Gamma(\frac{1}{2})\Gamma(\lambda)}{2\Gamma(\lambda + \frac{1}{2})} \sum_{q=0}^{\infty} \left( - \frac{u+v}{2} \right)^q \frac{(1-\mu+\nu)}{q!} \psi_q^{2q} \frac{\Gamma(\lambda + q + \frac{1}{2})}{q!}
$$

$$
= \frac{\pi^{1/2} \Gamma(\lambda)}{2\Gamma(\lambda + \frac{1}{2})} F\left( - \frac{u+v}{2}, \frac{1-\mu+\nu}{2}; \lambda + \frac{1}{2}; \psi^{2}\right)
$$

(A15)

and then

$$
\Xi_1 = \frac{\pi^{2}\Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})\Gamma(\frac{1}{2} - \frac{u}{2} - \frac{v}{2})\Gamma(1 - \frac{u}{2} + \frac{v}{2})} F\left( - \frac{u+v}{2}, \frac{1-\mu+\nu}{2}; \lambda + \frac{1}{2}; \psi^{2}\right)
$$

(A16)

This result for $\Xi_1$ is very similar to a formula in a standard reference. However, in the course of our present investigations we found this formula to be in error, most likely through a misprint. This led us to do the above derivation to find the correct formula. We would like to thank Prof. Fritz Oberhettinger of Oregon State University for our discussions with him on this problem.

In equations A2 and A16 let

\[ \lambda \equiv \frac{1}{2}, \quad \mu \equiv 1, \quad \nu \equiv n \]  \hspace{1cm} (A17)

Then we have

\[ \Lambda_n = \frac{1}{\pi} \Xi_1 \]

\[ = \frac{2\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} \operatorname{F}\left(-\frac{n+1}{2}, \frac{n}{2}; 1; \psi_0^2\right) \]  \hspace{1cm} (A18)

Since \((n+1)/2\) is a positive integer we have

\[ \Gamma\left(\frac{n+1}{2}\right) = \left(\frac{n-1}{2}\right)! \frac{(n-1)!!}{n} \]

\[ = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} = \left(-\frac{n}{2}\right)\left(-\frac{n}{2} + 1\right) \cdots \left(-\frac{1}{2}\right) = \frac{(-1)^{n+1}}{n} n!! \]  \hspace{1cm} (A19)

This gives

\[ \Lambda_n = (-1)^\frac{n+1}{2} \frac{n!!}{(n-1)!!} \operatorname{F}\left(-\frac{n+1}{2}, \frac{n}{2}; 1; \psi_0^2\right) \]  \hspace{1cm} (A20)

Using equation 71 this can be written

\[ \Lambda_n = \left(\psi_0^{1/2}\right)^n \operatorname{F}\left(-\frac{n+1}{2}, \frac{n}{2}; 1; \psi_0^2\right) \]  \hspace{1cm} (A21)

Writing out the series we have
\[ \Lambda_n = p_0^n(0) \sum_{q=0}^{\frac{n+1}{2}} \frac{(-\frac{n+1}{2})_{q} (\frac{n}{2})_{q}}{(q!)^2} \psi_0^{2q} \]  

(A22)

Note that since \((n+1)/2\) is a positive integer the series in equation A22 has only a finite number of terms, making \(\Lambda_n\) a polynomial in \(\psi_0\).
Appendix B: Properties of Admittance Series for Large $n$

The most difficult parts of the sensor response functions to calculate are the internal and external admittances because of the slow convergence of the series. In this appendix we consider the behavior of these series for large $n$.

From equation 105 the normalized internal admittance is

$$Y_{int} = \sum_{n=1}^{\infty} \pi \frac{n+1}{n(n+1)} \left[ p_n^1(n) \right]^2 F\left(-\frac{n+1}{2}, \frac{n}{2}; 1; \frac{\psi^2}{\nu^2}\right) \frac{kaj_n(ka)}{kaj_n^2(ka)}$$

(B1)

and from equation 110 the normalized external admittance is

$$Y_{ext} = \sum_{n=1}^{\infty} (-i) \pi \frac{n+1}{n(n+1)} \left[ p_n^1(n) \right]^2 F\left(-\frac{n+1}{2}, \frac{n}{2}; 1; \frac{\psi^2}{\nu^2}\right) \frac{kah_n(2)(ka)}{kah_n^2(2)(ka)}$$

(B2)

where from equation 71 we have

$$p_n^1(n) = (-1)^n \frac{n+1}{2n} \frac{n!}{(n-1)!!} \quad \text{for } n \text{ odd}$$

(B3)

Define the individual terms in the summation for $Y_{int}$ as $Y_{int_n}$ and in the summation for $Y_{ext}$ as $Y_{ext_n}$ so that equations B1 and B2 can be rewritten as

$$Y_{int} = \sum_{n=1}^{\infty} Y_{int_n}$$

(B4)

and

$$Y_{ext} = \sum_{n=1}^{\infty} Y_{ext_n}$$

(B5)

Define

$$n_1 = n_2 - 2 > 0$$

(B6)

where $n_1$ and $n_2$ are odd integers. Then write the normalized admittances as
\[ Y_{\text{int}} = \sum_{n=1}^{n_1,2} y_{\text{int}_n} + \Delta_{\text{int}} \]  
\[ Y_{\text{ext}} = \sum_{n=1}^{n_1,2} y_{\text{int}_n} + \Delta_{\text{ext}} \]  

Then \( \Delta_{\text{int}} \) and \( \Delta_{\text{ext}} \) are the errors introduced by truncating the two series with last terms given by \( n = n_1 \). These error terms are written as

\[ \Delta_{\text{int}} = \sum_{n=n_2}^{\infty} y_{\text{int}_n} \]  
\[ \Delta_{\text{ext}} = \sum_{n=n_2}^{\infty} y_{\text{ext}_n} \]

In this appendix we consider the individual terms in the sums and the error terms for large \( n \).

Consider the parts of the individual terms in the admittance sums. Define an integer as

\[ N = \frac{n + 1}{2} \]  

where only odd \( n > 0 \) are of interest. For the hypergeometric function we have

\[ F_n \equiv F \left( -\frac{n+1}{2}, \frac{n}{2}; 1; \psi_o^2 \right) = F \left( -N, N - \frac{1}{2}; 1; \psi_o^2 \right) \]

\[ = \sum_{q=0}^{\frac{n+1}{2}} \frac{(-\frac{n+1}{2}) (\frac{n}{2})}{(\frac{1}{2})^q q^q} \psi_o^{2q} \]  

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Now this particular hypergeometric function can be written as a Jacobi polynomial using\textsuperscript{1B}

\[
F(-N, \alpha+1+\beta+N; \alpha+1; x) = \sum_{q=0}^{N} \frac{(-N)^{N-q}}{(N-q)!} \psi_o^{2q} \psi_0^{N} (1 - 2x)
\]

which gives

\[
F_n = \sum_{n=0}^{N} \binom{N}{n} x^n (1 - 2x)
\]

The Jacobi polynomials can be written as\textsuperscript{2B}

\[
P_N^{(\alpha, \beta)}(x) = 2^{-N} \sum_{q=0}^{N} \binom{N+\alpha}{q} \binom{N+\beta}{N-q} x^q (1 - x)^{N-q} q (N+q+1)
\]

where binomial coefficients are given by

\[
\binom{N+\alpha}{q} = \frac{\Gamma(N+\alpha+1)}{q! \Gamma(N+\alpha-q+1)} = \frac{(N+\alpha-q+1)_q}{q!}
\]

\[
\binom{N+\beta}{N-q} = \frac{\Gamma(N+\beta+1)}{(N-q)!(\beta+q+1)} = \frac{(q+\beta+1)_{N-q}}{(N-q)!}
\]

The hypergeometric function of interest can then also be written

\[
F_n = 2^{-N} \sum_{q=0}^{N} \binom{N}{q} \binom{N-3}{N-q} \psi_o^{2q} (2 - 2\psi_o^2)^q
\]

\textsuperscript{1B.} Ref. 6, eqn. 15.4.6.

\textsuperscript{2B.} Ref. 9, p. 211.
\[
= \sum_{q=0}^{N} (-1)^{N-q} \binom{N}{q} \left(\frac{N}{N-q}\right)^{\frac{3}{2}} \left(\psi_o^2\right)^{N-q} \left(1 - \psi_o^2\right)^{q}
\]  
(B15)

Note that the Jacobi polynomial in equation B13 is a polynomial function of both \(a\) and \(\beta\) as well as its argument; the same is true of the hypergeometric function in equation B11. The range of \(a\) and \(\beta\) are often restricted to \(a > -1\) and \(\beta > -1\) in order to make the weight functions integrable. However, we are not concerned here with the orthogonality properties of the Jacobi polynomials. Since the Jacobi polynomials are polynomial functions of \(a\) and \(\beta\), as is the hypergeometric function, then both series are identical for all \(a\) and \(\beta\). Thus \(\beta = -(3/2)\) is allowed for the series represented in equation B15.

With the hypergeometric function represented as a Jacobi polynomial we look at the asymptotic form for large \(N\) (implying large \(n\)) of the Jacobi polynomials. As \(N \to \infty\) for \(0 < \xi < \pi\) we have

\[
P_N^{(\alpha, \beta)}(\cos(\xi)) = \frac{\cos \left[ \left(N + \frac{1}{2}(\alpha+\beta+1)\right)\xi - \frac{\pi}{4}(1+2\alpha) \right]}{(\pi N)^{1/2}} \left[\sin \left(\frac{\xi}{2}\right)\right]^{\alpha + \frac{1}{2}} \left[\cos \left(\frac{\xi}{2}\right)\right]^{\beta + \frac{1}{2}} + O \left(\frac{-3}{2}\right)
\]  
(B16)

Set

\[
\cos(\xi) = 1 - 2\psi_o^2
\]  
(B17)

giving

\[
\sin \left(\frac{\xi}{2}\right) = \left[\frac{1 - \cos(\xi)}{2}\right]^{1/2} = \psi_o
\]  
(B18)

\[
\cos \left(\frac{\xi}{2}\right) = \left[\frac{1 + \cos(\xi)}{2}\right]^{1/2} = \left[1 - \psi_o^2\right]^{1/2}
\]


4B. Ref. 9, p. 216.
Applying this to equation B12 for $0 < \psi < 1$ gives as $n \to \infty$

$$F_n = \left[\frac{1-\psi^2}{\pi N\psi}\right]^{1/2} \cos\left[(N - \frac{1}{4})\arccos\left(1-2\psi^2\right) - \frac{\pi}{4}\right] + O\left(n^{-\frac{3}{2}}\right) \quad (B19)$$

Thus as $n \to \infty$ the envelope of $F_n$ falls off proportional to $n^{-(1/2)}$. Also note as $\psi \to 0$ we have

$$\arccos\left(1 - 2\psi^2\right) = 2\psi + O\left(\psi^3\right) \quad (B20)$$

As a check $F_n$ was calculated from equation B10 and the asymptotic form in equation B19 was observed to approach the calculated values for large $n$ and for values of $\psi$ used for the graphs.

Now turn to the Legendre function. As $\nu \to \infty$ for fixed $\mu$ and $0 < \zeta < \pi$ we have\(^{5B}\)

$$P_{\nu}^\mu(\cos(\zeta)) = \frac{\Gamma(\nu+\mu+1)}{\Gamma\left(\nu + \frac{3}{2}\right)}\left(\frac{\pi}{2}\sin(\zeta)\right)^{-1/2} \cos\left[(\nu + \frac{1}{2})\zeta - \frac{\pi}{4} + \frac{\mu\pi}{2}\right] + O(\nu^{-1}) \quad (B21)$$

Thus as $n \to \infty$ and remembering that $n$ is odd we have

$$p_{n}^1(0) = \frac{\Gamma(n+2)}{\Gamma\left(n + \frac{3}{2}\right)}\left(\frac{2}{\pi}\right)^{1/2} \cos\left[(n + 1)\frac{\pi}{2}\right] + O(n^{-1})$$

$$= (-1)^{\frac{n+1}{2}} \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(n+2)}{\Gamma\left(n + \frac{3}{2}\right)} + O(n^{-1}) \quad (B22)$$

The gamma functions have an asymptotic form known as the Stirling approximation\(^{6B}\) which has as $\nu \to \infty$

\(^{5B}\) Ref. 6, eqn. 8.10.7.

\(^{6B}\) Ref. 6, eqn. 6.1.37.
\[
\Gamma(v) = e^{-v \nu} \nu^{-\frac{1}{2}} (2\pi)^{1/2} [1 + O(v^{-1})]
\] (B23)

Applying this to the gamma functions in equation B22 gives as \(n \to \infty\)

\[
\frac{\Gamma(n+2)}{\Gamma(n+3/2)} = e^{-n-2(n+2)} \frac{n+3}{n+1} [1 + O(n^{-1})]
\]

\[
e^{-\frac{1}{2}} e^{\left(n+\frac{3}{2}\right)\ln(n+2) - (n+1)\ln\left(n+\frac{3}{2}\right)} [1 + O(n^{-1})]
\]

\[
e^{\exp\left((n+1)\ln\left(n+\frac{3}{2}\right) + \frac{1}{2} \ln(n+2) - \frac{1}{2}\right) [1 + O(n^{-1})]}
\]

\[
e^{\exp\left[(n+1)\left[\frac{1/2}{n+3/2} + O(n^{-2})\right] + \frac{1}{2} \ln(n+2) - \frac{1}{2}\right] [1 + O(n^{-1})]}
\]

\[
e^{\exp\left[\ln\left((n+2)^{1/2}\right) + O(n^{-1})\right] [1 + O(n^{-1})]}
\]

\[
(n+2)^{1/2} e^{O(n^{-1})} [1 + O(n^{-1})]
\]

\[
= n^{1/2} [1 + O(n^{-1})]
\] (B24)

Thus for the Legendre function as \(n \to \infty\) we have

\[
P_n^1(0) = (-1)^n \frac{n+1}{2} \left(\frac{2}{\pi}\right)^{1/2} n^{1/2} [1 + O(n^{-1})]
\] (B25)

\[
\left[P_n^1(0)\right]^2 = \frac{2}{\pi} n [1 + O(n^{-1})]
\]
Next are the Bessel functions. As $n \to \infty$ we have

$$\frac{k a_n^j(k a)}{[k a_n^j(k a)]^\tau} = \left[ \left( \frac{ka}{(2n+1)!} \right)^{n+1} \left[ 1 + O(n^{-1}) \right] \right] \cdot \left[ \frac{(n+1)(ka)^n}{(2n+1)!} \left[ 1 + O(n^{-1}) \right] \right]^{-1}$$

$$= \frac{ka}{n+1} [1 + O(n^{-1})]$$

$$= \frac{ka}{n} [1 + O(n^{-1})]$$

and

$$\frac{k a_n^{(2)}(ka)}{[k a_n^{(2)}(ka)]^\tau} = \left[ \frac{i(2n-1)!}{(ka)^n} \left[ 1 + O(n^{-1}) \right] \right] \cdot \left[ \frac{i(2n-1)!}{(ka)^{n+1}} \left[ 1 + O(n^{-1}) \right] \right]^{-1}$$

$$= - \frac{ka}{n} [1 + O(n^{-1})]$$

Now we can consider the terms in the sums for the normalized admittances. As $n \to \infty$ we have

$$\gamma_{int_n} = i \pi \frac{2n+1}{n(n+1)} \frac{2}{\pi} n [1 + O(n^{-1})] \frac{ka}{n} [1 + O(n^{-1})]$$

$$\cdot \left[ \frac{[1-\psi_o^2]^{1/2}}{[\pi \psi_o]^{1/2}} \cos \left[ \left( N - \frac{1}{4} \right) \arccos \left( 1-2\psi_o^2 \right) - \frac{\pi}{4} \right] + O(n^{-\frac{3}{2}}) \right]$$

$$= i \frac{4ka}{n} [1 + O(n^{-1})] \left[ 1 - \psi_o^2 \right]^{1/2} \left[ \frac{2}{\pi n \psi_o} \right]^{1/2}$$

$$\cdot \left[ \cos \left[ \left( \frac{n}{2} + \frac{1}{4} \right) \arccos \left( 1 - 2\psi_o^2 \right) - \frac{\pi}{4} \right] + O(n^{-1}) \right]$$

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This last result is also the asymptotic form for $Y_{\text{ext}}$ for large $n$. Thus the individual terms in the sums fall off like $n^{-3/2}$ for large $n$.

Equation B8 gives the truncation error in stopping the admittance sums at $n = n_1$. Define an upper bound for these errors by

$$\Delta_1 = \max(|\Delta_{\text{int}}|, |\Delta_{\text{ext}}|)$$

Using the asymptotic form developed in equation B28 we can give an approximate value for $\Delta_1$ by setting the cosine to one and summing the magnitude of the dominant term giving

$$\Delta_1 \approx 4ka \left[1 - \psi_o^2\right]^{1/2} \left[\frac{1}{2\pi\psi_o}\right]^{1/2} \sum_{n=n_2}^{n_\infty} \frac{3}{2} n^{-3/2}$$

Replacing the sum by an integral we have

$$\Delta_1 = 4ka \left[1 - \psi_o^2\right]^{1/2} \left[\frac{1}{2\pi\psi_o}\right]^{1/2} \left[\frac{1}{2\pi\psi_o n_2}\right] \int_{n_2}^{n_\infty} \frac{3}{2} \frac{1}{\nu^{3/2}} d\nu$$

$$= 4ka \left[1 - \psi_o^2\right]^{1/2} \left[\frac{2}{\pi\psi_o n_2}\right]^{1/2} \frac{1}{n_2^{3/2}}$$

$$= 4ka \left[1 - \psi_o^2\right]^{1/2} \left[\frac{2}{\pi\psi_o n_2}\right]^{1/2}$$

which for small $\psi_o$ is

$$\Delta_1 \approx 4ka \left[\frac{2}{\pi\psi_o n_2}\right]^{1/2}$$

Actually this is a rather large overestimate of the truncation error since the cosine function alternates in sign and it was replaced by one.
Appendix C: Numerical Techniques for Computer Calculations
Joe P. Martinez, Dikewood

The numerical results plotted in figures 4 and 6 through 14 were calculated using the Control Data Corporation 6600 Computer in the Air Force Weapons Laboratory at Kirtland AFB.

The calculations for the short-circuit current transfer function and the admittances included the use of the spherical Bessel functions. These functions were computed by using both forward and backward recurrence techniques. During the initial attempts at producing the Bessel functions of the first kind, the \( j_n(ka) \), machine round-off error was encountered with the use of the forward recurrence relationship as well as by series representation. Backward recurrence was found to be one solution in overcoming this problem. In using this technique, where \( N \) is the largest order desired, the \( (N + 6) \)th function is set equal to zero, the \( (N + 5) \)th function is set equal to \( 10^{-50} \), and the recurrence relationship,

\[
\frac{2n + 1}{ka} \cdot j_n(ka) - j_{n+1}(ka) = 0
\]

is then used to determine the functions to \( n = 1 \). A ratio of the exact value to the calculated value at \( n = 1 \) is taken and all the calculated values to \( n = N \) are multiplied by this ratio. The Bessel functions of the second kind, \( y_n(ka) \), are calculated by finding the first two orders and then applying the recurrence relationship

\[
y_{n+1}(ka) = \frac{2n + 1}{ka} y_n(ka) - y_{n-1}(ka)
\]

By comparing the values produced by these methods against tables found in handbooks, it was determined that accuracy to eight significant figures was obtained over the range of \( n \) and \( ka \) used in the computations for this note.

Legendre functions of order \( 1 \) are used in the calculations of equation 72. These were calculated by computing the functions of degree 1 and 2 and then applying the recurrence formula

\[
\frac{p^1_{n+1}(x)}{p^1_n(x)} = \frac{2 + \frac{1}{n}}{2 + \frac{1}{n} - \left(1 + \frac{1}{n}\right)} p^1_{n-1}(x)
\]

where \( x = \cos(\theta_1) \). Results obtained by this method were compared with handbook tables and with calculations done by series representations and backward recurrence. These values were found to be
accurate to eight significant figures for the range of \( n \) used in the computations.

While performing the calculations for the short-circuit current transfer function (equation 72) it was determined that the series was a rapidly converging one and that it met the ratio test for convergence. This meant that an upper bound on the absolute error could be expressed as \(^{1}\)

\[
\varepsilon \leq \frac{|a_{N-2}|^2}{|a_{N-2}| - |a_N|} \quad \text{(C4)}
\]

where \( a_N \) is the last term included in the summation. As explained in Section IV, only the odd-numbered \( n \) are included in the summation. The relative error will then be

\[
\varepsilon_r = \left| \frac{2\varepsilon}{r} \right| \leq E \quad \text{(C5)}
\]

In the calculations, \( E \) was set at .001. The number of terms necessary to satisfy this criterion varied with \( ka \), the larger values requiring more terms. Generally the error criterion was satisfied in fewer than 100 orders (50 terms).

The calculation of the \( Y_{\text{int}} \) and \( Y_{\text{ext}} \) sums (equations 105 and 110) were handled in a different manner than that of the \( T \) sum due to their slow convergence. The behavior of the expressions involving the spherical Bessel functions and the hypergeometric function were studied separately to determine how the numerical calculations could be made.

It was shown in appendix B that, as \( n \to \infty \)

\[
\frac{kaj_n(ka)}{[kaj_n(ka)]'} \to \frac{ka}{n+1} \quad \text{(C6)}
\]

and

\[
\frac{kah_n^{(2)}(ka)}{[kah_n^{(2)}(ka)]'} \to -\frac{ka}{n} \quad \text{(C7)}
\]

\(^{1}\) W. Kaplan, Advanced Calculus, Addison-Wesley, 1952, pp. 328, 329.
Numerical calculations were made in order to determine at what point the asymptotic forms of these expressions could be used with minimum loss of accuracy. It was desirable to switch to the asymptotic form at some point to avoid round-off numerical errors due to the extremely small magnitude of the \( j_n(ka) \) at large \( n \). It was found that at \( n = 10|ka| + 30 \) the asymptotic forms may be used, with a relative error of .03% at \( ka = 1.0 \) introduced in the terms. Of course, as \( n \) gets larger this error will diminish with each term.

In performing the numerical calculations for the hypergeometric function of equations 105 and 110 it was found that numerical round-off error was introduced after the sum of equation 103 was carried to a large number of terms, the number depending on \( \psi_0 \). Going to double precision on the computer greatly improved the situation, as now the number of usable values was doubled. But, the round-off errors were still introduced before the \( \text{Yint} \) and \( \text{Yext} \) sums could be completed. As was mentioned in appendix B, the asymptotic form (equation B19) approaches \( F_n \) for large \( n \). So it was decided to switch to this form just before the numerical errors started to be significant. Table C1 is a summary of the pertinent values which occur immediately before the switch-over points.

<table>
<thead>
<tr>
<th>( \psi_0 )</th>
<th>( F \left( \frac{n+1, n}{2, \frac{1}{2}}; \psi_0 \right) )</th>
<th>Asymptotic</th>
<th>Absolute Difference</th>
<th>Relative Difference (Percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>.001</td>
<td>24999</td>
<td>.055569</td>
<td>.055816</td>
<td>.000246</td>
</tr>
<tr>
<td>.01</td>
<td>3199</td>
<td>.093564</td>
<td>.093394</td>
<td>.000170</td>
</tr>
<tr>
<td>.1</td>
<td>319</td>
<td>.095720</td>
<td>.096555</td>
<td>.000835</td>
</tr>
</tbody>
</table>

Table C1. Values of the hypergeometric function one term before the switch-over point

Now that methods have been established to calculate the terms involving the Bessel functions and the hypergeometric function, the calculation of the \( \text{Yint} \) and \( \text{Yext} \) sums may proceed. At some \( n_1 \) a switch to the asymptotic forms of equations C6 and C7 takes place, so that equation 105 becomes

\[
\text{Yint} = i\pi \sum_{n=1}^{n_1} \frac{2^{n+1}}{n(n+1)} \left[ \frac{n!!}{(n-1)!!} \right]^2 F \left( \frac{n+1, n}{2, \frac{1}{2}}; \psi_0 \right)^2 \frac{\text{ka}_n(ka)}{\text{ka}_n(ka)} + \text{ka} \sum_{n=n_1+2}^{\infty} \frac{2^{n+1}}{n} \left[ \frac{n!!}{(n+1)!!} \right]^2 F \left( \frac{n+1, n}{2, \frac{1}{2}}; \psi_0 \right)^2 \right]
\]  

\[\text{(C8)}\]
Similarly, equation 110 becomes

\[
Y_{\text{ext}} = i\pi \left[ -\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left[ \frac{n!!}{(n-1)!!} \right]^2 F \left( \frac{-n+1}{2}, \frac{n}{2}; 1; \psi_0^2 \right) \frac{\operatorname{kh}_n^{(2)}(ka)}{\operatorname{kh}_n^{(2)}(ka)} \right] + ka \sum_{n=n_1+2}^{\infty} \frac{2n+1}{n+1} \left[ \frac{(n-2)!!}{(n-1)!!} \right]^2 F \left( \frac{-n+1}{2}, \frac{n}{2}; 1; \psi_0^2 \right) \right]
\]

(C9)

Since the switch-over occurs when \( n > 10|ka| + 30 \), and the largest value of \( ka \) used in these calculations was 10.0, all changeovers to the asymptotic forms will take place by \( n > 131 \). It was found to be convenient to calculate the sums, with \( \pi\psi_0 ka \) factored out, from \( n = 201 \) to \( \infty \) (n odd) first, and then include these values in the calculations later. Since \( ka \) can be factored out from the last part of the summation, these values may be used with any \( ka \). The next problem was that of determining how far \( n \) should be carried. This was done numerically by running the sums out progressively further to larger \( n \) and noting the relative change. The sums for all \( \psi_0 \) were carried to \( n = 100001 \) and it was noted that they were still changing about 1 part in 1000, or .1%. It was further noted that there were peaks and minimums occurring as the sum progressed, forming a diminishing envelope converging at some number. The last few maxima and minima were then taken and the value to which they were converging was extrapolated to 5 significant digits. Table C2 gives these values. The relative error in the sum is then in the order of .05%. This error is, of course, multiplied by \( \pi ka \) when the sums are used in the admittance calculations.

<table>
<thead>
<tr>
<th>( \psi_0 )</th>
<th>( (i\pi ka)^{-1} \sum_{n=201}^{\infty} Y_{\text{int} n} )</th>
<th>( (i\pi ka)^{-1} \sum_{n=201}^{\infty} Y_{\text{ext} n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.001</td>
<td>( 1.0987 \times 10^0 )</td>
<td>( 1.1013 \times 10^0 )</td>
</tr>
<tr>
<td>.01</td>
<td>( -8.6355 \times 10^{-2} )</td>
<td>( -8.6605 \times 10^{-2} )</td>
</tr>
<tr>
<td>.1</td>
<td>( -2.0147 \times 10^{-3} )</td>
<td>( -2.0234 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

Table C2. Values to which admittance sums converge from \( n = 201 \), with \( i\pi ka \) factored out

Most of the other numerical calculations performed for the graphs in this note are straightforward and no explanation is required.