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The Small-Slope Approximation for Monostatic Scattering

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Abstract

By using a variational principle for scattering by rough interfaces, the functional derivative of the scattered part of an acoustic field originating from a point source is derived *directly* in the time domain. This general result is specialized to the case of scattering by surfaces on which fields satisfy Dirichlet boundary conditions. It is shown that when the small- slope appoximation is used for surface fields, a simple result can be obtained for monostatic scattering.

I   Introduction

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The small-slope approximation was developed by Voronovich to treat scattering of plane, monochromatic, scalar waves by rough interfaces [1]. Numerical experiments have shown that the zeroth order small-slope approximation, generally out-performs the conventional Kirchhoff approximation without added computational work [2]. The point of the present work is to show how the lowest order small-slope approximation is expressed when point, pulsed sources are considered. It turns out that in the case of monostatic scattering from a rough surface on which Dirichlet boundary conditions are satisfied, the zeroth order small-slope approximation takes on a particularly simple form. This result is readily compared to a result for monostatic scattering in the Kirchhoff approximation, Eqs.(31, 32) below. The latter was developed by Berry [3] for the purpose of inferring surface roughness from the shape of pulse echoes.

Recently Dashen and Wurmser [4] rediscovered a perturbation scheme for acoustic and electromagnetic scattering which was previously developed by Burrows [5] for electromagnetic scattering by surfaces. The small-slope and other approximation schemes are conveniently derived from this perturbation theory, at least for plane wave scattering [4, 6]. Rather than synthesize plane waves to obtain the present results for point sources, the perturbation result of Dashen and Burrows will be rederived directly for time-domain Green's functions. In the plane wave case, once a formula is obtained for the derivative of the scattered field with respect to upward rigid translations of the surface, it is easy to derive a result for the scattering amplitude. This is because the plane wave scattering amplitude changes by a simple phase factor when the scattering surface is rigidly translated [6]. There is no such simple way to obtain comparable formulas for Green's functions since each plane wave making up the Green's function changes phase by a different amount. However, as will be shown, for the case of monostatic scattering it is possible to integrate the functional derivative of the field with respect to the surface shape after the small-slope approximation is made for the surface fields. This proceedure results in a simple expression for the echo of a pulse heard at the location of the origin of the pulse.

In the Burrows-Dashen result, the perturbation of the scattered field is expressed in terms of products of the total field or its derivatives on the scattering surface. In contrast, the Helmholtz-Kirchhoff formula expresses the scattered field itself in terms of products

of unscattered and total fields on the scattering surface. In the Burrows-Dashen result, reciprocity is manifest, but, in general, only derivatives of scattered fields are obtained. In the Helmholtz-Kirchhoff result, reciprocity, though respected, is not manifest.

In section II the Burrows-Dashen perturbation result will be rederived directly for point sources in the time domain . In Section III the new result for monostatic scattering will be derived from the perturbation result. It will be compared to a corrected version of Berry's result based on the Kirchhoff approximation.

II   The Variational Principle and Perturbation Theory

The Burrows-Dashen perturbation scheme can be derived from the principle of least action. Kohn [7] and later Altshuler [8] used specializations of the least action principle to derive variational principles for scattering amplitudes and the field scattered by plane waves for the case of quantum potential scattering. Here these results are generalized to the case of point sources and point receivers in the time domain in media with variable densities and sound speeds. Furthermore, layered media with irregular fluid-fluid interfaces (across which pressure and normal velocity are continuous) will be considered. To treat point sources it will be assumed that trial functions vanish at infinity because there is a small dissipation. The general variational principle, which is just the expression of the principle of least action, is interesting because it embodies the applications of Green's theorem that go into a variety of approximations [9]. However, it will be used here only to treat the very special case of a single Dirichlet boundary, constant sound speed and density, and monostatic geometry. One advantage of approaching the perturbation formula through the variational principle is that some of the complications of continuing normal derivatives away from the unperturbed surface which were encountered by Dashen and Wurmser, can be avoided. It should also be pointed out that formulas similar to those presented by Dashen and Wurmser not only were given by Burrows, but also by Garabedjian in 1955 and apparently by Hadamard even earlier [10].

II.1   The Variational Principle

Consider acoustic waves in a medium with spatially variable ambient density, $\rho(\mathbf{r})$ and sound speed, $c(\mathbf{r})$. The Green's function G_e, (e for exact) for the pressure at a point $\mathbf{r'}$ at

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time t' arising from an impulsive source at $\mathbf{r}_o$ at time t_o satisfies the wave equation

$$\rho(\mathbf{r}')\nabla'\frac{1}{\rho(\mathbf{r}')}\nabla'G_e(\mathbf{r}',t';\mathbf{r}_o,t_o) - \frac{1}{c^2(\mathbf{r}')}\partial_{t'}^2 G_e(\mathbf{r}',t';\mathbf{r}_o,t_o) = \delta(\mathbf{r}'-\mathbf{r}_o)\delta(t'-t_o). \quad (1)$$

Boundary conditions are built into this equation: G_e and $\mathbf{n}\cdot\nabla G_e/\rho$ must be continuous. If they did exhibit jump discontinuities, the spatial gradient operators would give rise to δ functions which appear nowhere else in Eq.1. These boundary conditions express the requirement that the pressure G and the normal velocity, $(\mathbf{n}\cdot\partial_t\nabla G)/\rho$, be continuous. Causal Green's functions are assumed here, so that

$$G_e(\mathbf{r}',t';\mathbf{r}_o,t_o) = 0$$

if $t' < t_o$. Reciprocity in the time domain is expressed by [11]

$$\frac{1}{\rho(\mathbf{r}')}G_e(\mathbf{r}',t';\mathbf{r}_o,t_o) = \frac{1}{\rho(\mathbf{r}_o)}G_e(\mathbf{r}_o,-t_o;\mathbf{r}',-t'). \quad (2)$$

A variational estimator Γ for G_e is a functional of trial Green's functions, G_1 and G_2, $\Gamma(G_1,G_2)$, such that if $G_i = G_e + \epsilon g_i$, $i = 1,2$, (where g_i is arbitrary except for certain boundary conditions) then $\Gamma(G_1,G_2) - G_e = O(\epsilon^2)$ for small ϵ [12]. Consider two fluids occupying regions Ω^+ and Ω^- which are separated by an interface S. Within each region the sound speed and density are continuous, but across the interface they are allowed to have jump discontinuities. Trial functions and the exact Green's functions are assumed to vanish at infinity, since a small attentuation is assumed. Trial functions are assumed to be causal as well. In the following equation Γ is a variational estimator of the Green's function $\frac{1}{\rho}G_e$, which depends parametrically on the source location, $(\mathbf{r}_o,t_o)$ and the receiver (or f inal) location $(\mathbf{r}_f,t_f)$.

$$\Gamma(G_2,\mathbf{r}_f,t_f;G_1,\mathbf{r}_o,t_o) = \frac{1}{\rho(\mathbf{r}_o)}G_2(\mathbf{r}_o,-t_o;\mathbf{r}_f,-t_f) + \frac{1}{\rho(\mathbf{r}_f)}G_1(\mathbf{r}_f,t_f;\mathbf{r}_o,t_o)$$

$$+ \int_{-\infty}^{\infty}dt'\int_{\Omega^+\cup\Omega^-}d\mathbf{r}'[\nabla'G_2(\mathbf{r}',-t';\mathbf{r}_f,-t_f)\cdot\frac{1}{\rho(\mathbf{r}')}\nabla'G_1(\mathbf{r}',t';\mathbf{r}_o,t_o)$$

$$-\partial_{t'}G_2(\mathbf{r}',-t';\mathbf{r}_f,-t_f)\frac{1}{c^2(\mathbf{r}')\rho(\mathbf{r}')}\partial_{t'}G_1(\mathbf{r}',t';\mathbf{r}_o,t_o)]$$

$$+ \int_{-\infty}^{\infty}dt'\int_S dS[J[G_2(\mathbf{r}',-t';\mathbf{r}_f,-t_f)]A[\frac{1}{\rho(\mathbf{r}')}\mathbf{n}'\cdot\nabla'G_1(\mathbf{r}',t';\mathbf{r}_o,t_o)]$$

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$$+ A[\mathbf{n}' \cdot \nabla' G_2(\mathbf{r}', -t'; \mathbf{r}_f, -t_f)\frac{1}{\rho(\mathbf{r}')}]J[G_1(\mathbf{r}', t'; \mathbf{r}_o, t_o)]]], \qquad (3)$$

where n is the unit normal to the interface S from Ω^- to Ω^+. The operator J here is the jump across the interface S:

$$J[f(\mathbf{r})] = f(\mathbf{r}^+) - f(\mathbf{r}^-).$$

Likewise, the averaging operator A is

$$A[f(\mathbf{r})] = [f(\mathbf{r}^+) + f(\mathbf{r}^-)]/2.$$

When $G_1 = G_2 = G_{trial}$, the volume integral in Eq.(3) in just the negative of the Langrangian for free scalar fields obeying the Helmholtz equation [14]. The linear terms in G_1 and G_2 account for the fields here being driven by δ-function sources, and the surface integrals are designed to account for boundary conditions at an interface

To show that Γ is a variational estimator of G_e/ρ, write $G_i = G_e + \epsilon g_i$, $i = 1, 2$. Then integrate by parts, separately over $\Omega^\pm$ and over time, to remove the gradient operation and time derivatives from g_i where possible, to show

$$\Gamma(G_e, \mathbf{r}_f, t_f; G_e, \mathbf{r}_o, t_o) = \frac{1}{\rho(\mathbf{r}_f)}G_e(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o), \qquad (4)$$

$$\Gamma(G_e, \mathbf{r}_f, t_f; g_1, \mathbf{r}_o, t_o) = \frac{1}{\rho(\mathbf{r}_f)}G_e(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o), \qquad (5)$$

and

$$\Gamma(g_2, \mathbf{r}_f, t_f; G_e, \mathbf{r}_o, t_o) = \frac{1}{\rho(\mathbf{r}_o)}G_e(\mathbf{r}_o, -t_o; \mathbf{r}_f, -t_f). \qquad (6)$$

This means that

$$\Gamma(G_2, \mathbf{r}_f, t_f; G_1, \mathbf{r}_o, t_o) = \frac{1}{\rho(\mathbf{r}_f)}G_e(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o) + \bar{\Gamma}(\epsilon g_2, \mathbf{r}_f, t_f; \epsilon g_1, \mathbf{r}_o, t_o), \qquad (7)$$

where $\bar{\Gamma}$ is the bilinear form $\Gamma - G_1/\rho - G_2/\rho$. Since $\bar{\Gamma}$ is bilinear, the last term in Eq.(7) is order ϵ^2, and Eq.(7) shows that Γ is a variational estimator. Note that if the G_i are replaced by a single function ψ, the variational estimator Γ is the classical action for the field driven by δ-function sources. If ψ is an arbitrary superpostion of two fields, G_1 and G_2, the variational estimator is seen to follow from the principle least action. The integral along the surface S in the action represents a singular part of the Lagrange density.

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Physically it arises because there is work done if the interface moves under the influence of a pressure difference across the interface.

For the case of a Dirichlet surface replace

$$J[G_i] \rightarrow G_i$$

and

$$A[\mathbf{n} \cdot \nabla G_i / \rho] \rightarrow \mathbf{n} \cdot \nabla G_i / \rho,$$

and for the Neumann case drop the surface integral over S entirely. Note that in the integrals G_2 and G_1 always appear with the same arguments. Hence one could omit the arguments without danger of confusion. Note too, that any number of regions Ω could be considered as long as interface integrals are included in the variational estimator.

II.2 Perturbations

The change in the Green's function caused by a small change in the shape or location of the interface S is found from the variational estimator by choosing the trial functions G_1 and G_2 to be the Green's functions associated with a neighboring interface, S_0. It is assumed that the densities and sound speeds, $\rho^\pm(\mathbf{r})$ and $c^\pm(\mathbf{r})$ can be continued smoothly from $\Omega_0^\pm$ to $\Omega^\pm$ along with the Green's function on either side of the interface, $G_0^\pm$ [13]. Then the variational estimator shows that to first order in the displacement of the surface from S_0 to S, the Green's function G associated with S is given by

$$\frac{1}{\rho(\mathbf{r}_f)} G(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o) \approx \frac{1}{\rho(\mathbf{r}_o)} G_0(\mathbf{r}_o, -t_o; \mathbf{r}_f, -t_f) + \frac{1}{\rho(\mathbf{r}_f)} G_0(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o)$$

$$+ \int_{-\infty}^{\infty} dt' \int_{\Omega^+ \cup \Omega^-} d\mathbf{r}' [\nabla' G_0(\mathbf{r}', -t'; \mathbf{r}_f, -t_f) \cdot \frac{1}{\rho(\mathbf{r}')} \nabla' G_0(\mathbf{r}', t'; \mathbf{r}_o, t_o)$$

$$- \partial_{t'} G_0(\mathbf{r}', -t'; \mathbf{r}_f, -t_f) \frac{1}{c^2(\mathbf{r}')\rho(\mathbf{r}')} \partial_{t'} G_0(\mathbf{r}', t'; \mathbf{r}_o, t_o)]$$

$$+ \int_{-\infty}^{\infty} dt' \int_{S} dS [J[G_0(\mathbf{r}', -t'; \mathbf{r}_f, -t_f)] A[\frac{1}{\rho(\mathbf{r}')} \mathbf{n}' \cdot \nabla' G_0(\mathbf{r}', t'; \mathbf{r}_o, t_o)]$$

$$+ A[\mathbf{n}' \cdot \nabla' G_0(\mathbf{r}', -t'; \mathbf{r}_f, -t_f) \frac{1}{\rho(\mathbf{r}')}] J[G_0(\mathbf{r}', t'; \mathbf{r}_o, t_o)]]. \tag{8}$$

On the other hand, the unperturbed Green's function, G_0, is exactly given by the same formula with the regions $\Omega^\pm$ replaced by the regions $\Omega_0^\pm$. The interface integral vanishes because G_0 is continuous across S_0, by assumption. If this analogous expression for G_0 is subtracted from Eq.8, the left side is the perturbation of the Green's function. The linear terms cancel on the right. This leaves the difference of the volume integrals which is of order of the volume between the interfaces and the surface integral which is first order because the jump in G_0 vanishes on the neighboring surface S_0. Separating the integration over the upper + volumes from the − volumes gives

$$\frac{1}{\rho(\mathbf{r}_f)}\delta G(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o) \approx$$

$$+ \int_{-\infty}^{\infty} dt' \int_{\Omega^+} - \int_{\Omega_0^+} d\mathbf{r}'[\nabla' G_0 \cdot \frac{1}{\rho(\mathbf{r}')}\nabla' G_0 - \partial_{t'}G_0 \frac{1}{c^2(\mathbf{r}')\rho(\mathbf{r}')}\partial_{t'}G_0]^+$$

$$+ \int_{-\infty}^{\infty} dt' \int_{\Omega^-} - \int_{\Omega_0^-} d\mathbf{r}'[\nabla' G_0 \cdot \frac{1}{\rho(\mathbf{r}')}\nabla' G_0 - \partial_{t'}G_0 \frac{1}{c^2(\mathbf{r}')\rho(\mathbf{r}')}\partial_{t'}G_0]^-$$

$$+ \int_{-\infty}^{\infty} dt' \int_S dS[J[G_0]A[\frac{1}{\rho(\mathbf{r}')}\mathbf{n}' \cdot \nabla' G_0)] + A[\mathbf{n}' \cdot \nabla' G_0 \frac{1}{\rho(\mathbf{r}')}]J[G_0]]. \qquad (9)$$

In this equation, the arguments of the Green's functions have been omitted; they are to be understood as they appear in Eq.8. If the surface S is obtained from S_0 by normal displacement $\xi(\mathbf{r})$ into Ω_0^+, the volume Ω_0^+ is diminished by $\int dS\xi$ and the volume Ω_0^- is increased by this amount. The jump in the surface integral is found from

$$G_0^+ - G_0^-|_S \approx G_0^+ - G_0^-|_{S_0} + \xi(\mathbf{n} \cdot \nabla G_0^+ - \mathbf{n} \cdot \nabla G_0^-)|_{S_0} = \xi\mathbf{n} \cdot \nabla G_0^+(1 - \frac{\rho^-}{\rho^+}). \qquad (10)$$

Thus, using the boundary conditions satisfied by G_0, **the perturbation of the Green's function satisfies,**

$$\frac{1}{\rho(\mathbf{r}_f)}\delta G(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o) \approx$$

$$- \int_{-\infty}^{\infty} dt' \int_{S_0} dS\xi(\mathbf{r}')[\nabla' G_0^+ \cdot \frac{1}{\rho(\mathbf{r}'^+)}\nabla' G_0^+ - \nabla' G_0^- \cdot \frac{1}{\rho(\mathbf{r}'^-)}\nabla' G_0^-]$$

$$+ \int_{-\infty}^{\infty} dt' \int_{S_0} dS\xi(\mathbf{r}')[\partial_{t'}G_0^+ \frac{1}{c^2(\mathbf{r}'^+)\rho(\mathbf{r}'^+)}\partial_{t'}G_0^+ - \partial_{t'}G_0^- \frac{1}{c^2(\mathbf{r}'^-)\rho(\mathbf{r}'^-)}\partial_{t'}G_0^-]$$

$$+ \int_{-\infty}^{\infty} dt' \int_{S_0} dS\xi(\mathbf{r}')2\mathbf{n} \cdot \nabla G_0^+ \frac{1}{\rho^+}\mathbf{n} \cdot \nabla G_0^+(1 - \frac{\rho^-}{\rho^+}). \qquad (11)$$

To first order it doesn't matter if the integrations are taken over S or S_0. This result can be simplified still further by using

$$\nabla G_0 \cdot \nabla G_0 = \mathbf{n} \cdot \nabla G_0 \mathbf{n} \cdot \nabla G_0 + \nabla_{\parallel} G_0 \cdot \nabla_{\parallel} G_0$$

and the boundary conditions, which require that $\nabla_{\parallel} G_0$ and $\partial_t G_0$ be continuous. Furthermore, for infinite interfaces, as in layered media, it is convenient to refer the displacement of the surface to a horizontal surface. Then the displacment is described by its projection onto the vertical axis, $\delta h(\mathbf{R})$. Surface areas transform by $dS\xi = dR\delta h$. The perturbation of the Green's function is now given by

$$\frac{1}{\rho(\mathbf{r}_f)}\delta G(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o) \approx$$

$$+ \int_{-\infty}^{\infty} dt' \int d\mathbf{R}' \delta h(\mathbf{r}') \left[\mathbf{n} \cdot \nabla G_0^+ \frac{1}{\rho^+} \mathbf{n} \cdot \nabla G_0^+ (1 - \frac{\rho^-}{\rho^+}) - \nabla_{\parallel} G_0 \cdot \nabla_{\parallel} G_0 (\frac{1}{\rho^+} - \frac{1}{\rho^-}) \right.$$

$$\left. + \partial_{t'} G_0^+ \partial_{t'} G_0^+ \left(\frac{1}{c^2(\mathbf{r}'^+)\rho(\mathbf{r}'^+)} - \frac{1}{c^2(\mathbf{r}'^-)\rho(\mathbf{r}'^-)} \right) \right]. \tag{12}$$

The analogue of this result for plane wave scattering can be found in Refs.[4, 6]. In a layered medium it is clear that the perturbation of the Green's function will be a sum of contributions from the perturbation of each interface similar to that given above. To lowest order these perturbations simply add together. In the case of Dirichlet surfaces only the first term of the integrand, involving normal derivatives, contributes with $\rho^- = 0$.

It should be noted that the surface S_0 is not generally a flat surface and that as a consequence G_0 is not easily calculated. Nevertheless, Eq.(12) is useful in that it provides an exact, albeit formal, expression for the functional derivative of the Green's function with respect to the surface shape, $h(\mathbf{r})$, since Eq.(12) is exact to first order in δh.

III The Small-Slope Approximation

Consider now the special case of a Dirichlet surface. Let the scattering surface be described by

$$z = \epsilon h(\mathbf{R}),$$

where, as above, $\mathbf{R}$ is a two-dimensional vector in the $x - y$ plane. Denote the Green's function associated with this surface by G_ϵ and consider what happens when the surface is perturbed from $z = \epsilon h(\mathbf{R})$ to $z = (\epsilon + d\epsilon)h(\mathbf{R})$. Then δh is given by

$$\delta h(\mathbf{R}) = (d\epsilon)h(\mathbf{R}). \tag{13}$$

Dividing the perturbation of G_ϵ by $d\epsilon$ gives the following exact (because Eq.(12) is exact to first order in δh) result for the derivative of the Green's function with respect to ϵ.

$$\frac{dG_\epsilon}{d\epsilon}(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o) =$$

$$\int_{-\infty}^{\infty} dt' \int d\mathbf{R}' h(\mathbf{R}') \mathbf{n}_\epsilon \cdot \nabla G(\mathbf{r}_f, t_f; \mathbf{r}', t' | \{\epsilon h\}) \mathbf{n}_\epsilon \cdot \nabla G(\mathbf{r}', t'; \mathbf{r}_o, t_o | \{\epsilon h\}). \tag{14}$$

On the right side of this equation, the functional dependence of the Green's functions on the scattering surface ϵh has been made explicit. The Green's function for the desired surface, $h(\mathbf{R})$, namely G_1, is found by integrating from $\epsilon = 0$ to $\epsilon = 1$. The function $G_{\epsilon=0}$ is just the flat surface ($z = 0$) Green's function which can be found fairly simply if the density and sound speed depend only on the depth z. In performing the integration over ϵ, assume that the order of the integration can be changed as needed, and integrate over ϵ first. Then change the variable of integration from ϵ to

$$z' = \epsilon h(\mathbf{R}'). \tag{15}$$

One then obtains for the Green's function associated with the surface $h(\mathbf{R})$,

$$G(\mathbf{r}_f, t_f; r_o, t_o | h) - G(\mathbf{r}_f, t_f; r_o, t_o | 0) =$$

$$\int_{-\infty}^{\infty} dt' \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} \mathbf{n}_\epsilon \cdot \nabla G(\mathbf{r}_f, t_f; \mathbf{r}', t' | \{\frac{z'}{h(\mathbf{R}')}h\}) \mathbf{n}_\epsilon \cdot \nabla G(\mathbf{r}', t'; r_o, t_o | \{\frac{z'}{h(\mathbf{R}')}h\}). \tag{16}$$

In these expressions, the normal to the surface, $\mathbf{n}_\epsilon$ is given by

$$\mathbf{n}_\epsilon = \frac{\hat{z} - \epsilon \nabla h(\mathbf{R}')}{\sqrt{1 + (\epsilon \nabla h(\mathbf{R}'))^2}} = \frac{\hat{z}h(\mathbf{R}') - z' \nabla h(\mathbf{R}')}{\sqrt{h(\mathbf{R}')^2 + (z' \nabla h(\mathbf{R}'))^2}}. \tag{17}$$

In addition, in the last equation, the intermediate vector, $\mathbf{r}'$, is understood to be given by

$$\mathbf{r}' = (\mathbf{R}', z'). \tag{18}$$

It should be emphasized that Eq.(16) is an exact result for Dirichlet surfaces even when there are sound speed and density variations.

Now assume that, in fact, sound speed and density are constant. The small-slope approximation consists of replacing the normal gradient of the Green's function, G_ϵ, by twice the vertical derivative of the free-space Green's function, G_{free}, the Green's function in the absence of any boundaries. In the present case, G_{free} is given by

$$G_{free}(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o) = -\delta(t_f - t_o - |\mathbf{r}_f - \mathbf{r}_o|/c)/(4\pi|\mathbf{r}_f - \mathbf{r}_o|). \tag{19}$$

Note that this differs by a sign and a factor of 4π from the conventional Green's function, because, for the purposes of deriving the variational principle, it was convenient to have the wave equation driven by a simple δ-function. It is now convenient to consider the field $\psi(\mathbf{r}, t)$ produced by a pulse $F(t)$ originating at $\mathbf{r}_o$. In the small-slope approximation the difference between a pulse received when there is a rough surface and the pulse received when there is a flat surface located at $z = 0$ is given by

$$\psi(\mathbf{r}_f, t_f) - \psi_{flat}(\mathbf{r}_f, t_f) = \int_{-\infty}^{t_f} dt_o[G(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o|h) - G(\mathbf{r}_f, t_f; \mathbf{r}_o, t_o|0)]F(t_o)$$

$$= \frac{4}{16\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz' \partial_{z_f} \partial_{z_o} \frac{F(t_f - t_o - (|\mathbf{r}_f - \mathbf{r}'| + |\mathbf{r}' - \mathbf{r}_o|)/c)}{|\mathbf{r}_f - \mathbf{r}'||\mathbf{r}' - \mathbf{r}_o|}. \tag{20}$$

The point of this section is to show that the z' integral can be performed when the source and receiver coincide, i.e., when $\mathbf{r}_f = \mathbf{r}_o = \mathbf{r}$. The main result is given in Eq.31 below. It is a relatively simple expression but there is a good deal of algebra between here and there. To see how this works out, first carry out the differentiation and then set the source position equal to the receiver postion. Let

$$\rho = |\mathbf{r} - \mathbf{r}'|.$$

Without loss of generality, suppose that $\mathbf{r} = (0, z)$. Then

$$\dot{\psi}(\mathbf{r}_f, t_f) - \psi_{flat}(\mathbf{r}_f, t_f) = \frac{1}{4\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz'$$

$$\left[\frac{F''(t - 2\rho/c)}{c^2\rho^2} + \frac{2F'(t - 2\rho/c)}{c\rho^3} + \frac{F(t - 2\rho/c)}{\rho^4} \right] \frac{(z - z')^2}{\rho^2}. \tag{21}$$

To deal with this integral use the identities [3]

$$F'' = -\frac{\rho c}{2} \frac{\mathbf{R}' \cdot \nabla F'}{\mathbf{R}'^2} \tag{22}$$

and

$$F' = -\frac{\rho c}{2} \frac{\mathbf{R}' \cdot \nabla F'}{\mathbf{R}'^2}. \tag{23}$$

Let I_1 be the integral of the term involving the second derivative

$$I_1 = \frac{1}{4\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz' \frac{F''(t - 2\rho/c)}{c^2\rho^2} \frac{(z - z')^2}{\rho^2}.$$

$$= \frac{1}{4\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz' \frac{-1}{2\rho c} \frac{\mathbf{R}' \cdot \nabla F'}{\mathbf{R}'^2} \frac{(z - z')^2}{\rho^2}. \tag{24}$$

Write the last integrand as a divergence in two dimensions and use the identity

$$\nabla \cdot \frac{\mathbf{R}}{\mathbf{R}^2} = 2\pi\delta(\mathbf{R}). \tag{25}$$

When the divergence operator is passed outside the inner integral, a term in ∇h is produced from the upper limit of the inner integral. In this way I_1 can be shown to be given by

$$I_1 = \frac{1}{4\pi^2} \int d\mathbf{R}' \frac{\mathbf{R}' \cdot \nabla h(\mathbf{R}')}{\mathbf{R}'^2} \frac{F'}{2\rho c} \frac{(z - h(\mathbf{R}'))^2}{\rho^2}.$$

$$+ \frac{2\pi}{4\pi^2} \int_0^{h(0)} dz' \frac{F'(t - 2(z - z')/c)}{2c(z - z')}$$

$$- \frac{1}{4\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz' \frac{3F'(t - 2\rho/c)}{2c\rho^3} \frac{(z - z')^2}{\rho^2}. \tag{26}$$

The last term in this expression can be combined with the second integrand in Eq.(21) so that the scattered field at the receiver is now given by

$$\psi(\mathbf{r}_f, t_f) - \psi_{flat}(\mathbf{r}_f, t_f) = \frac{1}{4\pi^2} \int d\mathbf{R}' \frac{\mathbf{R}' \cdot \nabla h(\mathbf{R}')}{\mathbf{R}'^2} \frac{F'}{2\rho c} \frac{(z - h(\mathbf{R}'))^2}{\rho^2}.$$

$$+ \frac{2\pi}{4\pi^2} \int_0^{h(0)} dz' \frac{F'(t - 2(z - z')/c)}{2c(z - z')}$$

$$+ \frac{1}{4\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz' \frac{F'(t - 2\rho/c)}{2c\rho^3} \frac{(z - z')^2}{\rho^2}.$$

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$$+ \frac{1}{4\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz' \frac{F(t - 2\rho/c)}{\rho^4} \frac{(z - z')^2}{\rho^2}. \qquad (27)$$

The next to last term, which will be denoted by I_2, can be treated similarly to I_1, with the result

$$I_2 = \frac{1}{16\pi^2} \int d\mathbf{R}' \frac{\mathbf{R}' \cdot \nabla h(\mathbf{R}')}{R'^2} \frac{F(t - 2\rho/c)}{\rho^2} \frac{(z - h(\mathbf{R}'))^2}{\rho^2}.$$

$$+ \frac{2\pi}{16\pi^2} \int_0^{h(0)} dz' \frac{F(t - 2(z - z')/c)}{(z - z')^2}$$

$$- \frac{1}{4\pi^2} \int d\mathbf{R}' \int_0^{h(\mathbf{R}')} dz' \frac{F(t - 2\rho/c)}{\rho^4} \frac{(z - z')^2}{\rho^2}. \qquad (28)$$

It seems a bit magical, but when this expression for I_2 is used in Eq.(27), the integrands involving F cancel one another. Furthermore, since

$$\frac{d}{dz'} \frac{F(t - 2(z - z')/c)}{z - z'} = \frac{2F'}{c(z - z')} + \frac{F}{z - z'}, \qquad (29)$$

the integrals over z' alone can be performed. Finally, the scattered pulse is given by

$$\psi(\mathbf{r}_f, t_f) - \psi_{flat}(\mathbf{r}_f, t_f) = \frac{1}{4\pi} \left[\frac{F(t - 2(z - h(0))/c)}{2(z - h(0))} - \frac{F(t - 2z/c)}{2z} \right]$$

$$+ \frac{1}{4\pi^2} \int d\mathbf{R}' \frac{\mathbf{R}' \cdot \nabla h(\mathbf{R}')}{R'^2} \left[\frac{F(t - 2\rho/c)}{4\rho^2} + \frac{F'(t - 2\rho/c)}{2c\rho^2} \right] \frac{(z - h(\mathbf{R}'))^2}{\rho^2}. \qquad (30)$$

Because the sign of the free-space Green's function is negative, the pulse reflected from a flat surface at $z = 0$ is just

$$+ \frac{1}{4\pi} \frac{F(t - 2z/c)}{2z}.$$

This means that in the small-slope approximation, the scattered portion of the pulse

$$\psi^{SSA} = \psi - \psi_{direct}$$

is given by

$$\psi^{SSA}(\mathbf{r}_f, t_f) = \frac{1}{4\pi} \frac{F(t - 2(z - h(0))/c)}{2(z - h(0))}$$

$$+ \frac{1}{4\pi^2} \int d\mathbf{R}' \frac{\mathbf{R}' \cdot \nabla h(\mathbf{R}')}{R'^2} \left[\frac{F(t - 2\rho/c)}{4\rho^2} + \frac{F'(t - 2\rho/c)}{2c\rho^2} \right] \frac{(z - h(\mathbf{R}'))^2}{\rho^2}. \qquad (31)$$

The only approximation used in obtaining this result is the small-slope approximation.

A similar result can be obtained for the Kirchhoff approximation by using the same sorts of manipulations described here:

$$\psi^{Kir}(\mathbf{r}_f, t_f) = \frac{1}{4\pi} \frac{F(t - 2(z - h(0))/c)}{2(z - h(0))}$$

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$$+ \frac{1}{4\pi^2} \int d\mathbf{R}' \frac{\mathbf{R}' \cdot \nabla h(\mathbf{R}')}{R'^2} \left[\frac{F(t - 2\rho/c)}{4\rho^2} + \frac{F'(t - 2\rho/c)}{2c\rho^2} \right] . \qquad (32)$$

This last result differs slightly from Berry's expression [3] in that no far-field assumptions have been invoked.

Both the Kirchhoff and the small-slope approximations give the exact result for flat surfaces through $z = h(0)$. Both show that scattering takes place only because the gradient of the surface shape is non-vanishing. The two approximations differ only in that the small-slope approximation includes the factor

$$\frac{(z - h(\mathbf{R}'))^2}{\rho^2}.$$

This factor is the sine squared of the local angle of propagation at the surface. For plane waves, the Kirchhoff and Small-slope approximations differ by a factor of $k_z q_z / k^2$ [6]. In backscattering, vertical components of incident and out-going wavenumbers are equal. Thus the difference exhibited here between the Kirchhoff and Small-slope approximations for point sources is consistent with the difference observed for plane waves.

IV   Discussion

The primary results presented above are the variational principle and the subsequent derivation of what is, in effect, an exact expression for the functional derivative of the scattered field with respect to the surface shape. This was used in the previous section to derive a simple expression for the small-slope approximation for point sources in the time-domain. The only approximation used to derive Eq.(31) is Eq.(19). The variational principle should have general utility. It was shown in Ref.[9] that a variety of approximations can be based on the variational principle. If the trial Green's functions are the same, the variational expression is guaranteed to be reciprocal. Furthermore, scattering is described even if the trial Green's functions don't exhibit scattering. For example, one might try to describe reverberation in the ocean by using trial Green's functions calculated by the method of adiabatic normal modes.

What is surprising and annoying is that so much algebra is required to demonstrate the small-slope approximation. Such a simple result deserves a simpler derivation. Furthermore, this result does not seem to be generalizable to bistatic scattering, nor to variable

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sound speeds, at least not without further approximations. It may represent one of a very few situations in which there is something simple about the special case of backscattering.

Acknowledgements

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