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MODELING CUMULATIVE DAMAGE PROCESSES IN COMPOSITE LAMINATES

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Procedures for analysis of composite laminates are investigated with a view to modeling cumulative damage. Methods for stress and deformation analysis are examined and new methods necessary to model the behavior are developed. Two approaches, one based on modeling a thin laminate as a two-dimensional body and the other consisting of reducing the three-dimensional problem to a pseudo-two-dimensional one by utilizing the symmetries and the special loading conditions of a free-edge delamination specimen are explored. In the first approach, existing theories of combined bending and stretching of composite laminates based on assumed displacement or assumed stress variation patterns are carefully evaluated in respect of proper two-dimensional approximation of the three-dimensional problem. The existing theories, including the recently developed "layerwise" or discrete laminate theories are seen to be inadequate in properly allowing for shear deformation

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and free-edge effects as factors in damage by delamination. A new class of constitutive relationships between force resultants of the individual layers of a laminate and the deformations of the layers is proposed. Three different versions, based on different assumptions regarding patterns of stresses in equilibrium, are investigated. For theories based on assumption of stress distribution within each lamina and satisfaction of the equilibrium equations, the existing theory (Pagano's) is restated in a self-adjoint form suitable for systematic variational formulation and, hence, convenient application of Ritz type finite element procedures based on the general variational principle or appropriate extensions and specializations thereof. For the special case of free-edge delamination specimens, the problem reduces to a set of algebraic and ordinary differential equations. In the second approach, the symmetries and the loading patterns of a free-edge delamination specimen are used to reduce the three-dimensional problem to a pseudo-two-dimensional one in which there are three components of displacement but these are functions of only two independent spatial coordinates. The theories developed in both the approaches are implemented in effective finite element procedures. The computer codes are verified against available solutions and are applied to the problem of delamination of multilayer symmetric free-edge specimens.

FOREWORD

This report covers research carried out at The Ohio State University under Grant AFSC/ASD/F33615-85-C-3213. Dr. George P. Sendeckyj, Flight Dynamics Directorate (WL/FIBEC), Wright Laboratory, Wright-Patterson Air Force Base, Ohio was the Program Manager. At The Ohio State University, the work was performed under the supervision of Professors Ranbir S. Sandhu and William E. Wolfe of the Department of Civil Engineering. Professor Robert L. Sierakowski was also associated with the project during its early stages. Graduate students Ali Al-Ghothani, Cherng C. Chang, Soon Jo Hong, Hui-Huang Chyou, Harpal S. Chohan, Razan A. Dandan, Mehdi Moazzami, Gamal Saad, and Han-Ru Chu worked on different components of the project at various stages. The Ohio State University Instruction and Research Computer Center and The Ohio Supercomputer Center provided part of the computational and documentation facilities. Microcomputer facilities were available from an earlier Air Force Research grant: AFOSR-83-0055. These were further enhanced, as part of the present project, to meet the additional needs.

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SECTION I
INTRODUCTION

1.1 THE PROBLEM

Use of filamentary composites in aircraft components has been steadily growing. For satisfactory design it is imperative that the service behavior and the ultimate load capacity of the components be well understood. Experimental procedures cannot realistically be expected to cover all possible stress configurations for all possible designs. Analytical models of material behavior, which can be used to predict structural behavior under a variety of service conditions, are necessary to develop a firm basis for design.

1.2 RESEARCH OUTLINE

The purpose of the research described in this report was to develop reliable and efficient computational procedures for prediction of deformation and stress causing cumulative damage in structural components made of composite laminates. The proposed models were to be verified in an appropriate manner.

The work performed under the research program consisted primarily of analytical studies involving mathematical modelling of behavior of laminated plates under combined bending and stretching and of free-edge delamination specimens subjected to uniform extension.

The models developed were designed to be capable of representing simultaneous occurrence of flexure and stretching and have the capability of predicting interlaminar normal and shear stresses.

Review of available literature pointed out the inadequacy of the existing theories of mechanics of laminated composites. Theories based on assumption of a linear or higher order variation of the in-plane and the transverse components of displacement over the thickness of the laminate have been known to be incapable of representing local effects. Even the "layerwise" or discrete laminate theory could not properly predict interlayer stresses and there was no satisfactory procedure to allow for shear deformation effects. Pagano's [1978] assumed stress distribution approach is theoretically sound but its use to find numerical solutions to specific problems was limited to only a few layers. The finite element methods available could not accurately model the complete stress state especially at interfaces and free edges.

The research reported herein attempts to plug some of the gaps. An hierarchical approach to the construction of a sequence of refined discrete laminate theories to model combined bending and stretching of laminated plates, based upon assumed displacement patterns was developed. Coupled constitutive equations for force resultants were derived using an extension of Reissner's variational procedure. This eliminates the need for ad hoc "correction factors" to allow properly for shear. Pagano's theory was restated in a self-adjoint form with a reduction in the number of free field variables. This renders the approach convenient for variational formulation and application of finite element techniques. For analysis of free-edge delamination specimens, continuous traction finite elements were developed. Test programs, for which adequate documentation was available, were carefully examined. The predicted modes of failure and progressive damage sequence were compared with the test data. Figure 1 shows the various components of the research program and the individual reports, listed in Appendix B, wherein each is fully documented.

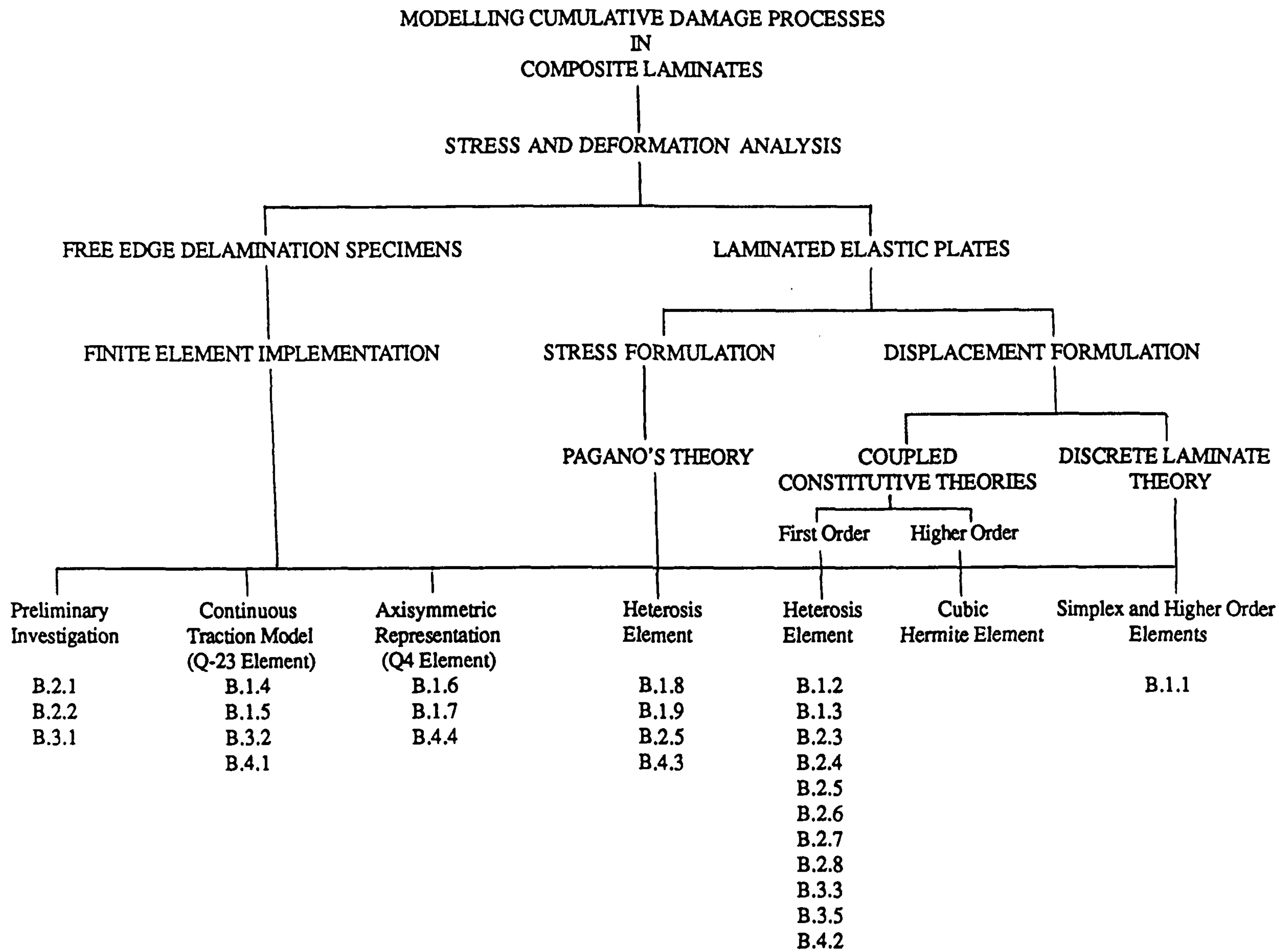


Fig ıre 1: SUMMARY OF THE RESEARCH EFFORT.

("B" refers to items in Appendix B)

1.3 STRUCTURE OF THE REPORT

This report describes the overall research effort and the principal results of the investigation. We first present, in Section II, a summary of previous work on modelling behavior of composite laminates. Section III contains a summary of the basic equations of linear elasticity and their specialization to the case of free-edge delamination specimens. Section IV presents an hierarchical approach to theory of laminated plates using assumed displacement patterns. Section V gives a summary of the new discrete laminate theories, developed in the present research program, allowing for coupled constitutive equations for the force resultants. Section VI contains a restatement of Pagano's assumed stress distribution theory. Section VII points out the self-adjoint form, useful for variational formulation, of laminate theories. The governing equations, for the collection of laminate theories, including the field and interface continuity equations are stated in a self-adjoint form and general variational principles are proposed. Extensions of the general formulation, to admit relaxation of continuity requirements on some of the field variables, are indicated as well as specializations to reduce the number of free field variables by requiring some of them to identically satisfy some of the field equations. This is often necessary to make the problem computationally tractable. Section VIII presents a summary of finite element studies. Appendix A contains a summary of the general procedure for setting up variational formulations for linear coupled problems. Appendix B lists the publications that have resulted from the research effort.

SECTION II

REVIEW OF PREVIOUS WORK

2.1 INTRODUCTION

This section briefly reviews previous work on modelling behavior of composite laminates with a view to identifying cumulative damage processes. Two-dimensional representation of the three-dimensional physical problem, and finite element solution procedures applied to evaluation of stress and deformation in composite laminates are discussed. Detailed reviews are available in the literature. As examples one may cite the recent reviews by Reddy [1990] and Hanna [1990]. The former is in the nature of a survey while the latter presents details of various theories along with applications.

2.2 TWO-DIMENSIONAL REPRESENTATION

2.2.1 Introduction

In order to simplify solution of the three-dimensional problem of stresses and deformations in composite laminates, it is desirable to take note of certain material and geometric symmetries that might exist and the fact that in most applications the thickness of the laminate is very small in comparison with the lateral dimensions. For homogeneous plates, it is customary to make the dependence of the field variables upon the transverse coordinate explicit. An assumption is usually made for the pattern of variation of the displacement or the stress components with this coordinate. For the case of free-edge delamination specimens under uniform axial strain, the three-dimensional problem reduces to a pseudo-two-dimensional one because of the dependence upon the longitudinal coordinate x_1 being eliminated. Specialization of the thin plate theories to the case of free-edge delamination specimens makes the problem one-dimensional.

2.2.2 Displacement-based Theories.

Existing displacement-based theories, as applied to laminated composite plates, can be classified into three categories, viz.,

1. Classical thin plate theory and the first order shear deformation theory.
2. Higher order theories.
3. Discrete laminate theories.

The first two categories assume the dependence of the components of the displacement vector to be polynomials in the coordinate normal to the surface of the plate. The third category treats each lamina as a homogeneous anisotropic plate and enforces continuity of components of traction and displacement at the interfaces.

2.2.2.1 Classical Thin Plate Theory [CPT].

A theory for a laminate was presented by Reissner and Stavsky [1961] for a two-layer system. Stavsky [1961] extended it to a multi-layer plate. Dong et al. [1962] extended the concepts to the analysis of anisotropic laminated shells. This theory assumes the in-plane components of displacement to be linear in the transverse coordinate and the out-of-plane component to be independent of it. Also, the rotation of any cross-section is assumed to be equal to the rotation of the tangent at the midsurface. This is equivalent to ignoring shear deformation and assuming the thickness of the plate to remain constant. The transverse strain is zero by the displacement asumptions and the transverse stress is also assumed zero. Implying infinite shear rigidity, it leads to an overestimation of plate stiffness resulting in underestimation of transverse deflection [Whitney, 1969; Pagano, 1969, 1970; Srinivas and Rao, 1970]. The effect is more pronounced in composite laminates due to the low ratio of shear modulus to Young's modulus. Also, the theory cannot predict local and interfacial behavior of the laminate which is critical to effectively model damage associated with delamination.

2.2.2.2 First Order Shear Deformation Theory [FSDT].

In the first order shear deformation theory [e.g., Yang et al., 1966; Whitney and Pagano, 1970] the rotation of the cross-sections is allowed to be different from the rotation of the tangent at the midsurface. This is an extension of Reissner's [1944, 1945] and Mindlin's [1951] ideas. It can allow for shear deformation and predict lateral deflections and fundamental frequencies of vibration reliably but does not give good quantitative results [Whitney 1969; Srinivas and Rao, 1970] for the transverse shear stress distribution. According to Lo et al. [1977], "Despite the increased generality of the shear deformation theory, the related flexural stress distributions show little improvement over those of the classical plate theory." To improve accuracy, Whitney and Pagano [1970] and Kulkarni and Pagano [1972] suggested use of a suitable shear correction factor K. However, the value of this factor was not a fixed constant [Whitney and Pagano 1970; Kulkarni and Pagano 1972] and different values were seen to be optimal for different layups and for various quantities of interest in the particular problem. Whitney and Pagano [1970] stated: "The evaluation of K in a specific problem depends on either the exact elasticity solution of the problem or experimental evidence. Owing to the scarcity of such information, we can say little about the evaluation of K at the present time." In Mindlin's approach [1951], thickness shear modes obtained by the approximate theory and by an exact elasticity solution for straight-crested flexural waves in an infinite plate were matched. Yang et al. [1966] applied the general theory to propagation of plane strain waves of Rayleigh-Lamb type in specific two-layer isotropic plate and showed that the value of

K depends upon the material properties of the layered plate. Nelson and Lorch [1974] presented a refined nine mode theory for laminated composite plates. They noted 41 independent nonzero stiffness quantities. Invariance of strain energy under translation of the reference surface limited the number of stiffness correction factors to a maximum of nine including two "in-plane" stiffness factors. An alternative approach has been to enforce equality of the shear energy expressions from an approximate laminate theory and from a stress field satisfying equilibrium. Chow [1971] used this method to obtain the shear correction factor for an orthotropic laminate of symmetric construction. Whitney [1973b] extended it to the case of an asymmetric orthotropic laminate. The approximate stress field was obtained by assuming linear variation of the in-plane stresses over the thickness of the plate and using the equilibrium equations to solve for the remaining components. Bert [1984] used this procedure to obtain the shear correction factor for a laminate version of Levinson's higher order theory. Reissner [1972, 1979] presented a procedure similar to the one used for homogeneous plates to obtain the transverse shear stiffness. Yang et al. [1966] also assumed the transverse direct stress component to be zero. Whitney [1970] assumed the integral of this stress to be zero over each layer and applied his theory to problems of cylindrical bending.

2.2.2.3 Higher Order Theories [HOT].

Whitney and Sun [1973a] proposed a higher order theory with quadratic polynomial functions for the in-plane displacement components and linear for the transverse in order to include the first antisymmetric shear mode and allow for nonzero normal strain in the transverse direction. A shear correction factor was used. In another application, to laminated shells, Whitney and Sun [1974] assumed the in-plane displacements to be linear in the transverse coordinate and the transverse displacement to be cubic. In both theories, a shear correction factor was used to enhance accuracy. Nelson and Lorch [1974] employed quadratic expressions for all the components of displacement. Lo et al. [1977] used cubic functions for the in-plane displacement components and quadratic for the transverse to have the transverse shear contributions from the two systems to be of the same order in the transverse coordinate. Levinson [1980] proposed using a cubic polynomial for the in-plane displacement components and constant transverse deflection for an homogeneous, isotropic plate. To satisfy the stress-free conditions on the faces of the plate, the quadratic term had to be dropped and the coefficient of the cubic term was related to that of the first order term and the transverse component of displacement. Bert [1984] extended this idea to laminates. Noting that the equilibrium equations in Levinson's formulation were inconsistent in the sense of a variational principle, Reddy [1984] derived the field equations from a virtual work equality. This simple higher order theory involves the same number of variables as the first order shear deformation theory but gives parabolic transverse shear strain variation through the thickness of the plate and also satisfies the stress-free conditions at the top and the bottom faces. This theory was used [Bert 1984; Reddy 1984] to analyze angle-ply and cross-ply laminated plates and shown to be more accurate than the first order theory. Table 1 summarizes some of the assumed displacement fields used in higher order theories. Use of higher order terms in the representation of displacement components increases the accuracy but the additional terms introduced into the problem formulation increase the cost and complexity of the solution process. Furthermore, the polynomial representation over the entire thickness of the plate cannot properly model the discontinuities in displacement gradients that generally exist at the interfaces between layers with different orientation of the material axes. Therefore, shear deformable theories, whether FSDT or higher order, cannot take into account the effects of local deformation accurately. This difficulty is more pronounced when the shear rigidities of the constituent layers are quite different [Whitney and Sun, 1973; Lo et al. 1977]. Also, these cannot satisfy

Table 1: THEORIES ALLOWING FOR SHEAR DEFORMATION

Reference	In-plane Displacement	Out-of-plane Displacement	Normal Strain	Correction Factors	Force Resultants
Reissner [1945]	Linear 4	Constant 1	Zero	1	8
Mindlin [1951]	Linear 4	Constant 1	Zero	1	8
Naghdi [1957]	Linear 4	Quadratic 3			
Whitney [1973]	Quadratic 6	Linear 2	Constant	5	14
Whitney [1974]	Linear 4	Quadratic 3	Linear		
Nelson [1974]	Quadratic 6	Quadratic 3	Linear	< 9	17
Lo [1977]	Cubic 8	Quadratic 3	Linear		20
Levinson [1980]	Cubic 4	Constant 1	Zero	1	8
Reddy [1984]	Cubic 6	Constant 1	Zero		13

interlayer traction continuity. None of the existing higher order theories based on assumed displacements can properly model interfacial stresses, especially near the traction-free edges.

2.2.2.4 Discrete Laminate Theories [DLT].

In a laminated composite plate, the deformation depends upon the stacking sequence of layers and upon the material properties of each [Sun and Whitney, 1973]. In order to allow for this properly, discrete laminate theories in which a layerwise linear variation of the in-plane displacement components is employed have been proposed [e.g., Sun and Whitney, 1973; Srinivas, 1973; Seide, 1980]. Typically, each layer is treated as an homogeneous, anisotropic plate and the governing equations of all these plates are coupled through interlaminar continuity equations. Evidently, to achieve greater accuracy of representation, it is possible to divide each lamina into a number of layers. Srinivas [1973] assumed a piecewise linear variation in the in-plane displacements with respect to the transverse coordinate and the transverse displacement was assumed to be constant over the thickness. This gave 2N+3 field equations for the N layers. The theory did not enforce continuity of traction across interfaces. Sun and Whitney [1973] used kinematic assumptions similar to Srinivas'. However, enforcement of continuity of shear stresses at interfaces reduced the number of field variables to the same as in the first order shear deformation theory regardless of the number of layers. The degree of the differential equation was seen to increase with N, the number of layers. Seide [1980], unlike Srinivas [1973] and Sun and Whitney [1973], did not use Hamilton's principle but directly combined the equilibrium equations of the layers using continuities of traction and displacement at the interfaces.

Murakami [1986] proposed a theory in which layerwise zigzag functions were added to the in-plane displacements adopted in the FSDT to account for local shear deformation effects. This theory was shown to yield more accurate in-plane stress predictions for symmetrically laminated thick orthotropic plates. Reddy [1987, 1989] proposed a layerwise theory as a generalization of two-dimensional theories by writing displacement components as piecewise smooth functions of the transverse coordinate in line with finite element concepts.

2.2.3 Stress-based Theories.

Pagano [1977a,b; 1978a,b] proposed a theory which assumed linear variation, with respect to the transverse coordinate, of the in-plane stresses. The remaining stress components were derived from the three-dimensional equilibrium equations. A variational principle was used to develop constitutive equations relating force resultants to "generalized displacements." Continuity of tractions as well as displacements at interfaces was explicitly satisfied. The boundary value problem was stated in terms of the generalized displacements and interfacial tractions. The only approximation made in this theory is the assumption of linear variation of the in-plane stress components over each layer thickness. As each layer can be subdivided into as many sublayers as one wishes, the theory ought to lead to a sequence of solutions of the problem which would converge to the exact solution with reduction in the sublayer thickness.

In Pagano's theory, the number of simultaneous partial differential equations to be solved is 13N. This has been the principle difficulty in applying this theory to practical problems. To overcome this, Pagano and Soni [1983] proposed a "global-local" model in which a laminate would be divided into a local zone and a global portion. A higher order displacement-based theory would be used to solve the global problem and then Pagano's theory would be applied to the local zone for better accuracy in that region.

Green and Naghdi [1981, 1982] proposed a generalized discrete laminate theory incorporating thermal and dynamical effects. Table 2 gives a comparison of the discrete laminate theories.

Table 2: DISCRETE LAMINATE THEORIES

Reference	Type	In-plane Displacements	Transverse Displacement	Field Variables
Seide [1980]	Displacement	Linear	Constant	2N + 3
Sun and Whitney [1973]	Displacement	Linear	Constant	2N + 3
Srinivas [1973]	Displacement	Linear	Constant	2N + 3
Pagano [1977,1978]	Stress	Linear	Quadratic Shear Cubic Normal Stress	13N

2.2.4 Free-Edge Delamination Specimens.

Free-edge delamination specimens are tested under uniform axial strain. This implies that the axial component of displacement is linear in the longitudinal coordinate. Away from the grips, it is reasonable to assume that the other two displacement components are independent of the longitudinal coordinate. This reduces the three-dimensional problem to one involving three variables (components of the displacement vector) but these are functions of only two independent spatial variables, viz., the transverse coordinate and the "other" of the in-plane coordinates.

Specialization of the plate theories, described earlier, to the case of a free-edge delamination specimen reduces the dimension of the problem from two to one. Thus, the dependence of various field variables is on the transverse in-plane coordinate only.

2.3 FINITE ELEMENT MODELLING.

Even highly simplified models of laminated composites lead to very complex sets of equations. Exact solutions to these equations have been possible only for cases of simplest geometry and load configurations. Use of numerical methods to obtain approximate solutions to the field equations is thus imperative. Both finite difference and finite element methods have been employed.

Hulbert and Rybicki [1971] used a boundary point least squares technique. Pipes and Paganao [1970] used the finite difference method. Most of the finite element studies have been for plane stress or three-dimensional models of composite laminates. Rybicki [1971] used Maxwell's stress functions approximated as fifth degree Hermite polynomials to get a stress field satisfying equilibrium. This gave 648 unknowns per element. Minimization of the complementary energy was used to evaluate the coefficients for the stress functions. Isakson and Levy [1971] used a shear interlayer element and a displacement-based formulation. Constant strain triangles were used to model the laminate layers. Wang and Crossman [1977a, 1977b] used constant strain elements and studied a plane strain slice of an angle-ply laminate specimen. Rybicki and Schmueser [1978] used Wilson's SAP IV program to carry out a three-dimensional analysis based on eight-point "brick" isoparametric elements. Most investigators confirmed Puppo and Evensen's [1970] and Pagano and Pipes [1973] findings. However, Bar-Yoseph [1983] noted that Whitcomb et al.'s [1982] findings are in contradiction of Pipes and Pagano's [1970] finite difference solution and the displacement-formulated perturbation solution by Hsu and Herakovich [1977]. All of the analyses were based on linear elastic material behavior and constant axial strain. None could allow for cumulative damage. Nishioka and Atluri [1980, 1982] used an assumed stress approach. Spilker and Chou [1980a,b] presented a special purpose hybrid-stress multi-layer finite element satisfying traction-free edge conditions exactly and applied it to a symmetric cross-ply laminate. Mixed (displacement-stress) formulations have been attempted [e.g., Labbe et al, 1982].

To admit crack-tip singularity into the analysis, special singularity elements have been used for some time. Wang et al. [1975a, 1975b], and Wang and Mandell [1977] used a two-dimensional hybrid stress element including a crack-tip singularity element embedded in a matrix interlayer between plies of the laminate. Muskhelishvilli's solution was the basis for the special element. Accuracy of one percent for the stress intensity factor calculation was claimed. Ueng et al. [1977] applied this procedure to study failure of a notched, unidirectional, boron-epoxy composite under simple tension along the fibers and perpendicular to the notch. Raju et al. [1980] and Whitcomb et al. [1982] tested the capability of the finite element method to solve for interlaminar stress in composite laminates. It was felt that with sufficient refinement, the method could give a highly accurate estimate of the interlayer stress field except in the two elements closest to a singularity or discontinuity. These conclusions were based on use of 8-point "brick" (trilinear lagrangian) three-dimensional elements and 8-point isoparametric quadrilateral plane strain elements. Stanton and Crain [1980] sought to economize on the computer effort. Linear constraints were applied to tricubic Lagrangian three-dimensional elements to get a lower order displacement along selected axes while retaining the higher order variation in material properties and higher order geometry. Transition elements were used to pass from discrete ply modelling to composite laminate modelling. Raju and Crews [1981] used a polar mesh near the intersection of the interface and the free edge along with a log-linear radial distance from the singular point. Bar-Yoseph [1982] presented a method based on composite expansion and assumed stress approach. Wang and Slomiana [1982] introduced finite elements allowing for fracture based on Griffith's criterion to simulate crack growth. This is an extension of earlier work by Wang and Crossman [1977a]. All three modes of crackin^n were considered. Rybicki et al. [1977] used an energy release rate criterion in finite element analysis of crack growth. Table 3 lists a comparison of various methods used for numerical solution of the free-edge delamination specimen problem.

In order to set up finite element solution procedures, variational formulations are generally used. For analysis of laminates, minimization of potential energy, minimization of complementary energy, Reissner's mixed variational principle, hvbrid variational formulations, global-local formulations etc., have been used.

All the investigators report success with whatever method they adopted. Good agreement with test data is claimed. However, it is noteworthy that, in general, no mention is made of computational efficiency. It is well known that triangular elements do not give good distribution of stress except in the simplest cases. Again, a small number of trilinear elements will not give good stress distribution which could be used as the basis for simulation of initiation and progresive growth of cracks and/or delamination. It has also been noted [e.g., Chang 1981; Jenq 1982] that mixed

Table 3: STRESS-DISTRIBUTION IN A FREE-EDGE DELAMINATION SPECIMEN.

Reference	Method of Analysis	Stress Components
Puppo & Evensen [1970]	Approximate Elastic Solution	$\sigma_{xx}, \sigma_{yy}, \sigma_{xz}, \sigma_{xy}$
Pipes & Pagano [1974]	Approximate Elastic Solution	$\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$
Hsu & Herakovich [1977]	Perturbation Technique	$\sigma_{zz}, \sigma_{xz}, \sigma_{yz}$
Pagano [1978]	Stress-Based Theory of Plates	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$
Pagano & Soni [1983]	Global-Local Model	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$
Pipes & Pagano [1970]	Finite Difference	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$
Wang & Crossman [1977a]	Finite Element Constant Strain Triangle	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$
Whitcomb [1982]	Finite Element 8-Noded Isoparametric Element	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$
Rybicki [1971]	Finite Element Method Equilibrium Stress Method	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$
Spilker [1980b]	Finite Element Method Hybrid Stress Model	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$
Wang & Yuan [1983]	Finite Element Singular Hybrid Element	$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}$

(displacement-stress) formulations result in poorly conditioned matrices and the solutions have spatially oscillatory errors. Most finite element formulations based on use of special singularity elements can adequately predict onset of fracture in the vicinity of the crack-tip but are not convenient for prediction of crack propagation and arrest. Also, they cannot predict the initiation of fracture in an initially uncracked material.

Some work [e.g., Pryor and Barker 1970; Mawenya and Davies 1973; Hinton 1976; Reddy 1982] has been done on flexure of composite laminates. In general, the finite element interpolation along the transverse axis is chosen to follow one of the plate theories. In-plane variation follows the usual finite element procedures. Lagrangian as well as isoparametric interpolations have been used. Engblom and Ochoa [1985] used through-the-thickness elements for restricted applications. Mawenya and Davies [1973] used a layerwise model. Recently, Reddy [1982, 1987] and Kuppuswamy and Reddy [1984] have used the layerwise theory for laminated plates.

2.4 THE PRESENT RESEARCH.

Review of the existing literature on the subject showed that a proper theory for modelling bending and stretching behavior of composite laminate plates, with the capability of providing reliable solutions for interlaminar stresses, was lacking. The available theories were based on a number of ad-hoc assumptions and were applicable to only certain special situations. Pipes and Pagano [1970] pointed out that the interlaminar normal stress is primarily responsible for delamination failures. It was extremely important to develop adequate theoretical models which could be used to predict stresses at the interfaces with the required degree of accuracy. Finite element models ensuring continuity of tractions across material interfaces are rare. For plate theories there was no concensus regarding consideration of shear deformation and most theories did not satisfy interlayer continuity of tractions and were inadequate for study of local effects causing damage. Pagano's theory, based on assumption of linear variation of the in-plane stress components over each layer, has given excellent results. However, its application has been restricted to a few layers. It appeared necessary to develop procedures which would permit application of Pagano's theory to multi-layer laminate configurations. This led to the following lines of investigation for development of the theoretical models:

1. Systematic development of models of combined bending and stretching of plates properly allowing for interlayer continuity of tractions, including revisiting existing theories;
2. Finite element implementation of the theoretical models developed under (1) above;
3. Development of finite element models for analysis of free-edge delamination specimens.

Under item 1 above, an hierarchical approach to the construction of assumed displacement type theories of laminated plates was identified. The constitutive equations in each case were derived from a general variational principle. These theories do not require the use of ad-hoc correction factors but directly and explicitly contain the influence of the layup and the layer properties upon the constitutive relations. Pagano's theory was restated in a form convenient for application of finite element procedures. It was seen that the constitutive equations for this theory need not be based upon any variational principle but could be derived directly by integrating the material constitutive relations. The first order coupled shear deformation theory and the restated Pagano's theory were implemented in finite element computer programs. Additionally, approximation of a free-edge delamination specimen as a segment of an annulus with a large radius was implemented in a finite element computer program. A significant result was the development of continuous traction finite elements using a cubic displacement representation for analysis of free-edge delamination specimens. In each case the influence of element size (mesh refinement) on accuracy was carefully examined. Wherever possible, microcomputer versions of the codes were prepared. All the computer codes were carefully documented. The codes have a modular structure to permit easy update and enhancement in the future. The validity of the proposed models was checked against available exact solutions and test data. The computer codes were used to predict stresses and deformations in multi-layer free-edge delamination specimens.

SECTION III

EQUATIONS OF THREE-DIMENSIONAL LINEAR ELASTICITY AND THEIR SPECIALIZATION TO FREE-EDGE DELAMINATION SPECIMENS

3.1 INTRODUCTION.

We start by summarizing the well-known equations of three-dimensional linear theory of elasticity. We shall then specialize these for the case of free-edge delamination specimens allowing for the special geometry and loading configuration. These consist of the fact that the specimens are uniformly stretched in the longtudinal direction and that the laminate possesses certain symmetries.

3.2 THREE-DIMENSIONAL ELASTICITY.

3.2.1 Kinematics.

If, using a rectangular cartesian frame of reference, u_i are the components of the vector describing the displacement of an arbitrary particle in an elastic solid, the cartesian components e_{ij} of the infinitesimal strain tensor are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

Here, and in the sequel, we use standard index notation. Roman indices take on the range of values from 1 to 3. Summation on repeated indices is implied unless otherwise indicated. A subscripted comma denotes differentiation with respect to the spatial coordinate indicated by the subscript following the comma. Parentheses around a pair of indices denote the symmetric part of the quantity with respect to the two indices. Greek indices take on values 1 and 2. Parentheses around a single index denote "no sum" on that index.

The strain tensor, defined by (1) is symmetric, i.e.

$$e_{ij} = e_{ji} = u_{(i,j)} \tag{2}$$

3.2.2 Equilibrium.

If f_i are the components of the body force vector acting upon an arbitrary unit volume of the solid, the static equilibrium equations for the solid are:

$$\sigma_{ij,i} + f_j = 0 \qquad \text{over} \qquad R \tag{3}$$

Here σ_{ij} are the components of the symmetric Cauchy stress tensor, R is the deformed configuration of the solid on which equilibrium is defined. However, for linear theory, no distinction is made between the undeformed and the deformed state for the purpose of defining the stress components and the equations of equilibrium. Often, the quantity on the left-hand side of (3) is referred to as the "disequilibrium force vector." Using d_i to represent components of the disequilibrium force vector, their vanishing everywhere implies pointwise equilibrium. However, if we consider the inner product

$$P = \int_R d_i\, g_i \, dR \tag{4}$$

the product vanishes for every bounded function g_i if and only if the quantity d_i is zero almost everywhere. Further noting that any bounded function can be approximated as closely as desired by a polynomial with rational coefficients, vanishing of d_i can be replaced by P = 0 for g_i, a polynomial with rational coefficients and of arbitrary degree in the spatial coordinates. This weak form is the basis of theories of mechanical behavior of surface structures e.g., plates and shells. We shall refer to this concept again in Section IV.

3.2.3 Constitutive Relations.

For a linear elastic material, the constitutive relations have the form

$$\sigma_{ij} = E_{klij} e_{kl} \tag{5}$$

Here E_{ijkl} are components of the isothermal elasticity tensor. Because of symmetry of the strain tensor,

$$E_{ijkl} = E_{jikl} \tag{6}$$

If there are no body force couples, the stress tensor is symmetric. This leads to

$$E_{ijkl} = E_{ijlk} \tag{7}$$

Further, if it is assumed that the stress is independent of the strain path and an energy density function of strain exists such that each component of the stress tensor is the gradient of this function along the "corresponding" component of the strain tensor,

$$E_{ijkl} = E_{klij} \tag{8}$$

Thus, the number of independent constants defining the elastic material reduces to 21. For a material having symmetry about a plane the number reduces to 13 and, for an orthotropic material, the number is 9. The relationship can then be written in the form

$$\sigma_{\alpha\beta} = E_{\gamma\delta\alpha\beta} e_{\gamma\delta} + E_{33\alpha\beta} e_{33} \tag{9}$$

$$\sigma_{\alpha 3} = 2 E_{\gamma 3\alpha 3} e_{\gamma 3} \tag{10}$$

$$\sigma_{33} = E_{\gamma\delta 33} e_{\gamma\delta} + E_{3333} e_{33} \tag{11}$$

The inverse relationships are

$$e_{\alpha\beta} = C_{\gamma\delta\alpha\beta} \sigma_{\gamma\delta} + C_{\alpha\beta 33} \sigma_{33} \tag{12}$$

$$e_{\alpha 3} = 2 C_{\gamma 3\alpha 3} \sigma_{\gamma 3} \tag{13}$$

$$e_{33} = C_{\gamma\delta 33} \sigma_{\gamma\delta} + C_{3333} \sigma_{33} \tag{14}$$

3.3 SPECIALIZATION TO FREE-EDGE DELAMINATION SPECIMENS.

3.3.1 The Free-Edge Delamination Specimen.

Figure 2 shows a symmetric laminate specimen under a uniform axial strain e_0 . For this case, away from the ends, the displacement field for any transverse section ($x_1 = \text{constant}$) has the following form [Pipes and Pagano 1970]

$$u_1(x_i) = e_0 x_1 + U_1(x_2, x_3) \tag{15}$$

$$u_2(x_i) = U_2(x_2, x_3) \tag{16}$$

and

$$u_3(x_i) = U_3(x_2, x_3) \tag{17}$$

3.3.2 Effect of Specimen Symmetries.

Symmetry of loading as well as geometry about the midplane ($x_3 = 0$) and rotational symmetry about the axis x_3 through the middle point result in the following relationships:

$$U_\alpha(x_2, -x_3) = U_\alpha(x_2, x_3) \tag{18}$$

$$U_3(x_2, -x_3) = -U_3(x_2, x_3) \tag{19}$$

$$U_2(-x_\beta, x_3) = -U_2(x_\beta, x_3) \tag{20}$$

$$U_3(-x_2, x_3) = U_3(x_2, x_3) \tag{21}$$

Using the chain rule of differentiation, (18), (19) and (21) yield

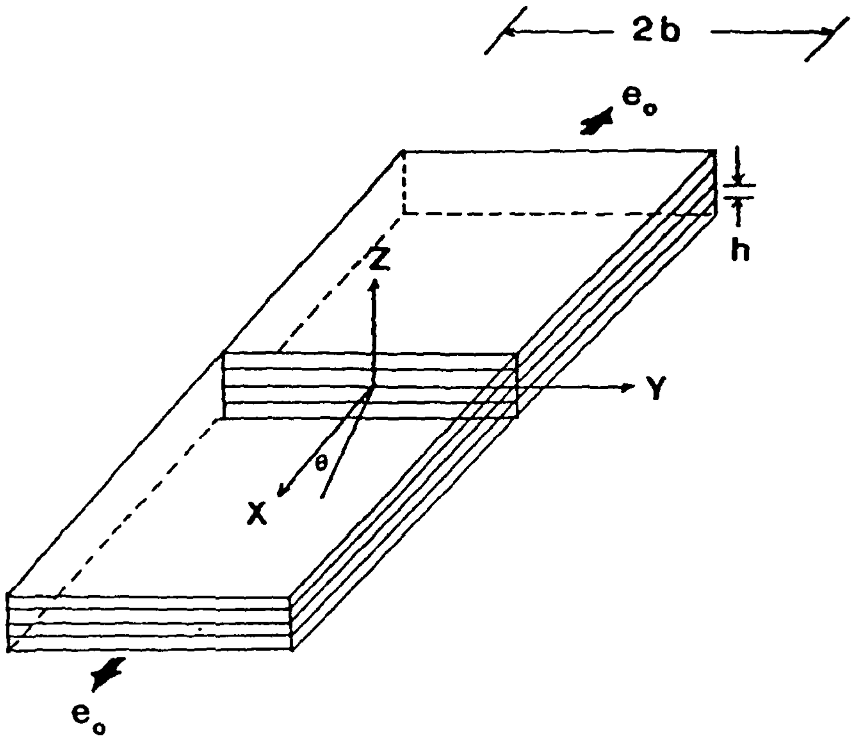
$$-U_{\alpha,3}(x_2, -x_3) = U_{\alpha,3}(x_2, x_3) \tag{22}$$

$$U_{3,2}(x_2, -x_3) = -U_{3,2}(x_2, x_3) \tag{23}$$

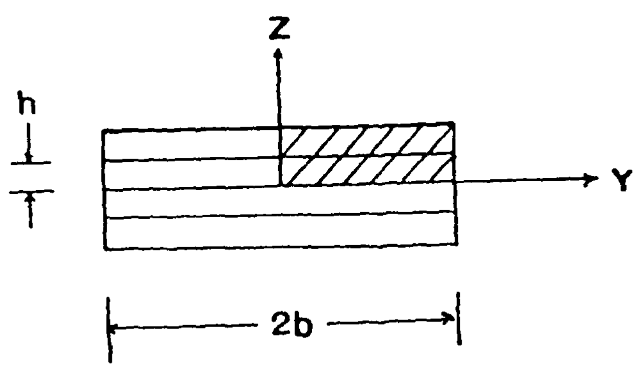
$$-U_{3,2}(-x_2, x_3) = U_{3,2}(x_2, x_3) \tag{24}$$

Setting $x_3 = 0$ in (19), (22), and (23),

$$U_3(x_2, 0) = 0 \tag{25}$$



(a) Symmetric Laminate



(b) x=constant Plane

Figure 2: FREE-EDGE DELAMINATION SPECIMEN

$$U_{3,2}(x_2, 0) = U_{\alpha,3}(x_2, 0) = 0 \tag{26}$$

Equation (26) also gives

$$(U_{3,2} - U_{\alpha,3})_{(x_2, 0)} = 0 \qquad \text{for all } x_2 \tag{27}$$

Equation (20) gives

$$U_2(0, x_3) = 0 \qquad \text{for all } x_3 \tag{28}$$

and, as a consequence,

$$U_{2,3}(0, x_3) = 0 \qquad \text{for all } x_3 \tag{29}$$

Equation (24) gives

$$U_{3,2}(0, x_3) = 0 \qquad \text{for all } x_3 \tag{30}$$

Equation (29) along with (30) leads to

$$\gamma_{23} = 0 \quad \text{for all } x_3 \text{ at } x_2 = 0 \tag{31}$$

and

$$(U_{3,2} - U_{2,3})_{(0, x_3)} = 0 \tag{32}$$

Summarizing, for the free-edge delamination specimen, the following constraints apply:

$$U_2(0, x_3) = U_3(x_2, 0) = 0 \tag{33}$$

$$(U_{3,2} - U_{2,3})_{(0, x_3)} = (U_{3,2} - U_{\alpha,3})_{(x_2, 0)} = 0 \tag{34}$$

$$\gamma_{23}(x_2, 0) = \gamma_{23}(0, x_3) = 0 \tag{35}$$

SECTION IV

EQUATIONS GOVERNING BENDING AND STRETCHING OF PLATES

4.1 INTRODUCTION.

To reduce the three-dimensional problem of a plate to one in two dimensions, the dependence of the field variables on the transverse coordinate is made explicit. This involves assumptions regarding dependence of stress or displacement on the coordinate x_3.

In the present research effort, using the discrete laminate concept, two approaches were considered. In one, assumptions were made regarding displacement variation with respect to the coordinate x_3. An innovative component of the present effort was that the constitutive relations were derived directly from a variational theorem using a generalization of Reissner's work [1984]. This revealed the existence of a coupling between the force resultants and deformations in various layers.

In the second approach, originally proposed by Pagano [1978], assumptions were made regarding stresses which were required to be in pointwise equilibrium. The stress-strain relations were satisfied pointwise as well. The strains were integrated using appropriate weighting functions to yield constitutive relations for the force resultants in terms of generalized displacements and their gradients. Continuity of displacements as well as of tractions was enforced across interlayer surfaces. No assumptions were necessary regarding dependence of the displacement components upon x_3.

In this Section we summarize the first approach. Details are given in reports listed as items B.1.1; B.1.2 and dissertations listed as items B.4.2 and B.4.5 in Appendix B.

The second approach is discussed in Section V.

4.2 GOVERNING EQUATIONS.

4.2.1 Kinematics.

In general, using a polynomial representation of the dependence of the components of displacement upon the coordinate x_3 through the thickness of a plate, one could write

$$u_i(x_j) = v_i(x_\beta) + x_3\,\phi_i(x_\beta) + x_3^2\,\psi_i(x_\beta) + \ldots \tag{36}$$

Most theories of bending and stretching of plates employ a polynomial up to a certain degree. For example, using terms up to the first order for the in-plane displacement components and only the constant term for u_3 gives the first order theory described by the assumptions

$$u_\alpha(x_i) = v_\alpha(x_\beta) + x_3\,\phi_\alpha(x_\beta) \tag{37}$$

$$u_3(x_i) = v_3(x_\beta) \tag{38}$$

Corresponding to these displacements, the strains are:

$$e_{\alpha\beta} = v_{(\alpha,\beta)}(x_\gamma) + x_3\,\phi_{(\alpha,\beta)}(x_\gamma) \tag{39}$$

$$\gamma_{\alpha 3} = v_{3,\alpha} + \phi_\alpha \tag{40}$$

$$e_{33} = 0 \tag{41}$$

For this case, the quantity $\phi_{(\alpha,\beta)}$ is referred to as the curvature of the plate. Using the general polynomial representation, the components of strain are

$$e_{\alpha\beta} = v_{(\alpha,\beta)}(x_\gamma) + x_3\,\phi_{(\alpha,\beta)}(x_\gamma) + x_3^2\,\psi_{(\alpha,\beta)}(x_\gamma) + \ldots\ldots\ldots \tag{42}$$

$$\gamma_{\alpha 3} = v_{3,\alpha} + x_3\phi_{3,\alpha} + x_3^2\,\psi_{3,\alpha} + x_3^3\,\chi_{3,\alpha} + \ldots\ldots\ldots + \phi_\alpha + 2\,x_3\,\psi_\alpha + 3\,x_3^2\,\chi_\alpha + \ldots\ldots\ldots \tag{43}$$

$$e_{33} = v_{3,3} + 2\,x_3\,\phi_3 + 3\,x_3^2\,\chi_3 + \ldots\ldots\ldots \tag{44}$$

For the discrete laminate theory, the above equations apply to each layer.

4.2.2 Equilibrium.

In line with the concepts outlined in Section 4.1, the "disequilibrium force vector" in Section 3.2.2 is orthogonal to an arbitrary function over the thickness of the layer. Noting that any bounded function over a finite domain can be approximated by a polynomial of sufficiently high degree and with rational coefficients, the condition for equilibrium, using a polynomial in x_3 is

$$\int_R (\sigma_{ij,i} + f_j) x_3^n dx_3 = 0, \text{ for } n = 0, 1, 2, ... \tag{45}$$

For a cylindrical solid with area A in coordinates x_α and thickness t in coordinate x_3, (45) is

$$\int_A \int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma_{ij,i} + f_j) x_3^n dx_3 \, dA = 0 \tag{46}$$

Here t could be a function of x_α If we require that the quantity on the left side of (46) vanish over arbitrary regions in the domain defined by x_α

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma_{ij,i} + f_j) x_3^n dx_3 = 0 \tag{47}$$

For n = 0, this leads to the equations

$$N_{\alpha\beta,\alpha} + \sigma^+_{\beta 3} - \sigma^-_{\beta 3} + F_\beta = 0 \tag{48}$$

$$Q_{\alpha,\alpha} + \sigma^+_{33} - \sigma^-_{33} + F_3 = 0 \tag{49}$$

Here we have introduced force resultants

$$(N_{\alpha\beta}, Q_\alpha, F_i) = \int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma_{\alpha\beta}, \sigma_{\alpha 3}, f_i) dx_3 \tag{50}$$

The superscripted + and - , respectively, denote the top and the bottom surfaces of the plate. Considering n = 1 and only the equilibrium equation corresponding to i = 1,2 we get

$$M_{\alpha\beta,\alpha} - Q_\beta + \frac{t}{2}(\sigma^+_{\beta 3} + \sigma^-_{\beta 3}) = 0 \tag{51}$$

where we define additional generalized forces

$$M_{\alpha\beta} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\alpha\beta} x_3 \, dx_3 \tag{52}$$

The quantities $Q_\alpha, N_{\alpha\beta}, M_{\alpha\beta}$, are, respectively, the shear force, the normal force and the bending moment resultants for the plate.

We note that for a consistent theory, the number of equilibrium equations must equal the number of free kinematic field variables. Thus, for the first order theory represented by the kinematic assumptions of (37) and (38), five equations of equilibrium are required. Equations (48) , (49) along with (51) provide these. If u_α is quadratic and u_3 linear in x_3, there are eight kinematic variables requiring eight equations of equilibrium. Following the argument given above, we shall use i = 1, 2, 3 for n = 0, 1 along with i = 1, 2 for n = 2 to furnish the eight equations. Extending the argument to the case of u_α cubic and u_3 quadratic in x_3, we have eleven kinematic variables and the corresponding equations of equilibrium will be set up using i = 1, 2, 3 for n = 0, 1, 2 and i = 1, 2 for n = 3. This provides a systematic approach to an hierarchical development of higher order theories of any order desired.

4.3 CONSTITUTIVE EQUATIONS FOR ELASTIC PLATES.

4.3.1 Introduction.

In setting up equations of equilibrium, force resultants i.e., quantities of the type

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{ij} x_3^n dx_3 \tag{53}$$

were introduced. For n = 0, there are five quantities viz., $N_{\alpha\beta}, Q_\alpha$. For n = 1, there are six additional quantities, viz., $M_{\alpha\beta}$, corresponding to i = 1, 2 and $M_{\alpha 3}$, $N_{3,3}$ defined by

$$M_{\alpha 3} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\alpha 3} x_3 dx_3 \tag{54}$$

$$N_{33} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{33} dx_3 \tag{55}$$

For n = 2 and i = 1, 2 there would be additional quantities

$$P_{\alpha\beta} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\alpha\beta} x_3^2 dx_3 \tag{56}$$

The number of force resultants is invariably larger than the number of equations of equilibrium. For this reason, the equations of equilibrium are not sufficient to define the resultants. However, if the equations of equilibrium are expressed in terms of the kinematic variables, the number of equations of equilibrium being equal to the number of kinematic variables, a solution could exist. The relationships, expressing force resultants in terms of kinematic variables, are the constitutive equations for the problem. For the first order theory, for five equations of equilibrium, there are eight force resultants needing as many constitutive relationships. For u_α quadratic and u_3 linear in x_3 there are eight equations of equilibrium in 14 force resultants needing 14 constitutive relations, and so on.

For a laminate there are kinematic variables, equilibrium equations and force resultants associated with each layer. In fact, additional field variables appear in the form of interlayer tractions which are a priori unknown and are of special importance in their influence on damage mechanisms. Corresponding to these interlayer tractions, equations of balance arise as continuity conditions for the components of displacement. For preexisting and specified delamination opening or slip, these would be the specified measures of discontinuity. Thus, the number of kinematic variables for the first order theory is 5N, where N is the number of layers, and there are 8N force resultants along with 3(N - 1) interlayer tractions with as many displacement continuity conditions. For a higher order theory based on u_α quadratic and u_3 linear in x_3, there are 8N kinematic variables with as many equilibrium equations, 14N force resultants along with 3(N - 1) interlayer tractions and as many displacement continuity equations.

To derive these constitutive relationships, one procedure would be to simply integrate the stress-strain relations for the material. Another is to proceed via appropriate variational formulations written for three-dimensional elasticity and then specialized to reflect the kinematic assumptions and the definitions of the force resultants.

Herein, we describe the traditional approach. An alternative, suggested by Reissner's work, used for the development of a hierarchical approach to constitutive theory of plates is discussed in Section V.

4.3.2 Direct Integration of Material Constitutive Relations.

The general linear elasticity relationships described by (5) upon specialization to the case of a lamina with material symmetry about the middle surface, reduce to (9) through (11) which, for the kth lamina, are:

$$\sigma^{(k)}_{\alpha\beta} = E^{(k)}_{\gamma\delta\alpha\beta}\, e^{(k)}_{\gamma\delta} + E^{(k)}_{33\alpha\beta}\, e^{(k)}_{33} \tag{57}$$

$$\sigma_{\alpha 3}^{(k)} = 2\, E_{\gamma 3 \alpha 3}^{(k)}\, e_{\gamma 3}^{(k)} \tag{58}$$

$$\sigma_{33} = E_{\gamma\delta 33}^{(k)}\, e_{\gamma\delta}^{(k)} + E_{3333}^{(k)}\, e_{33}^{(k)} \tag{59}$$

In developing theories of thin homogeneous plates, it is customary to eliminate e_{33} between (57) and (59) and then to assume a condition of plane stress to get

$$\sigma_{\alpha\beta}^{(k)} = \bar{E}_{\gamma\delta\alpha\beta}^{(k)}\, e_{\gamma\delta}^{(k)} \tag{60}$$

where

$$\bar{E}_{\gamma\delta\alpha\beta}^{(k)} = E_{\gamma\delta\alpha\beta}^{(k)} - \frac{E_{\gamma\delta 33}^{(k)}\, E_{33\alpha\beta}^{(k)}}{E_{3333}^{(k)}} \tag{61}$$

Noting (42) and (43), the material constitutive relations upon appropriate integration for the first order theory and an homogeneous layer give the following constitutive relationships for the force resultants:

$$N_{\alpha\beta}^{(k)} = t_k \bar{E}_{\gamma\delta\alpha\beta}^{(k)}\, v_{(\gamma,\delta)}^{(k)} \tag{62}$$

$$M_{\alpha\beta}^{(k)} = \frac{t_k^3}{12}\, \bar{E}_{\gamma\delta\alpha\beta}^{(k)}\, \phi_{(\gamma,\delta)}^{(k)} \tag{63}$$

$$Q_{\alpha}^{(k)} = t_k\, E_{\gamma 3\alpha 3}^{(k)}\, (v_{3,\gamma}^{(k)} + \phi_{\gamma}^{(k)}) \tag{64}$$

We note that the elimination of e_{33} is not consistent with the kinematic assumption of vanishing e_{33}. For the higher order theories in which e_{33} is nonvanishing, the assumption of generalized plane stress could be used. However, for laminates, the transverse stress at the interface is not negligible and indeed governs delamination. Here this assumption would not be admissible. The general relationships based on (42) through (44) would be

$$N_{\alpha\beta}^{(k)} = t_k\, E_{\gamma\delta\alpha\beta}^{(k)} \left| v_{(\gamma,\delta)}^{(k)} + \frac{t_k^2}{12}\, \psi_{(\gamma,\delta)}^{(k)} + \ldots\ldots \right| + t_k\, E_{33\alpha\beta}^{(k)} \left| v_{3,3}^{(k)} + \frac{t_k^2}{4}\, \chi_3^{(k)} + \ldots\ldots \right| \tag{65}$$

$$M_{\alpha\beta}^{(k)} = t_k^3 E_{\gamma\delta\alpha\beta}^{(k)} \left(\frac{1}{12}\, \phi_{\gamma\delta}^{(k)} + \ldots\ldots \right) + t_k^3 E_{33\alpha\beta}^{(k)} \left(\frac{1}{6} \phi_3^{(k)} + \ldots\ldots \right) \tag{66}$$

$$Q_{\alpha}^{(k)} = t_k\, E_{\gamma 3 \alpha 3}^{(k)} \left[v_{3,\gamma}^{(k)} + \frac{t_k^2}{12}\, \psi_{3,\gamma}^{(k)} + \ldots\ldots + \phi_{\gamma}^{(k)} + \frac{t_k^2}{4}\, \chi_{\gamma}^{(k)} + \ldots\ldots \right] \tag{67}$$

etc. The assumption of generalized plane stress was first used by Mindlin [1951] for homogeneous plate theory and later by Whitney {1970] for laminated plates. Wang [1972] showed that this assumption gave better accuracy than the assumption of plane stress i.e., $\sigma_{33} = 0$. For a laminate, (57) through (59) were directly used to set up the constitutive equations for the first order as well as the higher order theories.

It is known from elasticity solutions [e.g. Pagano 1969, 1970] that the transverse shear stress distribution is close to parabolic over the thickness of each layer. However, the kinematic assumptions (37) and (38) correspond to shear strain independent of x_3 within the layer. If, in addition, the interlayer continuity of tractions is enforced, the shear stress distribution would have to be constant over the entire thickness of the laminate. In general, the in-plane stress components are calculated using the constitutive equations (57) and the remaining components are derived from the stress-equilibrium equations for the three-dimensional configuration. Also, direct use of the constitutive relations as derived above will not allow for shear and transverse deformation properly and may result in erroneous plate stiffness. It has been customary to use a stiffness factor to account for this discrepancy. As part of the present research effort, consistent theories to allow for shear and transverse deformation were developed. This is briefly described in Section V. Detailed discussion is available in items B.1.2, B.2.6, B.2.7, B.3.3, B.4.2 and B.4.5 of Appendix B.

SECTION V

A VARIATIONAL APPROACH TO CONSTITUTIVE THEORY OF LAMINATED PLATES

5.1 PRELIMINARIES

Reissner [1984] derived consistent shear deformation constitutive equations from a mixed variational principle proposed on a somewhat ad hoc basis. Herein, we start by writing a general variational theorem for elastic bodies following Sandhu [1975, 1977] and then specialize it to the case of a laminated plate. Reissner's functional is seen to arise as a special case of the general formulation. Specialization to the case of a laminated plate using an equilibrium stress field yields the coupled constitutive relations between force resultants and laminar deformations.

The field equations of three-dimensional linear elasticity written as a self-adjoint operator matrix in the complementary formulation [Sandhu 1975] are

$$\begin{bmatrix} 0 & 0 & 0 & \delta_{\alpha\gamma}\dfrac{\partial}{\partial 3} & L_2 \\ 0 & 0 & \dfrac{\partial}{\partial 3} & \dfrac{\partial}{\partial \gamma} & 0 \\ 0 & -\dfrac{\partial}{\partial 3} & C_{3333} & 2C_{\gamma 333} & C_{\gamma\delta 33} \\ -\delta_{\alpha\gamma}\dfrac{\partial}{\partial 3} & -\dfrac{\partial}{\partial \alpha} & 2C_{33\alpha 3} & 4C_{\gamma 3\alpha 3} & 2C_{\gamma\delta\alpha 3} \\ -L_1 & 0 & C_{33\alpha\beta} & 2C_{\gamma 3\alpha\beta} & C_{\gamma\delta\alpha\beta} \end{bmatrix} \begin{Bmatrix} u_\gamma \\ u_3 \\ \sigma_{33} \\ \sigma_{\gamma 3} \\ \sigma_{\gamma\delta} \end{Bmatrix} = \begin{Bmatrix} -f_\alpha \\ -f_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{68}$$

Here $\delta_{\alpha\beta}$ is the identity tensor, $\dfrac{\partial}{\partial i} = \dfrac{\partial}{\partial x_i}$, and

$$L_1 = \frac{1}{2}\left(\delta_{\alpha\gamma}\frac{\partial}{\partial \beta} + \delta_{\beta\gamma}\frac{\partial}{\partial \alpha}\right) \tag{69}$$

$$L_2 = \frac{1}{2}\left(\delta_{\alpha\gamma}\frac{\partial}{\partial \delta} + \delta_{\alpha\delta}\frac{\partial}{\partial \gamma}\right) \tag{70}$$

He we have separated the "in-plane" displacements and stresses from the transverse and shear components. If the bilinear mapping is the inner product

$$<f,g> = \int_R f \cdot g \, dR \tag{71}$$

where f, g are functions defined over the region R, elements of the operator matrix in (68) satisfy self-adjointness in the sense of (A.21) of Appendix A to this report. The operators on the diagonal are symmetric and the off-diagonal operators constitute adjoint pairs as defined in Appendix A. Consistent boundary conditions associated with the field equations are

$$u_i n_j = \hat{u}_i n_j \qquad \text{on } S_1 \tag{72}$$

$$-\sigma_{ji} n_j = -\hat{t}_i \qquad \text{on } S_2 \tag{73}$$

Here a superposed circumflex denotes the prescribed value of the quantity over the portion of the boundary indicated in the equation, t_i are the cartesian components of the traction vector and n_i are components of the outward normal to the boundary. S_1 and S_2 constitute complementary subsets of the boundary of the solid. Besides the boundary conditions listed above, there could additionally be discontinuity conditions on the variables at surfaces interior to R. These can be stated as

$$(u_i n_j)' = g_{ij} \qquad \text{on} \qquad S_{1i} \tag{74}$$

$$-(\sigma_{ij} n_i)' = -h_j \qquad \text{on} \qquad S_{2i} \tag{75}$$

Here the superscripted prime denotes a jump in the quantity and S_{1i}, S_{2i} are contained in the union of surfaces in the interior of R. If there is no delamination and there are no discontinuities in tractions in the physical problem, the right-hand sides of (74) and (75) vanish, i.e., the continuity condition may be regarded as a "homogeneous" discontinuity equation.

5.2 A VARIATIONAL PRINCIPLE.

Using the definition (A.20), the governing functional for the field equations of linear elasticity along with the consistent boundary conditions and internal discontinuity conditions is

$$\Omega(u_i,\sigma_{ij}) = <u_i\,,\sigma_{ji,j} + 2f_i>_R + <\sigma_{ij}\,, -u_{i,j} + C_{klij}\sigma_{kl}>_R + <\sigma_{ij}\,, u_in_j - 2\hat{u}_in_j>_{S_1}$$

$$- <u_i\,, \sigma_{ji}n_j - 2\hat{t}_i>_{S_2} \tag{76}$$

It is easy to show that the Gateaux differential of this functional along arbitrary paths vanishes if and only if the field equations along with all the consistent boundary conditions and internal discontinuity conditions are satisfied. Noting that, for sufficiently smooth field variables, the following identity holds:

$$\int_R u_i\sigma_{ji,j}\,dR = -\int_R u_{i,j}\sigma_{ij}\,dR + \int_S u_i\sigma_{ji}n_j\,dS + \int_{S_i} (u_i\sigma_{ji}n_j)'\,dS \tag{77}$$

where S is the boundary of R and S_i is the set of internal surfaces imbedded in R over which the quantity $(u_i\sigma_{ji}n_j)$ is discontinuous. Extended forms of the governing functional (76) can be written through elimination of one or the other of the operators appearing on either side of the equality (77). Elimination of derivatives of all the components of stress leads to a generalization of the Hellinger-Reissner functional, viz.,

$$\Omega_1(u_i,\sigma_{ij}) = 2<u_i,f_i>_R + <\sigma_{ij},-2u_{i,j} + C_{klij}\sigma_{kl}>_R + 2<\sigma_{ij},u_in_j - \hat{u}_in_j>_{S_1} + 2<u_i,\hat{t}_i>_{S_2}$$

$$+ 2<\sigma_{ij},(u_in_j)' - g_{ij}>_{S_{1i}} + <u_i,h_i>_{S_{2i}} \tag{78}$$

Specialization of this functional, to the case where the in-plane strains are derived from stresses, leads to an extension of the functional recently postulated by Reissner [1984]. Assuming that the boundary conditions on S_i are identically satisfied, the functional reduces to

$$\Omega_2(u_\alpha,u_3,\sigma_{33},\sigma_{\alpha\beta},\sigma_{\gamma 3})$$

$$= \int_R \left(2u_i f_i + \sigma_{33}(-2u_{3,3} + \bar{e}_{33}) + 2\sigma_{\alpha 3}(-u_{3,\alpha} - u_{\alpha,3} + \bar{e}_{\alpha 3}) + \sigma_{\alpha\beta}(-2u_{\alpha,\beta} + \bar{e}_{\alpha\beta})\right) dR$$

$$+ 2\int_{S_2} u_i \hat{t}_i \, dS + 2\int_{S_{2i}} u_i h_i \, dS + 2\int_{S_{1i}} \sigma_{ij}((n_j u_i)' - g_{ij}) \, dS \tag{79}$$

Here we have again separated some of the in-plane quantities and introduced symbols

$$\bar{e}_{\alpha\beta} = C_{33\alpha\beta}\sigma_{33} + C_{\gamma 3\alpha\beta}\sigma_{\gamma 3} + C_{\gamma\delta\alpha\beta}\sigma_{\gamma\delta} \tag{80}$$

$$\bar{e}_{\alpha 3} = C_{33\alpha 3}\sigma_{33} + 2C_{\gamma 3\alpha 3}\sigma_{\gamma 3} + C_{\gamma\delta\alpha 3}\sigma_{\gamma\delta} \tag{81}$$

$$\bar{e}_{33} = C_{3333}\sigma_{33} + 2C_{\gamma 333}\sigma_{\gamma 3} + C_{\gamma\delta 33}\sigma_{\gamma\delta} \tag{82}$$

Vanishing of the Gateaux differential along arbitrary paths denoted by (v_i, τ_{ij}) gives, upon dropping the factor 2 throughout,

$$\Delta_{(\tau_{ij}, v_i)} \Omega_2 = \int_R \left(\tau_{\alpha\beta}(-u_{\alpha,\beta} + \bar{e}_{\alpha\beta}) + \tau_{\alpha 3}(-u_{\alpha,3} - u_{3,\alpha} + 2\bar{e}_{\alpha 3}) + \tau_{33}(-u_{3,3} + \bar{e}_{33})\right) dR$$

$$- \int_R (\sigma_{\alpha\beta} v_{\alpha,\beta} + \sigma_{\alpha 3}(v_{\alpha,3} + v_{3,\alpha}) + \sigma_{33} v_{3,3} - v_i f_i) \, dR$$

$$+ \int_{S_2} v_i \hat{t}_i \, dS + \int_{S_{2i}} v_i h_i \, dS + \int_{S_{1i}} \tau_{ij}((n_j u_i)' - g_{ij}) \, dS + \int_{S_{1i}} \sigma_{ij} n_j (v_i)' \, dS = 0 \tag{83}$$

Using the identity (77) and noting that $v_i = 0$ on S_1

$$\Delta_{(\tau_{ij}, v_i)} \Omega_2 = \int_R \left(v_i(\sigma_{ji,j} + f_i) + \tau_{\alpha\beta}(-u_{\alpha,\beta} + \bar{e}_{\alpha\beta}) + \tau_{\alpha 3}(-u_{\alpha 3} - u_{3,\alpha} + 2\bar{e}_{\alpha 3}) + \tau_{33}(-u_{3,3} + \bar{e}_{33})\right) dR$$

$$- \int_{S_2} v_i(t_i - \hat{t}_i) \, dS - \int_{S_{2i}} v_i\big((t_i)' - h_i\big) \, dS + \int_{S_{1i}} \tau_{ij}((n_j u_i)' - g_{ij}) \, dS \tag{84}$$

Vanishing of the Gateaux differential for arbitrary v_i, τ_{ij} implies equilibrium over R and S_2 , as well as the constitutive relations and the discontinuity conditions. For the special case where the components of stress σ_{ij} are restricted to satisfying exactly the relations

$$\int_R v_i(\sigma_{ji,j} + f_i)\,dR = 0 \tag{85}$$

$$\int_{S_2} v_i(t_i - \hat{t}_i)\,dS = 0 \tag{86}$$

$$\int_{S_{2i}} v_i(t_i' - h_{i)}\,dS = 0 \tag{87}$$

vanishing of $\Delta\Omega_2$ leads to

$$\int_R (\tau_{\alpha\beta}(-u_{\alpha,\beta} + \bar{e}_{\alpha\beta}) + \tau_{\alpha 3}(-u_{\alpha,3} - u_{3,\alpha} + 2\,\bar{e}_{\alpha 3}) + \tau_{33}(-u_{3,3} + \bar{e}_{33}))\,dR +$$

$$\int_{S_{1i}} \tau_{ij}((n_j u_i)' - g_{ij})\,dS = 0 \tag{88}$$

For a laminate of N layers the variational equation (88) becomes

$$\int_A \left\{ \sum_{k=1}^{N} \int_{-\frac{t_k}{2}}^{\frac{t_k}{2}} \left(\tau^{(k)}_{\alpha\beta}(-u^{(k)}_{\alpha,\beta} + \bar{e}^{(k)}_{\alpha\beta}) + \tau^{(k)}_{\alpha 3}(-u_{\alpha,3} + u^{(k)}_{3,\alpha} + 2\bar{e}^{(k)}_{\alpha 3}) + \tau^{(k)}_{33}(-u^{(k)}_{3,3} + \bar{e}^{(k)}_{33})\right) dx_3 \right\} dA$$

$$+ \int_{S_{2i}} \tau_{ij}((n_j u_i)' - g_{ij})\,dS = 0 \tag{89}$$

5.3 EQUILIBRIUM STRESS STATES

5.3.1 Preliminaries

Equation (85) through (87) can be satisfied by an infinite number of stress states. Two different approaches were studied in the course of the present research. These are:

1. Pointwise or "strong" satisfaction of equilibrium, i.e., choice of stress states such that

$$\sigma_{ji,j} + f_i \equiv 0 \qquad \text{on} \qquad R \tag{90}$$

In this case (85) would be satisfied regardless of the choice of the displacement field.

2. Global or "weak" satisfaction of equilibrium, i.e., assuming the components to be polynomials in the coordinate x_3 such that for a given choice of the displacement field, the disequilibrium force vector $\sigma_{ij,i}+f_j$ is orthogonal to v_j over R, $t_j - \hat{t}_j$ is orthogonal to v_j on S_2 , and $t_j - h_j$ is orthogonal to v_j on S_{21}. This will imply satisfaction of equilibrium of generalized force quantities.

Under (1) above, two alternatives, one based on the first order theory and another based on a higher order theory were studied. For (2) one case assuming displacements to correspond to the first order theory but the shear stresses to be cubic in x_3 was examined.

5.3.2 A First Order Theory

Assuming the in-plane stress components to be linear in the x_3 coordinate and proceeding to integrate the equations of equilibrium it can be shown [items B.1.2 and B.4.2 Appendix B] that, for vanishing body forces,

$$\sigma^{(k)}_{\alpha 3} = \zeta_1 Q^{(k)}_{\alpha} + \zeta_2 T^{(k-1)}_{\alpha} + \zeta_3 T^{(k)}_{\alpha} \tag{91}$$

Here $T^{(k)}_{\alpha}$ is the value of the stress component $\sigma_{\alpha 3}$ at the top of the kth layer, and

$$\zeta_1 = \frac{6}{t_k}\left(\frac{1}{4} - \theta^2\right) \tag{92}$$

$$\zeta_2 = -\frac{1}{4} - \theta + 3\theta^2 \tag{93}$$

$$\zeta_3 = -\frac{1}{4} + \theta + \theta^2 \tag{94}$$

and

$$\theta = \frac{x^{(k)}_3}{t_k} \tag{95}$$

5.3.3 A Higher Order Theory

Assuming the in-plane stress components to be quadratic in the x_3 coordinate, using definitions (50) (52), and (56) specialized to the kth layer, it can be shown [items B.2.6, B.2.7 and B.4.5 of Appendix B] that

$$\sigma_{\alpha\beta}^{(k)} = \eta_1^{(k)} N_{\alpha\beta}^{(k)} + \eta_2^{(k)} M_{\alpha\beta}^{(k)} + \eta_3^{(k)} P_{\alpha\beta}^{(k)} \tag{96}$$

where

$$\eta_1^{(k)} = \frac{1}{4t_k}(9 - 60\theta^2) \tag{97}$$

$$\eta_2^{(k)} = \frac{12}{t_k^2}\theta \tag{98}$$

$$\eta_3^{(k)} = \frac{15}{t_k^3}(-1 + 12\theta^2) \tag{99}$$

Using equilibrium equations (3) for i = 1, 2 along with specialization of (50) and (54) to the kth layer, for no body forces,

$$\sigma_{\alpha 3}^{(k)} = \zeta_1^{(k)} Q_\alpha^{(k)} + \zeta_2^{(k)} M_{\alpha 3}^{(k)} + \zeta_3^{(k)} T_\alpha^{(k-1)} + \zeta_4^{(k)} T_\alpha^{(k)} \tag{100}$$

where

$$\zeta_1^{(k)} = \frac{3}{2t_k}(1 - 4\theta^2) \tag{101}$$

$$\zeta_2^{(k)} = \frac{30\theta}{t_k^2}(1 - 4\theta^2) \tag{102}$$

$$\zeta_3^{(k)} = -\frac{1}{4} + \frac{3}{2}\theta + 3\theta^2 - 10\theta^3 \tag{103}$$

$$\zeta_4^{(k)} = -\frac{1}{4} - \frac{3}{2}\theta + 3\theta^2 + 10\theta^3 \tag{104}$$

Again, using equilibrium equation (3) for i = 3 along with specialization of (55) to the kth layer

$$\sigma_{33}^{(k)} = \xi_1^{(k)} N_{33}^{(k)} + \xi_2^{(k)} T_3^{(k-1)} + \xi_3^{(k)} T_3^{(k)} + \xi_4^{(k)} T_{\rho,\rho}^{(k-1)} + \xi_5^{(k)} T_{\rho,\rho}^{(k)} \tag{105}$$

Here

$$\xi_1^{(k)} = \frac{30}{16t_k}(1 - 8\,\theta^2 + 16\,\theta^4) \tag{106}$$

$$\xi_2^{(k)} = \frac{1}{16}(-7 - 24\,\theta + 120\,\theta^2 + 32\,\theta^3 - 240\,\theta^4) \tag{107}$$

$$\xi_3^{(k)} = \frac{1}{16}(-7 + 24\,\theta + 120\,\theta^2 - 32\,\theta^3 - 240\,\theta^4) \tag{108}$$

$$\xi_4^{(k)} = \frac{t_k}{32}(1 + 8\,\theta - 24\,\theta^2 - 32\,\theta^3 + 80\,\theta^4) \tag{109}$$

$$\xi_5^{(k)} = \frac{t_k}{32}(-1 + 8\,\theta + 24\,\theta^2 - 32\,\theta^3 - 80\,\theta^4) \tag{110}$$

5.3.4 An Alternative Approach: The Weak Form of Equilibrium.

In this approach, equilibrium is satisfied in the "weak" sense i.e., for a given choice of the displacement feld,

$$\int_R (\sigma_{ji,j} + f_i)\, u_i\, dR = 0 \tag{111}$$

For a first order theory, let the displacement field be given by

$$u_\alpha^{(k)} = v_\alpha^{(k)}(x_\beta) + x_3 \phi_\alpha^{(k)}(x_\beta) \tag{112}$$

$$u_3^{(k)} = v_3^{(k)}(x_\beta) + x_3^{(k)}\, \phi_3^{(k)}(x_\beta) \tag{113}$$

Let the stress field be such that $\sigma_{\alpha\beta}^{(k)}$, $\sigma_{\alpha 3}^{(k)}$, $\sigma_{33}^{(k)}$ are, repectively, linear, cubic, and quadratic in the coordinate $x_3^{(k)}$. Thus

$$\sigma_{\alpha\beta}^{(k)} = a_1 + b_1\, x_3^{(k)} \tag{114}$$

$$\sigma_{\alpha 3}^{(k)} = a_2 + b_2\, x_3^{(k)} + c_2\,(x_3^{(k)})^2 + d_2\,(x_3^{(k)})^3 \tag{115}$$

$$\sigma_{33}^{(k)} = a_3 + b_3\, x_3^{(k)} + c_3\,(x_3^{(k)})^2 \tag{116}$$

Evaluation of coefficients leads to the following expressions:

$$u_\alpha^{(k)} = v_\alpha^{(k)} + \theta\,(u_\alpha^{+(k)} - u_\alpha^{-(k)}) \tag{117}$$

$$u_3^{(k)} = v_3^{(k)} + \theta(u_3^{+(k)} - u_3^{-(k)}) \tag{118}$$

$$\sigma_{\alpha\beta}^{(k)} = \eta_1^{(k)} N_{\alpha\beta}^{(k)} + \eta_2^{(k)} M_{\alpha\beta}^{(k)} \tag{119}$$

$$\sigma_{\alpha 3}^{(k)} = \zeta_1^{(k)} Q_\alpha^{(k)} + \zeta_2^{(k)} M_{\alpha 3}^{(k)} + \zeta_3^{(k)} T_\alpha^{(k-1)} + \zeta_4^{(k)} T_\alpha^{(k)} \tag{120}$$

$$\sigma_{33}^{(k)} = \xi_1^{(k)} N_{33}^{(k)} + \xi_2^{(k)} T_3^{(k-1)} + \xi_3^{(k)} T_3^{(k)} \tag{121}$$

where

$$\eta_1^{(k)} = \frac{1}{t_k} \tag{122}$$

$$\eta_2^{(k)} = \frac{12}{t_k^2}\theta \tag{123}$$

$$\zeta_1 = \frac{3}{2t_k}(1 - 4\theta^2) \tag{124}$$

$$\zeta_2^{(k)} = \frac{30}{t_k^2}\theta(1 - 4\theta^2) \tag{125}$$

$$\zeta_3^{(k)} = -\frac{1}{4} + \frac{3}{2}\theta + 3\theta^2 - 10\theta^3 \tag{126}$$

$$\zeta_4^{(k)} = -\frac{1}{4} - \frac{3}{2}\theta + 3\theta^2 + 10\theta^3 \tag{127}$$

$$\xi_1^{(k)} = \frac{3}{2t_k}(1 - 4\theta^2) \tag{128}$$

$$\xi_2^{(k)} = -\frac{1}{4} - \theta + 3\theta^2 \tag{129}$$

$$\xi_3^{(k)} = -\frac{1}{4} + \theta + 3\theta^2 \tag{130}$$

Equation (111), using (117) through (130) yields the following equations of equilibrium for generalized forces:

$$N_{\beta\alpha,\beta}^{(k)} + \sigma_{\alpha 3}^{+(k)} - \sigma_{\alpha 3}^{-(k)} = 0 \tag{131}$$

$$M_{\beta\alpha,\beta}^{(k)} - Q_\alpha^{(k)} + \frac{t_k}{2}(\sigma_{\alpha 3}^{+(k)} + \sigma_{\alpha 3}^{-(k)}) = 0 \tag{132}$$

$$Q^{(k)}_{\alpha,\alpha} + \sigma^{+(k)}_{33} - \sigma^{-(k)}_{33} = 0 \tag{133}$$

$$M^{(k)}_{\alpha 3,\alpha} - N^{(k)}_{33} + \frac{t_k}{2}(\sigma^{+(k)}_{33} + \sigma^{-(k)}_{33}) = 0 \tag{134}$$

These are the same generalized force equilibrium equations as were obtained by direct evaluation of the integral in (47) for no body forces and n = 0, 1. Thus, the stress state selected satisfies the equilibrium equations in terms of generalized forces though pointwise equilibrium is not satisfied.

5.4 CONSTITUTIVE RELATIONS FOR FORCE RESULTANTS.

5.4.1 Preliminaries.

The variational equation (88) holds for every state satisfying (85) through (87) along with u_i identically equal to the specified value on the boundary S_1 and provided that (77) is valid. Substitution of the expressions (80) through (82) for $\bar{e}^{(k)}_{ij}$ and using equilibrium stress components defined in Section 5.3 would give constitutive relations between the force resultants and the gradients of the displacement vector. Using polynomial approximations (36), the constitutive equations relating the displacement parameters with force resultants are realized. Herein, we summarize the results of the three schemes. The first is based on the first order displacement expressions (37) and (38) along with the equilibrium stress field based on $\sigma^{(k)}_{\alpha\beta}$ linear in the coordinate $x^{(k)}_3$. The second is based on $u^{(k)}_\alpha$ quadratic and $u^{(k)}_3$ linear in $x^{(k)}_3$ with the equilibrium stress field described by (96), (100), and (105). The thrid approach uses the weak form of equilibrium described by (111) along with disp;lacement assumptions (112) and (113) leading to the stress state described by (119) through (121).

5.4.2 A First Order Shear Deformation Theory.

Assuming that the in-plane strains $e^{(k)}_{\alpha\beta} = u_{(\alpha,\beta)}$ are identically equal to $\bar{e}^{(k)}_{\alpha\beta}$ i.e., the in-plane stresses are defined by the displacement assumptions and the material constitutive laws, and noting that for the first order theory (38) implies $e_{33} = u_{3,3} = 0$, neglecting $\bar{e}_{33}$, the following specialization of (88) is realized for the case where S_{t_1} is identified as the set of internal surfaces on which a slip of magnitude $g_{\alpha 3}$ occurs:

$$\int_A \left| \sum_{k=1}^{N} \int_{-\frac{t_k}{2}}^{\frac{t_k}{2}} \tau^{(k)}_{\alpha 3}\left(- u^{(k)}_{\alpha,3} - u^{(k)}_{3,\alpha} + 2\,\bar{e}^{(k)}_{\alpha 3}\right) dx_3 \right| d\,A = 0 \tag{135}$$

Substituting (37), (38) for $u^{(k)}_{\alpha}$ and $u^{(k)}_{3}$, using (90) and $\sigma^{(k)}_{\alpha 3}$ in (80), and carrying out the integration over the thickness, for the equality to hold for arbitrary values of $\tau^{(k)}_{\alpha 3}$ satisfying equilibrium,

$$\phi^{(k)}_{\alpha} + v^{(k)}_{3,\alpha} = 4\,C_{\gamma 3\alpha 3}\left\{ \frac{6}{5t_k}\,Q^{(k)}_{\gamma} - \frac{1}{10}(T^{(k-1)}_{\gamma} + T^{(k)}_{\gamma}) \right\} \tag{136}$$

$$\begin{aligned} g^{(k)}_{\alpha 3} - n_3\, u'^{(k)}_{\alpha} - n_\alpha\, u'^{(k)}_{3} &= C^{(k)}_{\gamma 3\alpha 3}\left(- 3\,Q^{(k)}_{\gamma} - t_k\, T^{(k-1)}_{\gamma} + 4\,t_k\, T^{(k)}_{\gamma}\right) \\ &\quad + C^{(k+1)}_{\gamma 3\alpha 3}\left(- 3\,Q^{(k+1)}_{\gamma} - t_{k+1}\, T^{(k-1)}_{\gamma} + 4\,t_{k+1}\, T^{(k)}_{\gamma}\right) \\ &\qquad \text{for } k = 1,2,\ldots\ldots, N\text{-}1 \end{aligned} \tag{137}$$

These are $2(2N-1)$ equations in as many unknowns. Written in matrix form for all the layers, (136) and (137) constitute a set of narrow band equations in which the shear deformation of any layer depends upon the shearing force in that layer and the shearing stresses at the interfaces above and below that layer. Equations (137) represent displacement discontinuity equations for the interface. In case there is no internal slip i.e., $g^{(k)}_{\alpha 3} = 0$ and the displacements are continuous, i.e., $u'^{(k)}_{\alpha}$ and $u'^{(k)}_{3}$ vanish, (136) and (137) for all the layers can be written as

$$\begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \begin{Bmatrix} X_a \\ X_b \end{Bmatrix} = \begin{Bmatrix} R_a \\ 0 \end{Bmatrix} + \begin{Bmatrix} T_a \\ T_b \end{Bmatrix} \tag{138}$$

where

X_a is the set $\{Q_\alpha^{(k)}, k = 1,2,\dots N\}$, X_b is the set $\{\sigma_{\alpha3}^{(k)}, k = 1,2,\dots N\text{-}1\}$, R_a is the set $\{\phi_\alpha^{(k)} + u_{3,\alpha}, k = 1,2,\dots N\}$ and T_a, T_b depend upon the specified shear forces at the top and the bottom surfaces of the laminate. Elimination of X_b through static condensation yields

$$Q_\alpha^{(k)} = \sum_{j=1}^{N} \lambda_{\beta\alpha}^{(k)j} (\phi_\beta^{(k)j} + v_{3,\beta}^{j}) \tag{139}$$

and, conversely,

$$\phi_\alpha^{(k)} + v_{3,\alpha} = \sum_{j=1}^{N} \lambda_{\beta\alpha}^{(k)j} Q_{,}^{j} \tag{140}$$

where j denotes the layer number. The matrices with elements $\mu_{\alpha\beta}^{(k)j}$, and $\lambda_{\alpha\beta}^{(k)j}$ are full and symmetric. For a homogeneous plate, (139) gives

$$Q_\alpha = \frac{5}{6} t E_{\beta3\alpha3} (\phi_\beta + u_{3,\beta}) + \frac{1}{12} t (\sigma_{\alpha3}^{+} + \sigma_{\alpha3}^{-}) \tag{141}$$

where $E_{\alpha3\beta3} = C_{\alpha3\beta3}^{-1}$. For no surface shears, this reduces to Reissner's well-known shear deformation equation.

For the classical laminated plate theory, $\phi_\alpha^{(k)} = \phi_\alpha$ for all k. Then, defining

$$Q_\alpha = \sum_{k=1}^{N} Q_\alpha^{(k)},$$

$$Q_\alpha = \sum_{k=1}^{N} \sum_{j=1}^{N} \lambda_{\beta\alpha}^{kj} (\phi_\beta + u_{3,\beta}) \tag{142}$$

5.4.3 A Higher Order Coupled Constitutive Theory.

Using the kinematic assumptions of u_α quadratic and u linear in x_3 , substituting for strains in terms of the kinematic field variables, and writing $\bar{e}_{ij}$ in terms of force resultants using (96) to (110) , the variational equality (88) yields the constitutive relations for the force resultants. Explicitly, the variational equality has the form

$$\sum_{k=1}^{N}\int_{-\frac{t_k}{2}}^{\frac{t_k}{2}}\left(\left(<\overline{Q}_{\alpha}^{(k)},\overline{M}_{\alpha 3}^{(k)},\overline{T}_{\alpha}^{(k-1)},\overline{T}_{\alpha}^{(k)}>\begin{pmatrix}\zeta_1^{(k)}\\ \zeta_2^{(k)}\\ \zeta_3^{(k)}\\ \zeta_4^{(k)}\\ \zeta_5^{(k)}\end{pmatrix}\left(<1\,,x_3^{(k)}>\begin{pmatrix}\phi_{\alpha}^{(k)}+v_{3,\alpha}^{(k)}\\ 2\,\psi_{\alpha}^{(k)}+\phi_{3,\alpha}^{(k)}\end{pmatrix}\right.\right.\right.$$

$$\left.\left.-<\zeta_1^{(k)},\zeta_2^{(k)},\zeta_3^{(k)},\zeta_4^{(k)}>4\,C_{\gamma 3\alpha 3}\begin{pmatrix}Q_{\gamma}^{(k)}\\ M_{\gamma 3}^{(k)}\\ T_{\gamma}^{(k-1)}\\ T_{\gamma}^{(k)}\end{pmatrix}+(g_{\alpha 3}^{(k)}+g_{3\alpha}^{(k)})\right)\right)$$

$$+\left(<\overline{N}_{\alpha\beta}^{(k)},\overline{M}_{\alpha\beta}^{(k)},\overline{P}_{\alpha\beta}^{(k)}>\begin{pmatrix}\eta_1^{(k)}\\ \eta_2^{(k)}\\ \eta_3^{(k)}\end{pmatrix}\left(<1,x_3^{(k)},(x_3^{(k)})^2>\begin{pmatrix}v_{(\alpha,\beta)}^{(k)}\\ \phi_{(\alpha,\beta)}^{(k)}\\ \psi_{(\alpha,\beta)}^{(k)}\end{pmatrix}\right.\right.$$

$$-<\eta_1^{(k)},\eta_2^{(k)},\eta_3^{(k)}>C_{\gamma\delta\alpha\beta}^{(k)}\begin{pmatrix}N_{\gamma\delta}^{(k)}\\ M_{\gamma\delta}^{(k)}\\ P_{\gamma\delta}^{(k)}\end{pmatrix}$$

$$\left.\left.-<\xi_1^{(k)},\xi_2^{(k)},\xi_3^{(k)},\xi_4^{(k)},\xi_5^{(k)}>C_{33\alpha\beta}^{(k)}\begin{pmatrix}N_{33}^{(k)}\\ T_3^{(k-1)}\\ T_3^{(k)}\\ T\rho{,}\rho^{(k-1)}\\ T\rho{,}\rho^{(k)}\end{pmatrix}+g_{\alpha\beta}^{(k)}-(n_{\beta}u_{\alpha})'^{(k)}\right)\right)$$

$$+\left(<\overline{N}_{33}^{(k)},\overline{T}_3^{(k-1)},\overline{T}_3^{(k)},\overline{T}_{\rho\rho}^{(k-1)},\overline{T}_{\rho\rho}^{(k)}>\begin{pmatrix}\xi_1^{(k)}\\ \xi_2^{(k)}\\ \xi_3^{(k)}\\ \xi_4^{(k)}\\ \xi_5^{(k)}\end{pmatrix}\left(\phi_3^{(k)}\right.\right.$$

$$-\langle \eta_1^{(k)}, \eta_2^{(k)}, \eta_3^{(k)} \rangle C^{(k)}_{\gamma\delta 33} \begin{Bmatrix} N^{(k)}_{\gamma\delta} \\ M^{(k)}_{\gamma\delta} \\ P^{(k)}_{\gamma\delta} \end{Bmatrix}$$

$$-\langle \xi_1^{(k)}, \xi_2^{(k)}, \xi_3^{(k)}, \xi_4^{(k)}, \xi_5^{(k)} \rangle C^{(k)}_{3333} \begin{Bmatrix} N^{(k)}_{33} \\ T_3^{(k-1)} \\ T_3^{(k)} \\ T^{(k-1)}_{\rho,\rho} \\ T^{(k)}_{\rho,\rho} \end{Bmatrix} + g^{(k)}_{33} - (n_3 u_3)^{\prime(k)} \Bigg\}\Bigg\}\Bigg\} dx_3 = 0 \quad (143)$$

Carrying out the integrations over the thickness and writing

$$\gamma_\alpha^{(k)} = 2\, e_{\alpha 3} = \phi_\alpha^{(k)} + v^{(k)}_{3,\alpha} \quad (144)$$

$$\kappa_\alpha^{(k)} = 2\, \psi_\alpha^{(k)} + \phi^{(k)}_{3,\alpha} \quad (145)$$

the equality holding for arbitrary values of $\overline{Q}^{(k)}_\alpha, \overline{M}^{(k)}_{\alpha 3}, \overline{T}^{(k)}_\alpha, \overline{N}^{(k)}_{\alpha\beta}, \overline{M}^{(k)}_{\alpha\beta}, \overline{P}^{(k)}_{\alpha\beta}, \overline{N}^{(k)}_{33}, \overline{T}^{(k)}_3$ leads to the following constitutive relations for the case where $g^{(k)}_{\alpha\beta}$ identically equals $(u_\alpha n_\beta)^{\prime(k)}$

$$\gamma_\alpha^{(k)} = \frac{2}{5} C^{(k)}_{\gamma 3\alpha 3} \left[\frac{12}{t_k} Q_\gamma^{(k)} - (T_\gamma^{(k-1)} + T_\gamma^{(k)}) \right] \quad (146)$$

$$\kappa_\alpha^{(k)} = \frac{4}{7 t_k} C^{(k)}_{\gamma 3\alpha 3} \left[\frac{120}{t_k} M^{(k)}_{\gamma 3} + \frac{3}{2} (T_\gamma^{(k-1)} - T_\gamma^{(k)}) \right] \quad (147)$$

$$v^{(k)}_{(\alpha,\beta)} = C^{(k)}_{\gamma\delta\alpha\beta} \left[\frac{1}{4 t_k} N^{(k)}_{\gamma\delta} - \frac{15}{t_k^3} P^{(k)}_{\gamma\delta} \right]$$

$$+ C^{(k)}_{33\alpha\beta} \left[\frac{60}{t_k} N^{(k)}_{33} - \frac{5}{2} (T_3^{(k-1)} + T_3^{(k)}) + \frac{t_k}{56} (T^{(k-1)}_{\rho,\rho} - T^{(k)}_{\rho,\rho}) \right] \quad (148)$$

$$\phi^{(k)}_{(\alpha,\beta)} = C^{(k)}_{\gamma\delta\alpha\beta} \frac{12}{t_k^3} M^{(k)}_{\gamma\delta} + C^{(k)}_{33\alpha\beta} \left[-\frac{6}{5 t_k} (T_3^{(k-1)} - T_3^{(k)}) + \frac{1}{10} (T^{(k-1)}_{\rho,\rho} + T^{(k)}_{\rho,\rho}) \right] \quad (149)$$

$$\psi^{(k)}_{(\alpha,\beta)} = C^{(k)}_{\gamma\delta\alpha\beta} \left[-\frac{15}{t_k^3} N^{(k)}_{\gamma\delta} + \frac{180}{t_k^5} P^{(k)}_{\gamma\delta} \right]$$

$$+ C^{(k)}_{33\alpha\beta}\left[-\frac{60}{7t_k^3}N^{(k)}_{33} + \frac{30}{7t_k^2}(T^{(k-1)}_3 + T^{(k)}_3) - \frac{15}{2t_k}(T^{(k-1)}_{\rho,\rho} - T^{(k)}_{\rho,\rho})\right] \tag{150}$$

$$\phi^{(k)}_3 = C^{(k)}_{\gamma\delta 33}\left[\frac{60}{t_k}N^{(k)}_{\gamma\delta} - \frac{60}{7t_k^2}P^{(k)}_{\gamma\delta}\right]$$

$$+ C^{(k)}_{3333}\left[\frac{10}{7t_k}N^{(k)}_{33} - \frac{3}{14}(T^{(k-1)}_3 + T^{(k)}_3) + \frac{t_k}{84}(T^{(k-1)}_{\rho,\rho} - T^{(k)}_{\rho,\rho})\right] \tag{151}$$

$$g^{(k)}_{33} - n_3 u'^{(k)}_3 = C^{(k)}_{\gamma\delta 33}\left[-\frac{5}{2}N^{(k)}_{\gamma\delta} + \frac{6}{5t_k}M^{(k)}_{\gamma\delta} + \frac{30}{t_k^2}P^{(k)}_{\gamma\delta}\right]$$

$$+ C^{(k+1)}_{\gamma\delta 33}\left[-\frac{5}{2}N^{(k+1)}_{\gamma\delta} - \frac{6}{5t_{k+1}}M^{(k+1)}_{\gamma\delta} + \frac{30}{7t_{k+1}^2}P^{(k+1)}_{\gamma\delta}\right]$$

$$+ C^{(k)}_{3333}\left[-\frac{3}{14}N^{(k)}_{33} - \frac{t_k}{70}T^{(k-1)}_3 + \frac{8t_k}{35}T^{(k)}_3 + \frac{t_k^2}{210}T^{(k-1)}_{\rho,\rho} + \frac{t_k^2}{60}T^{(k)}_{\rho,\rho}\right]$$

$$+ C^{(k+1)}_{3333}\left[-\frac{3}{14}N^{(k+1)}_{33} + \frac{8t_{k+1}}{35}T^{(k)}_3 - \frac{t_{k+1}}{70}T^{(k+1)}_3 - \frac{t_k^2}{60}T^{(k)}_{\rho\rho} - \frac{t_k^2}{210}T^{(k+1)}_{\rho\rho}\right] \tag{152}$$

$$g^{(k)}_{\alpha 3} + g^{(k)}_{3\alpha} =$$

$$\frac{4}{70}C^{(k)}_{\gamma 3\alpha 3}\left[-7Q^{(k)}_\gamma - \frac{15}{t_k}M^{(k)}_{\gamma 3} + t_k T^{(k-1)}_\gamma + 6t_k T^{(k)}_\gamma\right]$$

$$+ \frac{4}{70}C^{(k+1)}_{\gamma 3\alpha 3}\left[-7Q^{(k+1)}_\gamma + \frac{15}{t_{k+1}}M^{(k+1)}_{\gamma 3} + 6t_{k+1}T^{(k)}_\gamma + t_{k+1}T^{(k+1)}_\gamma\right]$$

$$+ \frac{1}{2}C^{(k)}_{\gamma\delta 33}\left[\frac{1}{28}N^{(k)}_{\gamma\delta,\alpha} - \frac{1}{5t_k}M^{(k)}_{\gamma\delta,\alpha} - 15P^{(k)}_{\gamma\delta,\alpha}\right]$$

$$+ \frac{1}{2}C^{(k+1)}_{\gamma\delta 33}\left[-\frac{1}{28}N^{(}_{\gamma\delta,\alpha}k+1) - \frac{1}{5t_{k+1}}M^{(k+1)}_{\gamma\delta,\alpha} + 15P^{(k+1)}_{\gamma\delta,\alpha}\right]$$

$$+ \frac{1}{6}C^{(k)}_{3333}\left[\frac{t_k}{14}N^{(k)}_{33,\alpha} + \frac{t_k^2}{35}T^{(k-1)}_{3,\alpha} - \frac{t_k^2}{10}T^{(k)}_{3,\alpha} - \frac{t_k^3}{210}T^{(k-1)}_{\rho,\rho\alpha} - \frac{t_k^3}{210}T^{(k)}_{\rho,\rho\alpha}\right]$$

$$+ \frac{1}{6}C^{(k+1)}_{3333}\left[-\frac{t_{k+1}}{14}N^{(k+1)}_{33,\alpha} + \frac{t_{k+1}^2}{10}T^{(k)}_{3,\alpha} - \frac{t_{k+1}^2}{35}T^{(k+1)}_{3,\alpha} - \frac{t_{k+1}^3}{105}T^{(k)}_{\rho,\rho\alpha} - \frac{t_{k+1}^3}{210}T^{(k+1)}_{\rho,\rho\alpha}\right] \tag{153}$$

Equations (146) through (151) are the 14 constitutive relations for the 14 force resultants for each layer. Equations (152) and (153) are the continuity equations. If interlayer slip and delamination were not present, the left-hand sides of these equations would be zero.

5.4.4 An Alternative Theory.

Using the kinematic assumptions (112) and (113), substituting for strains in terms of the kinematic field variables and writing $\bar{e}_{ij}^{(k)}$ in terms of the force resultants using (114) through (116) along with the material constitutive relations (80) through (82), the variational equality yields the constitutive relations for the force resultants. Explicitly, the equality has the form:

$$\sum_{k=1}^{N}\int_{-\frac{t_k}{2}}^{\frac{t_k}{2}}\Bigg\{\Bigg[\langle\bar{N}_{\alpha\beta}^{(k)},\bar{M}_{\alpha\beta}^{(k)}\rangle\begin{Bmatrix}\eta_1^{(k)}\\ \eta_2^{(k)}\end{Bmatrix}\left(\langle 1,x_3^{(k)}\rangle\begin{Bmatrix}v_{(\alpha,\beta)}^{(k)}\\ \phi_{(\alpha,\beta)}^{(k)}\end{Bmatrix}\right)$$

$$-\langle\eta_1^{(k)},\eta_2^{(k)}\rangle C_{\gamma\delta\alpha\beta}^{(k)}\begin{Bmatrix}N_{\gamma\delta}^{(k)}\\ M_{\gamma\delta}^{(k)}\end{Bmatrix}-\langle\xi_1^{(k)},\xi_2^{(k)},\xi_3^{(k)}\rangle C_{\gamma\delta 33}^{(k)}\begin{Bmatrix}N_{33}^{(k)}\\ T_3^{(k-1)}\\ T_3^{(k)}\end{Bmatrix}\Bigg]$$

$$+\left[\langle\bar{Q}_\alpha^{(k)},\bar{M}_{\alpha 3}^{(k)},\bar{T}_\alpha^{(k-1)},\bar{T}_\alpha^{(k)}\rangle\begin{Bmatrix}\zeta_1^{(k)}\\ \zeta_2^{(k)}\\ \zeta_3^{(k)}\\ \zeta_4^{(k)}\end{Bmatrix}\left(\langle 1,x_3\rangle\begin{Bmatrix}\phi_\alpha^{(k)}+v_{3,\alpha}^{(k)}\\ \phi_{3,\alpha}^{(k)}\end{Bmatrix}\right)\right.$$

$$\left.-\langle\zeta_1^{(k)},\zeta_2^{(k)},\zeta_3^{(k)},\zeta_4^{(k)}\rangle\, 4\, C_{\gamma 3\alpha 3}^{(k)}\begin{Bmatrix}Q_\alpha^{(k)}\\ M_{\alpha 3}^{(k)}\\ T_\alpha^{(k-1)}\\ T_\alpha^{(k)}\end{Bmatrix}\right]$$

$$+\left(<\overline{N}_{33}^{(k)},\overline{T}_3^{(k-1)},\overline{T}_3^{(k)}> \begin{Bmatrix} \xi_1^{(k)} \\ \xi_2^{(k)} \\ \xi_3^{(k)} \end{Bmatrix} \left(\phi_3 - <\eta_1^{(k)},\eta_2^{(k)}> C_{\gamma\delta 33} \begin{Bmatrix} N_{\gamma\delta}^{(k)} \\ M_{\gamma\delta}^{(k)} \end{Bmatrix} \right.\right.$$

$$\left.\left. - <\xi_1^{(k)},\xi_2^{(k)},\xi_3^{(k)}> C_{3333}^{(k)} \begin{Bmatrix} N_{33}^{(k)} \\ T_3^{(k-1)} \\ T_3^{(k)} \end{Bmatrix} \right)\right) dx_3^{(k)} = 0 \tag{154}$$

Carrying out the integrations over the thickness, for the equality to hold for arbitrary values of the quantities $\overline{N}_{\alpha\beta}^{(k)},\overline{M}_{\alpha\beta}^{(k)},\overline{Q}_{\alpha}^{(k)},\overline{M}_{\alpha 3}^{(k)},\overline{N}_3^{(k)},\overline{T}_{\alpha}^{(k)}$ and $\overline{T}_3^{(k)}$ leads to the following constitutive relations:

$$v_{(\alpha,\beta)}^{(k)} = \frac{1}{t_k}\left(C_{\gamma\delta\alpha\beta}^{(k)} N_{\gamma\delta}^{(k)} - C_{33\alpha\beta}^{(k)} N_{33}^{(k)}\right) \tag{155}$$

$$\phi_{(\alpha,\beta)}^{(k)} = \frac{12}{t_k^3} C_{\gamma\delta\alpha\beta}^{(k)} M_{\gamma\delta}^{(k)} - \frac{1}{t_k} C_{33\alpha\beta}^{(k)}(T_3^{(k-1)} - T_3^{(k)}) \tag{156}$$

$$\phi_{\alpha}^{(k)} + v_{3,\alpha}^{(k)} = 4\, C_{\gamma 3\alpha 3}^{(k)}\left(\frac{6}{5t_k} Q_{\gamma}^{(k)} - \frac{1}{10}(T_{\gamma}^{(k-1)} + T_{\gamma}^{(k)})\right) \tag{157}$$

$$\phi_{3,\alpha}^{(k)} = 4\, C_{\gamma 3\alpha 3}^{(k)}\left(\frac{120}{7t_k^3} M_{\gamma 3}^{(k)} + \frac{3}{7t_k}(T_{\gamma}^{(k-1)} - T_{\gamma}^{(k)})\right) \tag{158}$$

$$\phi_3^{(k)} = \frac{1}{t_k} C_{\gamma\delta 33}^{(k)} N_{\gamma\delta}^{(k)} + C_{3333}^{(k)}\left(\frac{6}{5t_k} N_{33}^{(k)} - \frac{1}{10}(T_3^{(k-1)} + T_3^{(k)})\right) \tag{159}$$

$$g_{\alpha 3}^{(k)} + g_{3\alpha}^{(k)} = 4\, C_{\gamma 3\alpha 3}^{(k)}\left(-\frac{1}{10} Q_{\gamma}^{(k)} - \frac{3}{7t_k} M_{\gamma 3}^{(k)} + \frac{t_k}{70}(T_{\gamma}^{(k-1)} + 6T_{\gamma}^{(k)})\right)$$

$$+ 4\, C_{\gamma 3\alpha 3}^{(k+1)}\left(-\frac{1}{10} Q_{\gamma}^{(k+1)} + \frac{3}{7t_{k+1}} M_{\gamma 3}^{(k+1)} + \frac{t_{k+1}}{70}(6T_{\gamma}^{(k)} + T_{\gamma}^{(k+1)})\right) \tag{160}$$

$$g_{33}^{(k)} - n_3 u_3^{\prime(k)} = \frac{1}{t_k} C_{\gamma\delta 33}^{(k)} M_{\gamma\delta}^{(k)} + C_{3333}^{(k)}\left(-\frac{1}{10} N_{33}^{(k)} - \frac{t_k}{30} T_3^{(k-1)} + \frac{2t_k}{15} T_3^{(k)}\right)$$

$$- \frac{1}{t_{k+1}} C_{\gamma\delta 33}^{(k+1)} M_{\gamma\delta}^{(k+1)} + C_{3333}^{(k+1)}\left(-\frac{1}{10} N_{33}^{(k+1)} + \frac{2t_{k+1}}{15} T_3^{(k)} - \frac{t_{k+1}}{30} T_3^{(k+1)}\right) \tag{161}$$

Equations (155) through (159) are 11 N constitutive equations for the mechanical variables $N_{\alpha\beta}^{(k)}, M_{\alpha\beta}^{(k)}, Q_{\alpha}^{(k)}, M_{\alpha 3}^{(k)}$, and $N_{33}^{(k)}$. The equations (160) and (161) are the displacement continuity conditions. In case there are no displacement discontinuities in the physical problem, the left-hand sides of these two equations will vanish.

5.5 DISCUSSION.

A comparison of the three alternative approaches described above shows that the constitutive relationships are identical wherever the field variables involved are the same. For example, (136), (146) and (157) are identical relationships for shear deformation. Similarly, (147) and (158) for the bending stiffness are identical. Other relationships have different form because different field variables are used to describe the deformation.

These theories explicitly allow for the laminated nature of the system and do away with the need for use of ad hoc stiffness factors. The relations are compliance type i.e., they express deformations in terms of force resultants. Thus they can be directly included in a complementary type variational formulation. Direct formulation is possible by inverting the constitutive relations. Indeed, for the first order theory, (139) represents the stiffness relations for the shearing force. It should be noted that even though the complementary relationships are narrow band, the stiffness relations will be full matrices i.e., the force in any layer will cause deformations in all the layers. For the higher order theories, inversion of the constitutive relations would obviously be more complicated. On the free edge, some of the traction components are specified. These cannot be condensed out as could be done for the interior points. Here the full form of the constitutive matrix has to be retained. The computer implementation of the application of finite element methods to laminates described in Section VIII allowed for this feature.

SECTION VI

A RESTATEMENT OF PAGANO'S THEORY

6.1 INTRODUCTION

Solving three-dimensional equations of linear elasticity for a homogeneous orthotropic laminated plate, Pagano [1969, 1970] observed that the in-plane displacements were layerwise almost linear, and the shear stresses in each layer were nearly parabolic. Following Reissner's procedure for plates, assuming the in-plane stresses to be linear over each layer, Pagano [1978] solved the equations of equilibrium to obtain shearing stresses which were quadratic and transverse normal stress which was cubic in the transverse coordinate. Thus all the six components of stress were considered in the analysis. The constitutive equations were derived using a variational approach. Pagano satisfied the continuity of displacements as well as tractions at the interlayer surfaces. The accuracy of results from Pagano's theory depends solely upon the accuracy of the assumption of linear in-plane stresses over each layer. Thus, the procedure can be made as accurate as desired by subdividing each lamina into a number of layers. However, the theory yields 13N (where N is the number of layers) coupled equations which are difficult to solve. To overcome this difficulty, Pagano and Soni [1983] proposed a "global-local" approach to the problem. In this the laminate would be modelled by a higher order theory, e.g., Whitney's, to get approximate values of stresses and deformations. This result would then be used as input to a second stage solution, based on Pagano's theory, for the local problem. However, the global solution being grossly in error, the final solution is still generally unreliable [e.g., Hsiao and Soni 1987]. Numerical solutions, using the finite element or finite difference methods, are possible. For application of the finite element method in a systematic fashion it is desirable to write a variational principle for the problem.

As part of the present research effort, Pagano's theory was carefully examined. The constitutive relations for the force resultants were obtained by direct integration of strains and their moments without resorting to any variational principles. The number of field variables was reduced by elimination of the quantities $N_{33}^{(k)}$ and $M_{33}^{(k)}$. Consistent boundary conditions were identified and variational principles developed for the problem. Extended variational formulations admitting relaxation of the continuity requirements on some of the variables were introduced along with suitable specializations to make the problem tractable. One of the specializations was implemented in a finite element computer program. Details of this investigation are contained in items B.1.8, B.1.9, and B.4.3 of Appendix B. Herein, we merely present an outline of the principal concepts and summarize the resulting equations. Variational formulation of Pagano's theory as well as the displacement-based theories described in Sections IV and V is discussed in Section VII.

6.2 PAGANO'S THEORY.

6.2.1 Equations for Stress Components.

Assuming the in-plane stresses to be linear over a layer, using the midsurface as the origin for the local transverse coordinate x_3, we have:

$$\sigma_{\alpha\beta} = \frac{N_{\alpha\beta}}{t} + 12\,\frac{x_3 M_{\alpha\beta}}{t^3} \tag{162}$$

Substituting this expression in the equations of equilibrium, Pagano [1978] evaluated the remaining components of stress as:

$$\sigma_{\alpha 3} = \frac{3}{2t}(Q_\alpha - \frac{t}{2}\,S_\alpha)\left|1 - (\frac{2\,x_3}{t})^2\right| + T_\alpha^- + (x_3 + \frac{t}{2})\,\frac{P_\alpha}{t} \tag{163}$$

$$\sigma_{33} = \frac{1}{2}(T_{33}^- + T_{33}^+) - \frac{1}{2}(T_{\alpha,\alpha}^+ - T_{\alpha,\alpha}^-)\left(\frac{x_3^2}{t} - \frac{t}{4}\right) - (T_{\alpha,\alpha}^+ + T_{\alpha,\alpha}^-)\left(\frac{x_3^3}{t^2} - \frac{x_3}{4}\right)$$

$$- \frac{3}{2}(T_3^- - T_3^+)\left(\frac{x_3}{t} - \frac{4x_3^3}{3t^3}\right) \tag{164}$$

where

$$P_\alpha = T_\alpha^+ - T_\alpha^- \tag{165}$$

and

$$S_\alpha = \frac{t}{2}(T_\alpha^+ + T_\alpha^-) \tag{166}$$

Pagano's results are identical to Reissner's though the form is somewhat different.

6.2.2 Equilibrium Equations.

Substitution of the expressions for components of stress in (55) followed by integration leads to

$$N_{33} = \frac{t}{2}(T_3^+ + T_3^-) + \frac{t^2}{12}(T_{\alpha,\alpha}^+ - T_{\alpha,\alpha}^-) \tag{167}$$

Similarly, defining

$$M_{33} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{33} x_3 \, dx_3 \tag{168}$$

We have, upon appropriate substitutions,

$$M_{33} = \frac{t^3}{120}(T_{\alpha,\alpha}^+ + T_{\alpha,\alpha}^-) + \frac{t^2}{10}(T_3^+ - T_3^-) \tag{169}$$

Pagano combined (169) with the equation of transverse force equilibrium to write the equations of equilibrium for the plate as (48) , (51) and

$$Q_{\alpha,\alpha} + \frac{20}{t^2} M_{33} + T_3^- - T_3^+ - \frac{t}{6}(T_{\alpha,\alpha}^+ + T_{\alpha,\alpha}^-) = 0 \tag{170}$$

Some manipulation of (169) and (170) leads to the following alternative equation:

$$Q_{\alpha,\alpha} + \frac{60M_{33}}{t^2} + 5(T_3^- - T_3^+) - \frac{t}{2}(T^+_{\alpha,\alpha} + T^-_{\alpha,\alpha}) = 0 \tag{171}$$

This equation can be solved for M_{33} and the relationship used to replace (169) above.

6.2.3 Constitutive Equations for Force Resultants.

Pagano [1978] set up a mixed variational principle to derive the constitutive equations for the problem. This required introduction of quantities defined as follows:

$$(\bar{f}, f', \hat{f}) = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\frac{2}{t}, \frac{4x}{t^2}, \frac{8x^2}{t^3} \right) f(x)\,dx \tag{172}$$

where f is any function of x. In the context of a lamina, components of the displacement vector as functions of the transverse coordinate x_3 are integrated with appropriate weights indicated in (172). In our statement of Pagano's theory, we do not use any variational principle but, instead, directly integrate strain-stress relations (9) through (11). Using (1) to define the strain components, and carrying out the appropriate integration, we get the following results.

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} e_{\alpha\beta}\,dx_3 \quad \text{gives}$$

$$\bar{u}_{(\alpha,\beta)} = \frac{2}{t}(C_{\gamma\delta\alpha\beta}N_{\gamma\delta} + C_{33\alpha\beta}N_{33}) \tag{173}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} x_3\, e_{\alpha\beta}\,dx_3 \quad \text{gives}$$

$$u^{*}_{(\alpha,\beta)} = \frac{4}{t^2}(C_{\gamma\delta\alpha\beta}M_{\gamma\delta} + C_{33\alpha\beta}M_{33}) \tag{174}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} e_{33}\,dx_3 \quad \text{gives}$$

$$u_3^+ - u_3^- = C_{\alpha\beta 33} N_{\alpha\beta} + C_{3333} N_{33} \tag{175}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} x_3 e_{33}\, dx_3 \qquad \text{gives}$$

$$\bar{u}_3 = u_3^+ + u_3^- - \frac{2}{t}\left(C_{\alpha\beta 33} M_{\alpha\beta} + C_{3333} M_{33} \right) \tag{176}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} x_3^2 e_{33}\, dx_3 \qquad \text{gives}$$

$$6\, u_3' = -C_{\alpha\beta 33} N_{\alpha\beta} - \frac{t}{5} C_{3333} (T_3^+ + T_3^-) - \frac{3}{5} C_{3333} N_{33} + 3(u_3^+ - u_3^-) \tag{177}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} x_3^3 e_{33}\, dx_3 \qquad \text{gives}$$

$$\hat{u}_3 = \frac{1}{3}(u_3^+ + u_3^-) - \frac{2}{5t} C_{\alpha\beta 33} M_{\alpha\beta} - \frac{2}{7t} C_{3333} M_{33} - \frac{t}{105} C_{3333} (T_3^+ - T_3^-) \tag{178}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} e_{\alpha 3}\, dx_3 \qquad \text{gives}$$

$$\bar{u}_{3,\alpha} = \frac{8}{t} C_{\beta 3\alpha 3} Q_\beta - \frac{2}{t}(u_\alpha^+ - u_\alpha^-) \tag{179}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} x_3 e_{\alpha 3}\, dx_3 \qquad \text{gives}$$

$$\frac{t}{4} u_{3,\alpha}' - \frac{1}{2} \bar{u}_\alpha = \frac{t}{3} C_{\beta 3\alpha 3}(T_3^+ - T_3^-) - \frac{1}{2}(u_\alpha^+ + u_\alpha^-) \tag{180}$$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} x_3^2 e_{\alpha 3}\, dx_3 \qquad \text{gives}$$

$$\acute{u}_{3,\alpha} = \frac{8}{15} C_{\beta 3\alpha 3}(T_3^+ + T_3^-) + \frac{8}{5t} C_{\beta 3\alpha 3} Q_\beta + \frac{4}{t} u_\alpha' - \frac{2}{t}(u_\alpha^+ - u_\alpha^-) \tag{181}$$

Elimination of the surface displacements from pairs of equations leads to

$$3\hat{u}_3 - \bar{u}_3 = \frac{4}{5T} C_{\alpha\beta 33} M_{\alpha\beta} + \frac{8}{7t} C_{3333} M_{33} - \frac{t}{35} C_{3333}(T_3^+ - T_3^-) \tag{182}$$

$$6u_3^{*} = 2 C_{\alpha\beta 33} N_{\alpha\beta} + \frac{12}{5} C_{3333} N_{33} - \frac{t}{5} C_{3333}(T_3^+ + T_3^-) \tag{183}$$

$$\hat{u}_{3,\alpha} - \bar{u}_{3,\alpha} - \frac{4}{t} u_\alpha^{*} = \frac{8}{15} C_{\beta 3\alpha 3}(T_3^+ + T_3^-) - \frac{32}{5t} C_{\beta 3\alpha 3} Q_\beta \tag{184}$$

Eliminating N_{33} and M_{33} using (55) and (168) , the constitutive equations (173) and (174) become

$$\frac{1}{2}\Gamma_2\bar{v}_\gamma = \frac{1}{2}\bar{u}_{(\alpha,\beta)} = \frac{1}{t} C_{\mu\rho\alpha\beta} N_{\mu\rho} + \frac{1}{2} C_{33\alpha\beta}\left(T_3^+ + T_3^- + \frac{t}{6}(T_{\rho,\rho}^+ - T_{\rho,\rho}^-)\right) \tag{185}$$

$$\frac{1}{2}\Gamma_2\bar{\phi}_\gamma = \frac{3}{t} u_{(\alpha,\beta)}^{*} = \frac{12}{t^3} C_{\mu\rho\alpha\beta} M_{\mu\rho} + \frac{1}{10}\left(\frac{12}{t}(T_3^+ - T_3^-) + T_{\rho,\rho}^+ + T_{\rho,\rho}^-\right) \tag{186}$$

where

$$\Gamma_2 \equiv \delta_{\mu\rho}\frac{\partial}{\partial\rho} + \delta_{\rho\gamma}\frac{\partial}{\partial\mu} \tag{187}$$

and

$$\bar{v}_\rho \equiv \frac{1}{2}\bar{u}_\rho \tag{188}$$

Equation (184) can be written as

$$\bar{v}_{3,\rho} + \bar{\phi}_\rho = \frac{3}{4}(\bar{u}_{3,\rho} - \hat{u}_{3,\rho}) + \frac{3}{t} u_\rho^{*} = \frac{2}{5t}(12 C_{\gamma 3\rho 3} Q_\gamma - t(T_3^+ + T_3^-)) \tag{189}$$

where we have introduced the definitions

$$\bar{v}_3 \equiv \frac{3}{4}(\bar{u}_3 - \hat{u}_3) \tag{190}$$

$$\bar{\phi}_\rho \equiv \frac{3}{t} u_\rho^{*} \tag{191}$$

Equations (185) , (188) along with (182) and (183) are the constitutive equations of Pagano's theory.

6.2.4 Interface Displacements.

The displacements at the surfaces of the plate can be obtained from (177) through (181) . Thus

$$u_\alpha^+ = -t(\frac{3}{8}\hat{u}_{3,\alpha} - \frac{1}{8}\bar{u}_{3,\alpha} - \frac{3}{2t}u_\alpha^*) - (\frac{t}{4}u_{3,\alpha}^* - \frac{1}{2}\bar{u}_\alpha)$$

$$+ \frac{4}{30}C_{\beta 3\alpha 3}\Big(t(4T_\beta^+ - T_\beta^-) - 3Q_\beta\Big) \tag{192}$$

$$u_\alpha^- = t(\frac{3}{8}\hat{u}_{3,\alpha} - \frac{1}{8}\bar{u}_{3,\alpha} - \frac{3}{2t}u_\alpha^*) - (\frac{t}{4}u_{3,\alpha}^* - \frac{1}{2}\bar{u}_\alpha)$$

$$+ \frac{4}{30}C_{\beta 3\alpha 3}\Big(t(4T_\beta^- - T_\beta^+) - 3Q_\beta\Big) \tag{193}$$

$$u_3^+ = \frac{3}{4}(5\hat{u}_3 - \bar{u}_3) + \frac{3}{2}u_3^* + \frac{C_{3333}}{70t}\Big(t^2(6T_3^+ + T_3^-) - 7tN_{33} - 30M_{33}\Big) \tag{194}$$

$$u_3^- = \frac{3}{4}(5\hat{u}_3 - \bar{u}_3) - \frac{3}{2}u_3^* - \frac{C_{3333}}{70t}\Big(t^2(6T_3^+ + T_3^-) - 7tN_{33} + 30M_{33}\Big) \tag{195}$$

6.2.5 Continuity Conditions.

Continuity of tractions and displacements across the interfaces between the kth and the (k+1)th layer of a laminate simply implies

$$\sigma_{i3}^{-(k)} = \sigma_{i3}^{+(k+1)} \tag{196}$$

$$u_i^{(k)}(-\frac{1}{2}t_k) = u_i^{(k+1)}(\frac{1}{2}t_{k+1}) \tag{197}$$

6.3 THE FIELD EQUATIONS IN OPERATOR MATRIX FORM.

6.3.1 Equilibrium and Constitutive Relationships.

The equations of equilibrium (48), (51) and (171) along with the constitutive equations (182), (183), (185) and (189) can be written collectively in the form

$$[A]^{(k)}\{u\}^{(k)} + [B]^{(k)}\{\sigma\}^{-(k)} + [C]^{(k)}\{\sigma\}^{+(k)} = 0 \quad \text{for } k = 1, 2, \;\ldots\ldots\; N \tag{198}$$

where

$$[A]^{(k)} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}\Gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\Gamma_1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial\gamma} \\ -\frac{1}{2}\Gamma_2 & 0 & 0 & \frac{1}{t}C^{(k)}_{\mu\rho\alpha\beta} & 0 & 0 \\ 0 & -\frac{1}{2}\Gamma_2 & 0 & 0 & \frac{12}{t_k^3}C^{(k)}_{\mu\rho\alpha\beta} & 0 \\ 0 & -\delta_{\gamma\rho} & -\frac{\partial}{\partial\rho} & 0 & 0 & \frac{24}{5t_k}C^{(k)}_{\rho 3\gamma 3} \end{bmatrix} \tag{199}$$

$$[B]^{(k)} = \begin{bmatrix} -1 & 0 \\ \frac{t_k}{2} & 0 \\ 0 & -1 \\ -\frac{t_k}{12}C^{(k)}_{\mu\rho 33}\frac{\partial}{\partial\gamma} & \frac{1}{2}C^{(k)}_{\mu\rho 33} \\ \frac{1}{10}C^{(k)}_{\mu\rho 33}\frac{\partial}{\partial\gamma} & -\frac{6}{5t_k}C^{(k)}_{\mu\rho 33} \\ -\frac{2}{5}C^{(k)}_{\rho 3\gamma 3} & 0 \end{bmatrix} \tag{200}$$

$$[C]^{(k)} = \begin{bmatrix} 1 & 0 \\ \frac{t_k}{2} & 0 \\ 0 & 1 \\ \frac{1}{12}t_k C^{(k)}_{\mu\rho 33}\frac{\partial}{\partial\gamma} & \frac{1}{2}C^{(k)}_{\mu\rho 33} \\ \frac{1}{10}C^{(k)}_{\mu\rho 33}\frac{\partial}{\partial\gamma} & \frac{6}{5t_k}C^{(k)}_{\mu\rho 33} \\ -\frac{2}{5}C^{(k)}_{\rho 3\gamma 3} & 0 \end{bmatrix} \tag{201}$$

$$\{\sigma\}^{\pm(k)} = \begin{Bmatrix} \sigma^{\pm(k)}_{\gamma 3} \\ \sigma^{\pm(k)}_{33} \end{Bmatrix} \tag{202}$$

and

$$\{u\}^{(k)} = \begin{Bmatrix} \bar{v}_{\gamma}^{(k)} \\ \bar{\phi}_{\gamma}^{(k)} \\ \bar{v}_{3}^{(k)} \\ N_{\mu\rho}^{(k)} \\ M_{\mu\rho}^{(k)} \\ Q_{\rho}^{(k)} \end{Bmatrix} \tag{203}$$

6.3.2 Displacement Continuity Equations.

Substitution of (192) through (195) into (196) gives

$$[\Lambda]^{(k)}\{\sigma\}^{(k-1)} + [\overline{B}]^{(k)}\{u\}^{(k)} + [\Xi]^{(k)}\{\sigma\}^{-(k)} + [\overline{C}]^{(k-1)}\{u\}^{(k+1)} + [\overline{\Lambda}]^{(k-1)}\{\sigma\}^{(k+1)} = 0$$

$$k = 1, 2, 3, ... N-1 \tag{204}$$

where

$$[\Lambda]^{(k)} = \begin{bmatrix} \Lambda_{11}^{(k)} & \Lambda_{21}^{(k)} \\ \Lambda_{21}^{(k)} & \Lambda_{22}^{(k)} \end{bmatrix} \tag{205}$$

with

$$\Lambda_{11}^{(k)} = \frac{1}{140} C_{3333}^{(k)} t_k^3 \frac{\partial^2}{\partial\rho\partial\gamma} - \frac{2}{15} t_k C_{\gamma 3\rho 3}^{(k)}$$

$$\Lambda_{12}^{(k)} = \frac{13}{420} C_{3333}^{(k)} t_k^2 \frac{\partial}{\partial\rho}$$

$$\Lambda_{21}^{(k)} = \frac{13}{420} C_{3333}^{(k)} t_k^2 \frac{\partial}{\partial\gamma}$$

$$\Lambda_{22}^{(k)} = \frac{9}{70} C_{3333}^{(k)} t_k$$

$$[\overline{B}]^{(k)} = \begin{bmatrix} -1 & \frac{1}{2}t_k & 0 & \frac{1}{12} t_k C_{\mu\rho 33}^{(k)} \frac{\partial}{\partial\rho} & -\frac{1}{10} C_{\mu\rho 33}^{(k)} \frac{\partial}{\partial\rho} & -\frac{2}{5} C_{\rho 3\gamma 3}^{(k)} \\ 0 & 0 & -1 & \frac{1}{2} C_{\mu\rho 33}^{(k)} & -\frac{6}{5t_k} C_{\mu\rho 33}^{(k)} & 0 \end{bmatrix} \tag{206}$$

$$[\Xi]^{(k)} = \begin{bmatrix} \Xi_{11}^{(k)} & \Xi_{12}^{(k)} \\ \Xi_{21}^{(k)} & \Xi_{22}^{(k)} \end{bmatrix} \tag{207}$$

with

$$\Xi_{11}^{(k)} = -\frac{1}{105}\left(C_{3333}^{(k)} t_k^3 + C_{3333}^{(k+1)} t_{k+1}^3 \right)\frac{\partial^2}{\partial\gamma\partial\rho} + \frac{2}{15}\left(t_k C_{\gamma 3\rho 3}^{(k)} + t_{k+1} C_{\gamma 3\rho 3}^{(k+1)} \right)$$

$$\Xi_{12}^{(k)} = \frac{11}{210}\left(C_{3333}^{(k)} t_k^2 - C_{3333}^{(k+1)} t_{k+1}^2 \right)\frac{\partial}{\partial\rho}$$

$$\Xi_{21}^{(k)} = \frac{11}{210}\left(-C_{3333}^{(k)} t_k^2 + C_{3333}^{(k+1)} t_{k+1}^2 \right)\frac{\partial}{\partial\rho}$$

$$\Xi_{22}^{(k)} = \frac{13}{35}\left(C_{3333}^{(k)} t_k + C_{3333}^{(k+1)} t_{k+1} \right)$$

$$[\overline{C}]^{(k)} = \begin{vmatrix} 1 & \frac{1}{2}t_k & 0 & -\frac{1}{12}t_k C_{\mu\rho 33}^{(k)}\frac{\partial}{\partial\rho} & -\frac{1}{10}C_{\mu\rho 33}^{(k)}\frac{\partial}{\partial\rho} & -\frac{2}{5}C_{\rho 3\gamma 3}^{(k)} \\ 0 & 0 & 1 & \frac{1}{2}C_{\mu\rho 33}^{(k)} & \frac{6}{5t_k}C_{\mu\rho 33}^{(k)} & 0 \end{vmatrix} \tag{208}$$

and

$$[\overline{\Lambda}]^{(k+!)} = \begin{vmatrix} \overline{\Lambda}_{11}^{(k+1)} & \overline{\Lambda}_{12}^{(k+1)} \\ \overline{\Lambda}_{21}^{(k+1)} & \overline{\Lambda}_{22}^{(k+1)} \end{vmatrix} \tag{209}$$

with

$$\overline{\Lambda}_{11}^{(k+1)} = \frac{1}{140}C_{3333}^{(k+1)} t_{k-1}^3 \frac{\partial^2}{\partial\gamma\partial\rho} - \frac{2}{15}t_{k+1} C_{\rho 3\gamma 3}^{(k+1)}$$

$$\overline{\Lambda}_{12}^{(k+1)} = -\frac{13}{420}C_{3333}^{(k+1)} t_{k+1}^2 \frac{\partial}{\partial\rho}$$

$$\overline{\Lambda}_{21}^{(k+1)} = -\frac{13}{420}C_{3333}^{(k+1)} t_{k+1} \frac{\partial}{\partial\gamma}$$

$$\overline{\Lambda}_{22}^{(k+1)} = \frac{9}{70}C_{3333}^{(k+1)} t_{k+1}$$

6.3.3 The Forcing Functions.

The quantities $\{\sigma\}^{+(1)}$ and $\{\sigma\}^{-(N)}$ are known for a problem and, therefore, can be moved to the right side of the equalities (198) and (204) for k corresponding to the top and the bottom layers. These will appear as forcing functions for the problem. Defining these as $[Q]^{(1)}$, $[Q]^{(N)}$, $[P]^{(1)}$, $[P]^{(N)}$, we have

$$[Q]^{(1)} = [\Lambda]^{(1)}\{\sigma\}^{(0)} = \begin{Bmatrix} \frac{1}{140}C^{(1)}_{3333}t_1^{\,3}\hat{\sigma}^{(0)}_{\gamma 3,\gamma\rho} - \frac{2}{15}t_1C^{(1)}_{\gamma 3\rho 3}\hat{\sigma}^{(0)}_{\gamma 3} + \frac{13}{420}C^{(1)}_{3333}t_1^2\hat{\sigma}^{(0)}_{33,\rho} \\ \frac{13}{420}C^{(1)}_{3333}t_1^2\hat{\sigma}^{(0)}_{\gamma 3,\gamma} + \frac{9}{70}C^{(1)}_{3333}t_1\hat{\sigma}^{(0)}_{33} \end{Bmatrix} \tag{210}$$

$$[Q]^{(N-1)} = [\overline{\Lambda}]^{(N)}\{\sigma\}^{-(N)} = \begin{Bmatrix} \frac{1}{140}C^{(N)}_{3333}t_N^3\hat{\sigma}^{(N)}_{\gamma 3,\gamma\rho} - \frac{2}{15}t_NC^{(N)}_{\gamma 3\rho 3}\hat{\sigma}^{(0)}_{\gamma 3} - \frac{13}{420}C^{(N)}_{3333}t_N^2\hat{\sigma}^{(N)}_{33,\rho} \\ -\frac{13}{420}C^{(N)}_{3333}t_N^2\hat{\sigma}^{(N)}_{\gamma 3,\gamma} + \frac{9}{70}C^{(N)}_{3333}t_N\hat{\sigma}^{(N)}_{33} \end{Bmatrix} \tag{211}$$

$$[P]^{(1)} = [C]^{(1)}\{\hat{\sigma}\}^{(0)} = \begin{Bmatrix} \hat{\sigma}^{(0)}_{\gamma 3} \\ \frac{1}{2}t_1\hat{\sigma}^{(0)}_{\gamma 3} \\ \hat{\sigma}^{(0)}_{33} \\ \frac{1}{12}C^{(12)}_{\mu\rho 33}(t_1\hat{\sigma}^{(0)}_{\gamma 3,\gamma} + 6\,\hat{\sigma}^{(0)}_{33}) \\ \frac{1}{10}C^{(1)}_{\mu\rho 33}(\hat{\sigma}^{(0)}_{\gamma 3,\gamma} + \frac{12}{t_1}\hat{\sigma}^{(0)}_{33}) \\ -\frac{2}{5}C^{(1)}_{\gamma 3\rho 3}\hat{\sigma}^{(0)}_{\gamma 3} \end{Bmatrix} \tag{212}$$

and

$$[P]^{(N)} = [B]^{(N)}\{\hat{\sigma}\}^{(N)} = \begin{Bmatrix} -\hat{\sigma}^{(N)}_{\gamma 3} \\ \frac{1}{2}t_N\hat{\sigma}^{(N)}_{\gamma 3} \\ -\hat{\sigma}^{(N)}_{33} \\ \frac{1}{12}C^{(N)}_{\mu\rho 33}\left(-t_N\hat{\sigma}^{(N)}_{\gamma 3,\gamma} + 6\hat{\sigma}^{(N)}_{33}\right) \\ \frac{1}{10}C^{(N)}_{\mu\rho 33}\left[\hat{\sigma}^{(N)}_{\gamma 3,\gamma} - \frac{12}{t_N}\hat{\sigma}^{(N)}_{33}\right] \\ -\frac{2}{5}C^{(N)}_{\rho 3\gamma 3}\hat{\sigma}^{(N)}_{\gamma 3} \end{Bmatrix} \tag{213}$$

In (210) through (213) the superposed hat denotes the specified value of the quantity.

6.4 DISCUSSION.

The equations of Pagano's theory stated above are self-adjoint in the sense of (A.21). To complete the statement of the boundary value problem, we need to identify consistent boundary operators. This feature is discussed, along with the broader problem of variational formulation of the governing equations, in Section VII.

SECTION VII

VARIATIONAL FORMULATION OF THEORIES OF LAMINATED PLATES

7.1 INTRODUCTION.

In developing the constitutive theory for laminated plates in Section V, care was exercised to assume the path of the Gateaux differential to lie in the same space as the one in which the solution for stresses lies. This resulted in constitutive relations, for the force resultants, which are symmetric and a strain energy function or complementary energy function can be written for the plate. Similarly, the equations of Pagano's theory are self-adjoint in the sense of (A.21). In both the assumed displacement and the assumed stress approaches, the complete set of field equations can be written in the form

$$[X]\{y\} = \{z\} \tag{214}$$

where $[X]$ is the matrix of field operators, $\{y\}$ is the set of field variables, and $\{z\}$ is the set of forcing functions. These field equations include the equations of equilibrium, the constitutive equations as well as the continuity conditions. For each of the theories described in Sections V and VI, the system of coupled equations is self-adjoint. If consistent boundary conditions can be identified, general variational formulations along with appropriate extensions and suitable specializations can be systematically generated following the procedures outlined in Appendix A and detailed in references [Sandhu 1970, 1971, 1975, 1976]. Detailed discussion of application to laminated plates, using different theories as the starting point, is given in items B.1.1, B.1.2, B.1.8, B.4.2, and B.4.3 of Appendix B. Herein, we merely present an outline of the essential features.

7.2 THE FIELD EQUATIONS.

In (214) the operator matrix [X] consists of sets of operators applicable to each layer. If the field variables are arranged as

$$\{y\} = \begin{Bmatrix} \{u\}^{(1)} \\ \{\sigma\}^{-(1)} \\ \{u\}^{(2)} \\ \{\sigma\}^{-(2)} \\ \{u\}^{(3)} \\ \{\sigma\}^{-(3)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{u\}^{(N-1)} \\ \{\sigma\}^{-(N-1)} \\ \{u\}^{(N)} \end{Bmatrix} \tag{215}$$

then the matrix [X] has the form:

$$[X] = \left[\begin{array}{ccccccccccc} [A]^{(1)} & [B]^{(1)} & 0 & 0 & 0 & 0 & & & & & \\ [\overline{B}]^{(1)} & [\Xi]^{(1)} & [\overline{C}]^{(2)} & [\overline{\Lambda}]^{(2)} & 0 & 0 & & & & & \\ 0 & [C]^{(2)} & [A]^{(2)} & [B]^{(2)} & 0 & 0 & & & & & \\ 0 & [\Lambda]^{(2)} & [\overline{B}]^{(2)} & [\Xi]^{(2)} & [\overline{C}]^{(3)} & [\overline{\Lambda}]^{(3)} & & & & & \\ 0 & 0 & 0 & [C]^{(3)} & [A]^{(3)} & [B]^{(3)} & & & & & \\ 0 & 0 & 0 & [\Lambda]^{(3)} & [\overline{B}]^{(3)} & [\Xi]^{(3)} & & & & & \\ & & & 0 & 0 & [C]^{(4)} & & & & & \\ & & & 0 & 0 & [\Lambda]^{(4)} & . & . & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & . & . & 0 & 0 & 0 \\ & & & & & & . & [C]^{(N-1)} & [\overline{C}]^{(N-1)} & [\overline{\Lambda}]^{(N-1)} & 0 \\ & & & & & & 0 & [\Lambda]^{(N-1)} & [A]^{(N-1)} & [B]^{(N-1)} & 0 \\ & & & & & & 0 & 0 & [\overline{B}]^{(N-1)} & [\Xi]^{(N-1)} & [\overline{C}]^{(N)} \\ & & & & & & & & 0 & [C]^{(N)} & [A]^{(N)} \end{array} \right] \tag{216}$$

This operator matrix includes the equations of equilibrium as well the constitutive equations for each layer and also the continuity equations across interfaces. Explicitly, the equations of equilibrium and the constitutive equations have been stated in (198), those of displacement continuity in (205), and (210) through (213) define the forcing functions for each layer. Collectively, the forcing function {z} has the form:

$$\{z\} = \begin{Bmatrix} -\{P\}^{(1)} \\ -\{Q\}^{(1)} \\ 0 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\{Q\}^{(N-1)} \\ -\{P\}^{(N)} \end{Bmatrix} \tag{217}$$

In (216) the operators $[A]^{(k)}$ for $k = 1,2, \ldots\ldots N$ and $[\Xi]^{(k)}$ for $k = 1, 2 \ldots\ldots N\text{-}1$ are self-adjoint with respect to inner product and the sets of operators $\{[\Lambda]^{(k)},[\overline{\Lambda}]^{(k)}\}$; $\{[C]^{(k)},[\overline{C}]^{(k)}\}$; $\{[B]^{(k)},[\overline{B}]^{(k)}\}$ constitute adjoint pairs. This is sufficient to satisfy the self-adjointness condition, described in Appendix A to this report, proposed by Sandhu [1976] for linear coupled problems.

7.3 CONSISTENT BOUNDARY CONDITIONS.

Consistent sets of boundary conditions must be identified for systematic development of variational principles for a given set of field equations. For tha case of laminated plate theories described herein, the form of the boundary conditions was found to be

$$[W]\{y\} = \{g\} \tag{218}$$

where

$$[W] = \begin{bmatrix}
[D]^{(1)} & 0 & 0 & 0 & 0 & & & & & & \\
[\bar{E}]^{(1)} & [\psi]^{(1)} & [\bar{F}]^{(2)} & [\bar{\theta}]^{(2)} & 0 & & & & & & \\
0 & 0 & [D]^{(2)} & 0 & 0 & & & & & & \\
0 & [\theta]^{(2)} & [\bar{E}] & [\psi]^{(2)} & [\bar{F}]^{(3)} & & & & & & \\
0 & 0 & 0 & 0 & [D]^{(3)} & & & & & & \\
0 & 0 & 0 & [\theta]^{(3)} & [\bar{E}]^{(3)} & [\bar{\theta}]^{(4)} & & & & & \\
 & & & 0 & 0 & 0 & & & 0 & 0 & 0 \\
 & & & 0 & 0 & [\psi]^{(4)} & . & . & 0 & 0 & 0 \\
 & & & 0 & 0 & . & . & . & [\bar{F}]^{(N-1)} & [\bar{\theta}]^{(N-1)} & 0 \\
 & & & & & . & . & 0 & [D]^{(N-1)} & 0 & 0 \\
 & & & & & & 0 & [\theta]^{(N-1)} & [\bar{E}]^{(N-1)} & [\psi]^{(N-1)} & [\bar{F}]^{(N)} \\
 & & & & & & 0 & 0 & 0 & 0 & [D]^{(N)}
\end{bmatrix} \tag{219}$$

$$
\{y\} = \begin{Bmatrix} \{u\}^{(1)} \\ \{\sigma\}^{-(1)} \\ \{u\}^{(2)} \\ \{\sigma\}^{-(2)} \\ \{u\}^{(3)} \\ \{\sigma\}^{-(3)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{u\}^{(N-1)} \\ \{\sigma\}^{-(N-1)} \\ \{u\}^{(N)} \end{Bmatrix} \quad \text{and} \quad \{g\} = \begin{Bmatrix} \{g_u\}^{(1)} \\ \{g_\sigma\}^{(1)} \\ \{g_u\}^{(2)} \\ \{g_\sigma\}^{(2)} \\ \{g_u\}^{(3)} \\ \{g_\sigma\}^{(3)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \{g_u\}^{(N-1)} \\ \{g_\sigma\}^{(N-1)} \\ \{g_u\}^{(N)} \end{Bmatrix} \tag{220}
$$

In this set, the operator $[D]^{(k)}$ describes the boundary conditions for the field variables (displacements and force resultants) for the kth layer. The other set of equations, viz.,

$$
[\theta]^{(k-1)}\{\sigma\}^{-(k-1)} + [\overline{E}]^{(k)}\{u\}^{(k)} + [\psi]^{(k)}\{\sigma\}^{-(k)}\overline{F}]^{(k+1)}\{u\}^{(k+1)} + [\overline{\theta}]^{(k+1)}\{\sigma\}^{-(k+1} = \{g_\sigma\}^{(k)} \tag{221}
$$

denotes the displacement continuity equations at the boundary. The right-hand side of (218) lists the specified values of the field variables or their appropriate functions at the boundary. Detailed form of indvidual operators is given in appropriate items listed in Appendix B. As an illustration, we list herein the operators as they appear in Pagano's theory.

7.3.1 CONSISTENT BOUNDARY OPERATORS IN PAGANO'S THEORY

The operators and quantities appearing in the vectors of field variables and forcing functions in (219) and (220), for Pagano's theory, are:

$$\{u\}^{(k)} = \begin{Bmatrix} v_\gamma^{(k)} \\ \phi_\gamma^{(k)} \\ v_3^{(k)} \\ N_{\alpha\beta}^{(k)} \\ M_{\alpha\beta}^{(k)} \\ Q_\gamma^{(k)} \end{Bmatrix} \tag{222}$$

$$[D]^{(k)} = \begin{bmatrix} 0 & 0 & 0 & -\eta_\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\eta_\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta_\beta \\ \eta_\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_\beta & 0 & 0 & 0 \end{bmatrix} \tag{223}$$

$$[E]^{(k)} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{12} t_k C_{\alpha\beta 33}^{(k)} \eta_\gamma & \frac{1}{10} C_{\alpha\beta 33}^{(k)} \eta_\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{224}$$

$$[F]^{(k)} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{12} t_k C_{\alpha\beta 33}^{(k)} \eta_\gamma & \frac{1}{10} C_{\alpha\beta 33}^{(k)} \eta_\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{225}$$

$$[\theta]^{(k)} = \begin{bmatrix} -\frac{1}{140} t_k^3 C_{3333}^{(k)} \eta_\rho \frac{\partial}{\partial \gamma} & -\frac{13}{420} t_k^2 C_{3333}^{(k)} \eta_\gamma \\ 0 & 0 \end{bmatrix} \tag{226}$$

$$[\bar{\theta}]^{(k)} = \begin{bmatrix} -\frac{1}{140} t_k^3 C_{3333}^{(k)} \eta_\rho \frac{\partial}{\partial \gamma} & \frac{13}{420} t_k^2 C_{3333}^{(k)} \eta_\gamma \\ 0 & 0 \end{bmatrix} \tag{227}$$

$$[\psi]^{(k)} = \begin{bmatrix} \frac{1}{105}[t_k^3 C_{3333}^{(k)} + t_{k+1}^3 C_{3333}^{(k+1)}]\eta_\rho \frac{\partial}{\partial \gamma} & \frac{11}{210}[t_{k+1}^2 C_{3333}^{(k+1)} - t_k^2 C_{3333}^{(k)}]\eta_\rho \\ 0 & 0 \end{bmatrix} \tag{228}$$

and

$$\{\sigma\}^{-(k)} = \begin{Bmatrix} \sigma_{\gamma 3}^{-(k)} \\ \sigma_{33}^{-(k)} \end{Bmatrix} \tag{229}$$

Jump discontinuities in the field variables can be introduced as conditions on internal surfaces. Indeed, as the finite element bases generally have limited smoothness across interelement boundaries, even in cases where there are no discontinuities in the physical problem, the smoothness condition of the physical problem needs to be introduced as a set of "homogeneous" discontinuity conditions. Similar to the format for the conditions at the external boundary, the discontinuity conditions are:

$$\begin{Bmatrix} -(N_{\alpha\beta}^{(k)})'\eta_\beta \\ -(M_{\alpha\beta}^{(k)})'\eta_\beta \\ -(Q_\alpha^{(k)})'\eta_\alpha \\ (v_\alpha^{(k)})'\eta_\alpha \\ (\phi_\alpha^{(k)})'\eta_\beta \\ (v_3^{(k)})'\eta_\alpha \end{Bmatrix} = \begin{Bmatrix} g_1^{'(k)} \\ g_3^{'(k)} \\ g_5^{'(k)} \\ g_2^{'(k)} \\ g_4^{'(k)} \\ g_6^{'(k)} \end{Bmatrix} \quad \text{on} \quad \begin{Bmatrix} S_{1i}^{(k)} \\ S_{3i}^{(k)} \\ S_{5i}^{(k)} \\ S_{2i}^{(k)} \\ S_{4i}^{(k)} \\ S_{6i}^{(k)} \end{Bmatrix} \tag{230}$$

$$\begin{aligned}
&[-\frac{1}{140} t_k^3 C_{3333}^{(k)} \eta_\gamma](\sigma_{\rho 3,\rho}^{-(k-1)})' + [-\frac{13}{420} t_k^2 C_{3333}^{(k)} \eta_\gamma](\sigma_{33}^{-(k-1)})' \\
&+ C_{\alpha\beta 33}^{(k)} \eta_\gamma [-\frac{1}{12}(N_{\alpha\beta}^{(k)})' + \frac{1}{10}(M_{\alpha\beta}^{(k)})'] \\
&+ \frac{1}{105}(t_{k+1}^3 C_{3333}^{(k+1)} + t_k^3 C_{3333}^{(k)}) \eta_\gamma (\sigma_{\rho 3,\rho}^{-(k)})' \\
&+ \frac{11}{210}(t_{k+1}^2 C_{3333}^{(k+1)} - t_k^2 C_{3333}^{(k)}) \eta_\gamma (\sigma_{33}^{-(k)})' + \frac{1}{12} t_{k+1} C_{\alpha\beta 33}^{(k+1)} \eta_\gamma (N_{\alpha\beta}^{(k+1)})' \\
&+ \frac{1}{10} C_{\alpha\beta 33}^{(k+1)} \eta_\gamma (M_{\alpha\beta}^{(k+1)})' - \frac{1}{140} t_{k+1}^3 C_{3333}^{(k+1)} \eta_\gamma (\sigma_{\rho 3,\rho}^{-(k+1)})' \\
&+ \frac{13}{420} t_{k+1}^2 C_{3333}^{(k+1)} \eta_\gamma (\sigma_{33}^{-(k+1)})' = g_\sigma^{'(k)} \qquad \text{on} \qquad S_i^{(k)}
\end{aligned} \tag{231}$$

The subscripted i indicates an internal surface. The "homogeneous" discontinuity condition, i.e., vanishing $g_i^{'(k)}$, represents the internal continuity constraint for the

7.4 A VARIATIONAL PRINCIPLE.

For the set of field equations, including the continuity equations, along with the boundary conditions and the jump discontinuity conditions given by (214), (218), (230), and (231), a variational principle can be constructed following the procedure outlined in Appendix A. The governing functional for the problem, following (A.20), is:

$$
\begin{aligned}
\Omega(\{u\},\{\sigma\}) &= 2<\{u\}^{(1)},\{P\}^{(1)}>_{S^{(1)}} + 2<\{u\}^{(N)},\{P\}^{(N)}>_{R^{(N)}} \\
&+ \sum_{k=2}^{N} <\{u\}^{(k)},[C]^{(k)}\{\sigma\}^{(k-1)}>_{R^{(k)}} + \sum_{k=1}^{N} <\{u\}^{(k)},[A]^{(k)}\{u\}^{(k)}>_{R^{(k)}} \\
&+ \sum_{k=1}^{N-1} <\{u\}^{(k)},[B]^{(k)}\{\sigma\}^{(k)}>_{R^{(k)}} \\
&+ \sum_{k=1}^{N-1} <\{\sigma\}^{(k)},[\bar{B}]^{(k)}\{u\}^{(k)}>_{R^{(k)}} + \sum_{k=1}^{N-1} <\{\sigma\}^{-(k)},[\Xi]^{(k)}\{\sigma\}^{-(k)}>_{R^{(k)}} \\
&+ \sum_{k=1}^{N-1} <\{\sigma\}^{-(k)},[\bar{C}]^{(k+1)}\{u\}^{(k+1)}>_{R^{(k+1)}} + \sum_{k=2}^{N-1} <\{\sigma\}^{-(k)},[\Lambda]^{(k)}\{\sigma\}^{-(k-1)}>_{R^{(k)}} \\
&+ \sum_{k=1}^{N-2} <\{\sigma\}^{-(k)},[\bar{\Lambda}]^{(k+1)}\{\sigma\}^{-(k+1)}>_{R^{(k+1)}} \\
&+ 2<\{\sigma\}^{-(N-1)},[Q]^{(N-1)}>_{R^{(N)}} + 2<\{\sigma\}^{-(1)},[Q]^{(1)}>_{R^{(1)}} \\
&+ \sum_{k=1}^{N} <\{u\}^{(k)},[D]^{(k)}\{u\}^{(k)}-2\{g_u\}^{(k)}>_{S_u^{(k)}} \\
&+ <\{\sigma\}^{-(1)},[\bar{E}]^{(1)}\{u\}^{(1)} + [\psi]^{(1)}\{\sigma\}^{-(1)} + [\bar{F}]^{(1)}\{u\}^{(2)} + [\bar{\theta}]^{(2)}\{\sigma\}^{-(2)} - 2\{g_\sigma\}^{(1)}>_{S^{(1)}} \\
&+ \sum_{k=2}^{N-2} <\{\sigma\}^{-(k)},[\theta]^{(k)}\{\sigma\}^{-(k-1)} + [\bar{E}]^{(k)}\{u\}^{(k)} + [\psi]^{(k)}\{\sigma\}^{-(k)} \\
&+ [\bar{F}]^{(k+1)}\{u\}^{(k+1)} + [\bar{\theta}]^{(k+1)}\{\sigma\}^{-(k-1)} - 2\{g_\sigma\}^{(k)}>_{S^{(k)}} \\
&+ <\{\sigma\}^{-(N-1)},[\theta]^{(N-1)}\{\sigma\}^{-(N-2)} + [\bar{E}]^{(N-1)}\{u\}^{(N-1)} + [\psi]^{(N-1)}\{\sigma\}^{-(N-1)} + [\bar{F}]^{(N)}\{u\}^{(N)} \\
&- 2\{g_\sigma\}^{(N-1)}>_{S^{(N-1)}}
\end{aligned}
$$

$$+\sum_{k=1}^{N}<\{u\}^{(k)},[D]^{(k)}(\{u\}^{(k)})'-2\{g_u{}'\}^{(k)}>_{S_{ui}^{(k)}}$$

$$+<\{\sigma\}^{-(1)},[E]^{(1)}(\{u\}^{(1)})'+[\psi]^{(1)}(\{\sigma\}^{-(1)})'+[F]^{(1)}(\{u\}^{(2)})'+[\theta]^{(2)}(\{\sigma\}^{-(2)})'$$

$$-2\{g_\sigma{}'\}^{(1)}>_{S_i^{(1)}}$$

$$+\sum_{k=2}^{N-2}<\{\sigma\}^{-(k)},[\theta]^{(k)}(\{\sigma\}^{-(k-1)})'+[E]^{(k)}(\{u\}^{(k)})'+[\psi]^{(k)}(\{\sigma\}^{-(k)})'$$

$$+[F]^{(k+1)}(\{u\}^{(k+1)})'+[\theta]^{(k+1)}(\{\sigma\}^{-(k+1)})'-2\{g_\sigma{}'\}^{(k)}>_{S_i^{(k)}}$$

$$+<\{\sigma\}^{-(N-1)},[\theta]^{(N-1)}(\{\sigma\}^{-(N-2)})'+[E]^{(N-1)}(\{u\}^{(N-1)})'+[\psi]^{(N-1)}(\{\sigma\}^{-(N-1)})'$$

$$+[F]^{(N)}(\{u\}^{(N)})'-2\{g_\sigma{}'\}^{(N-1)}>_{S_i^{(N-1)}} \tag{232}$$

Here $R^{(k)}$ is the two-dimensional region occupied by the kth lamina and $S_u^{(k)}$ symbolically represents appropriate portions of the boundary of $R^{(k)}$. $\{u\}^{(k)},\{\sigma\}^{-(k)}$ denote the sets $\{u\}^{(k)}=\{v_\alpha^{(k)},\phi_\alpha^{(k)},v_3^{(k)},N_{\alpha\beta}^{(k)},M_{\alpha\beta}^{(k)},Q_\alpha^{(k)}\}$ and $\{\sigma\}^{-(k)}=\{\sigma_{\gamma 3}^{-(k)},\sigma_{33}^{-(k)}\}$, k= 1,2, ... N respectively. $S_{ui}^{(k)}$ represents appropriate subsets of internal boundaries in the spatial region. $S^{(k)}$ is the boundary of $R^{(k)}$ and $S^{'(k)}$ is the collection of a finite number of internal surfacesacross which the field variables or the path of Gateaux differentiation may be discontinuous.

It can be shown that the Gateaux differential of this functional along arbitrary paths in the appropriate space of vector-valued functions vanishes if and only if all the field equations, the boundary conditions, and the discontinuity conditions in the interior and on the boundary are satisfied. The details of variational formulation and Gateaux differentiation for various theories developed in the course of the present research program are contained in appropriate items listed in Appendix B.

7.5 EXTENDED VARIATIONAL PRINCIPLES.

In the operator matrix [X] in (216) the sets $\{[B]^{(k)}, [\overline{B}]^{(k)}\}$; $\{[C]^{(k)}, [\overline{C}]^{(k)}\}$; and $\{[\Lambda]^{(k)}, [\overline{\Lambda}]^{(k)}\}$ constitute pairs of adjoint operator matrices. Thus

$$<\{\tilde{u}\}^{(k)}, [C]^{(k)}\{\sigma\}^{-(k-1)}> = <\{\sigma\}^{-(k-1)}, [\overline{C}^{(k)}]\{\tilde{u}\}^{(k)}$$

$$+ <\sigma_{\gamma 3}^{-(k-1)}\eta_\gamma, C_{\alpha\beta 33}^{(k)}(\frac{t_k}{12}\tilde{N}_{\alpha\beta}^{(k)} + \frac{1}{10}\tilde{M}_{\alpha\beta}^{(k)})>_{S^{(k)}} \tag{233}$$

$$<\tilde{\sigma}^{-(k-1)}, \overline{\Lambda}^{(k)}\sigma^{-(k)}>_{R^{(k)}} = <\sigma^{-(k)}, \Lambda^{(k)}\tilde{\sigma}^{-(k-1)}>_{R^{(k)}}$$

$$+ <\tilde{\sigma}_{\gamma 3}^{(k-1)}\eta_\gamma, \frac{1}{140}t_k^3 C_{3333}^{(k)}\sigma_{\rho 3,\rho}^{-(k)}>_{S^{(k)}} - <\sigma_{\gamma 3}^{-(k)}\eta_\gamma, \frac{1}{140}t_k^3 C_{3333}^{(k)}\tilde{\sigma}_{\rho 3,\rho}^{-(k-1)}>_{S^{(k)}}$$

$$+ <\sigma_{33}^{(k)}, -\frac{13}{420}t_k^2 C_{3333}^{(k)}\tilde{\sigma}_{\gamma 3}^{(k-1)}\eta_\gamma>_{S^{(k)}} + <\tilde{\sigma}_{33}^{-(k-1)}, -\frac{13}{420}t_k^2 C_{3333}^{(k)}\sigma_{\gamma 3}^{-(k)}\eta_\gamma>_{S^{(k)}} \tag{234}$$

and

$$<\{\tilde{u}\}^{(k)}, [B]^{(k)}\{\sigma\}^{-(k)}> = <\{\sigma\}^{-(k)}, [\overline{B}]^{(k)}\{\tilde{u}\}^{(k)}>$$

$$+ <\sigma_{\gamma 3}^{-(k)}\eta_\gamma, C_{\alpha\beta 33}^{(k)}(\frac{1}{10}\tilde{M}_{\alpha\beta}^{(k)} - \frac{t_k}{12}\tilde{N}_{\alpha\beta}^{(k)})>_{S^{(k)}} \tag{235}$$

Also, the operator matrices $[\Xi]^{(k)}$ and $[A]^{(k)}$ are self-adjoint. This implies that some of the operators in these matrices can be replaced by their adjoints with appropriate changes in the boundary terms. Thus, it is possible to eliminate the operators requiring differentiation of the generalized force quantities. This extends the class of admissible functions to include force resultants which may not be differentiable. Specifically, elimination of operators $A_{14}^{(k)}$, $A_{25}^{(k)}$, and $A_{36}^{(k)}$ from the operator matrix $[A]^{(k)}$ can be accomplished through the following identities:

$$<\tilde{v}_\gamma^{(k)}, A_{14}^{(k)}N_{\alpha\beta}^{(k)}>_{R^{(k)}} = <N_{\mu\rho}^{(k)}, A_{41}^{(k)}\tilde{v}_\gamma^{(k)}>_{R^{(k)}} + <N_{\alpha\beta}^{(k)}\eta_\alpha, \tilde{v}_\beta^{(k)}>_{S^{(k)}} \tag{236}$$

$$<\tilde{\phi}^{(k)}, A_{25}^{(k)}M_{\alpha\beta}^{(k)}>_{R^{(k)}} = <M_{\mu\rho}^{(k)}, A_{52}^{(k)}\tilde{\phi}^{(k)}>_{R^{(k)}} + <M_{\alpha\beta}^{(k)}\eta_\alpha, \tilde{\phi}_\beta^{(k)}>_{S^{(k)}} \tag{237}$$

$$<\tilde{v}_3^{(k)}, A_{36}^{(k)}Q_\alpha^{(k)}>_{R^{(k)}} = <Q_\alpha^{(k)}\eta_\alpha, \tilde{v}_3^{(k)}>_{S^{(k)}} \tag{238}$$

Using (236) through (238) to eliminate terms containing derivatives of force resultants leads to functionals with extended domain of definition. Some of these functionals can be specialized to yield formulations in which the number of field variables is reduced by requiring some of the field equations to be satisfied identically. One such specialization is stated below.

7.6 A SPECIAL VARIATIONAL PRINCIPLE.

Assuming the variational formulation to be extended through elimination of derivatives of the force resultants, and further assuming that the physical problem has no displacement and force discontinuities, i.e., $\{g_u\}^{(k)}$ vanish, and that the displacement boundary conditions are satisfied, the governing functional reduces to the one used by Chyou [items B.1.8 and B.4.3, Appendix B] to set up a finite element solution scheme for the problem. This functional, in explicit form, is:

$$\Omega_1(u,\sigma) = 2<\bar{v}_\gamma^{(1)}, \sigma_{\gamma 3}^{(0)}>_{R^{(1)}} + <\bar{\phi}_\gamma^{(1)}, t_1\sigma_{\gamma 3}^{(0)}>_{R^{(1)}} + 2<\bar{v}_3^{(1)}, \sigma_{33}^{(0)}>_{R^{(1)}}$$

$$-2<\bar{v}_\gamma^{(N)}, \sigma_{\gamma 3}^{(N)}>_{R^{(N)}} + <\bar{\phi}_\gamma^{(N)}, t_N\sigma_{\gamma 3}^{(N)}>_{R^{(N)}} + 2<\bar{v}_3^{(N)}, -\sigma_{33}^{(N)}>_{R^{(N)}}$$

$$+\sum_{k=2}^{N} 2\{<\bar{v}_\gamma^{(k)}, \sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + <\bar{\phi}_\gamma^{(k)}, \frac{1}{2}t_k\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + <\bar{v}_3^{(k)}, \sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<N_{\alpha\beta}^{(k)}, \frac{1}{12}C_{\alpha\beta 33}^{(k)}(t_k\sigma_{\gamma 3}^{-(k-1)} + 6\sigma_{33}^{-(k-1)}>_{R^{(k)}} + <M_{\alpha\beta}^{(k)}, \frac{1}{10}C_{\alpha\beta 33}^{(k)}(\sigma_{\gamma 3,\gamma}^{-(k-1)} + \frac{12}{t_k}\sigma_{33}^{-(k-1)}>_{R^{(k)}}$$

$$-<Q_\rho^{(k)}, \frac{2}{5}C_{\gamma 3\rho 3}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}}\}$$

$$-\sum_{k=1}^{N}\{<N_{\alpha\beta}^{(k)}, \bar{v}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}} + <M_{\alpha\beta}^{(k)}, \bar{\phi}_{(\alpha,\beta)}^{(k)}>_{R^{(k)}} + <Q_\gamma^{(k)}, \bar{v}_{3,\gamma}^{(k)} + \bar{\phi}_\gamma^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N}\{<N_{\alpha\beta}^{(k)}, \frac{1}{12}C_{\alpha\beta 33}^{(k)}(6\sigma_{33}^{-(k)} - t_k\sigma_{\gamma 3,\gamma}^{-(k)})>_{R^{(k)}} + <M_{\alpha\beta}^{(k)}, \frac{1}{10}C_{\alpha\beta 33}^{(k)}(\sigma_{\gamma 3,\gamma}^{-(k)} - \frac{12}{t_k}\sigma_{33}^{-(k)}>_{R^{(k)}}$$

$$-<Q_\rho^{(k)}, \frac{2}{5}C_{\gamma 3\rho 3}^{(k)}\sigma_{\gamma 3}^{(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1} 2\{<\overline{v}_\gamma^{(k)}, -\sigma_{\gamma 3}^{(k)}>_{R^{(k)}} + <\overline{\phi}_\gamma^{(k)}, \frac{t_k}{2}\sigma_{\gamma 3}^{(k)}>_{R^{(k)}} + <\overline{v}_3^{(k)}, -\sigma_{33}^{-(k)}>_{R^{(k)}}\}$$

$$+\sum_{k=1}^{N-1}\{<\sigma_{\rho 3}^{-(k)}, \Xi_{11}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} + 2<\sigma_{33}^{-(k)}, \Xi_{21}^{(k)}\sigma_{\gamma 3}^{-(k)}>_{R^{(k)}} + <\sigma_{33}^{-(k)}, \Xi_{22}^{(k)}\sigma_{33}^{-(k)}>_{R^{(k)}}\}$$

$$+2\sum_{k=2}^{N-1}\{2<\sigma_{\rho 3}^{-(k)}, \Lambda_{11}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + <\sigma_{33}^{-(k)}, \Lambda_{21}^{(k)}\sigma_{\gamma 3}^{-(k-1)}>_{R^{(k)}} + <\sigma_{33}^{-(k)}, \Lambda_{22}^{(k)}\sigma_{33}^{-(k-1)}>_{R^{(k)}}\}$$

$$+2\sum_{k=1}^{N-2}\{<\sigma_{33}^{-(k)}, \overline{\Lambda}_{21}^{(k+1)}\sigma_{\gamma 3}^{(k+1)}>_{R^{(k)}}\}$$

$$+2<\{\sigma\}^{-(N-1)}, [\overline{\Lambda}]^{(N)}\{\sigma\}^{(N)}>_{R^{(N)}} + 2<\{\sigma\}^{(1)}, [\overline{\Lambda}]^{(1)}\{\sigma\}^{(0)}>_{R^{(1)}}$$

$$-2\sum_{k=1}^{N}\{<\overline{v}_\alpha^{(k)}, g_1^{(k)}>_{S_1^{(k)}} + <\overline{\phi}_\alpha^{(k)}, g_3^{(k)}>_{S_3^{(k)}} + <\overline{v}_3^{(k)}, g_5^{(k)}>_{S_5^{(k)}}\}$$

$$+2<\sigma_{\gamma 3}^{-(1)}, \frac{1}{210}(t_1^3 C_{3333}^{(1)} + t_2^3 C_{3333}^{(2)})\eta_\gamma(\sigma_{\rho 3,\rho}^{-(1)})'>_{S_1^{(1)}}$$

$$+\sum_{k=2}^{N-2}\{2<\sigma_{\gamma 3}^{-(k)}, -\frac{1}{140}t_k^3 C_{3333}^{(k)}\eta_\gamma(\sigma_{\rho 3,\rho}^{-(k-1)})'$$

$$+\frac{1}{210}(t_k^3 C_{3333}^{(k)} + t_{k+1}^3 C_{3333}^{(k+1)})\eta_\gamma(\sigma_{\rho 3,\rho}^{-(k-1)})'>_{S_1^{(k)}}\}$$

$$+2<\sigma_{\gamma 3}^{-(N-1)}, -\frac{1}{140}t_{N-1}^3 C_{3333}^{(N-1)}\eta_\gamma(\sigma_{\rho 3,\rho}^{-(N-2)})'$$

$$+\frac{1}{210}(t_{N-1}^3 C_{3333}^{(N-1)} + t_N^3 C_{3333}^{(N)})\eta_\gamma(\sigma_{\rho 3,\rho}^{(N-1)})'>_{S_1^{(N-1)}} \tag{239}$$

Because the mechanical variables $N_{\alpha\beta}^{(k)}$, $M_{\alpha\beta}^{(k)}$, and $Q_\alpha^{(k)}$ are completely defined by the constitutive relations (185), (186) and (189), the above specialization has only the field variables, $\overline{v}_\alpha^{(k)}$, $\overline{\phi}_\alpha^{(k)}$, $\overline{v}_3^{(k)}$ and $\sigma_{i3}^{-(k)}$. The number of field variables, therefore, is 8N, where N is the number of layers. This is less than the 13N in Pagano's original theory. Other specializations, allowing for elimination of only some of the terms containing derivatives of the force resultants, are possible. Alternatively, terms containing derivatives of the kinematic variables can be eliminated using the self-adjointness of the field operator matrix. The particular specialization described by (239) was used, in the present research program, to develop finite element solution schemes for Pagano's theory as well as the assumed-displacement layerwise theories of laminated composite plates. Detailed discussion of the finite element implementation is given in relevant items listed in Appendix B.

SECTION VIII

FINITE ELEMENT STUDIES

8.1 INTRODUCTION.

The theoretical development described in the previous chapters was used to develop computational models for stress and deformation analysis of laminated composites. The work consisted of three groups of studies based on the finite element method, viz.,

1. Prliminary studies directed towards accurate determination of stresses from displacement solutions determined by finite element methods.
2. Elastic analysis of free-edge delamination specimens.
3. Analysis of laminated composites including free-edge delamination specimens using laminated plate theories.

In this section we briefly describe each of the three components.

8.2 PRELIMINARY STUDIES.

The finite element method for analysis of linear elastic solids, based on minimization of potential energy using displacement interpolation functions which do not have continuous derivatives across element boundaries, yields discontinuous approximation to the stress field. Stresses across interfaces as well as at the surface of the solid are, therefore, difficult to determine. To ensure stress continuity across element boundaries, Hinton and Campbell [1974] proposed a smoothing function approach. A continuous stress pattern over the entire domain was obtained. Oden and Brauchli [1971] derived improved stresses based on the theory of conjugate approximations. However, the determination of the conjugate functions involved the inversion of a rather large matrix. An approximate implementation of the concept, using consistent nodal averages, was proposed by Oden and Reddy. Zienkiewicz et al. [1974] proposed use of a least squares procedure with a mixed formulation to arrive at a continuous stress approximation. Mirza and Olson [1980] studied the performance of the mixed finite element method. Chang [1981] and Jenq [1982] tried several different interpolation schemes and found that these procedures were expensive and not always sufficiently accurate. Moazzami [1984] extended Cook's work [1982] on Loubignac's method to plane elasticity. This method gave good approximation for stresses but was quite expensive because of increased bandwidth. Cook and Huang [1986] proposed that instead of fitting the average nodal stress to the equilibrium equation over each element by using Loubignac iteration, an approximate nodal strain that made use of a finite difference scheme be used. In the present research effort, a scheme for calculation of nodal point stresses using the method of undetermined coefficients applied to the displacement solution from a reliable finite element analysis was considered. Interpolation between the nodal points would define the continuous field. The method was applied to several plane elasticity problems for which the exact solutions are known. A four-point isoparametric quadrilateral element modified to eliminate the spurious shear modes was used to get the displacement field. Two approximation schemes, one based on making the formula exact for quadratic polynomial interpolation for displacement and the other for cubic polynomials, were used. Averaging procedures based on different selections of the nodal points in any group used for the determination of the formula were tried. It was found that the procedure could yield good estimates of stresses even for relatively coarse meshes.

Another study considered the use of higher order elements with continuous gradients across element boundaries. Such elements were originally introduced by Clough and Felippa [1968] for analysis of plate bending. Tocher and Hartz [1967] extended their application to finite element analysis of plane elasticity and showed the effectiveness of "conforming" elements. In the present research, Tocher and Hartz' work was revisited. Clough and Felippa's triangular elements were combined into quadrilaterals. Figure 3 shows Felippa's LCCT-12 cubic displacement triangular element. The quantities w_y and w_z denote derivatives of the field variable w with respect to the spatial coordinates y and z, respectively. Assuming normal gradients to vary linearly along an edge the midside nodal point on that edge can be eliminated. This results in elements designated as LCCT-11 or LCCT-9 depending upon whether one or all three of the midside nodal points are eliminated in this manner. LCCT-9 is the element used by Tocher and Hartz [1967]. Figure 4 shows quadrilaterals built up from four LCCT-12 elements and its variants, the LCCT-11 and the LCCT-9. These quadrilaterals are referred to as Q-23, Q-19, and Q-15, respectively. For homogeneous materials, these elements yielded stress distribution close to the exact even for very coarse meshes. Figure 5 and Figure 6 show finite element models of two example problems solved. Figure 7 through Figure 10 show the results of the finite element analyses compared to the Euler-Bernoulli beam theory in one case and a Ritz solution obtained by Hiremath [1985] in the other. Results from the conventional four-point isoparametric element, designated Q-4, are also included. In the present research the Q-23 element was developed for analysis of laminated composites. However, in that context, it has to be kept in mind that the tractions, and not strains, are continuous across the inter-laminar boundaries.

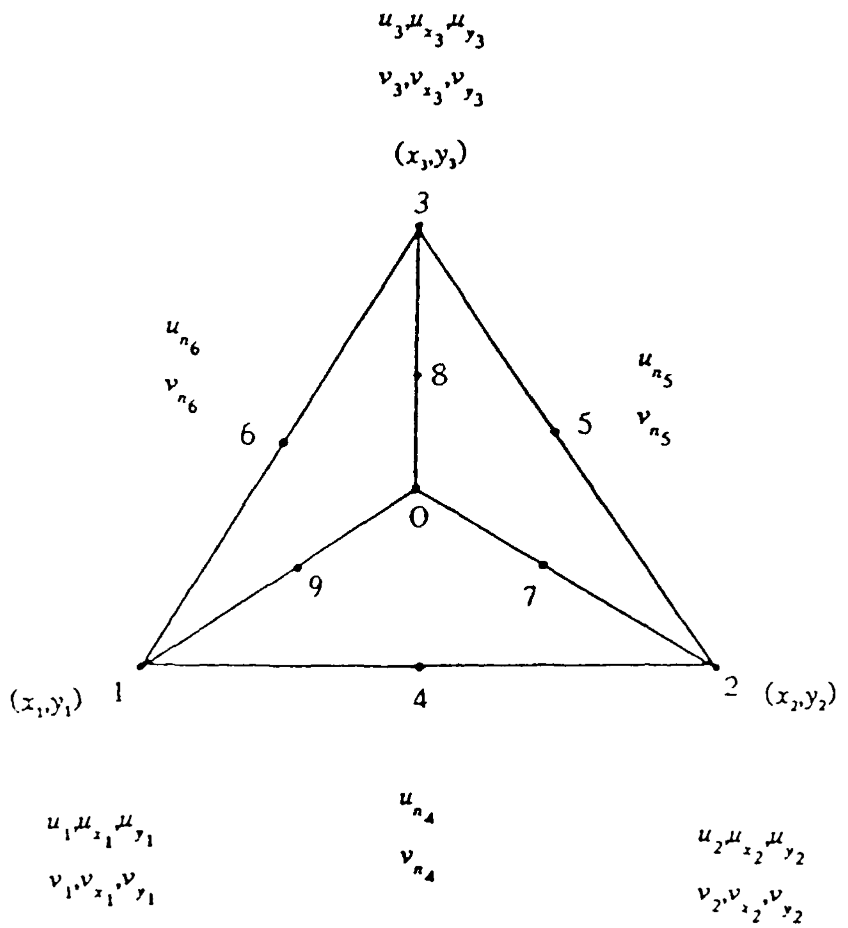
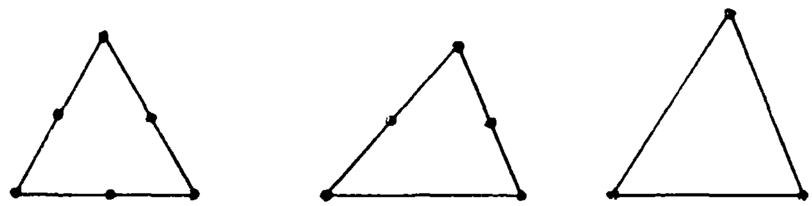
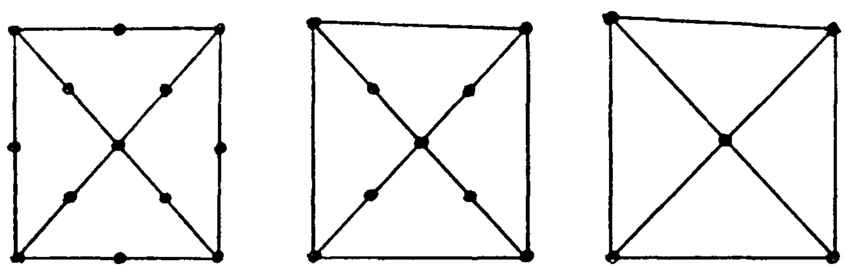


Figure 3: APPLICATION OF CLOUGH & FELIPPA'S LCCT-12 ELEMENT TO PLANE ELASTICITY.

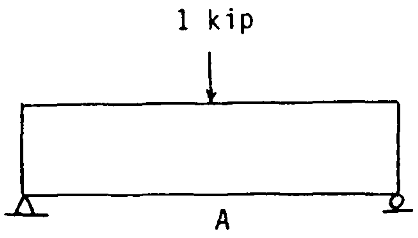


(a) LCCT-12 (b) LCCT-11 (c) LCCT-9

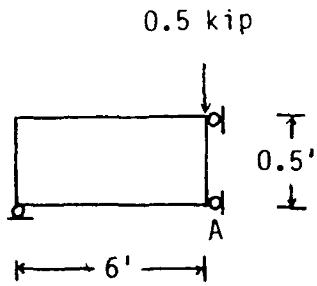


(a) Q-23 (b) Q-19 (c) Q-15

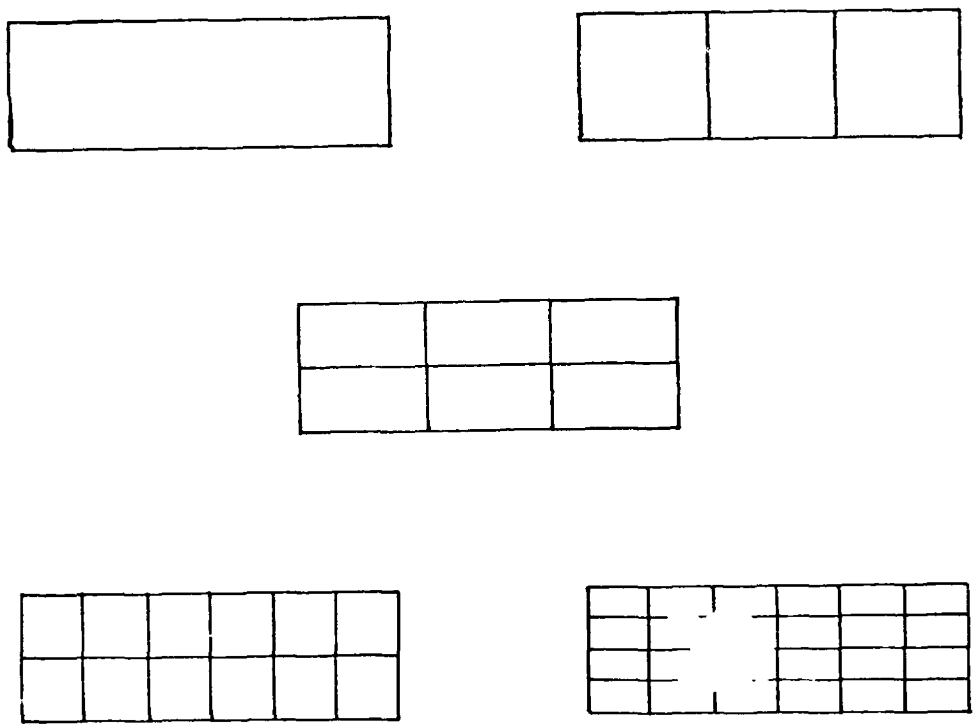
Figure 4: QUADRILATERAL ELEMENTS: Q-23, Q-19, AND Q-15.



(a) Loading

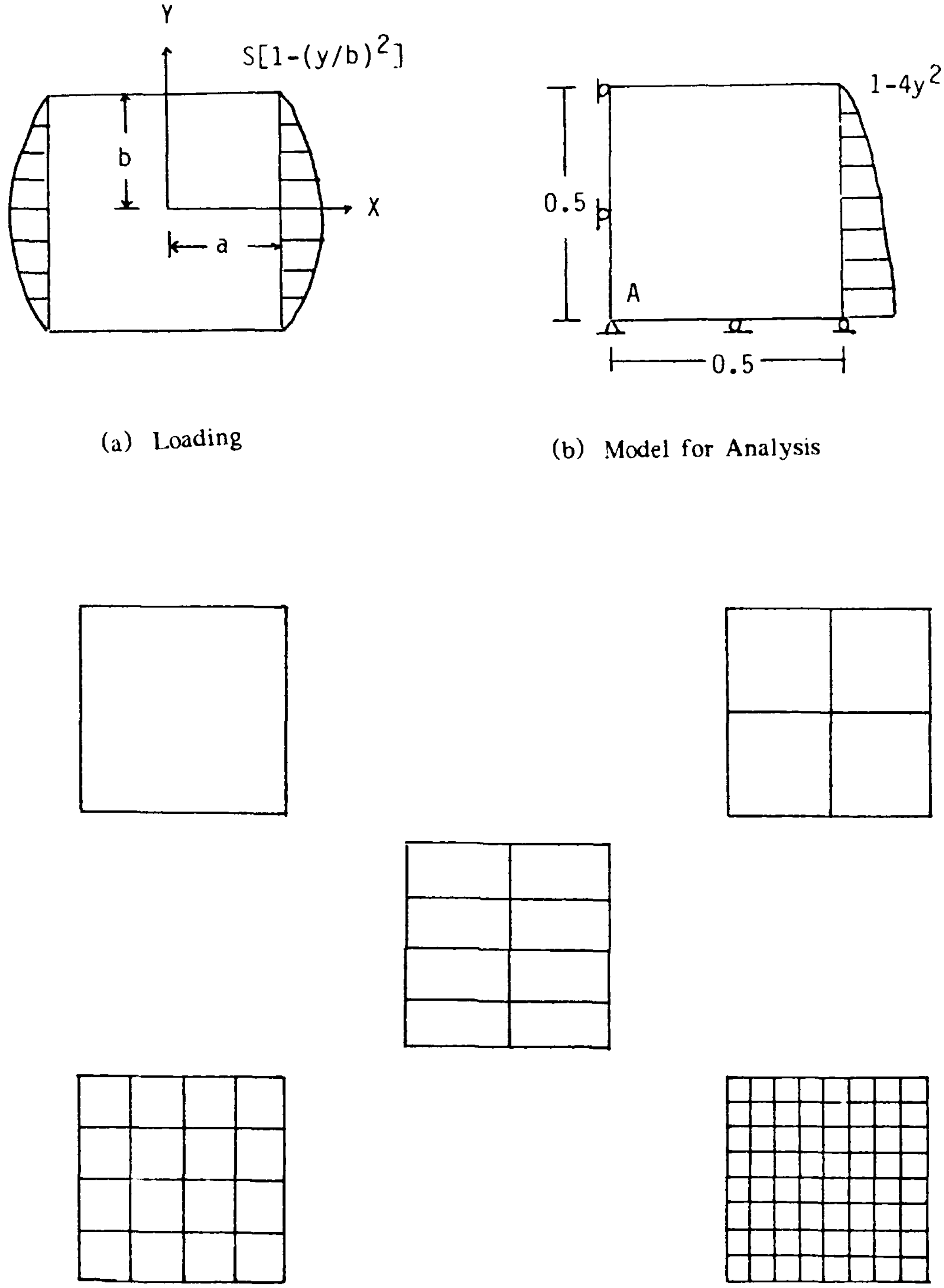


(b) Model for Analysis



(C) Five Finite Element Meshes

Figure 5: FINITE ELEMENT MODELS OF SIMPLE BEAM.



(a) Loading

(b) Model for Analysis

(C) Five Finite Element Meshes

Figure 6: FINITE ELEMENT MODELS OF A PLATE UNDER INPLANE LOADING.

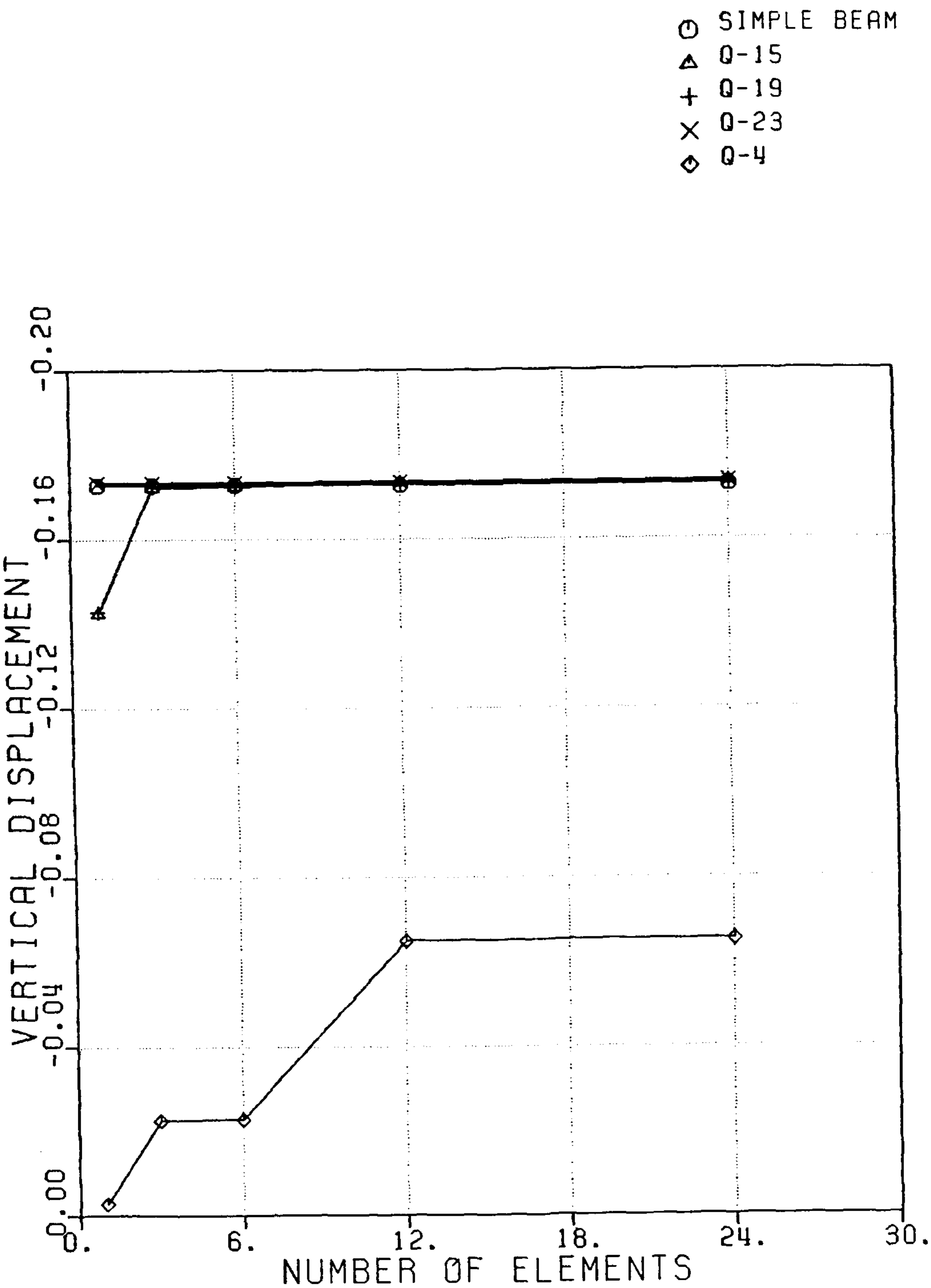


Figure 7: SIMPLE BEAM: Y-DISPLACEMENT AT POINT A.

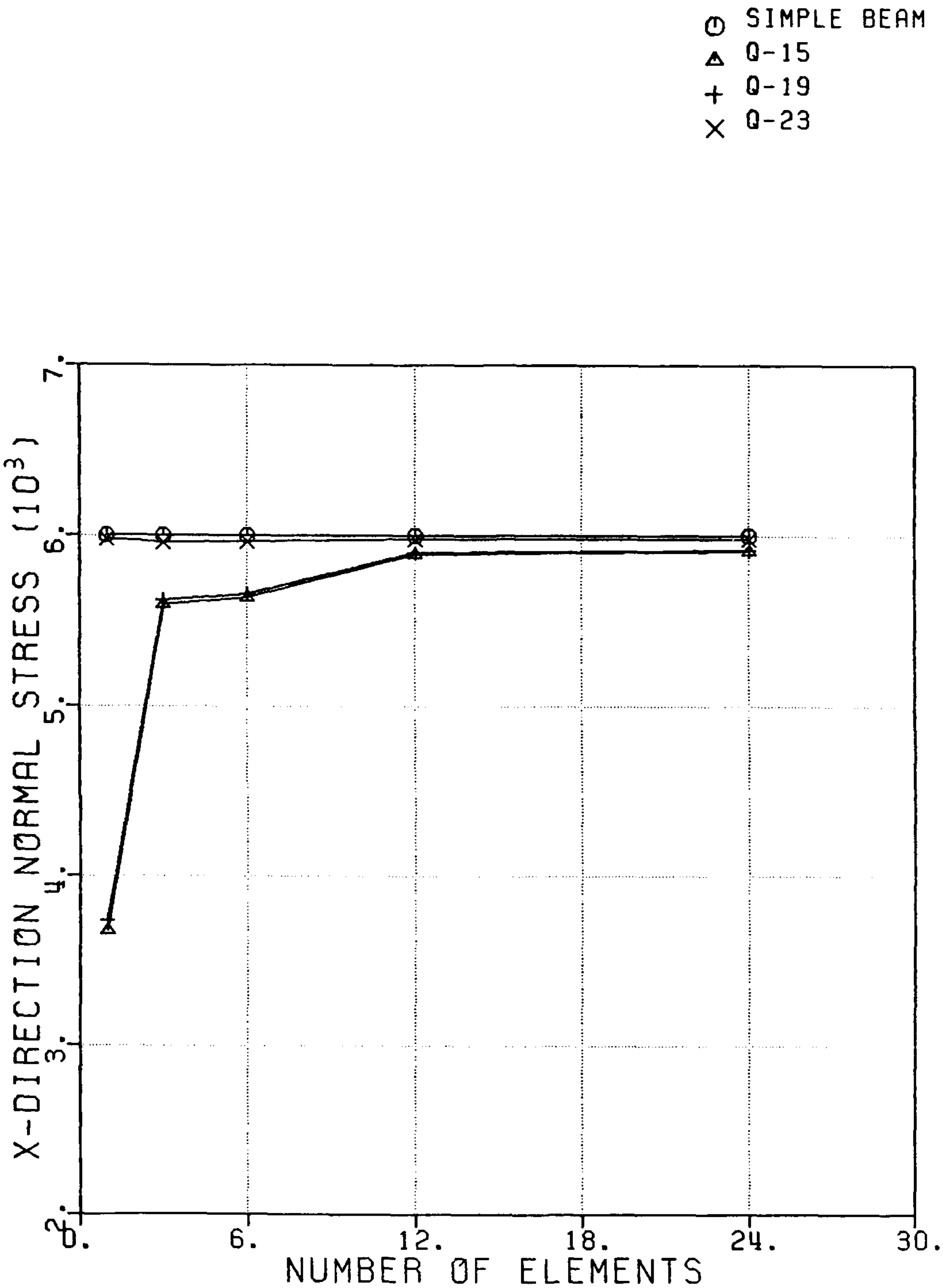


Figure 8: SIMPLE BEAM: LONGITUDINAL STRESS AT POINT A.

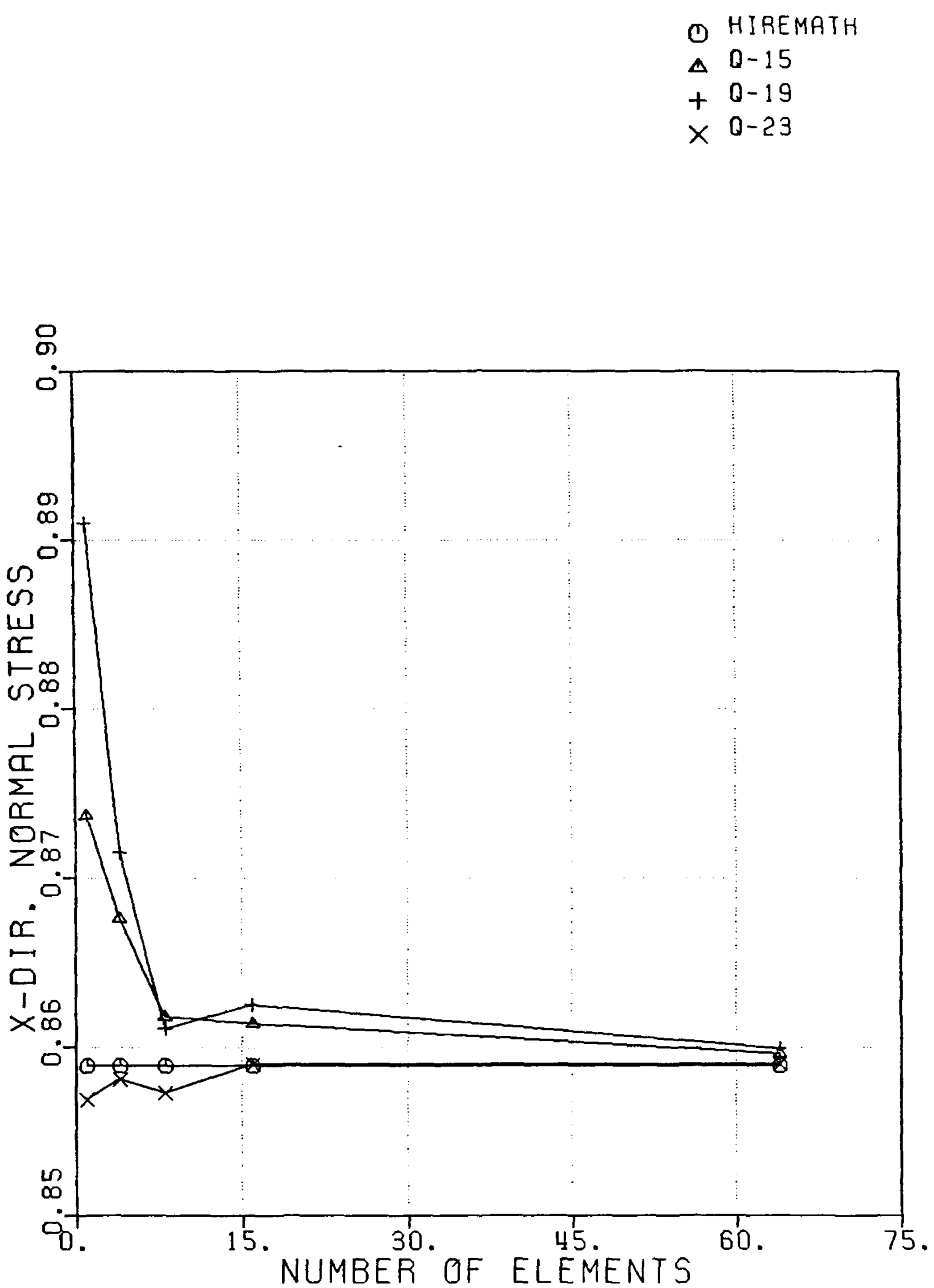


Figure 9: PLATE UNDER PARABOLIC LOAD: X-STRESS AT POINT A.

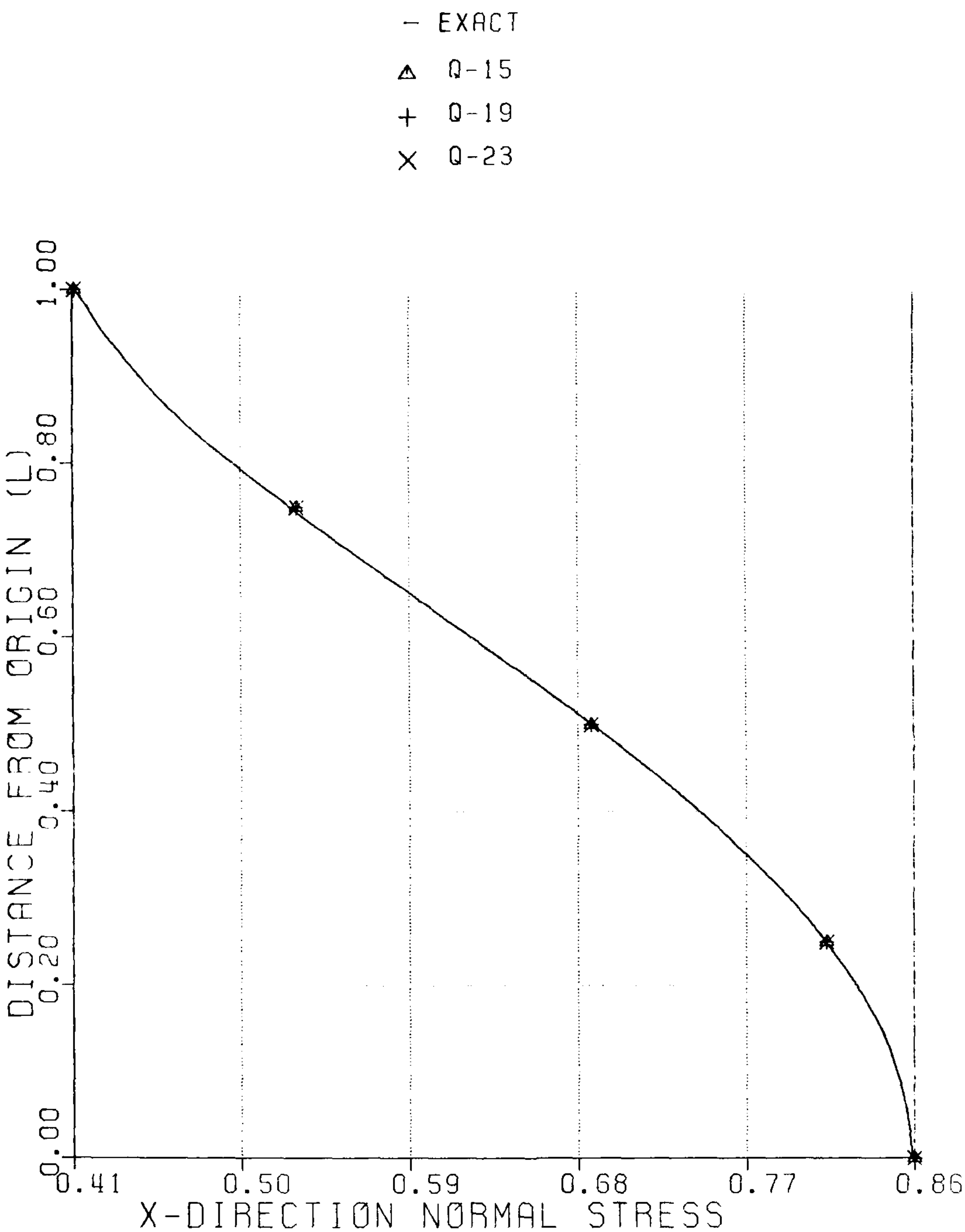


Figure 10: PLATE UNDER PARABOLIC LOAD: X-STRESS ALONG X=0.

8.3 ANALYSIS OF FREE-EDGE DELAMINATION SPECIMENS.

8.3.1 Introduction.

Success of the continuous strain finite element model in solving problems of plane elasticity motivated its extension to the problem of free-edge delamination. Specialization of the three-dimensional elasticity problem to the case of free-edge delamination specimens results in three components of displacement depending upon only two spatial variables. For this pseudo-two-dimensional problem, the continuous strain element can be used with only minor changes to yield a fully three-dimensional stress state. However, in the actual specimen, the tractions are continuous at interlaminar surfaces but not so the strains. The continuous-strain representation would result in discontinuous tractions. In the present research, this observation led to the development of a continuous-traction finite element representation. This procedure involves cubic interpolation of the displacement components over each element and can be quite expensive to use for large problems involving a large number of layers. The Air Force had previously used an axisymmetric model to represent a free-edge delamination specimen. This is because an annulus of large internal radius, subjected to a uniform internal radial pressure of small magnitude, will stretch essentially uniformly in the tangential direction. Thus, a small segment along the circumferential direction will be a practically straight coupon of uniform width under uniform longitudinal strain. In the present research effort, this procedure was also revisited. The method was extended to include a complete three-dimensional state of stress by permitting circumferential axisymmetric displacement in addition to the radial and transverse displacements.

In the following sections, we describe the finite element implementation of the continuous strain and continuous traction finite element analysis as well as the axisymmetric modelling of free-edge delamination specimens.

8.3.2 Continuous Strain Finite Element Modelling of Free-Edge Delamination Specimens.

For plate bending analysis, Clough and Felippa [1968] used the values of w, the transverse displacement of the plate, and its derivatives w_x and w_y as the generalized coordinates in the finite element representation. For the free-edge delamination specimen, the displacement field involves three components of displacement dependent on two spatial coordinates. Thus, nodal values of u, u_y, u_z; v, v_y, v_z would be required in addition to w, w_y, and w_z. Derivation of the interpolation functions is given in items B.1.5, and B.3.1 of Appendix B. Figure 11 shows the nodal point degrees of freedom for the element LCCT-12 for the pseudo-two-dimensional elasticity problem. Elements of quadrilateral shape can be set up as assemblages of triangular elements as discussed in Section 8.2. The element Q-23 has 23 degrees of freedom for each component of displacement. However, the interior nodal points can be eliminated by static condensation to yield an element with 48 degrees of freedom represented by the values of the three components of displacement along with their derivatives at the four corners and the normal derivatives of the displacement components at the four midside nodes. Application of the procedure to cross-ply and angle-ply free-edge delamination specimens showed [item B.3.1, Appendix B] that the element could provide good estimates of stress-distribution over the specimen except for the component σ_{yz}. This was because the element could not satisfy the traction-free condition at the free edges. This was the motivation for the development of the continuous traction procedure described in the next section.

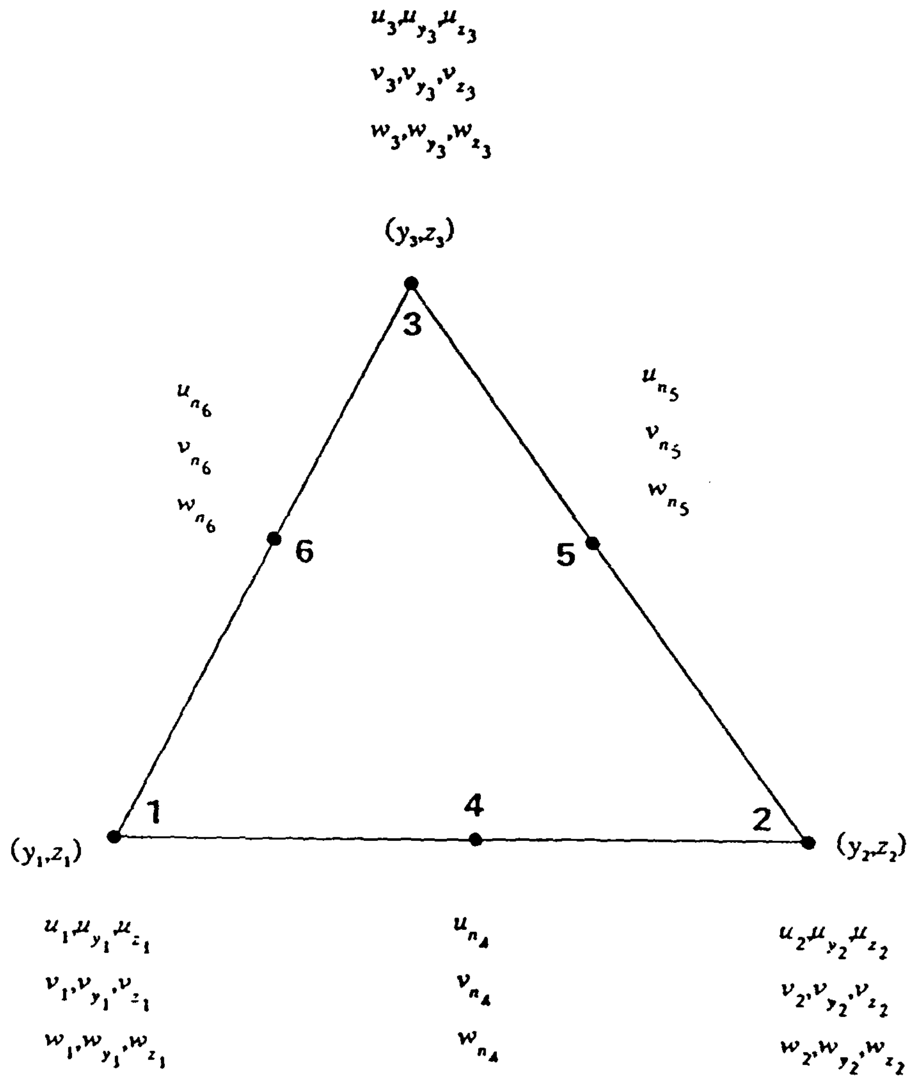


Figure 11: CONTINUOUS-STRAIN TRIANGULAR ELEMENT

8.3.3 Continuous Traction Finite Element Modelling of Free-Edge Delamination Specimens.

High interlaminar stresses near the free edge are primarily responsible for the onset of delamination in free-edge delamination specimens. In order to use stress-based criteria for onset and propagation of damage, it is necessary that a reliable estimate of interlaminar stresses be available for a given situation. The continuous-strain element discussed in the previous section is useful for homogeneous materials but cannot give correct results for layered materials because in these systems, it is the interlayer tractions and not the strains that are continuous. For the continuous-strain element, the components of displacement and their derivatives at the nodal points were selected as the degrees of freedom. Assuming that the interlayer surfaces will lie along element boundaries, it is reasonable to still use the continuous-strain representation within the element. However, for connection with the adjoining elements, the quantities associated with the nodal points should be the traction components. If the y-axis is parallel to the interfaces in a laminate, the displacement being continuous across the interface, derivatives with respect to y would be continuous. Thus the derivatives of u, v, and w with respect to the z-coordinate need to be replaced by the components of traction on the interface. For nodal points located on the interfaces corresponding to z = constant, the components of traction equal the stress components σ_{xz}, σ_{yz}, and σ_{zz}. For midside nodal points and nodal points on other inter-element boundary orientations, the components will equal the stresses σ_{nx}, σ_{nn}, and σ_{ns}. This requires a transformation relationship between the degrees of freedom to be used for "global" assembly and those to be used for "local" or intra-element interpolation.

8.3.3.1 Transformation for the Corner Nodal Points.

The stress-strain relationship for an orthotropic material with reference to the cartesian reference frame described by x,y,z axes is:

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{31} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{54} & \bar{C}_{55} & 0 \\ \bar{C}_{61} & \bar{C}_{62} & \bar{C}_{62} & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} \tag{240}$$

Here $\bar{C}_{ij}$ are components of the compliance tensor, written in the reduced notation, for monoclinic materials with symmetry with respect to the x-y plane. These components are related to the material compliance coefficients C_{ij} through the following relationships:

$$\begin{aligned}
\bar{C}_{11} &= C_{11}m^4 + (2C_{12} + C_{66})m^2n^2 + C_{22}n^4 \\
\bar{C}_{12} &= (C_{11} + C_{22} - 2C_{12} - C_{66})m^2n^2 + C_{12} \\
\bar{C}_{13} &= C_{13}m^2 + C_{23}n^2 \\
\bar{C}_{16} &= [2C_{11}m^2 - 2C_{22}n^2 + (2C_{12} + C_{66})(n^2 - m^2)]mn \\
\bar{C}_{22} &= C_{11}n^4 + (2C_{12} + C_{66})m^2n^2 + C_{22}n^4 \\
\bar{C}_{23} &= C_{13}n^2 + C_{23}m^2 \\
\bar{C}_{26} &= [2C_{11}n^2 - 2C_{22}m^2 + (2C_{12} + C_{66})(m^2 - n^2)]mn \\
\bar{C}_{33} &= C_{33} \\
\bar{C}_{36} &= 2(C_{13} - C_{23})mn \\
\bar{C}_{44} &= C_{44}m^2 + C_{55}n^2 \\
\bar{C}_{45} &= (C_{44} + C_{55})mn \\
\bar{C}_{55} &= C_{44}n^2 + C_{55}m^2
\end{aligned} \tag{241}$$

$$\bar{C}_{66} = 4(C_{11} + C_{22} - 2C_{12} - C_{66})m^2n^2 + C_{66}$$

where

$$C_{11} = \frac{1}{E_{11}},\ C_{12} = -\frac{\nu_{21}}{E_{22}},\ C_{13} = -\frac{\nu_{31}}{E_{33}}$$

$$C_{21} = -\frac{\nu_{12}}{E_{11}},\ C_{22} = \frac{1}{E_{22}},\ C_{23} = -\frac{\nu_{23}}{E_{33}} \tag{242}$$

$$C_{33} = \frac{1}{E_{33}},\ C_{44} = \frac{1}{G_{23}},\ C_{55} = \frac{1}{G_{13}},\ C_{66} = \frac{1}{G_{12}}$$

and $m = \cos\theta$, and $n = \sin\theta$. Here θ is the angle between the ply orientation (the material 1-axis) and the system x-axis. E_{kk}, G_{ij}, ν_{ij}, are moduli of elasticity in extension, the shear moduli, and Poisson's ratios in material coordinates.

Replacing the strains by appropriate expressions in terms of gradients of displacement components, allowing for the special case of derivatives along the longitudinal axis (the x-axis) being zero, (240) gives:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \tag{243}$$

where

$$\{\epsilon_1\} = \begin{Bmatrix} u_x \\ v_y \\ u_y \end{Bmatrix},\ \{\epsilon_2\} = \begin{Bmatrix} w_z \\ w_y + v_z \\ u_z \end{Bmatrix} \tag{244}$$

$$\{\sigma_1\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix},\ \{\sigma_2\} = \begin{Bmatrix} \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \tag{245}$$

and

$$[D_{11}] = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix},\ [D_{12}] = \begin{bmatrix} \bar{C}_{13} & 0 & 0 \\ \bar{C}_{23} & 0 & 0 \\ \bar{C}_{36} & 0 & 0 \end{bmatrix} \tag{246}$$

$$[D_{21}] = \begin{bmatrix} \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [D_{22}] = \begin{bmatrix} \bar{C}_{33} & 0 & 0 \\ 0 & \bar{C}_{44} & \bar{C}_{45} \\ 0 & \bar{C}_{45} & \bar{C}_{55} \end{bmatrix} \tag{247}$$

Elimination of $\{\sigma_1\}$ between the two equations in (243) gives

$$\{\epsilon^*\} = [C^*]\{\sigma_2\} \tag{248}$$

where

$$\{\epsilon^*\} = \{\epsilon_2\} - [D_{21}]\,[D_{11}]^{-1}\{\epsilon_1\} \tag{249}$$

$$[C^*] = [D_{22}] - [D_{21}]\,[D_{11}]^{-1}[D_{12}] \tag{250}$$

Explicitly, (248) is:

$$\begin{Bmatrix} w_z - B_1 u_x - B_2 v_y - B_3 u_y \\ w_y + v_z \\ u_z \end{Bmatrix} = \begin{bmatrix} \bar{C}_{33} - X & 0 & 0 \\ 0 & \bar{C}_{44} & \bar{C}_{45} \\ 0 & \bar{C}_{45} & \bar{C}_{55} \end{bmatrix} \begin{Bmatrix} \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \tag{251}$$

where

$$B_1 = \bar{C}_{11}\bar{Q}_{11} + \bar{C}_{23}\bar{Q}_{12} + \bar{C}_{36}\bar{Q}_{16} \tag{252}$$

$$B_2 = \bar{C}_{13}\bar{Q}_{12} + \bar{C}_{23}\bar{Q}_{22} + \bar{C}_{36}\bar{Q}_{26} \tag{253}$$

$$B_3 = \bar{C}_{13}\bar{Q}_{16} + \bar{C}_{23}\bar{Q}_{26} + \bar{C}_{36}\bar{Q}_{66} \tag{254}$$

$$X = B_1\bar{C}_{13} + B_2\bar{C}_{23} + B_3\bar{C}_{36} \tag{255}$$

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4) \\ \bar{Q}_{16} &= mn^3Q_{22} + m^3nQ_{11} - mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \\ \bar{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4 \\ \bar{Q}_{26} &= m^3nQ_{22} + mn^3Q_{11} + mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12})m^2n^2 + Q_{66}(m^2 - n^2)^2 \end{aligned} \tag{256}$$

and

$$Q_{11} = \frac{E_{11}}{1 - \nu_{21}\nu_{12}}, \ Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}, \ Q_{12} = \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}}, \ Q_{66} = G_{12} \tag{257}$$

Solving (251) for u_z, v_z, and w_z in terms of the remaining quantities gives the following transformation from the global degrees of freedom, which include components of traction at the nodal points, to the displacement degrees of freedom which are used to describe the variation of displacement within the element:

$$
\begin{Bmatrix} u \\ v \\ w \\ u_y \\ v_y \\ w_y \\ u_z \\ v_z \\ w_z \end{Bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \overline{C}_{55} & \overline{C}_{45} & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \overline{C}_{45} & \overline{C}_{44} & 0 \\
0 & 0 & 0 & B_3 & B_2 & 0 & 0 & 0 & \overline{C}_{33} - X
\end{bmatrix}
\begin{Bmatrix} u \\ v \\ w \\ u_y \\ v_y \\ w_y \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix}
+
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ B_1 e_0 \end{Bmatrix}
\tag{258}
$$

8.3.3.2 Transformation for Midside Nodes.

The transformation from stress components normal to the element boundary at the midside points and the normal displacement gradients used for interpolation within the element is:

$$
\begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix}_{i+3} = [T_1]_i^{-1} \begin{Bmatrix} \sigma_{nx} \\ \sigma_{nn} \\ \sigma_{ns} \end{Bmatrix}_{i+3} - [T_1]_i^{-1} [T_2]_i \{r'\}_{i+3} - [T_1]_i^{-1} \{T_3\}_i \tag{259}
$$

Here

$$
[T_1]_i = \left[\begin{matrix}
m^2\overline{S}_{66} + n^2\overline{S}_{55} & m^2\overline{S}_{26} + n^2\overline{S}_{45} \\
m^3\overline{S}_{26} + mn^2\overline{S}_{36} + 2mn^2\overline{S}_{45} & m^3\overline{S}_{22} + mn^2\overline{S}_{23} + 2mn^2\overline{S}_{44} \\
(\overline{S}_{36} - \overline{S}_{26})m^2n + (m^2n - n^3\overline{S}_{45} & (\overline{S}_{23} - \overline{S}_{22})m^2n + (m^2n - n^3)\overline{S}_{44}
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
(\overline{S}_{45} + \overline{S}_{36})mn \\
2m^2n\overline{S}_{44} + m^2n\overline{S}_{23} + n^3\overline{S}_{33} \\
(m^3 - mn^2)\overline{S}_{44} + (\overline{S}_{33} - \overline{S}_{23})mn^2
\end{matrix}\right] \tag{260}
$$

$$
\{r'\}_{i+3} = \{u_j, v_j, w_j, u_{y_j}, v_{y_j}, w_{y_j}, \sigma_{xz_j}, \sigma_{yz_j}, \sigma_{zz_j}, u_k, v_k, w_k, u_{y_k}, v_{y_k}, w_{y_k}, \sigma_{xz_k}, \sigma_{yz_k}, \sigma_{zz_k}\}^T \tag{261}
$$

$$\{T_3\}_i = \left\{ \begin{matrix} \frac{1}{2}\bar{S}_{45}mn^2B_1 - \frac{1}{2}\bar{S}_{36}m^3B_1 + m\bar{S}_{16} \\ \bar{S}_{44}m^2n^2B_1 - \frac{1}{2}(m^4\bar{S}_{23} + m^2n^2\bar{S}_{33})B_1 + m^2\bar{S}_{12} + n^2\bar{S}_{13} \\ \frac{1}{2}\bar{S}_{44}(m^3n - mn^3)B_1 - \frac{1}{2}(\bar{S}_{33} - \bar{S}_{23})m^3nB_1 + (\bar{S}_{13} - \bar{S}_{12})mn \end{matrix} \right\} e_0 \tag{262}$$

and

$$[T_2]_i = [TA][TB] \tag{263}$$

where

$$[TA] = \left[\begin{matrix} m\bar{S}_{66} & m\bar{S}_{26} & n\bar{S}_{45} & n\bar{S}_{55} & n\bar{S}_{45} & m\bar{S}_{36} \\ m^2\bar{S}_{26} + n^2\bar{S}_{36} & m^2\bar{S}_{22} + n^2\bar{S}_{23} & 2mn\bar{S}_{44} & 2mn\bar{S}_{45} & 2mn\bar{S}_{44} & m^2\bar{S}_{23} + n^2\bar{S}_{33} \\ (\bar{S}_{36} - \bar{S}_{26})mn & (\bar{S}_{23} - \bar{S}_{22})mn & (m^2 - n^2)\bar{S}_{44} & (m^2 - n^2)\bar{S}_{45} & (m^2 - n^2)\bar{S}_{44} & (\bar{S}_{33} - \bar{S}_{23})mn \end{matrix} \right] \tag{264}$$

$$[TB] = \left[\begin{matrix} \frac{3n}{2l_i} & 0 & 0 & -\frac{n^2}{4} & 0 & 0 & \frac{mn}{4}\bar{C}_{55} & \frac{mn}{4}\bar{C}_{45} & 0 \\ 0 & \frac{3n}{2l_i} & 0 & 0 & -\frac{n^2}{4} & -\frac{mn}{4} & \frac{mn}{4}\bar{C}_{45} & \frac{mn}{4}\bar{S}_{44} & 0 \\ 0 & 0 & \frac{3n}{2l_i} & \frac{mn}{4}B_3 & \frac{mn}{4}B_2 & -\frac{n^2}{4} & 0 & 0 & \frac{mn}{4}(\bar{C}_{33} - X) \\ -\frac{3m}{2l_i} & 0 & 0 & \frac{mn}{4} & 0 & 0 & -\frac{m^2}{4}\bar{C}_{55} & -\frac{m^2}{4}\bar{C}45 & 0 \\ 0 & -\frac{3m}{2l_i} & 0 & 0 & \frac{mn}{4} & \frac{m^2}{4} & -\frac{m^2}{4}\bar{C}_{45} & -\frac{m^2}{4}\bar{C}_{44} & 0 \\ 0 & 0 & -\frac{3m}{2l_i} & -\frac{mn}{4}B_3 & -\frac{m^2}{4}B_2 & \frac{mn}{4} & 0 & 0 & -\frac{m^2}{4}(\bar{C}_{33} - X) \end{matrix} \right]$$

$$\left.\begin{matrix}
-\frac{3n}{2l_i} & 0 & 0 & -\frac{n^2}{4} & 0 & 0 & \frac{mn}{4}\bar{C}_{55} & \frac{mn}{4}\bar{C}_{45} & 0 \\
0 & -\frac{3n}{2l_i} & 0 & 0 & -\frac{n^2}{4} & -\frac{mn}{4} & \frac{mn}{4}\bar{C}_{45} & \frac{mn}{4}\bar{C}_{44} & 0 \\
0 & 0 & -\frac{3n}{2l_i} & \frac{mn}{4}B_3 & \frac{mn}{4}B_2 & -\frac{n^2}{4} & 0 & 0 & \frac{mn}{4}(\bar{C}_{33}-X) \\
\frac{3m}{2l_i} & 0 & 0 & \frac{mn}{4} & 0 & 0 & -\frac{m^2}{4}\bar{C}_{55} & -\frac{m^2}{4}\bar{C}_{45} & 0 \\
0 & \frac{3m}{2l_i} & 0 & 0 & \frac{mn}{4} & \frac{m^2}{4} & -\frac{m^2}{4}\bar{C}_{45} & -\frac{m^2}{4}\bar{C}_{44} & 0 \\
0 & 0 & \frac{3m}{2l_i} & -\frac{m^2}{4}B_3 & -\frac{m^2}{4}B_2 & \frac{mn}{4} & 0 & 0 & -\frac{m^2}{4}(\bar{C}_{33}-X)
\end{matrix}\right| \tag{265}$$

$$m = -\frac{b_i}{l_i}, \quad n = -\frac{a_i}{l_i}, \quad a_i = y_k - y_j, \quad b_i = z_j - z_k, \quad l_i = \sqrt{a^2 + b^2} \tag{266}$$

y_i, z_i are the coordinates of the corner nodal points of the triangular finite element and $\bar{S}_{ij}$ are elements of the material stiffness matrix $[\bar{S}]$ which is inverse of the compliance matrix $[\bar{C}]$ with elements $\bar{C}_{ij}$.

8.3.3.3 Traction-free Boundary Conditions.

The traction-free boundary conditions for one quadrant of the free-edge delamination specimen are:

$$\sigma_{xz}(y,H) = \sigma_{yz}(y,H) = \sigma_{zz}(y,H) = 0 \tag{267}$$

at the top surface along with

$$\sigma_{yy}(B,z) = \sigma_{xy}(B,z) = \sigma_{zy}(B,z) = 0 \tag{268}$$

at the lateral free edge. Here 2B and 2H are the total width and the total thickness of the specimen. The Q-23 element has the tractions at the top surface included in the list of field variables for the finite element model. Therefore, these conditions can be explicitly satisfied simply by specifying the values at the nodal points located on the top surface. Similarly, for the midside nodal points on the lateral free edge, the tractions are field variables for the finite element model and their values can be specified directly. At the corner points of the elements, the component σ_{yz} can be directly specified. However, the components σ_{yy} and σ_{yx} have to be specified as linear constraints on the problem. Expressing these in terms of the kinematic variables, we have:

$$\sigma_{yy} = 0 = \bar{S}_{12}e_0 + \bar{S}_{23}w_z + \bar{S}_{22}v_y + \bar{S}_{26}u_y \tag{269}$$

$$\sigma_{xy} = 0 = \bar{S}_{16}e_0 + \bar{S}_{36}w_z + \bar{S}_{26}v_y + \bar{S}_{66}u_y \tag{270}$$

Solving for v_y and u_y in terms of the remaining quantities:

$$\begin{Bmatrix} v_y \\ u_y \end{Bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \begin{Bmatrix} e_0 \\ w_z \end{Bmatrix} \tag{271}$$

where

$$\begin{aligned} I_{11} &= \frac{1}{\alpha}(\bar{S}_{26}\bar{S}_{16} - \bar{S}_{66}\bar{S}_{12}) \\ I_{12} &= \frac{1}{\alpha}(\bar{S}_{26}\bar{S}_{36} - \bar{S}_{66}\bar{S}_{23}) \\ I_{21} &= \frac{1}{\alpha}(\bar{S}_{26}\bar{S}_{12} - \bar{S}_{22}\bar{S}_{16}) \\ I_{22} &= \frac{1}{\alpha}(\bar{S}_{26}\bar{S}_{23} - \bar{S}_{22}\bar{S}_{36}) \end{aligned} \tag{272}$$

and

$$\alpha = \bar{S}_{22}\bar{S}_{66} - \bar{S}_{26}^2 \tag{273}$$

Using (251) to write w_z in terms of σ_{zz}, (271) can be incorporated into the transformation (258).

8.3.4 An Axisymmetric Model.

8.3.4.1 Introduction.

A computer program previously used by the Air Force to model free-edge delamination specimens was applicable to an axisymmetric configuration. The model assumed the tangential shearing strains to be zero i.e., the deformation is purely radial and/or axial. It was based on the use of a bilinear Lagrange interpolation or optional use of a reduced integration procedure proposed by Singh [1977] and Sandhu and Singh [1978]. In the present research, the program was enhanced to include "twist" in the analysis. The enhanced version has the capability, which the earlier version lacked, to model the complete stress state in angle-ply specimens. Herein, we give an outline of the theoretical development. Details are available in items B.1.6 and B.1.7 of Appendix B.

8.3.4.2 The Axisymmetric Model.

The equations of equilibrium, in cylindrical coordinates, for linear elastic deformation, are:

$$\begin{bmatrix} \frac{1}{r}+\frac{\partial}{\partial r} & 0 & -\frac{1}{r} & 0 & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & 0 & \frac{1}{r}\frac{\partial}{\partial \theta} & 0 & \frac{1}{r}+\frac{\partial}{\partial r} \\ 0 & 0 & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} & \frac{2}{r}+\frac{\partial}{\partial r} & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{z\theta} \\ \sigma_{\theta r} \\ \sigma_{rz} \end{Bmatrix} = \begin{Bmatrix} -f_r \\ -f_z \\ -f_\theta \end{Bmatrix} \tag{274}$$

The strain-displacement relationships in cylindrical coordinates are:

$$\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta\theta} \\ \gamma_{z\theta} \\ \gamma_{\theta r} \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ \frac{1}{r} & 0 & \frac{1}{r}\frac{\partial}{\partial \theta} \\ 0 & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{r}\frac{\partial}{\partial \theta} & 0 & \frac{\partial}{\partial r}-\frac{1}{r} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} & 0 \end{bmatrix} \begin{Bmatrix} u_r \\ u_z \\ u_\theta \end{Bmatrix} \tag{275}$$

where, r, z, and θ are, respectively, the radial, the axial and the tangential coordinates. u_r, u_z, and u_θ are the displacements corresponding to the three coordinate axes. For axisymmetry, all quantities are independent of θ, and the equations (274) and (275) reduce to

$$\begin{bmatrix} \frac{1}{r}+\frac{\partial}{\partial r} & 0 & -\frac{1}{r} & 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & 0 & 0 & 0 & \frac{1}{r}+\frac{\partial}{\partial r} \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{2}{r}+\frac{\partial}{\partial r} & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{z\theta} \\ \sigma_{\theta r} \\ \sigma_{rz} \end{Bmatrix} = \begin{Bmatrix} -f_r \\ -f_z \\ -f_\theta \end{Bmatrix} \tag{276}$$

$$\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta\theta} \\ \gamma_{z\theta} \\ \gamma_{\theta r} \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial r}-\frac{1}{r} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} & 0 \end{bmatrix} \begin{Bmatrix} u_r \\ u_z \\ u_\theta \end{Bmatrix} \tag{277}$$

These stress-strain relations need appropriate transformation from the material to the local axes and then, if necessary, to global axes. For cylindrical systems the local and global axes, in general, coincide.

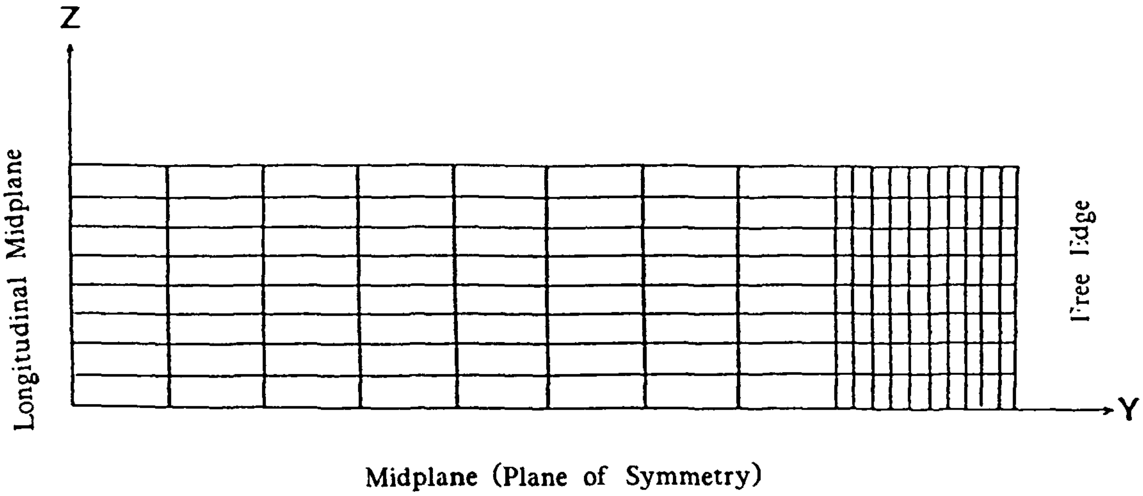
8.3.5 Computer Implementation.

The schemes described in Sections 1.3.3 and 1.3.4 were implemented in finite element computer programs. The codes were verified by application to Pagano's problem [1978] and then applied to analyze stresses and deformations in selected 22-layer delamination specimens.

8.3.5.1 Continuous Traction Finite Element Analysis.

The finite element discretization of a quadrant of the cross-section of a free-edge delamination specimen is shown in Figure 12. Each layer was modelled by four sublayers. In the transverse direction, the elements were of width B/10 in the middle part and B/50 for the ten elements nearest the free edge. The cross-ply specimen was $[0/90]_s$ and the angle-ply specimen was $[\pm 45]_s$. Comparison of results from the continuous traction model with Pagano's theoretical solution is given in Figure 13 through Figure 20. The agreement is excellent.

The program was also used to solve a 22-layer problem. Several different layups, for which test data were available, were examined. Details are given in items B.1.5 of Appendix B. Here, we show the results for $[(\pm 49.8)_5/90]_s$ specimen. Figure 21 shows the finite element model used for one quadrant of the specimen. Figure 22 shows the distribution of σ_{zz} along the mid-plane of the specimen in the vicinity of the free edge. The "constant strain element" denotes results of a pseudo-two-dimensional finite element analysis, ignoring σ_{xz} and σ_{xy} and using Q1 elements, reported by Sandhu and Sendeckyj [1987]. Through-the-thickness distribution of σ_{zz} and σ_{xz} at the free edge is shown in Figure 23. The plot shows considerable irregularity in the distribution of σ_{zz} at the free edge. Figure 24 shows the influence of distance from the free edge on through-the-thickness distribution of σ_{zz}. The irregularity in stress distribution is confined to the free edge and the distribution is much smoother even a short distance away ($\frac{y}{B} = 0.995$) from the free edge. The distribution of σ_{zz} at the free edge was carefully examined by refining the mesh in the vicinity. Finite element models, based upon refinement along each of the y and the z axes and along both the axes, were used. Along the y-axis, the refinement consisted of subdividing each edge element into



144 (8x18) F. E. Mesh - Continuous Traction Analysis

Figure 12: FINITE ELEMENT MODEL OF FREE-EDGE DELAMINATION SPECIMENS.

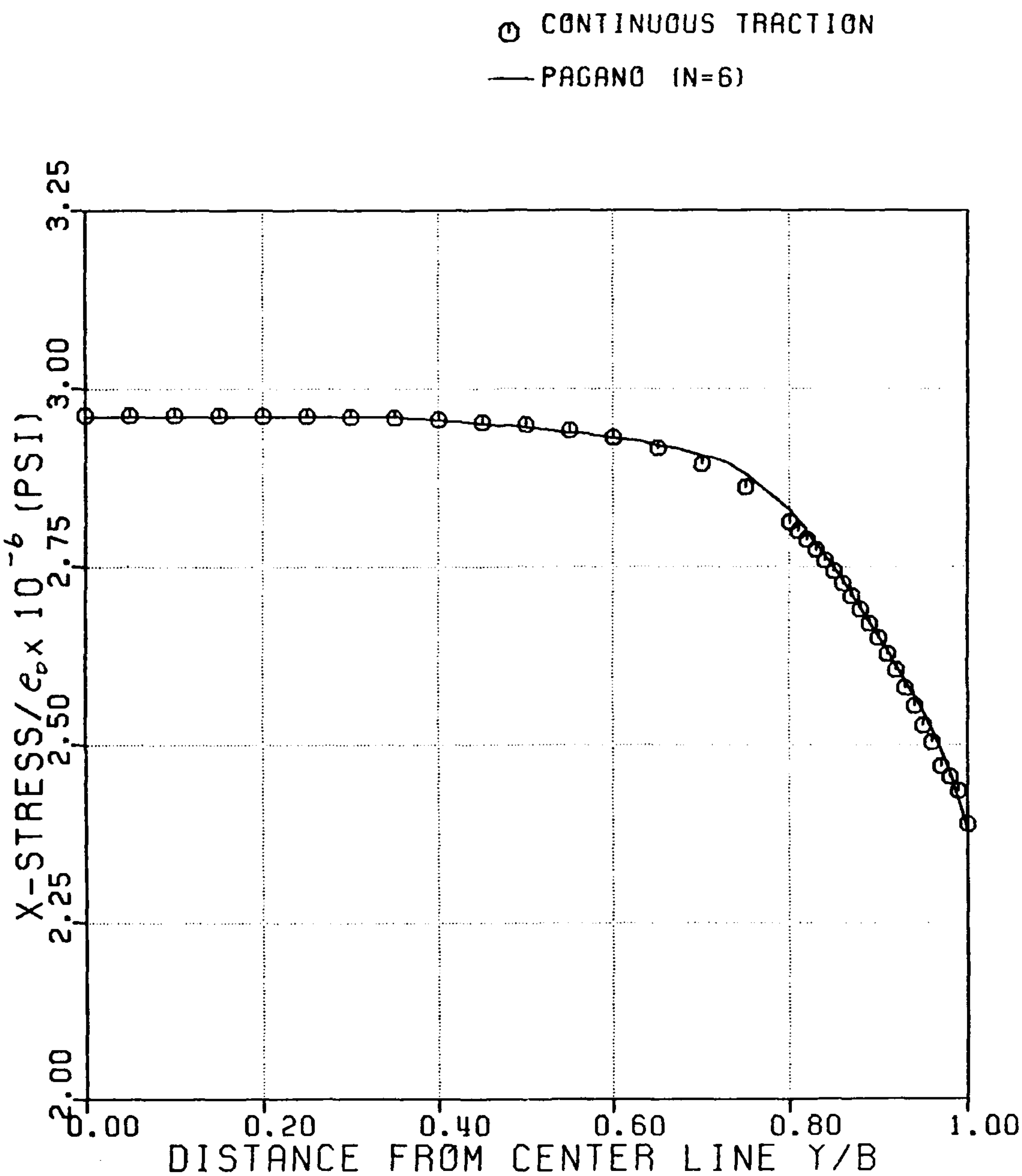


Figure 13: DISTRIBUTION OF X-STRESS ALONG CENTER OF TOP LAYER (45 DEGREES) IN A [45/-45] SYMMETRIC LAMINATE.

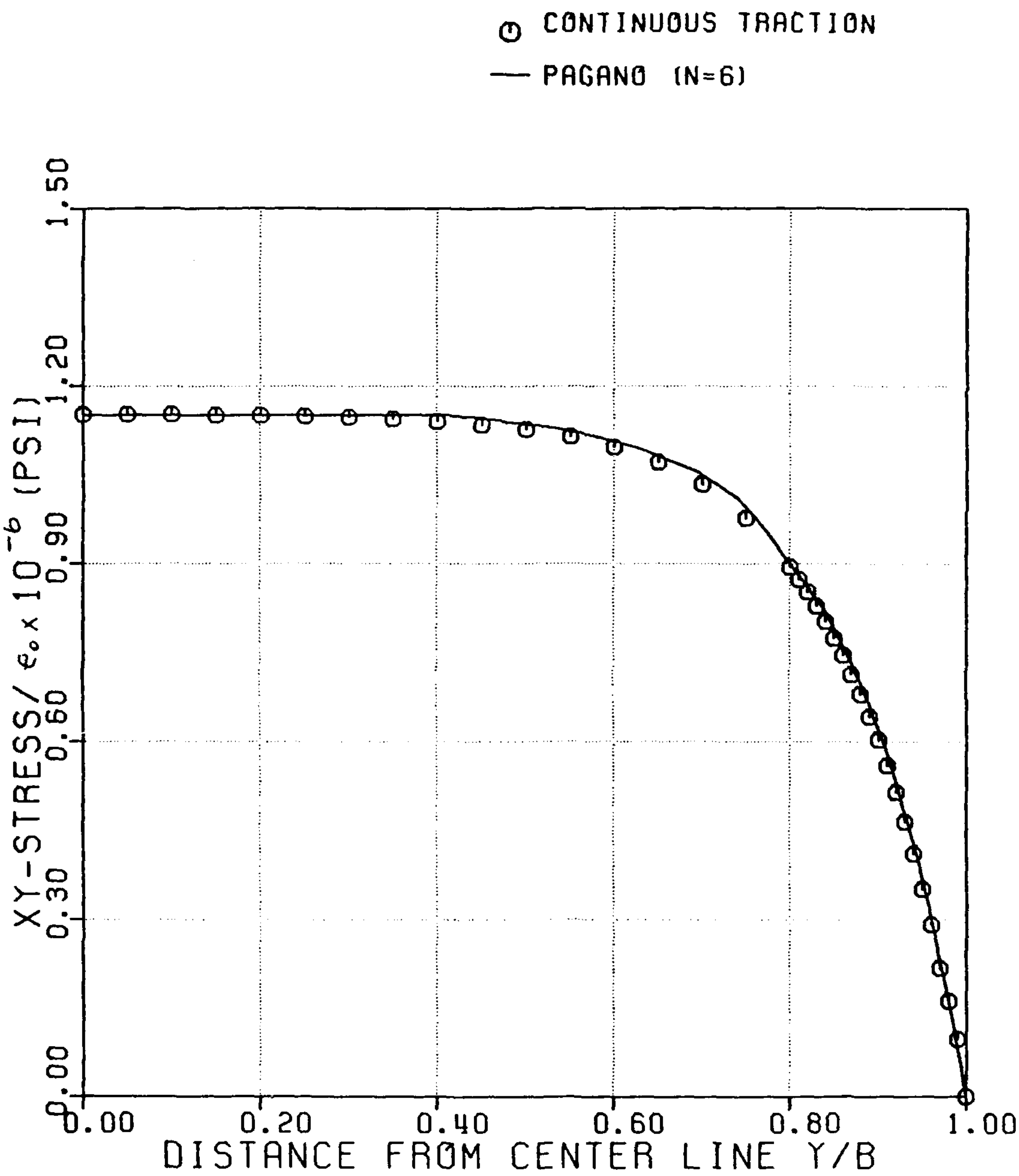


Figure 14: DISTRIBUTION OF XY-STRESS ALONG CENTER OF THE TOP LAYER (45 DEGREES) IN A [45/-45] SYMMETRIC LAMINATE.

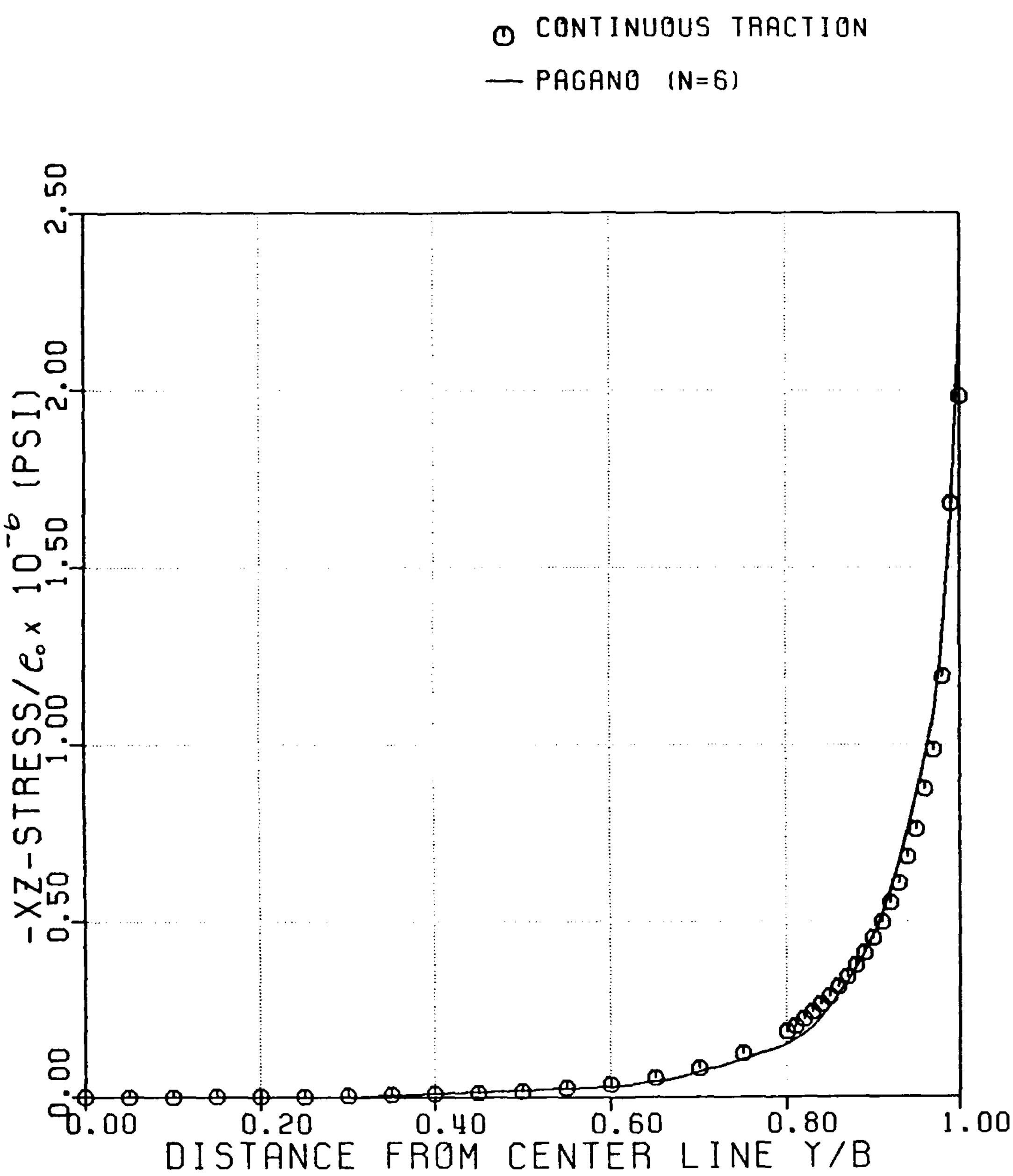


Figure 15: DISTRIBUTION OF XZ-STRESS ALONG 45/-45 INTERFACE OF A [45/-45] SYMMETRIC LAMINATE.

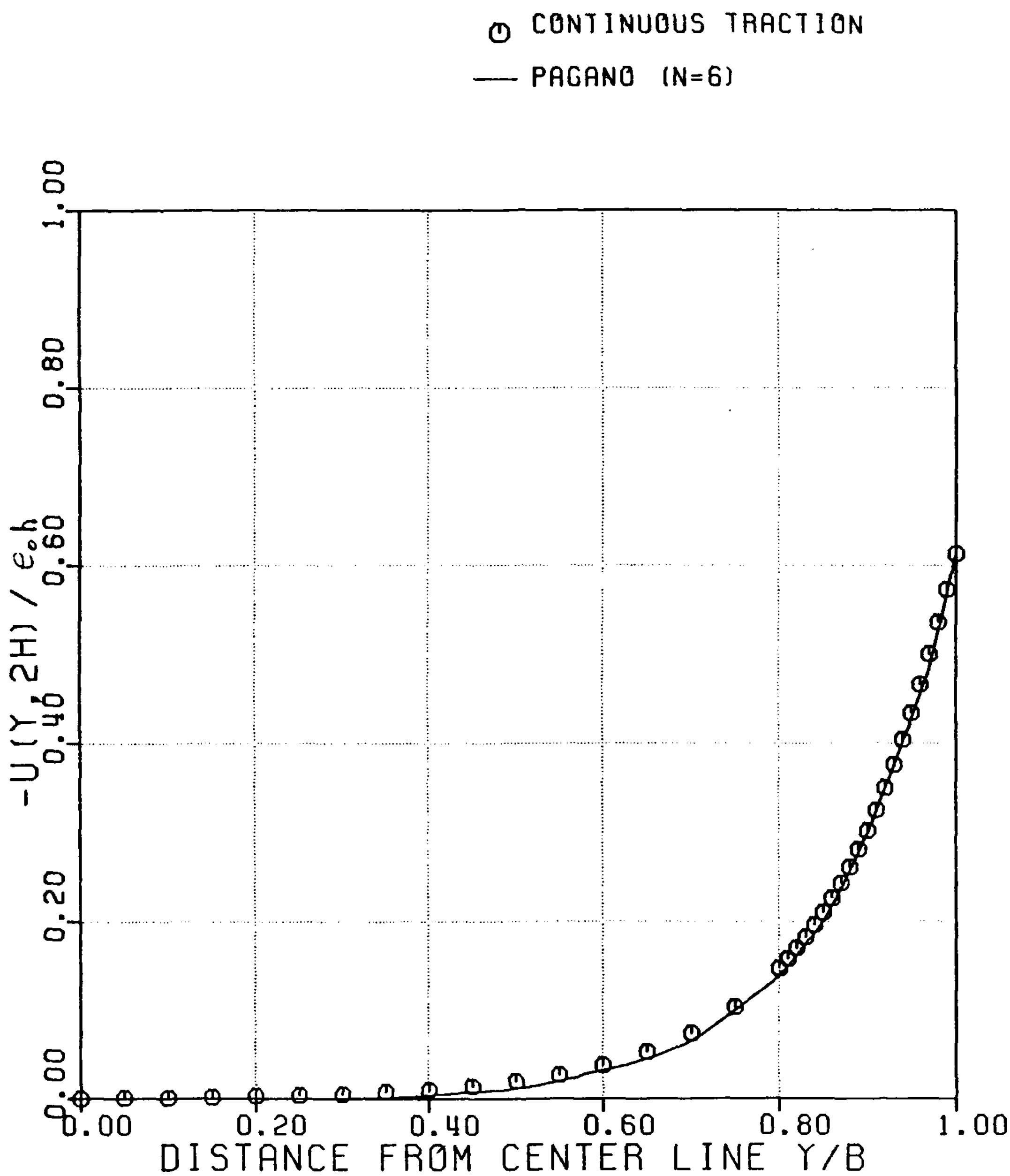


Figure 16: AXIAL DISPLACEMENT ACROSS THE TOP SURFACE OF [45/-45] SYMMETRIC LAMINATE.

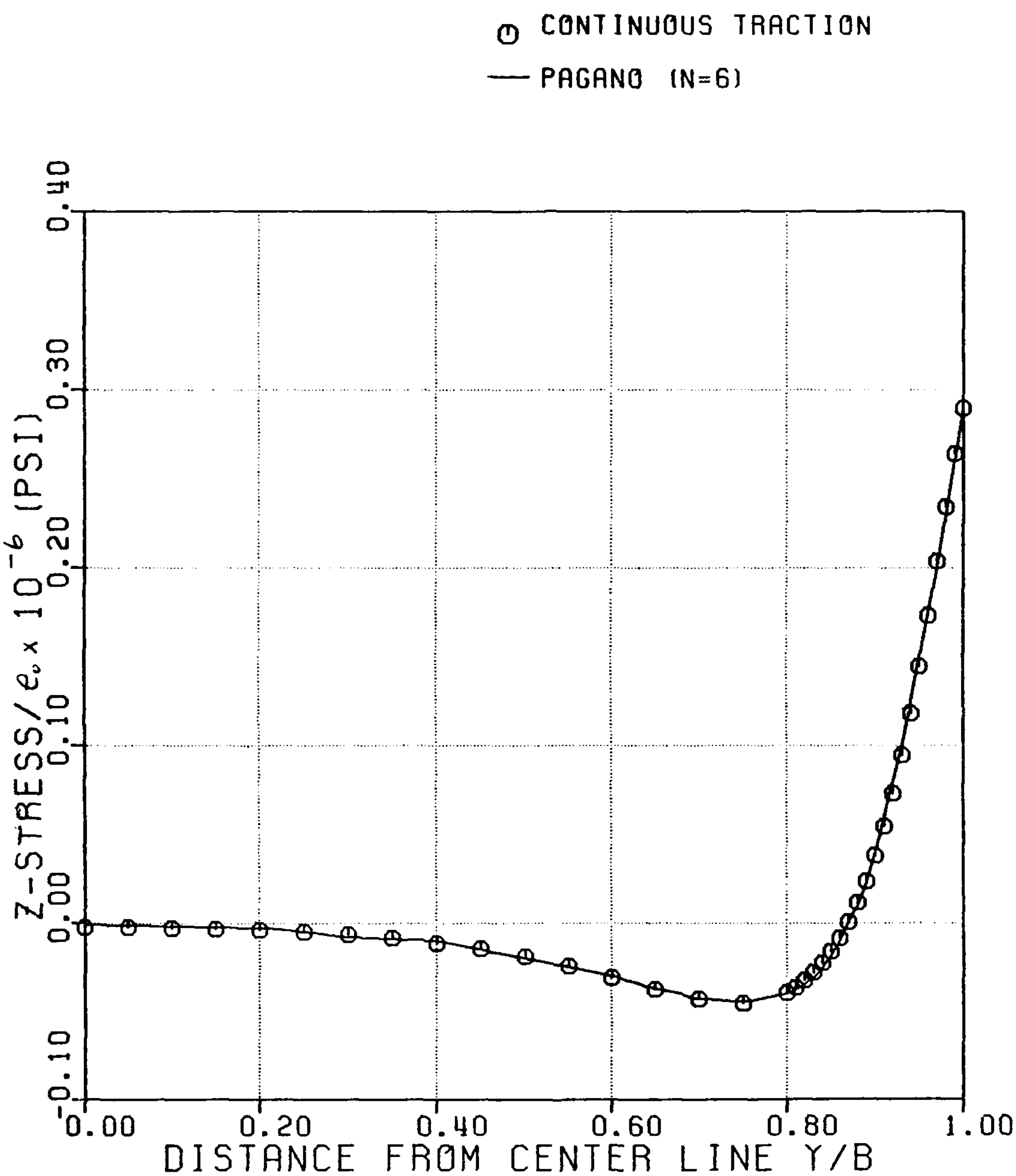


Figure 17: DISTRIBUTION OF Z-STRESS ALONG MIDPLANE OF [0/90] SYMMETRIC LAMINATE.

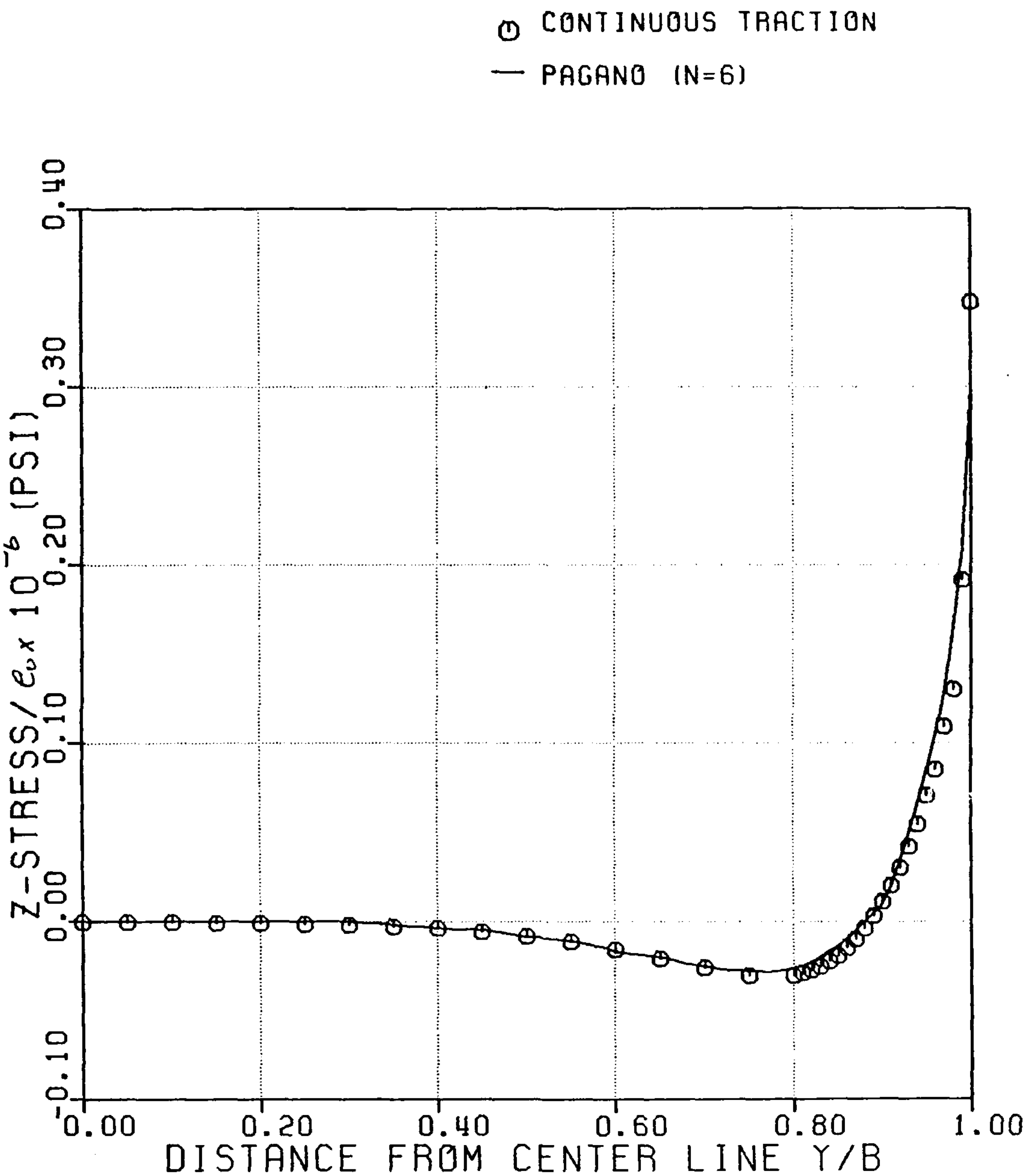


Figure 18: DISTRIBUTION OF Z-STRESS ALONG 0/90 INTERFACE OF A [0/90] SYMMETRIC LAMINATE.

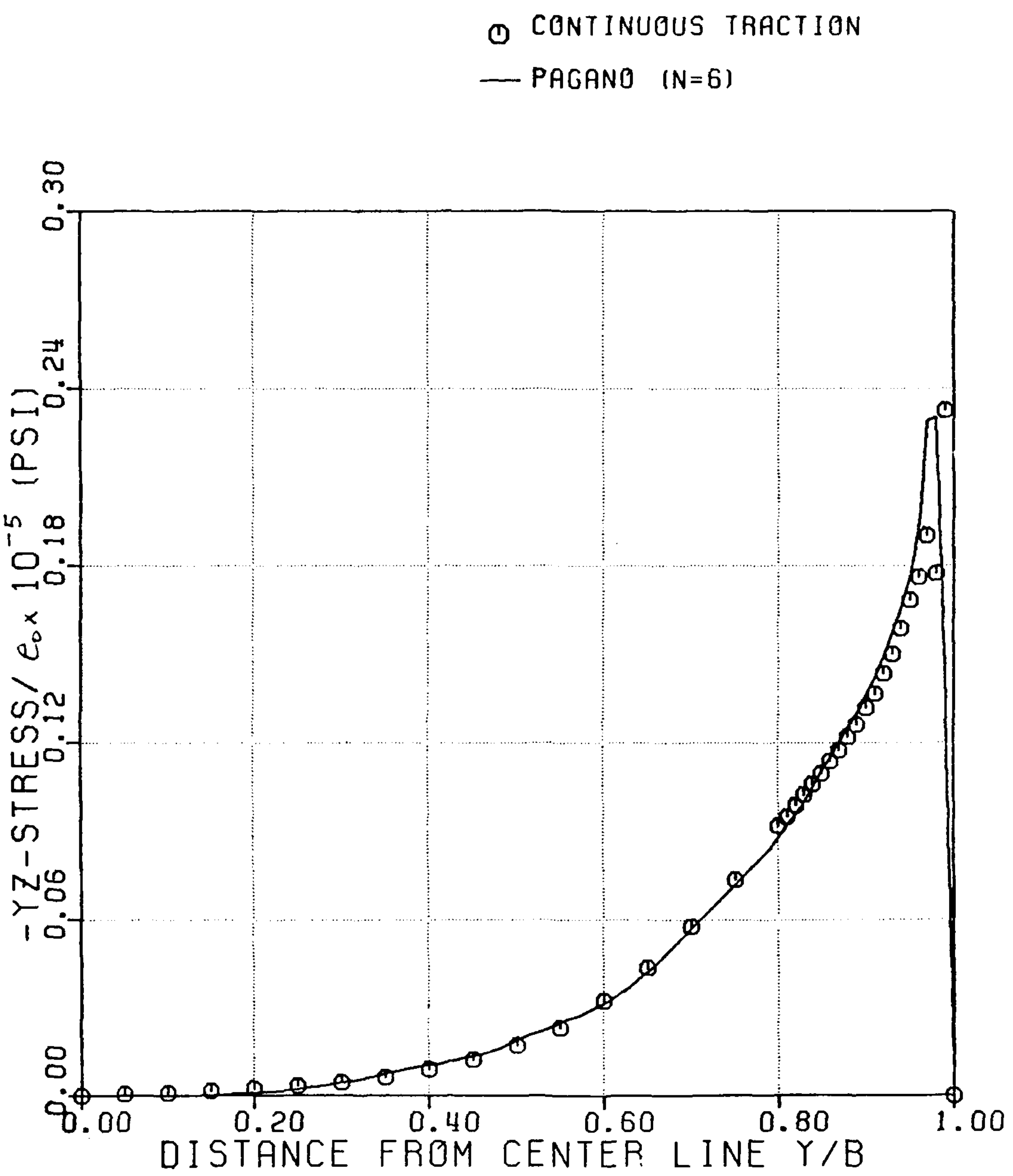


Figure 19: DISTRIBUTION OF YZ-STRESS ALONG 0/90 INTERFACE OF A [0/90] SYMMETRIC LAMINATE.

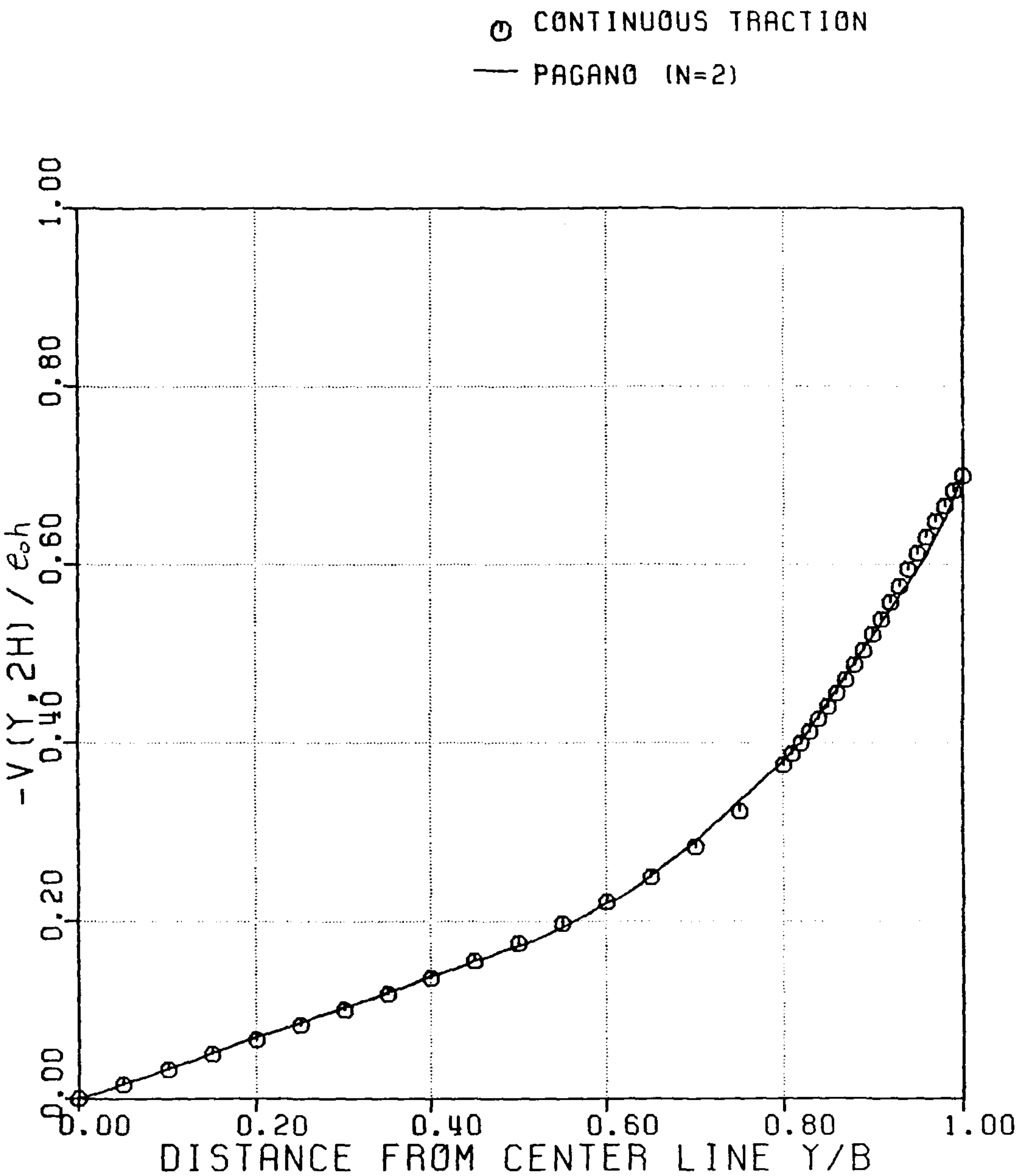
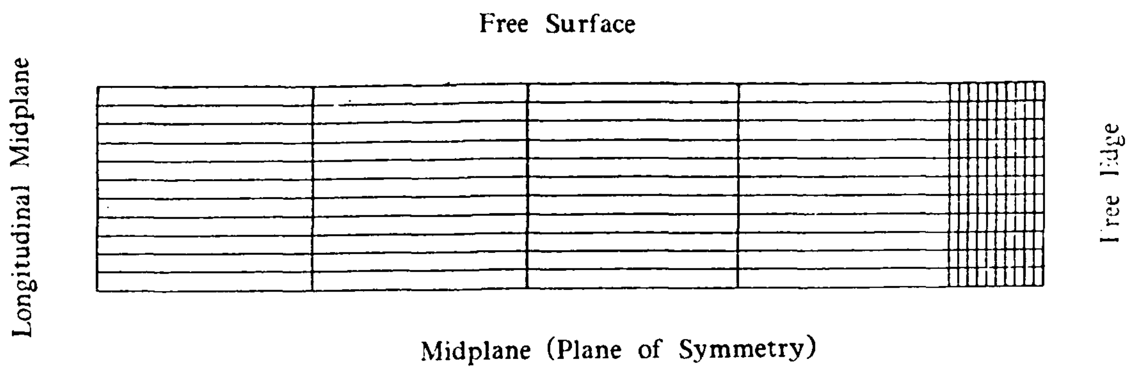
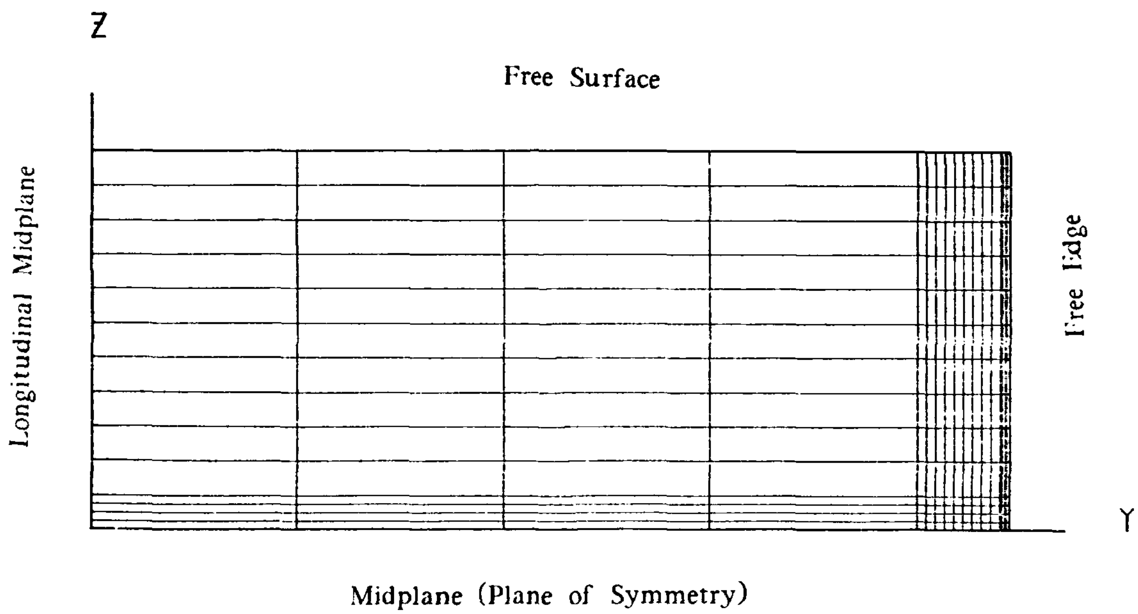


Figure 20: TRANSVERSE DISPLACEMENT ACROSS THE TOP SURFACE OF A [0/90] SYMMETRIC LAMINATE.



154 (11x14) F. E. Mesh - Continuous Traction Analysis



238 (14x17) F. E. Mesh - Continuous Traction Analysis

Figure 21: FINITE ELEMENT MODEL OF 22-LAYER DELAMINATION SPECIMEN.

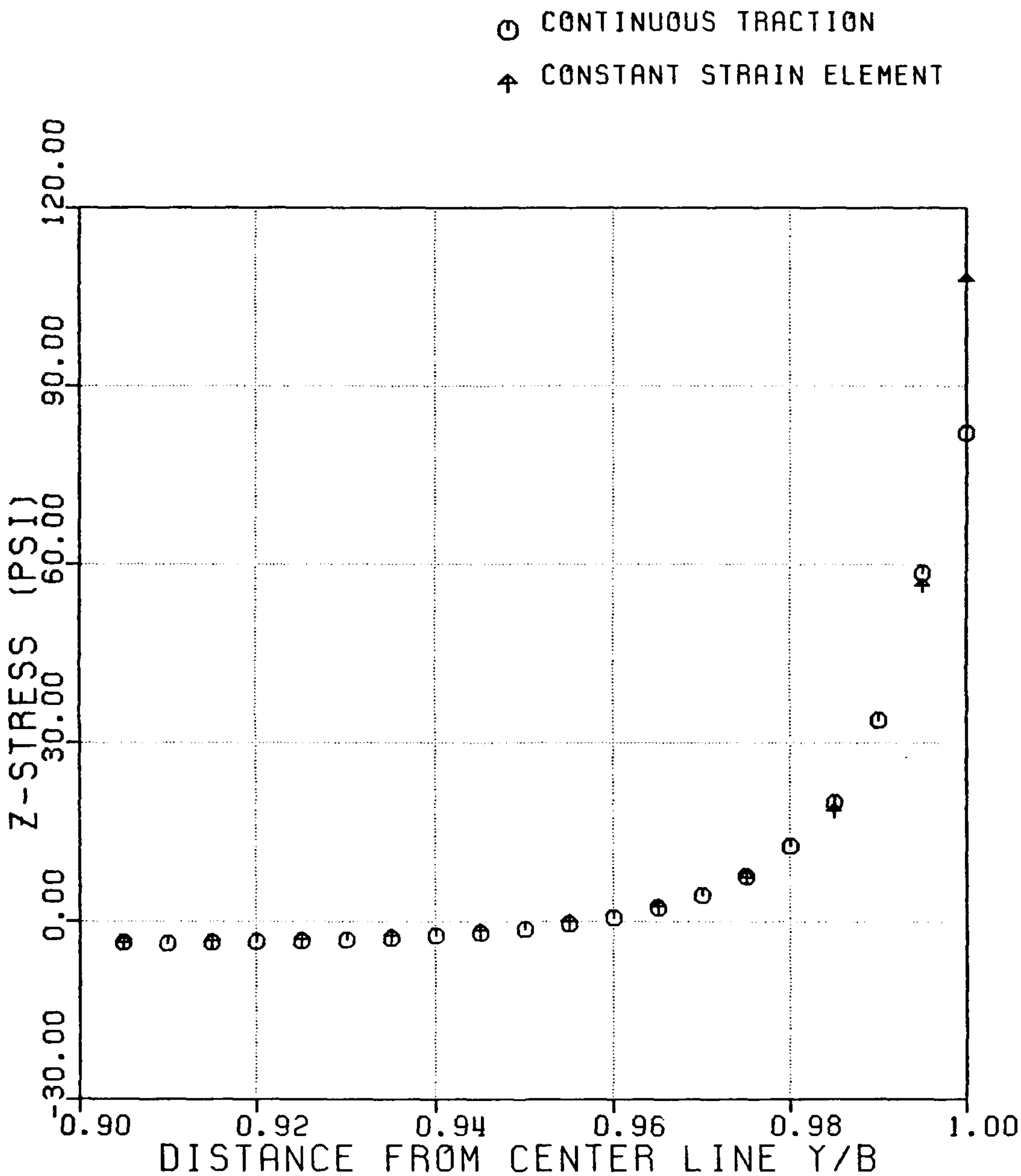


Figure 22: DISTRIBUTION OF Z-STRESS ALONG MIDPLANE OF 22-LAYER DELAMINATION SPECIMEN.

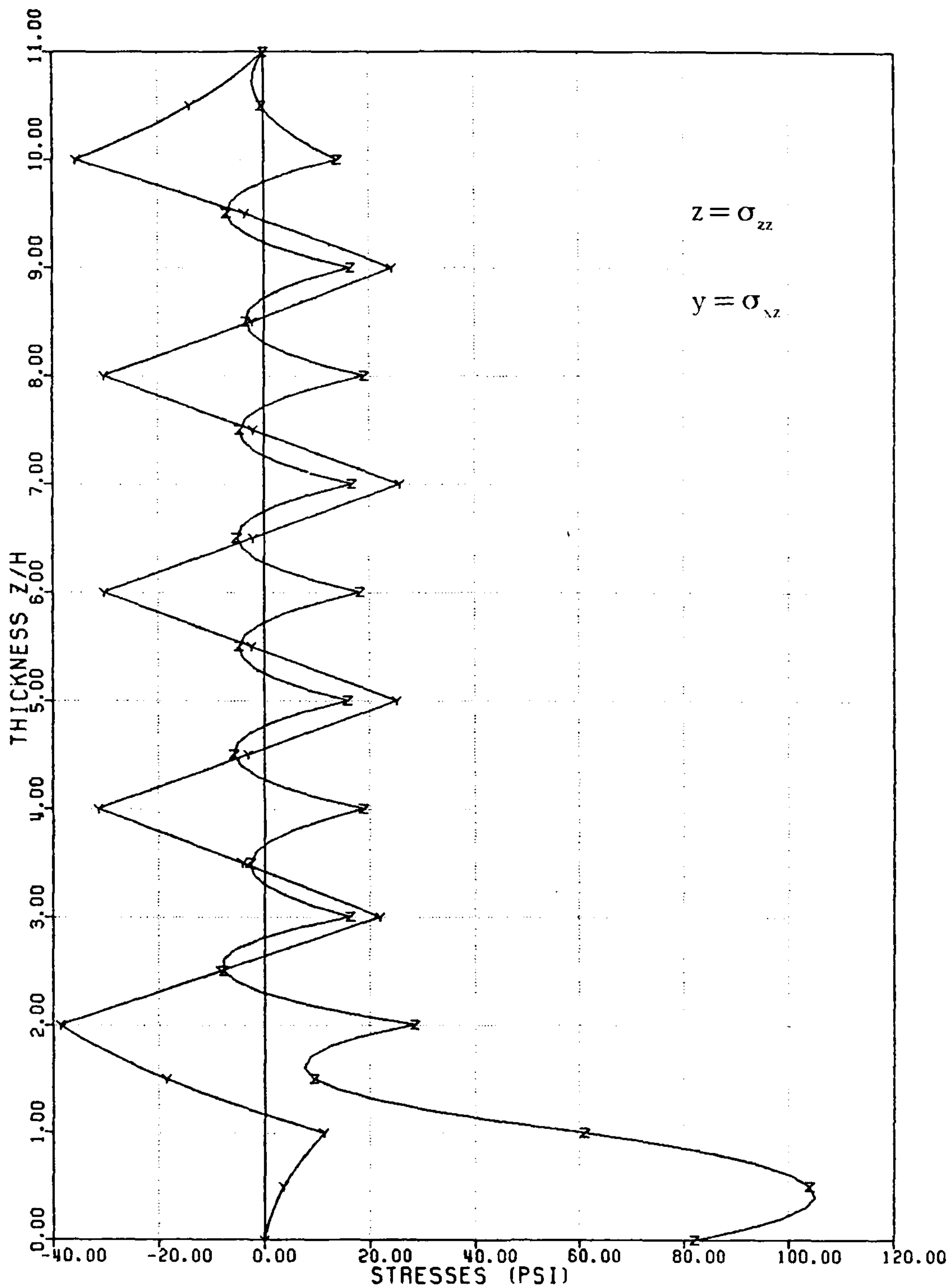


Figure 23: THROUGH-THE-THICKNESS STRESS DISTRIBUTION AT THE FREE-EDGE OF THE 22-LAYER FEDS.

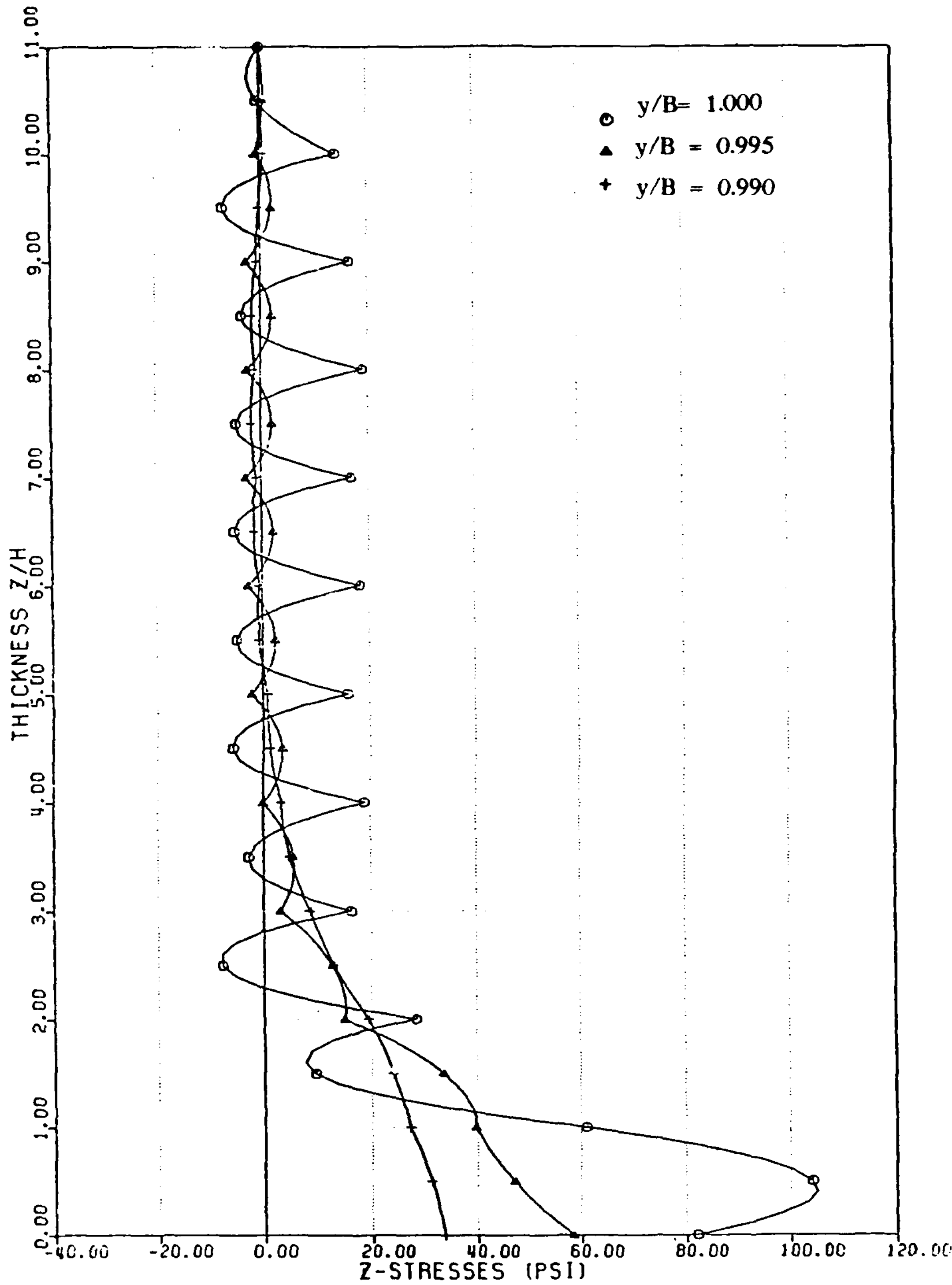


Figure 24: THROUGH-THE-THICKNESS DISTRIBUTION OF Z-STRESS NEAR THE FREE EDGE OF THE 22-LAYER FEDS.

four. Along the z-axis, refinements only for the 90-degree ply near the midplane of the specimen as well as for all the layers were carried out. Selected results are plotted in Figure 25 and Figure 26. Refinement along the z-axis over the 90° layer improved the solution over that layer without much change in the stress patterns elsewhere. Refinement over the y-axis, on the other hand, improved the results over the layers other than the 90° layer with little effect on the stresses in that layer. To improve stress solution all along the free edge, it was necessary to refine the mesh along both the axes. Figure 27 and Figure 28 show plots of the through-the-thickness variation of σ_{zz} at $\frac{y}{B} = 0.99$. Refinement of the mesh has little effect on the stress distribution at this location. Thus, the influence of mesh refinement is entirely confined to the immediate vicinity of the free edge.

Figure 29 shows the variation of σ_{yz} along the center of the 90-degree layer for a distance of eight ply thicknesses from the free edge.

The computer program is quite efficient. For the 154-element mesh with 513 nodal points, the total number of algebraic equations was 2311 with a band-width of 186. The execution time for this case was 121 seconds on an IBM 3081 mainframe computer.

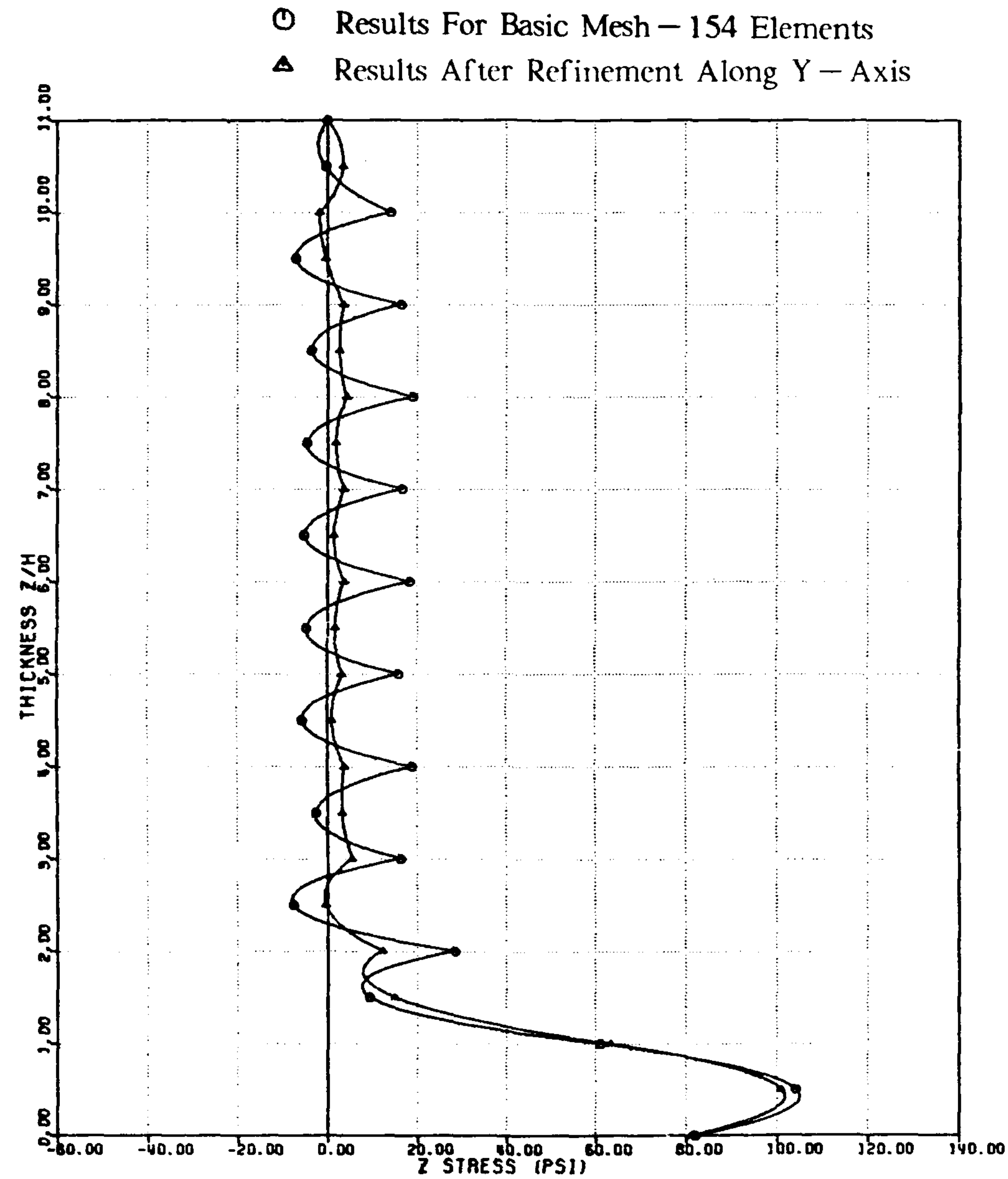


Figure 25: THROUGH-THE-THICKNESS Z-STRESS DISTRIBUTION AT THE FREE-EDGE OF THE 22-LAYER FEDS. (REFINEMENT ALONG Y-AXIS ONLY).

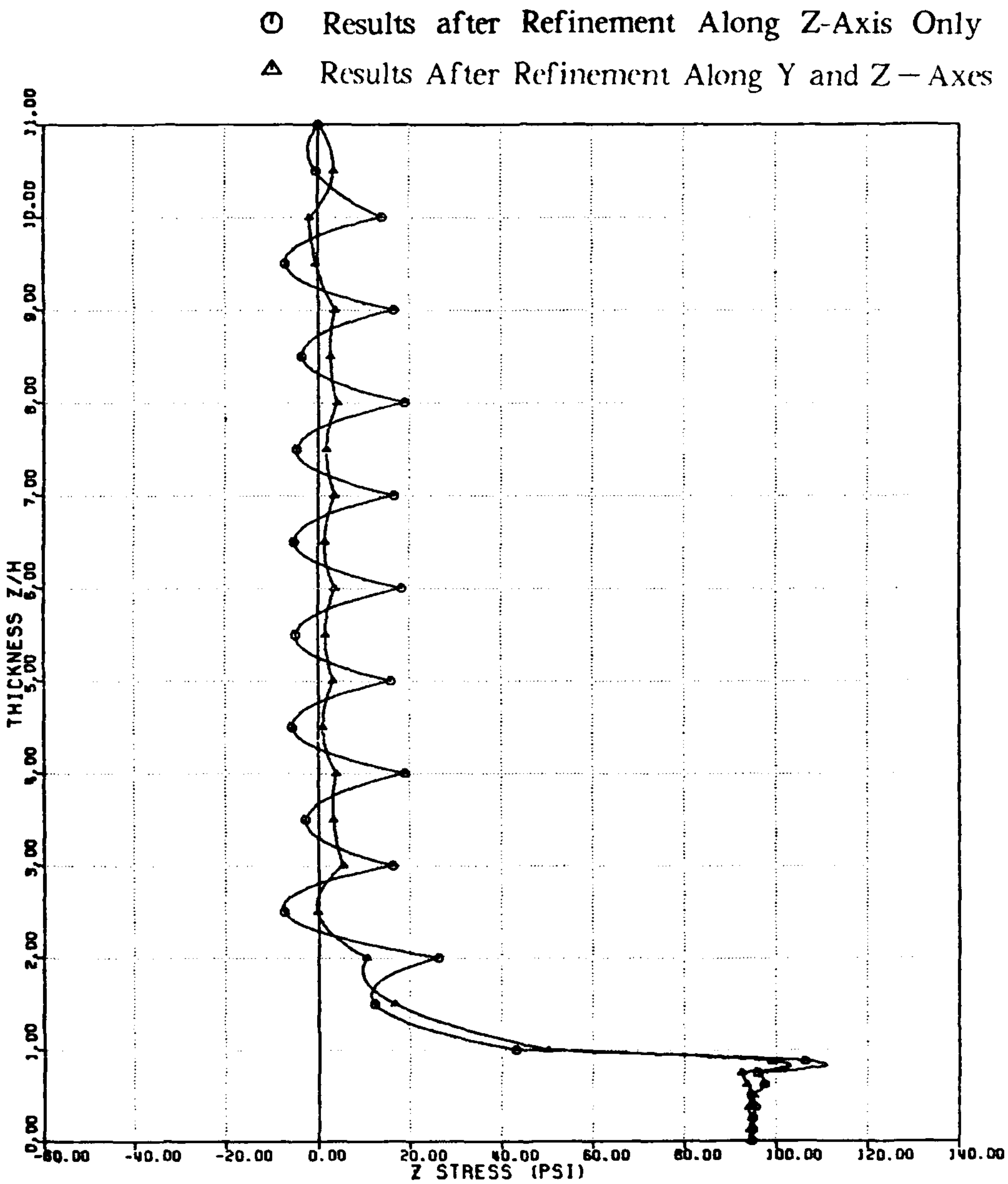


Figure 26: THROUGH-THE-THICKNESS DISTRIBUTION OF Z-STRESS NEAR THE FREE EDGE OF THE 22-LAYER FEDS. (REFINEMENT ALONG Y- AND Z-AXES).

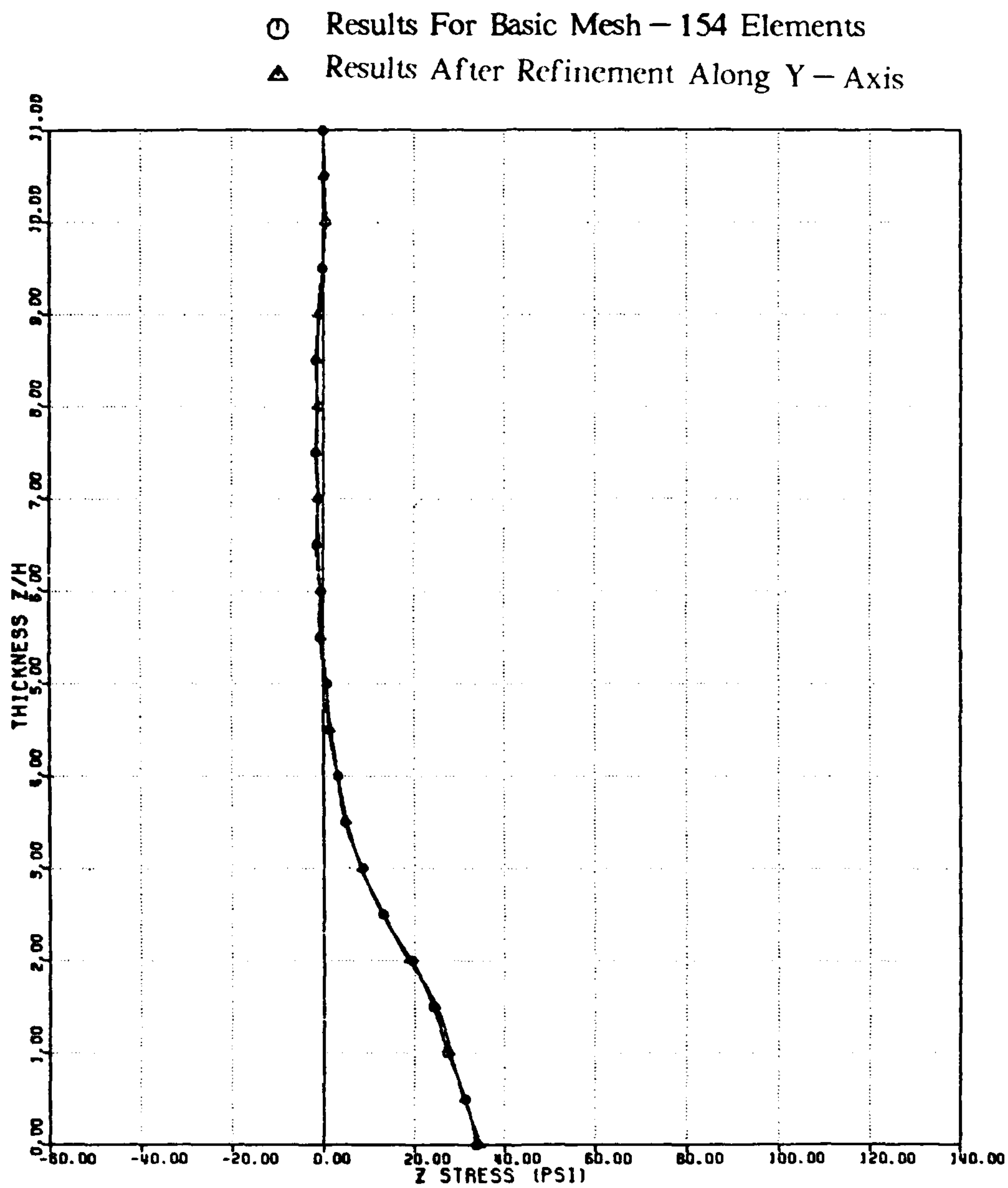


Figure 27: THROUGH-THE-THICKNESS Z-STRESS DISTRIBUTION AT Y/B = .99 FOR THE 22-LAYER FEDS. (REFINEMENT ALONG Y-AXIS ONLY).

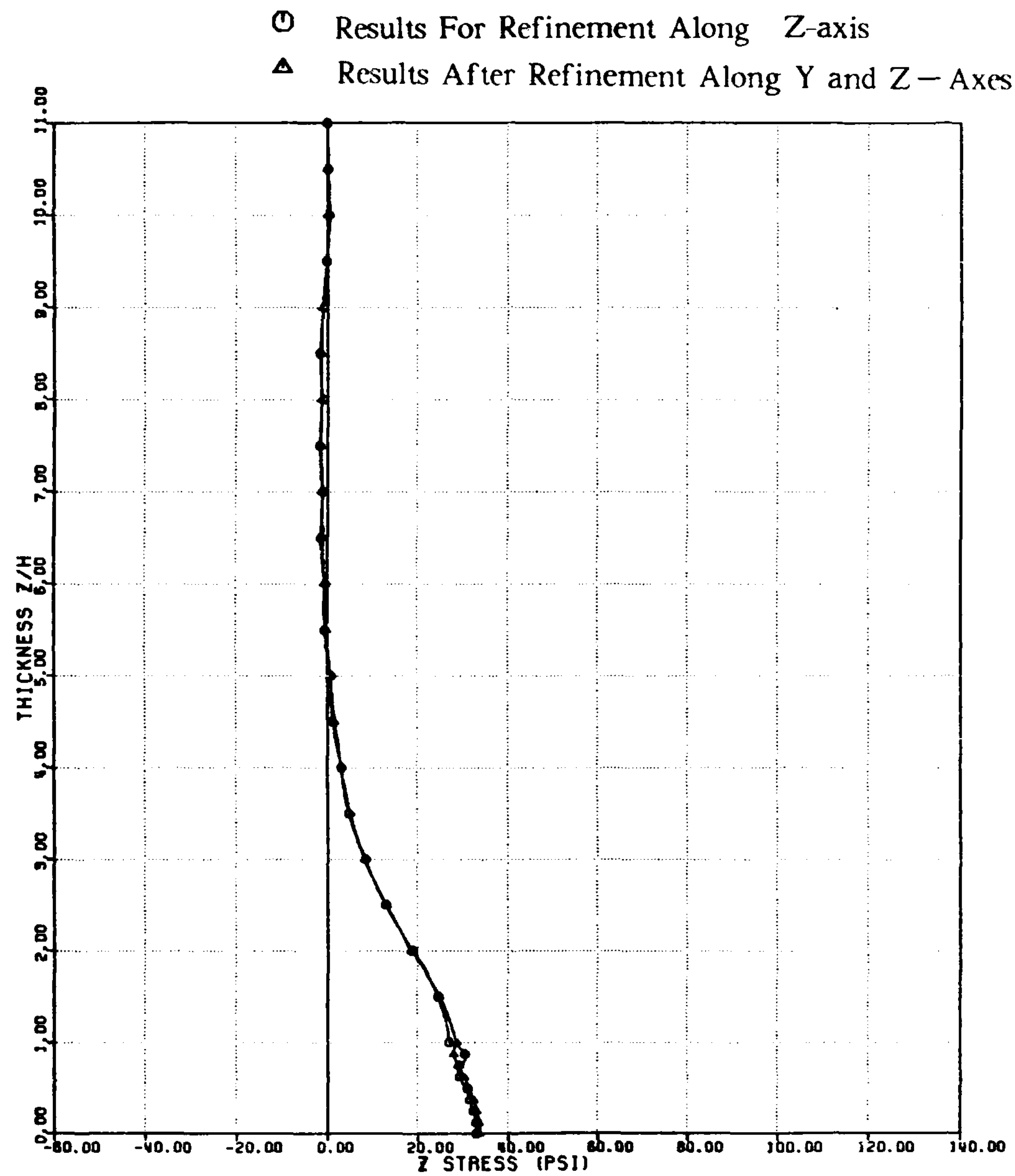


Figure 28: THROUGH-THE-THICKNESS Z-STRESS DISTRIBUTION AT Y/B = .99 FOR THE 22-LAYER FEDS. (REFINEMENT ALONG Y- AND Z-AXES)

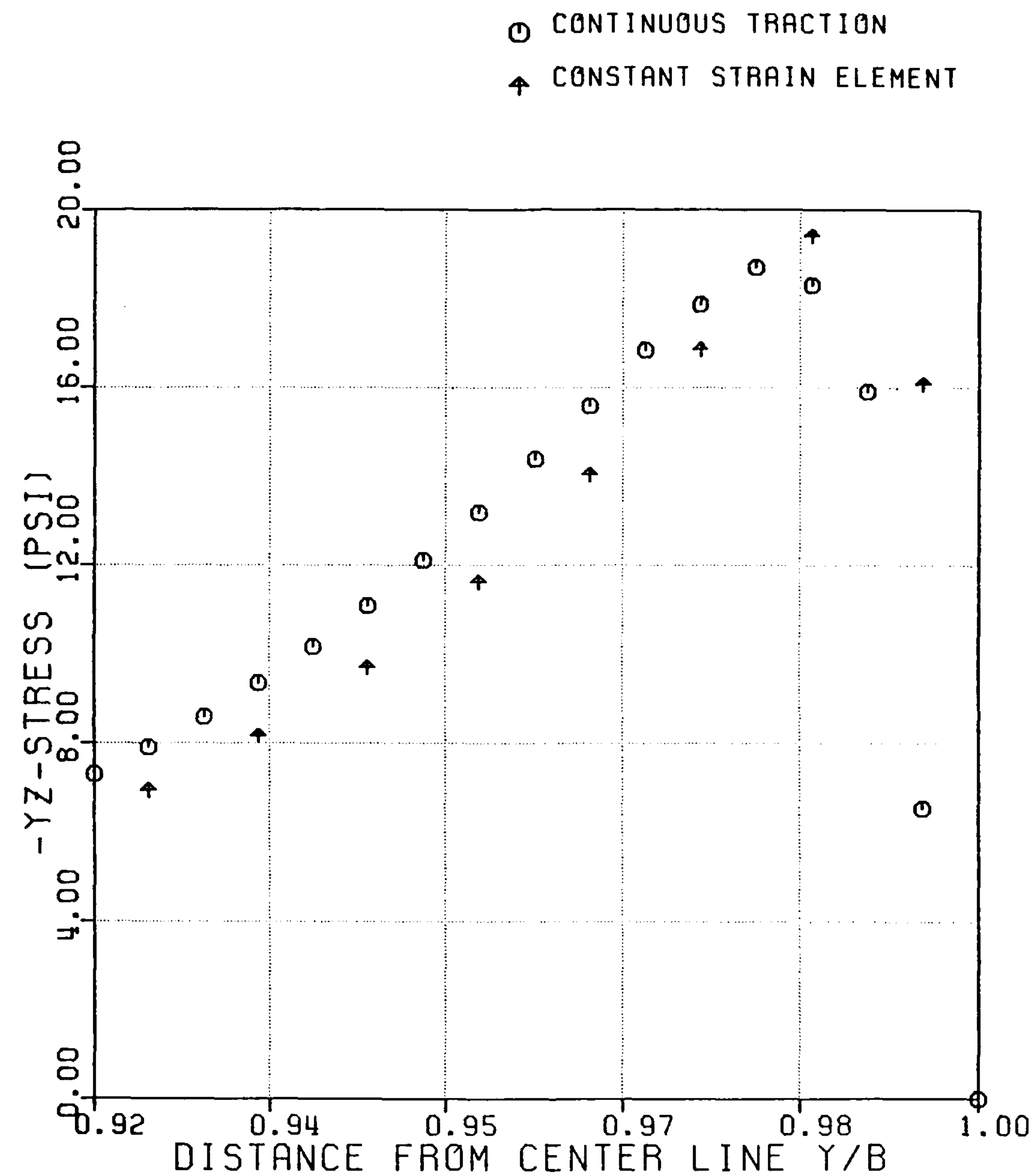


Figure 29: DISTRIBUTION OF YZ-STRESS NEAR THE FREE EDGE OF THE 22-LAYER SPECIMEN.

8.3.5.2 The Axisymmetric Model.

For the axisymmetric model, the modified four-point isoparametric element was used. The modification, introduced by Doherty et al. [1969] and Zienkiewicz et al. [1972] consists of evaluating the contribution of shearing strains to the energy using one-point Gauss quadrature. This amounts to the assumption of constant shearing strain in the element. Use of this "selective" reduced integration procedure has been found to give better estimates of stress in thin-walled structural systems. Comparison of this procedure with Sandhu and Singh's [1978] reduced integration scheme showed that the results from the two analyses were comparable but the "selective" scheme was the more economical in computational effort. To ensure uniformity of extensional strain over the delamination specimen, different values of the internal radius of the annulus were tried. It was found that using an internal radius exceeding 10^4 times the thickness of the specimen would give essentially uniform tangential strain in the annulus corresponding to uniform extension in the specimen.

Figure 30 shows the correspondence between the free-edge delamination specimen and a segment from a large-radius ring under radial pressure. A 576-element mesh shown in Figure 31 was used to model the delamination specimens analyzed by Pagano [1978]. It should be noted that, for the model used, the complete thickness of the laminate has to be modelled instead of the half-thickness used in the continuous traction element analysis described in the previous section. Comparison of results from the axisymmetric model and the continuous traction element analyses showed that the numerical results were in close agreement while the computational effort was much smaller for the axisymmetric model which uses a lower order element. Of course, for comparable accuracy, a finer mesh would, in general, be required for the simpler model. As an illustration, Figure 32 shows through-the-thickness distribution of σ_{zz} at

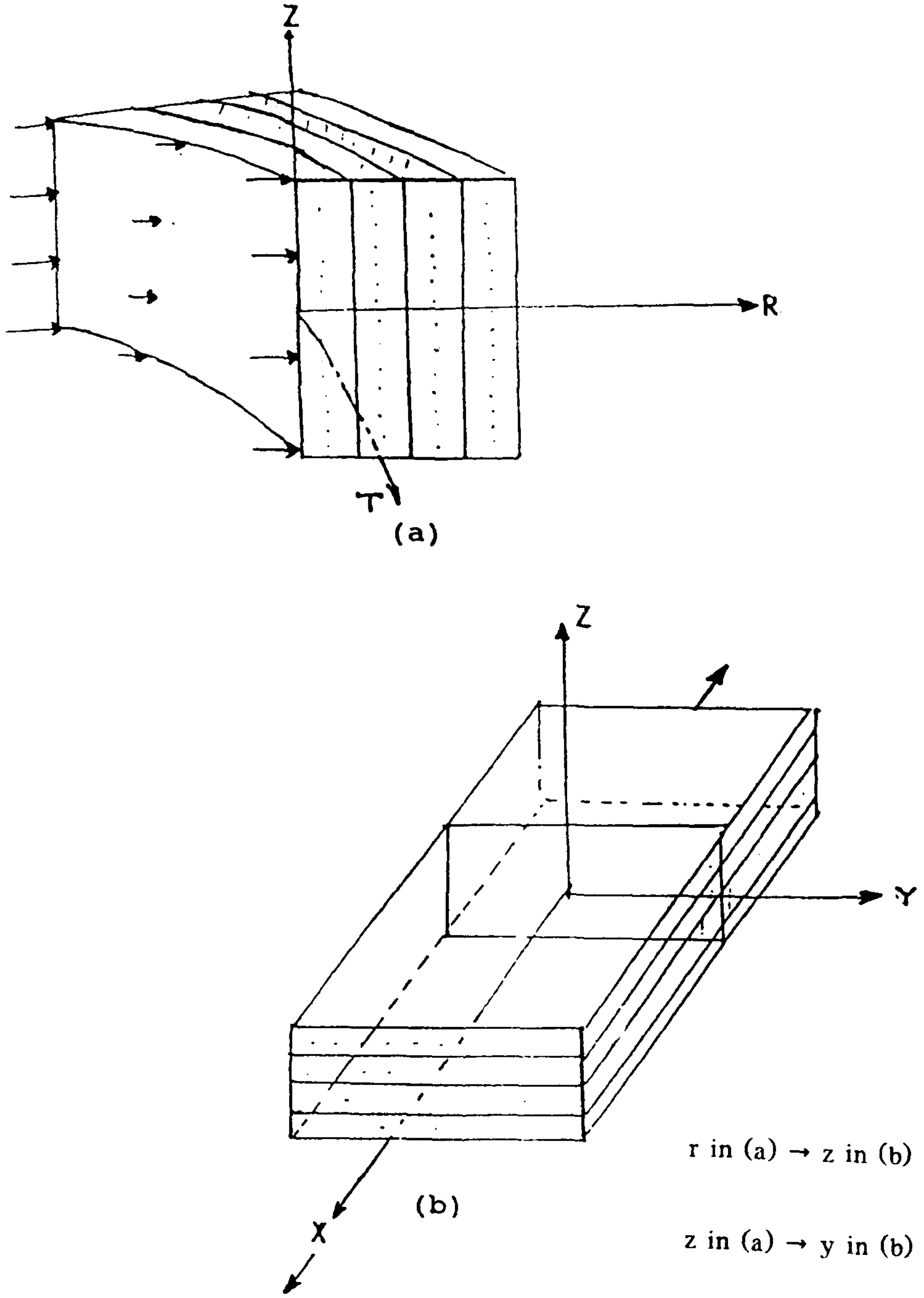
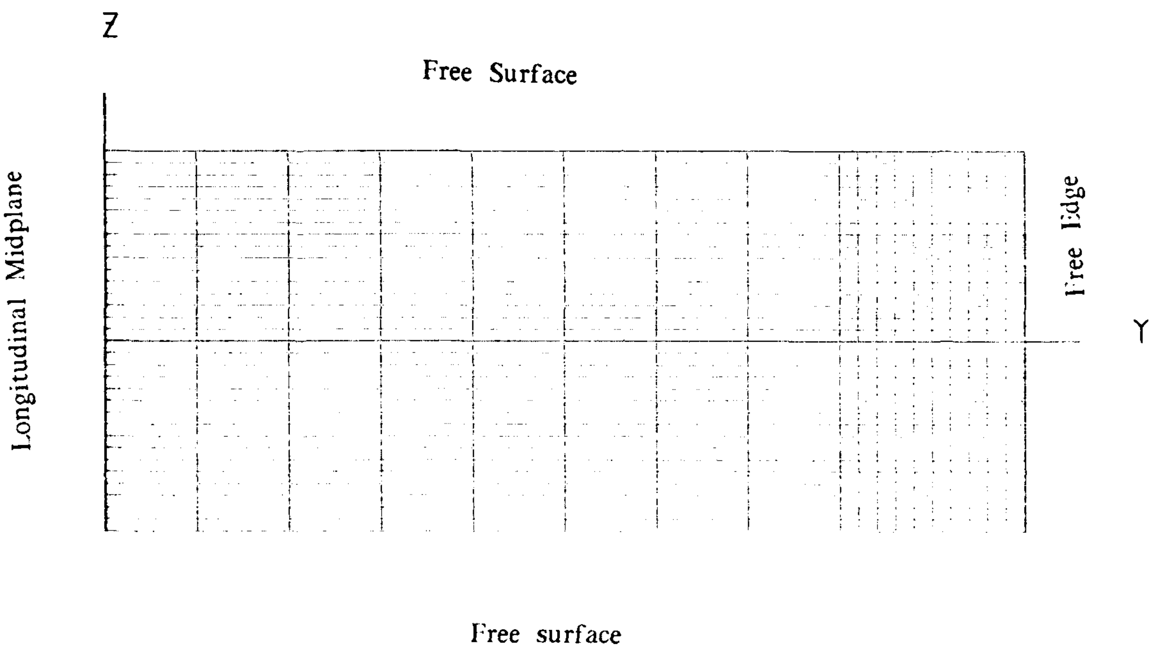


Figure 30: CORRESPONDENCE BETWEEN THE FREE-EDGE DELAMINATION SPECIMEN AND A SEGMENT OF AN ANNULUS.



576 (32x18) Finite Element Model For Axisymmetric Analysis.

Figure 31: FINITE ELEMENT MODEL OF THE AXISYMMETRIC PROBLEM.

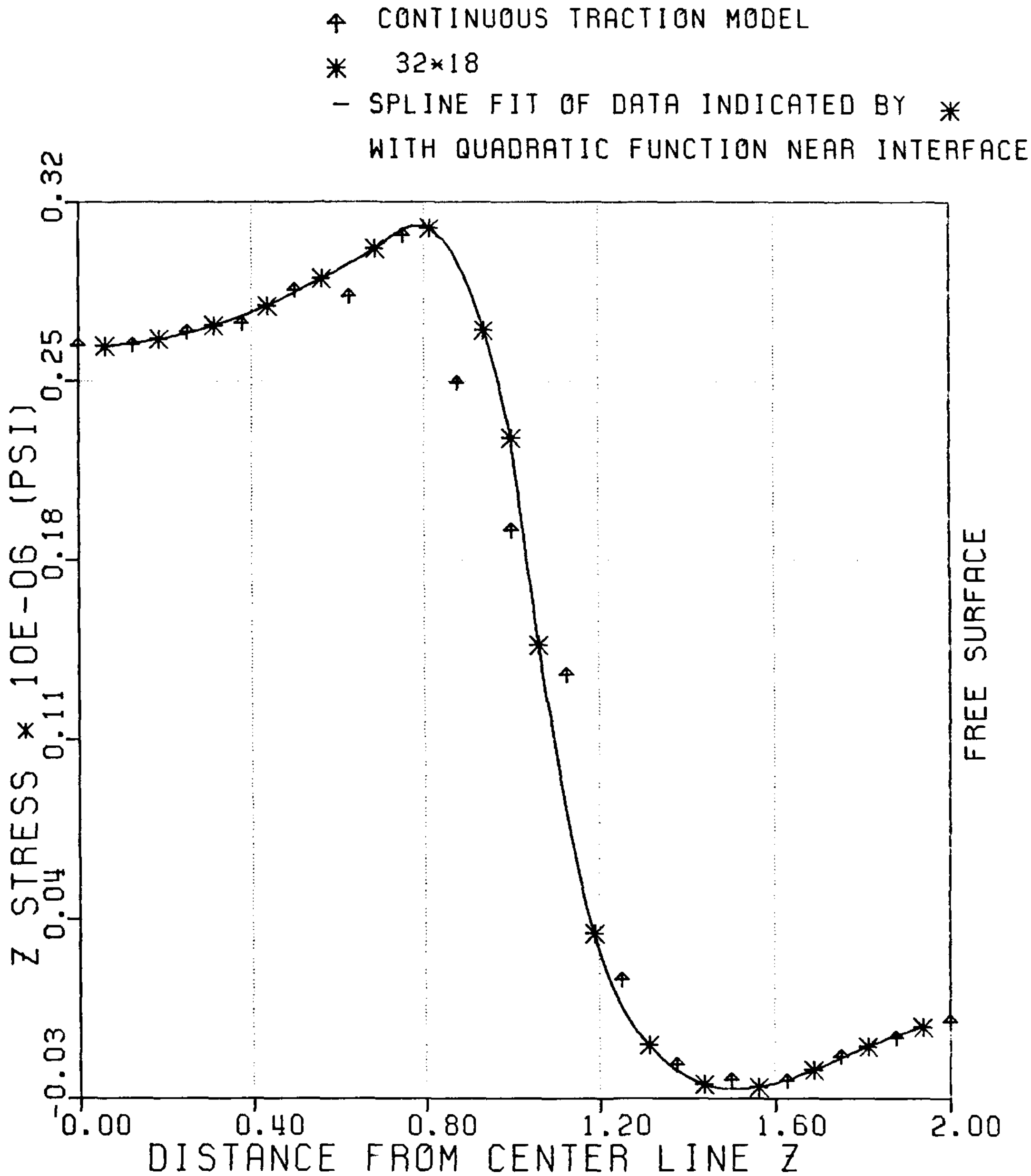


Figure 32: THROUGH-THE-THICKNESS Z-STRESS DISTRIBUTION AT Y/B = .99 FOR THE [0/90] SYMMETRIC LAMINATE.

$\frac{y}{B} = 0.99$ for a cross-ply specimen over half the thickness, using 8 equal subdivisions over the thickness of each layer and 18 elements over half the width as shown in Figure 31. Similar plots for the stress component σ_{yz} at $\frac{y}{B} = 0.99$ are shown in Figure 33. Figure 34 through Figure 36 show the through-the-thickness variation in stress components σ_{xz}, σ_{zz}, and σ_{yz}, respectively, for the four-layer $[\pm 45]_s$ specimen at the plane $\frac{y}{B} = 0.99$ The results shown were obtained using 1152 elements for the axisymmetric model (complete cross-section) and 166 elements for the continuous traction model.

The procedure was applied to a 22-layer delamination specimen for which test data were available. Several mesh refinements were used. Figure 37 shows the finite element mesh used to get the plots - Figure 38 and Figure 39 - showing through-the-thickness distribution of σ_{zz} and σ_{yz} at $\frac{y}{B} = 0.995$ for a $[(\pm 25)_5/90]_s$ delamination specimen for the axisymmetric model as well as the continuous traction model discussed earlier.

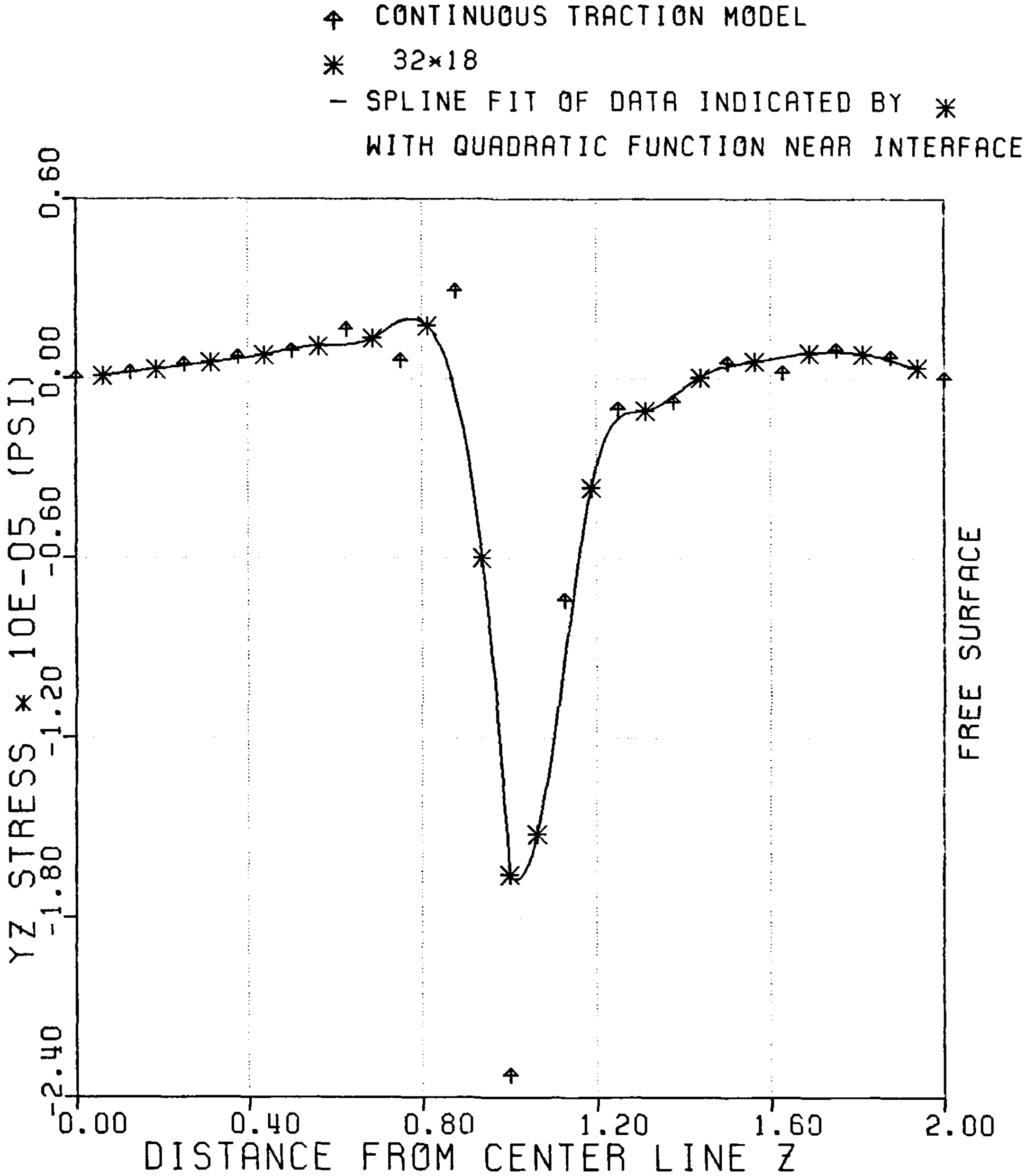


Figure 33: THROUGH-THE-THICKNESS YZ-STRESS DISTRIBUTION AT Y/B = .99 FOR THE [0/90] SYMMETRIC LAMINATE.

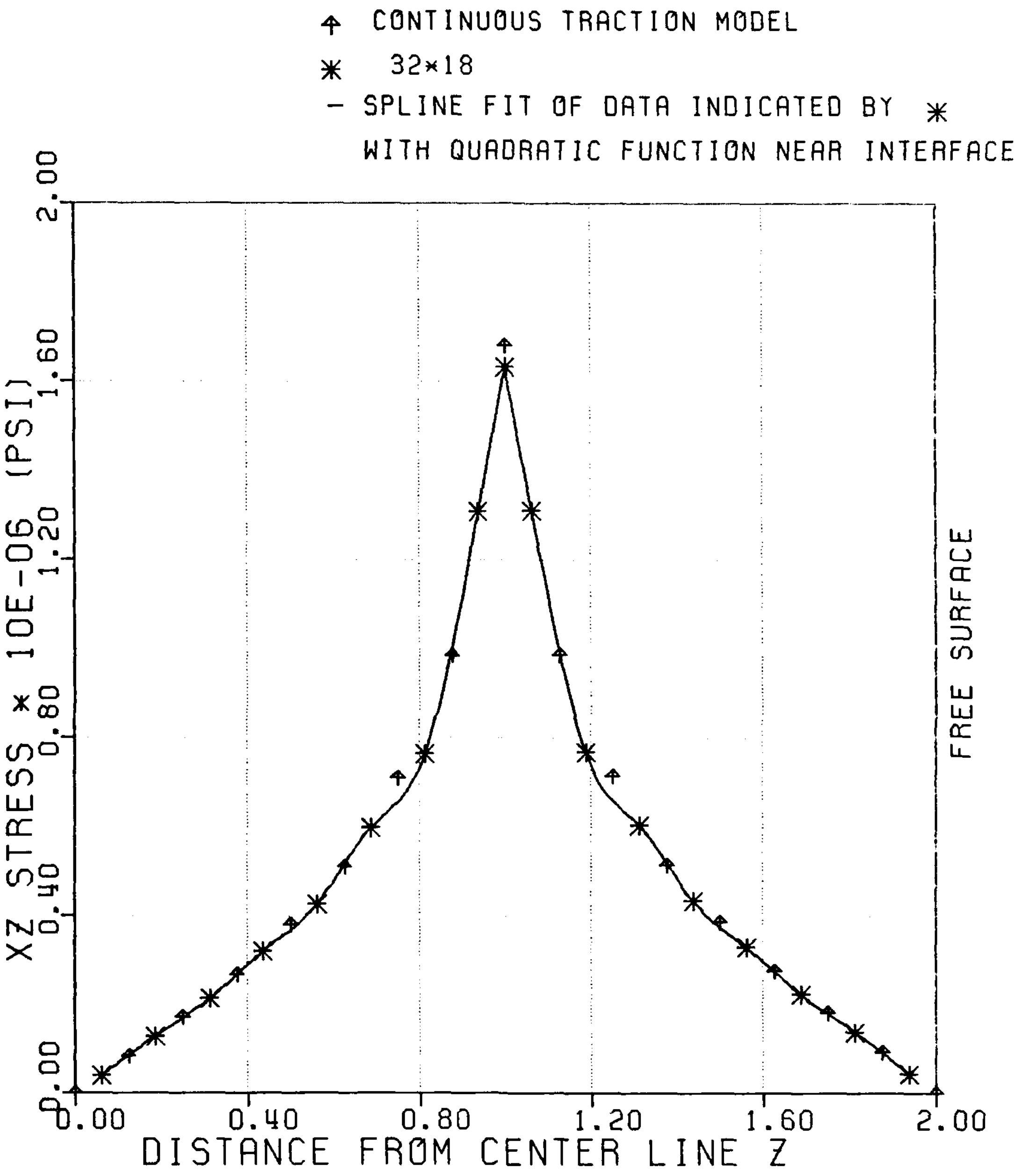


Figure 34: THROUGH-THE-THICKNESS DISTRIBUTION OF XZ-STRESS AT Y/B = .99 FOR THE [45/-45] SYMMETRIC LAMINATE.

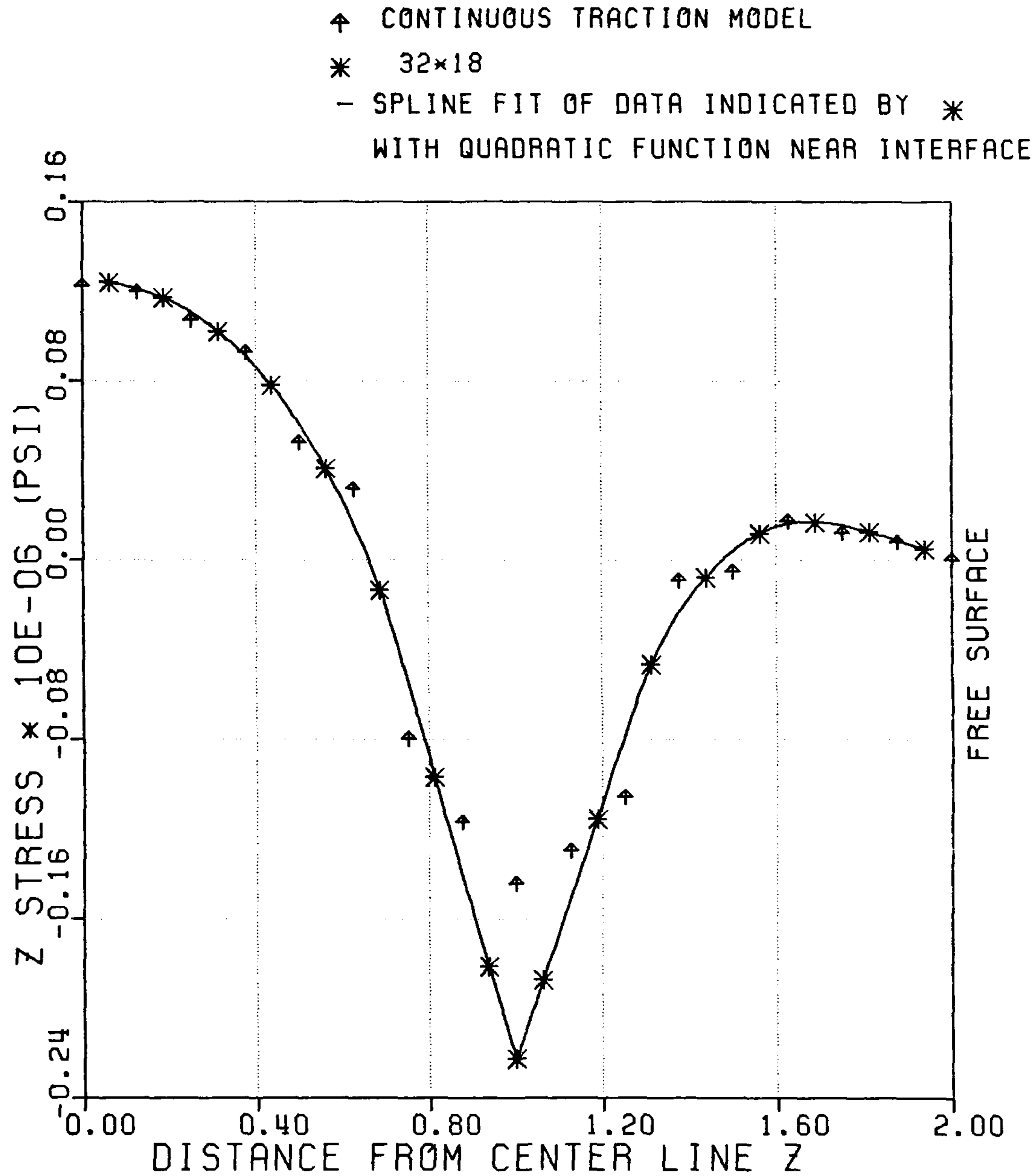


Figure 35: THROUGH-THE-THICKNESS DISTRIBUTION OF Z-STRESS AT Y/B = .99 FOR THE [45/-45] SYMMETRIC LAMINATE.

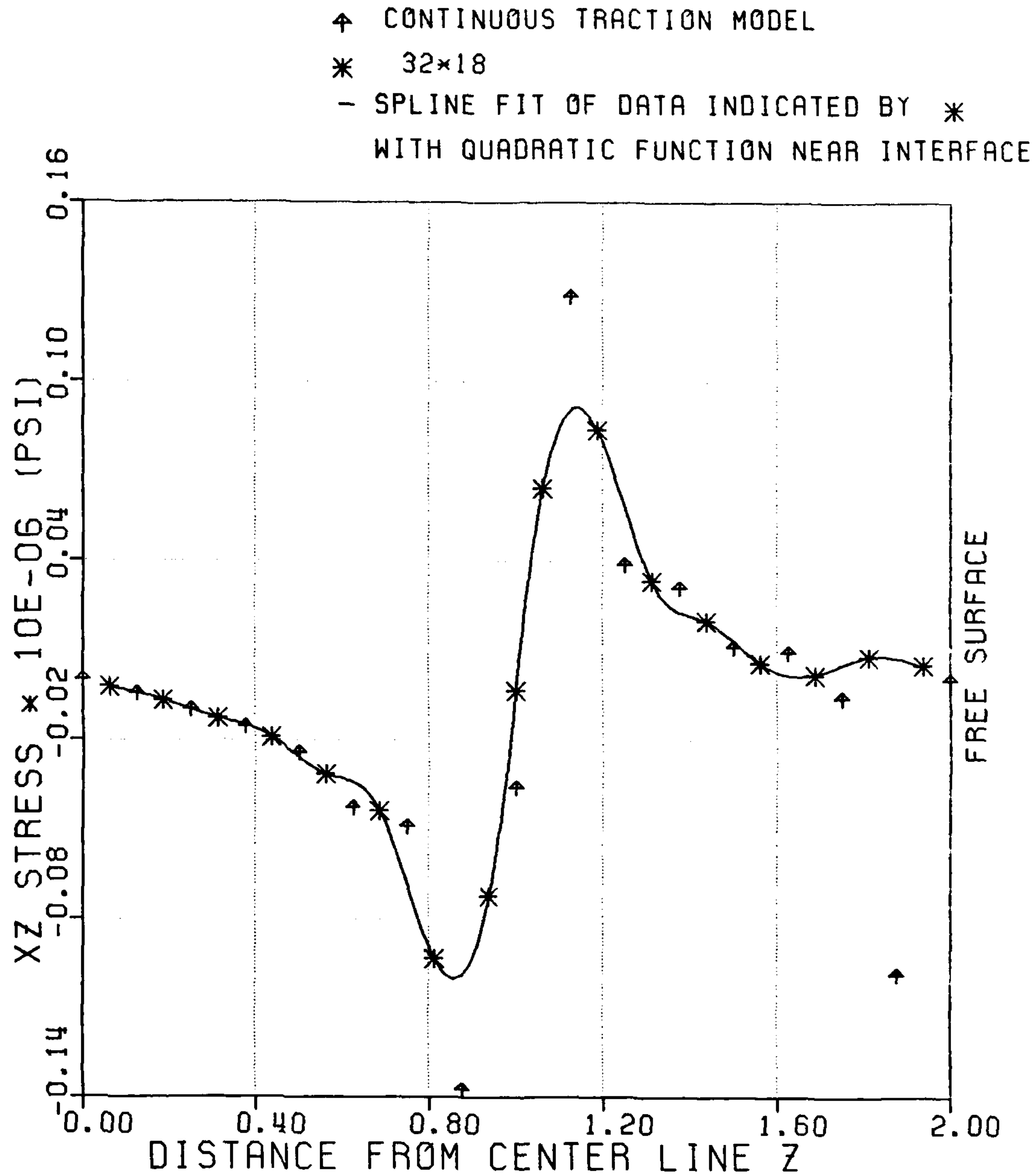
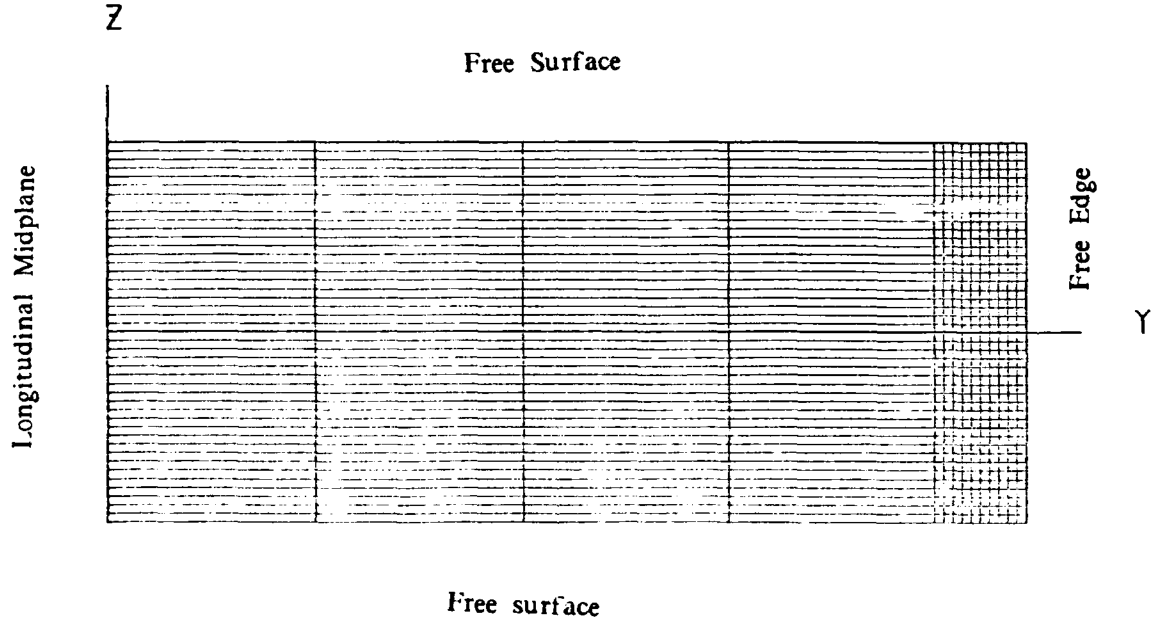


Figure 36: THROUGH-THE-THICKNESS DISTRIBUTION OF YZ-STRESS AT Y/B = .99 FOR THE [45/-45] SYMMETRIC LAMINATE.



616 (44x14) Finite Element Model For Axisymmetric Analysis.

Figure 37: FINITE ELEMENT MODEL OF 22-LAYER DELAMINATION SPECIMEN.

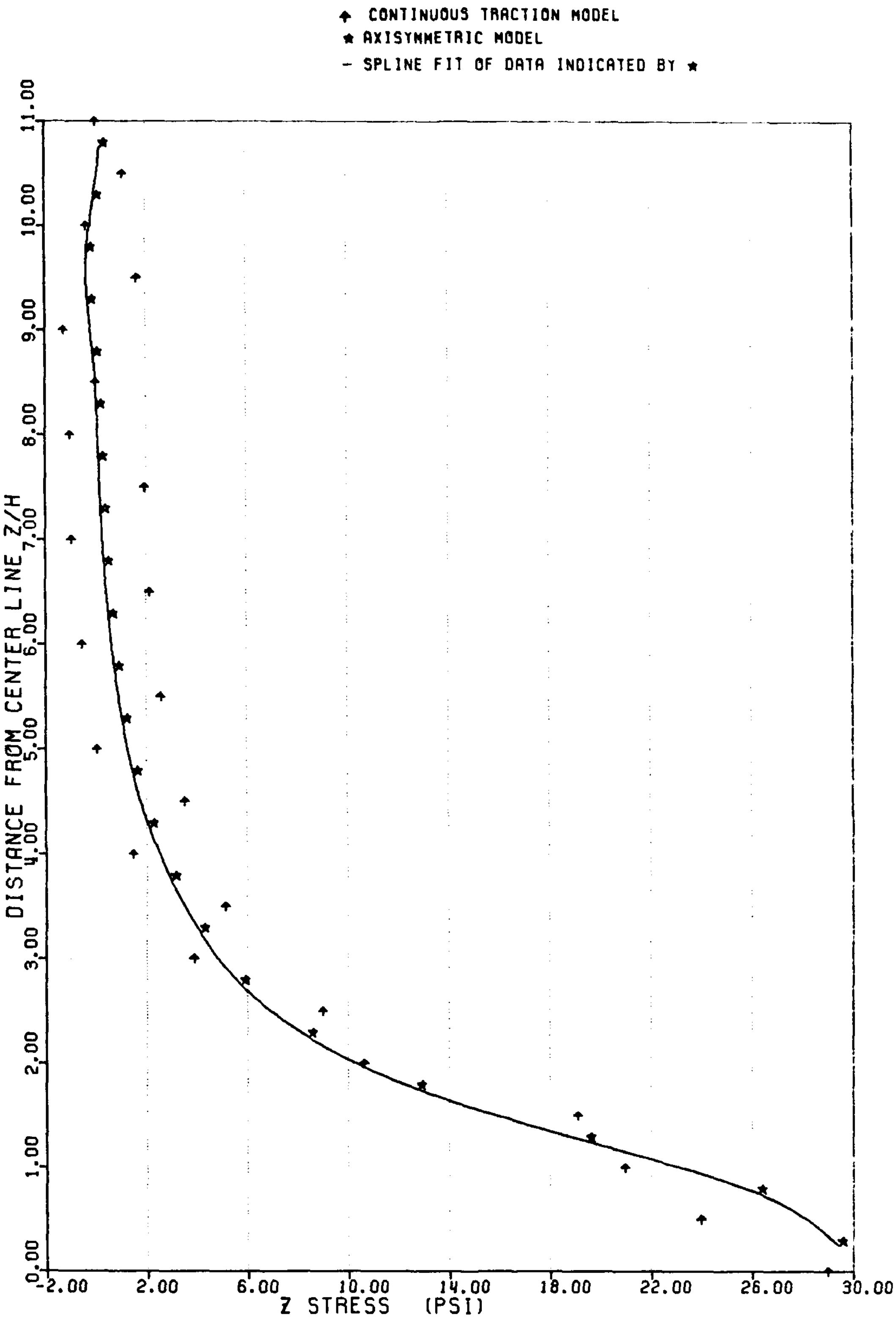


Figure 38: THROUGH-THE-THICKNESS DISTRIBUTION OF Z-STRESS AT Y/B = .995 OF THE 22-LAYER FEDS.

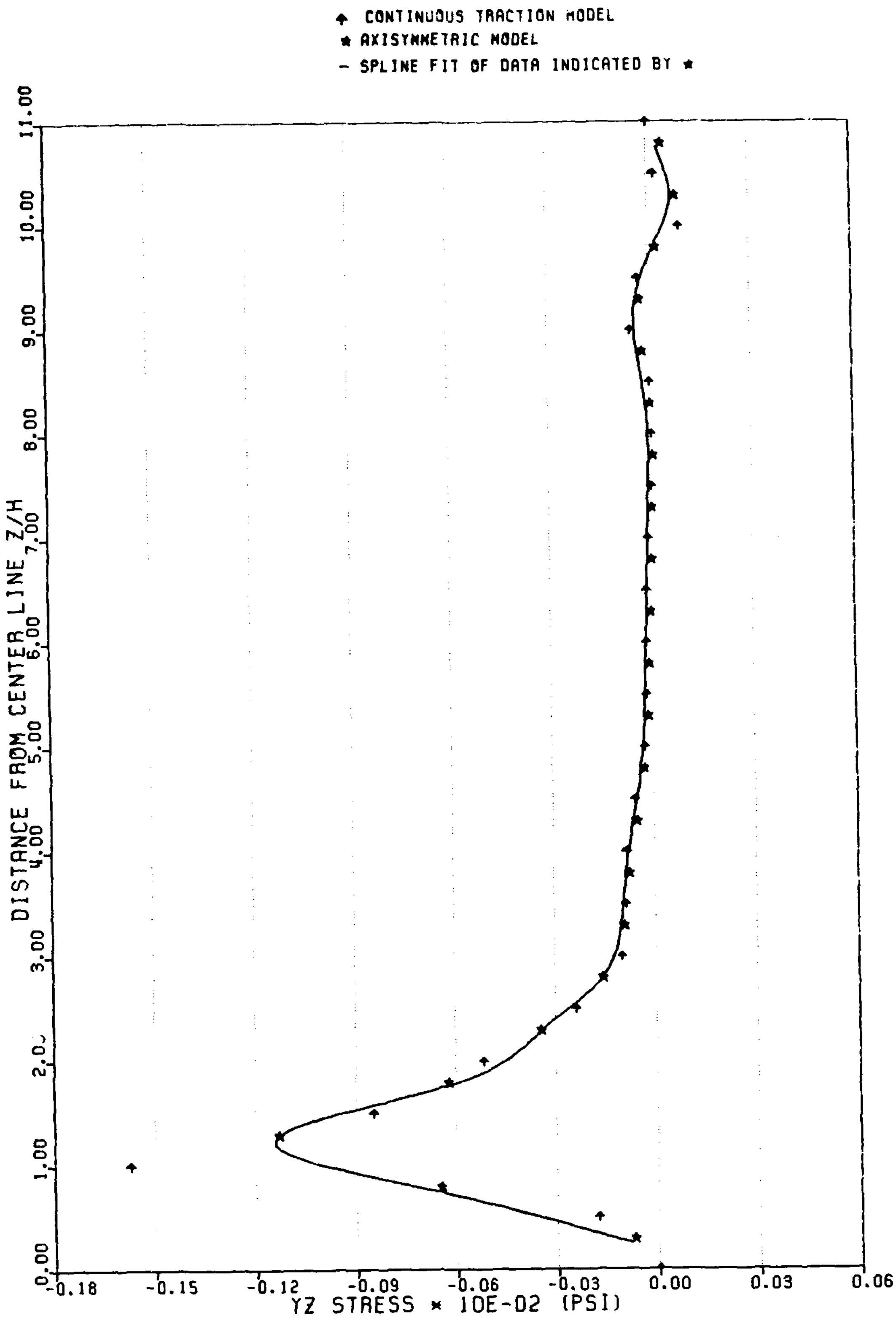


Figure 39: THROUGH-THE-THICKNESS DISTRIBUTION OF YZ-STRESS AT Y/B = .995 OF THE 22-LAYER FEDS.

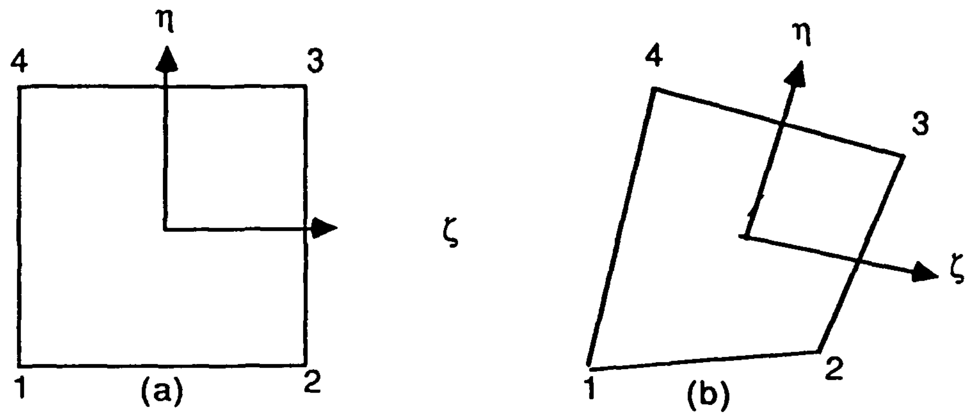
8.4 ANALYSES BASED ON LAMINATED PLATE THEORIES.

8.4.1 Introduction.

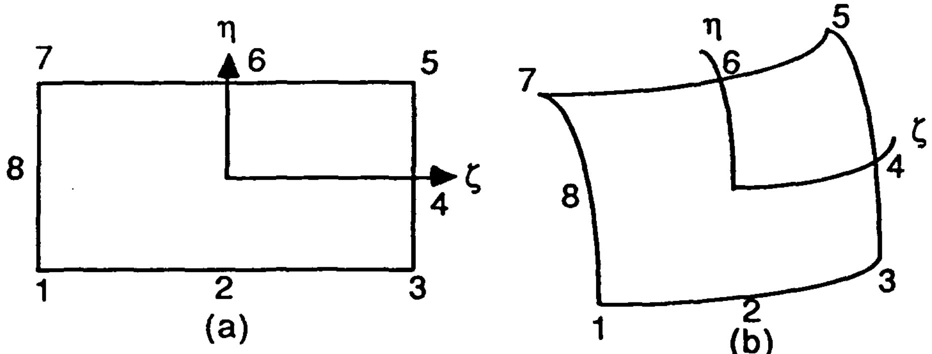
A more general approach to the problem would be applicable not only to free-edge delamination specimens but also to plates of general configuration. For the present research effort, this involved implementation of appropriate theories of bending and stretching of laminated composite plates. To start, the existing layerwise theory of laminated plates was implemented in a finite element computer program. Limitations of this theory were immediately noticed, and subsequent research effort was directed towards development of comprehensive theoretical basis for the constitutive and variational theory of laminated composite plates and its implementation in finite element computer models. The theory has been described in Sections IV through VII. In this section we summarize the finite element implementation and list some results.

8.4.2 Discrete Laminate Theory.

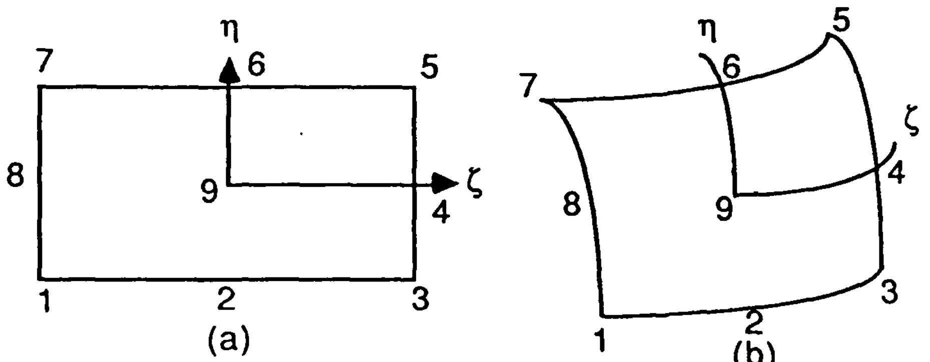
Mawenya and Davis [1974] proposed a general finite element formulation using quadratic isoparametric multilayer plate elements. However, they did not give details of their theoretical and numerical formulation. In the present research program, the existing theory of laminated plates was restated in a self-adjoint form and implemented in a finite element computer program. The program is capable of using the bilinear isoparametric interpolation as well as biquadratic Lagrangian or 8-node serendipity interpolation. Details of this investigation are given in item B.1.1 of Appendix B. Figure 40 shows the three element types available in the program. Application to sandwich as well as homogeneous plates showed very good performance in modelling deflections. However, results for σ_{zz}, σ_{xz}, and σ_{yz} were quite inaccurate near the free edge. It was found that refinement of mesh over the thickness resulted in improvement in stress predictions while refinement in the y-direction did not have a



1. Element Q4: (a)Rectangular parent (b)Isoparametric counterpart



2. Element Q8: (a)Rectangular parent (b)Isoparametric counterpart



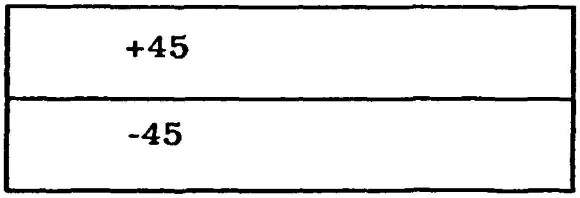
3. Element Q9: (a)Rectangular parent (b)Isoparametric counterpart

Figure 40: FINITE ELEMENT INTERPOLATION SCHEMES USED FOR DISCRETE LAMINATE THEORY.

significant effect. This was specially noticeable near the free edge. Figure 41 shows the sequence of refinement of the model along the z-axis i.e., over the thickness of the specimen. Figure 42 and Figure 43 show influence of refinement upon σ_{xz} and σ_{yz} along the interface of a $[\pm 45]_s$ specimen.

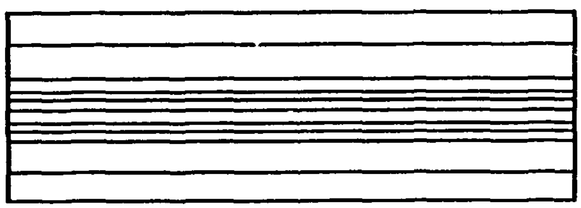
8.4.3 A First Order Theory.

The first order theory for constitutive relations for force resultants in laminated plates described in Section IV was written in a self-adjoint form, appropriate variational formulations were developed and a specialization was implemented in finite element computer program. Items B.1.2, B.1.3 B.2.3, B.2.4, B.2.6, B.3.3 and B.4.2 give details of the investigation. The finite element implementation used Hughes' [1978] Heterosis element shown in Figure 44. The traction-free boundary condition could not be explicitly satisfied because the interlaminar tractions had been condensed out of the constitutive equations. This condition could be introduced as specified tractions in the constitutive theory. This would make the constitutive relations different at the surface and in the interior necessitating the assumption of a pattern of variation from the free edge towards the inside of the specimen. Numerical results from the analyses showed that the influence of the shear coupling indicated by the theoretical investigation was not significant. This was probably because the theory is first order, based on the assumption of $e_{33} = 0$, and ignores the coupling between the other force resultants. The numerical results practically coincided with those from the discrete laminate theory which did not allow for the constitutive coupling. It was noticed that refinement of each layer into sublayers resulted in noticeable improvement in the solution. Figure 45 and Figure 46 show the results for subdivision of each of the two layers in one-half of a $[\pm 45]_s$ free-edge delamination specimen into three and five sublayers. This remarkable improvement with refinement was the motivation for the search for a higher order theory.



(a) Two Sub-layers (N=2)

(b) Six Sub-layers (N=6)



(c) Ten Sub-layers (N=10)

Figure 41: SEQUENCE OF REFINEMENTS OVER THE THICKNESS OF ONE-HALF OF THE [45/-45] SYMMETRIC SPECIMEN.

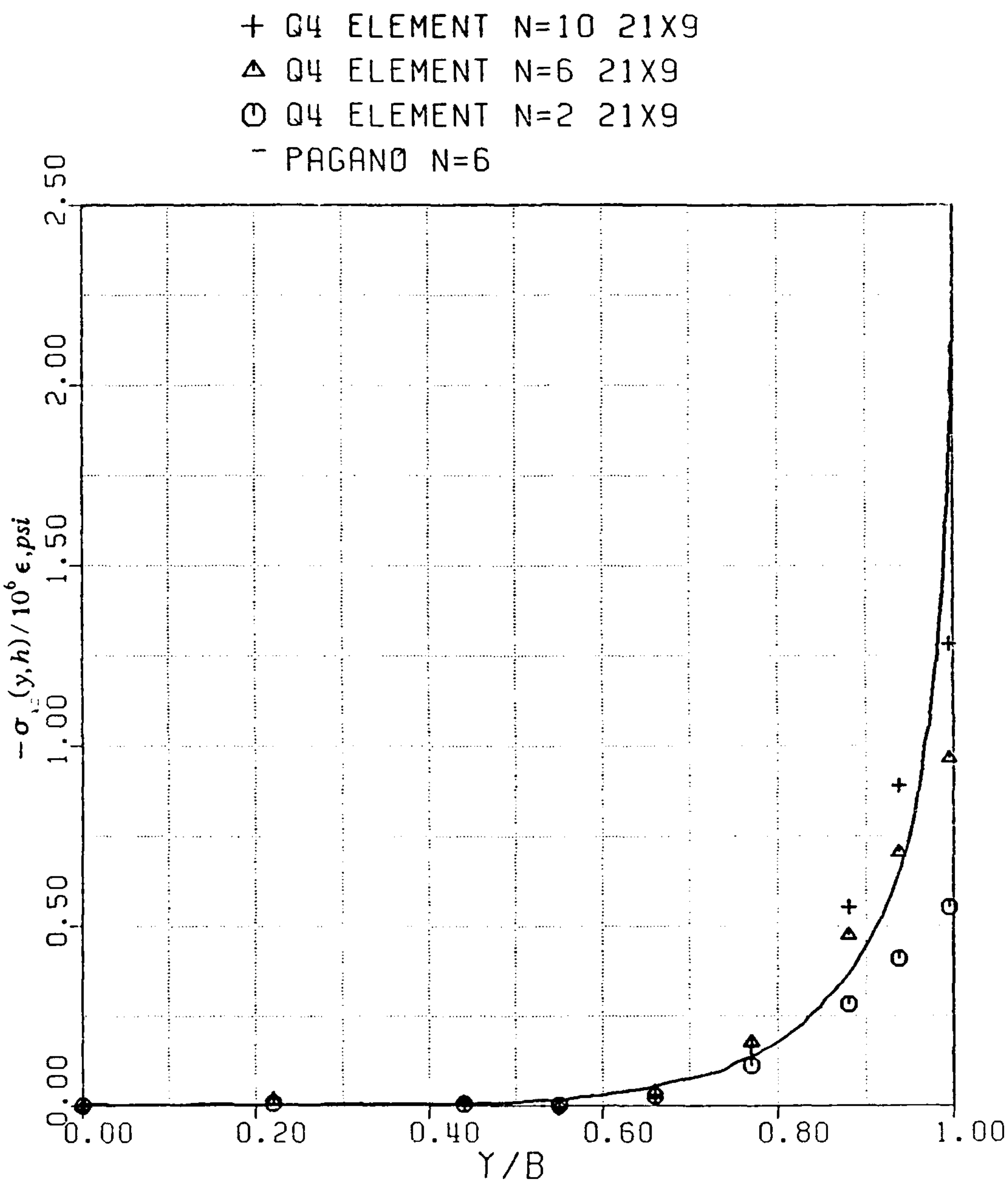


Figure 42: DISTRIBUTION OF XZ-STRESS AT INTERFACE - INFLUENCE OF REFINEMENT OVER THE THICKNESS.

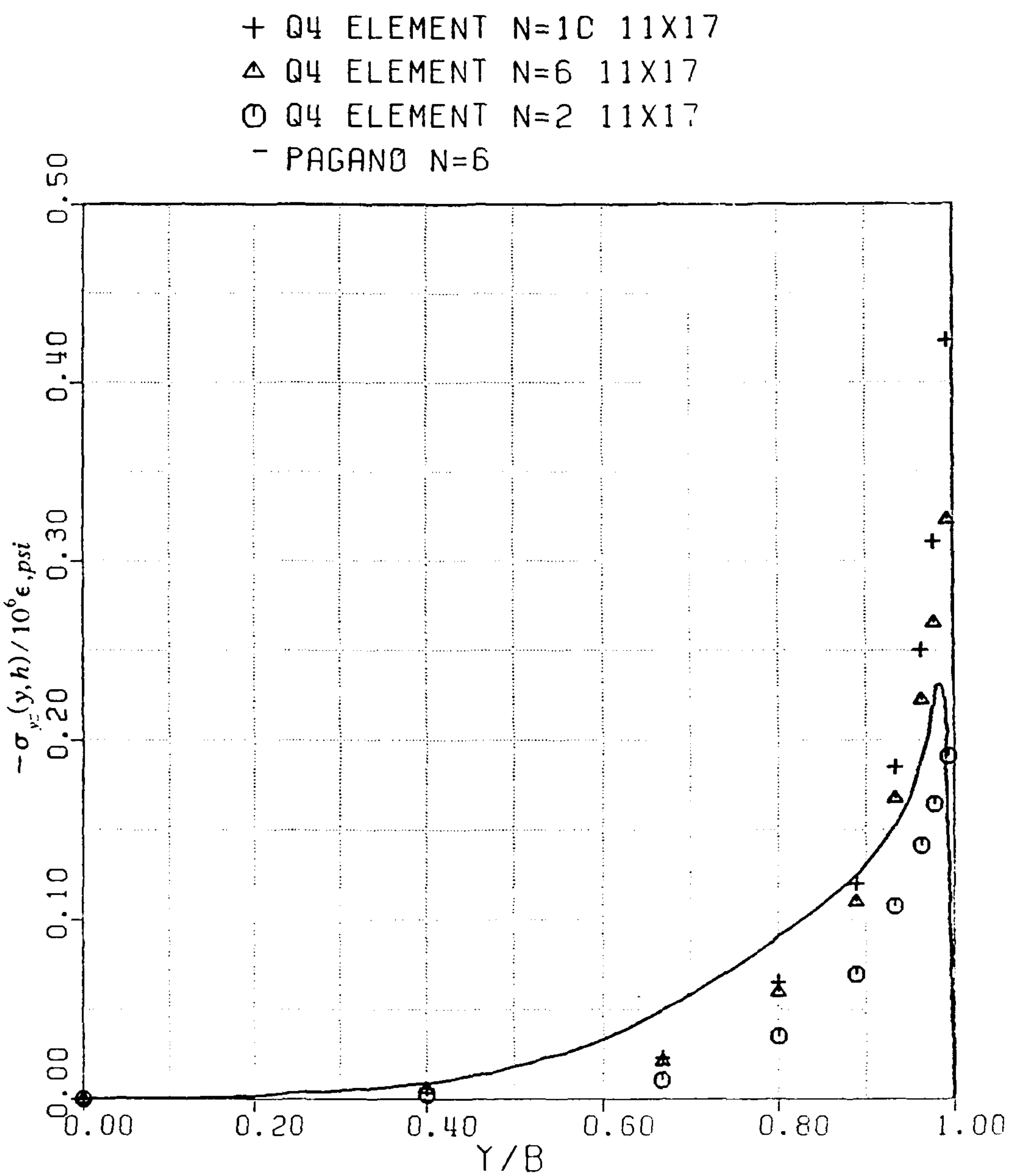
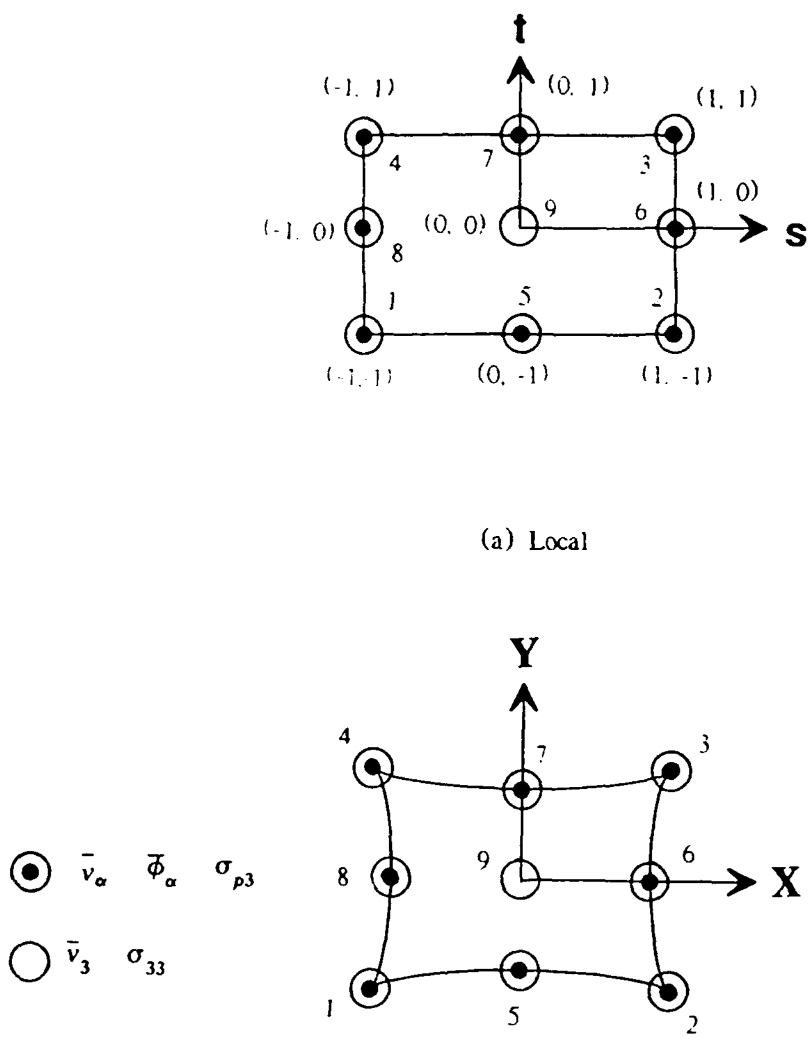


Figure 43: DISTRIBUTION OF YZ-STRESS AT INTERFACE OF THE [45/-45] SPECIMEN - INFLUENCE OF REFINEMENT OVER THE THICKNESS.



(a) Local

(b) Global

Figure 44: THE HETEROSIS ELEMENT.

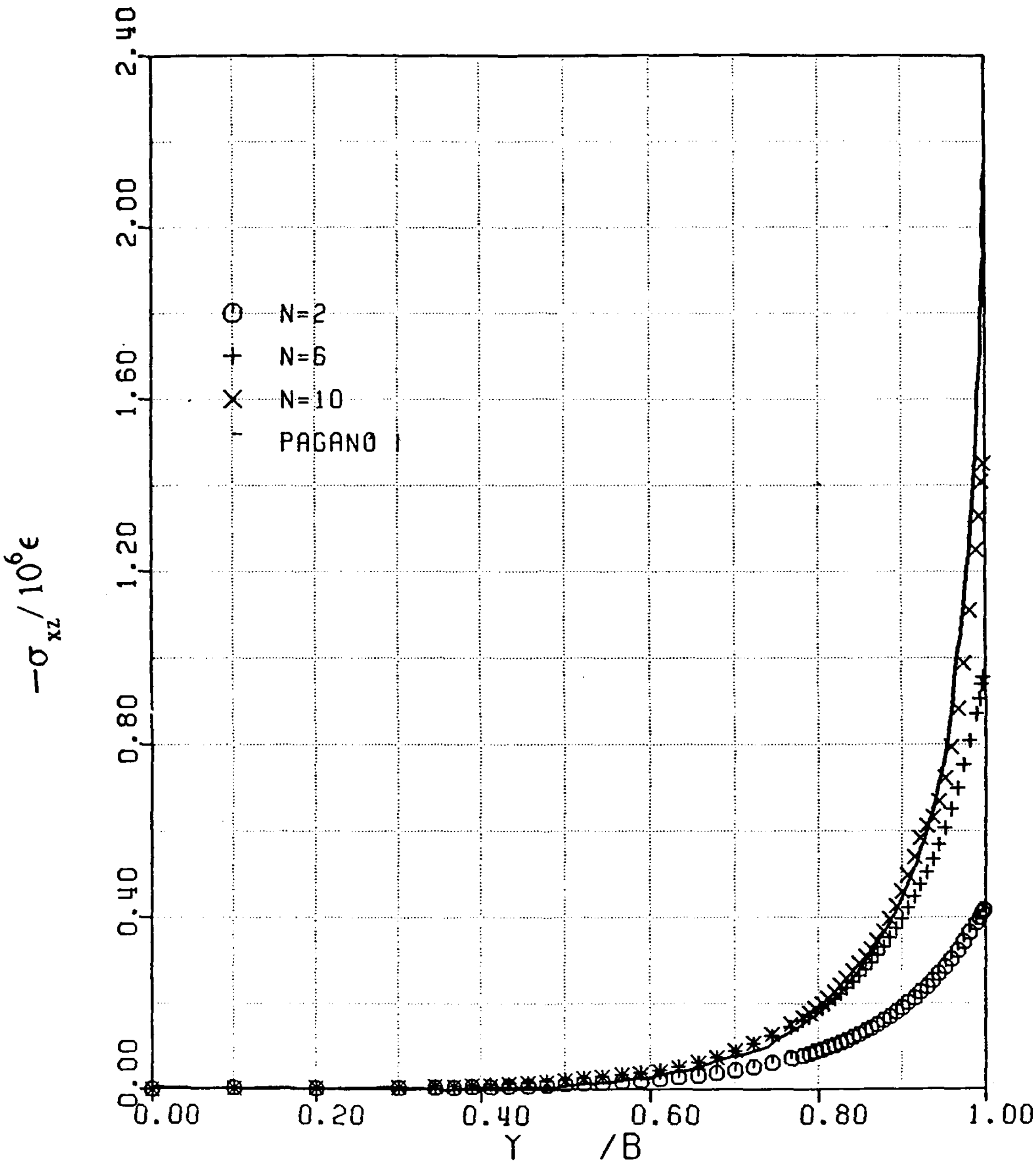


Figure 45: DISTRIBUTION OF XZ-STRESS AT INTERFACE - INFLUENCE OF REFINEMENT OVER THE THICKNESS.

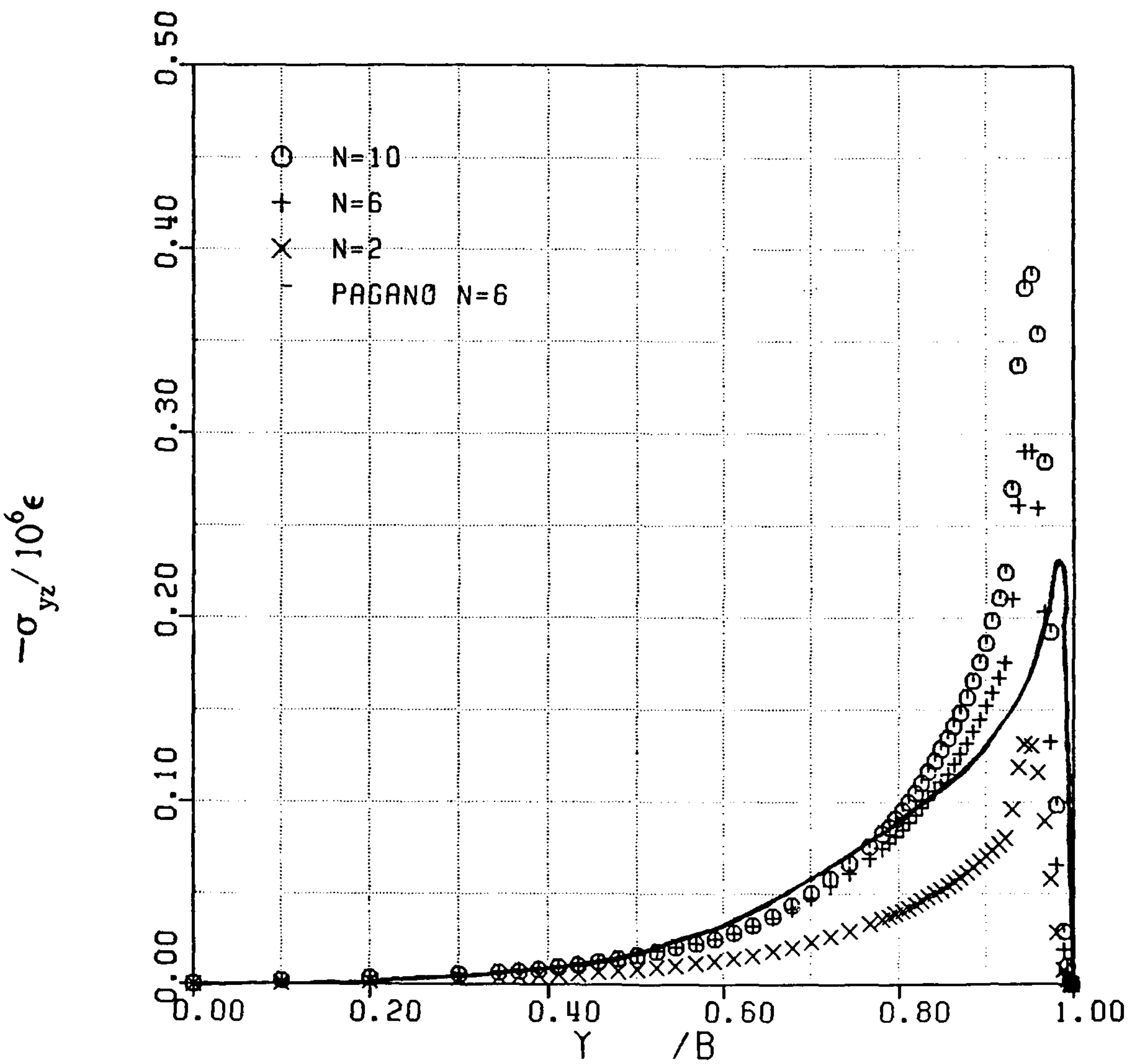


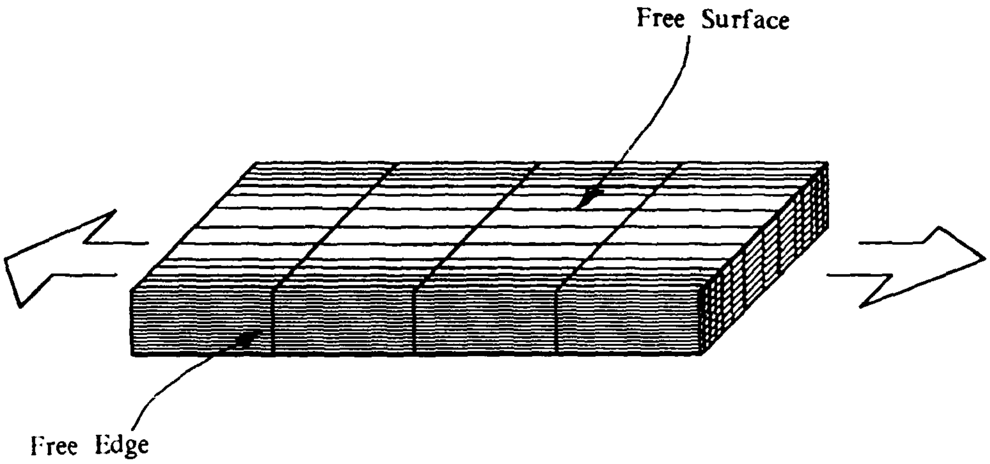
Figure 46: DISTRIBUTION OF YZ-STRESS AT INTERFACE OF THE [45/-45] SPECIMEN - INFLUENCE OF REFINEMENT OVER THE THICKNESS.

8.4.4 A Higher Order Theory.

Investigation of the existing discrete laminate theory and the first order theory showed that refinement over the thickness of the laminate resulted in improvement of accuracy of the predicted stresses. This was the motivation for the development of higher order theories described in Section VI. A specialization of the higher order theory to the free-edge delamination specimens was implemented in a finite element computer program using cubic Hermite polynomial interpolation for all the field variables. The solution showed rapid convergence with refinement of the mesh. Even very coarse mesh (six elements over the entire width of the element) gave reasonable results for four-layer cross-ply and $[\pm 45]_s$ angle-ply specimens.

8.4.5 Pagano's Theory.

Pagano's theory as restated in Section VI was implemented in a finite element computer program using Heterosis elements. Application to Pagano's problems [1978] of four-layer cross-ply and angle-ply specimens showed excellent agreement. The procedure was then applied to a 22-layer free-edge delamination specimen $[(\pm 25.5)_5/90]_s$. For a rather coarse 64-element model shown in Figure 47, the results for σ_{zz} along the midplane and along the free edge are plotted in Figure 48 and Figure 49. The results agree very well with those from the axisymmteric model described earlier.



256 (4x8x8) F. E. Mesh - Pagano's Theory

Figure 47: FINITE ELEMENT MODEL OF THE 22-LAYER DELAMINATION SPECIMEN.

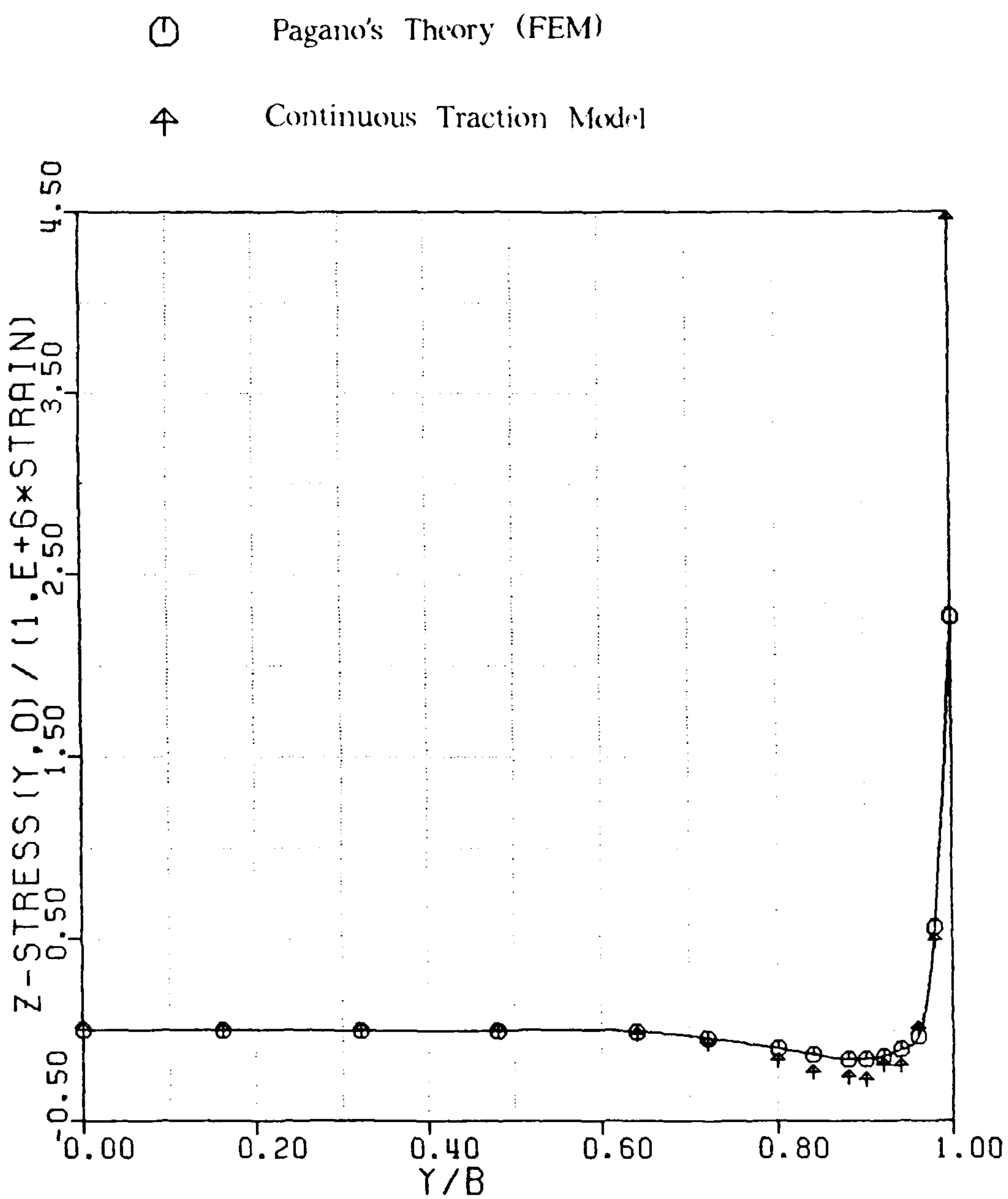


Figure 48: DISTRIBUTION OF Z-STRESS ALONG MIDPLANE OF THE 22-LAYER SPECIMEN.

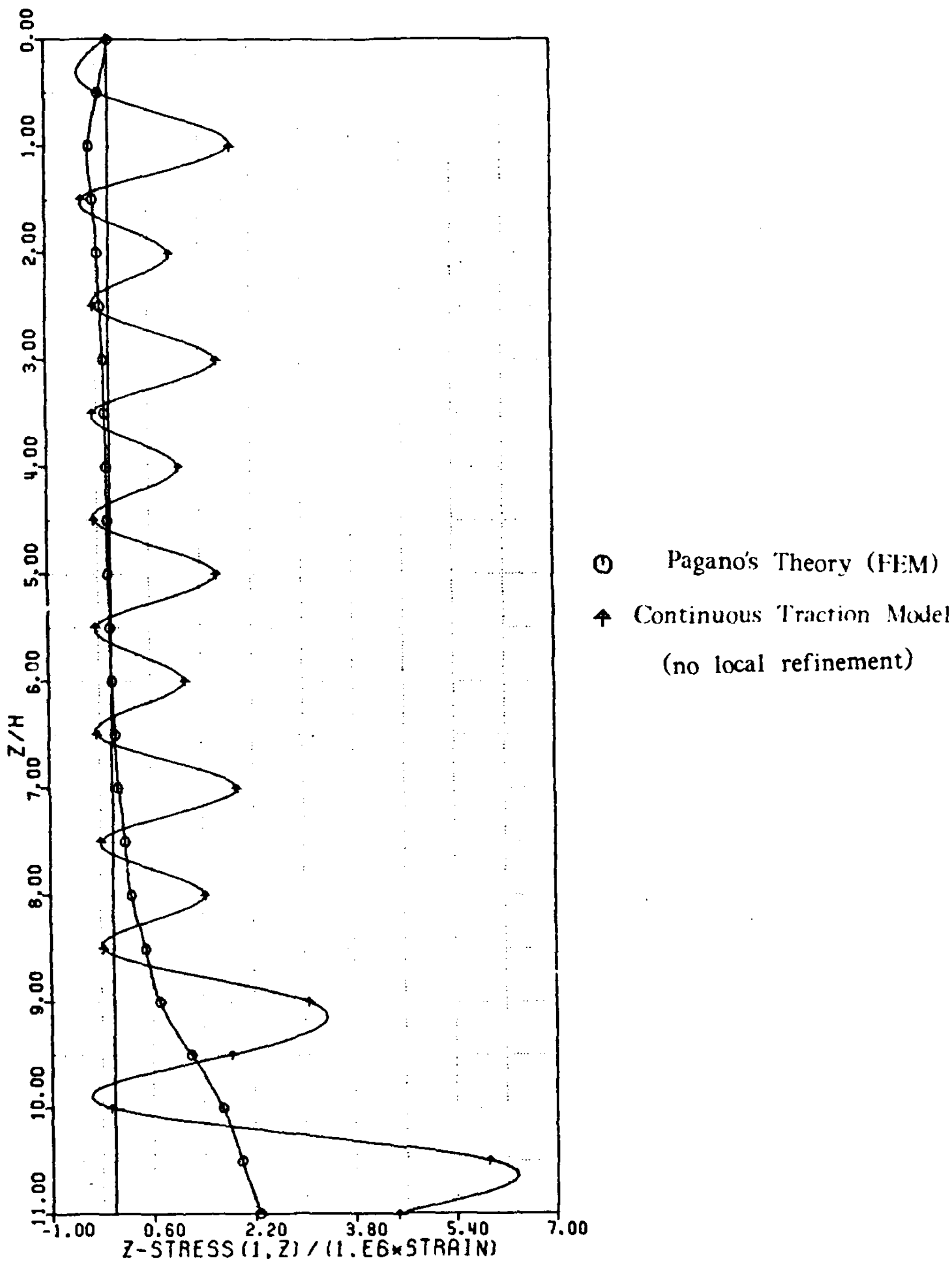


Figure 49: DISTRIBUTION OF Z-STRESS ALONG THE FREE EDGE OF THE 22-LAYER SPECIMEN..

SECTION IX
DISCUSSION

For analysis of stresses and deformations in a free-edge delamination specimen, the continuous traction cubic interpolation finite element, with appropriate refinement of the mesh in the vicinity of the free edge and interfaces, will yield reliable predictions. The representation by a segment of an annulus does not give continuous tractions but is much more economical in computational effort required to obtain the solution. Even the pseudo-two-dimensional representation ignoring σ_{xy} and σ_{xz} gives reasonable accuracy except near the free edge.

Existing theories of laminated plates, including the discrete laminate theory, cannot allow for the traction-free edge conditions. Thus, they are not useful for delamination prediction even though they are adequate for representation of gross behavior like deflection under load or frequency of vibration. Coupled constitutive theory, allowing for the shear coupling between layers, is mechanically consistent and independent of the need for ad hoc correction factors. However, it too cannot properly model the stresses in the vicinity of the free edge and the interfaces between plies. The higher order theory, developed during this reasearch program, based on quadratic variation of $\sigma_{\alpha\beta}$ over the thickness of each layer or sublayer along with use of consistent constitutive relationships, gave reasonable accuracy even for coarse meshes (six elements over the entire width of the delamination specimen). The computer program has been written for a specialization of the theory to free-edge delamination specimens. To solve the problem of an arbitrary plate configuration involving free edges, further work on development of a comprehensive computer program would be necessary. Finite element implementation of Pagano's theory shows that this theory, based on the sole assumption of linear variation of the in-plane stresses over each layer, is capable of yielding accurate estimates of stress and deformation throughout the laminate. As the laminae can be subdivided into as many sublayers as desired, it should be possible to set up sequences of solutions converging to the correct solution. However, this procedure is quite expensive in computational effort as well as storage requirements. An alternative solution procedures for this formulation, based on reduction of the problem of determining roots of the characteristic polynomial to a matrix eigenvalue problem, appears to hold good promise of combining accuracy with economy of effort and extending the application of Pagano's theory to multi-layer laminates.

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Appendix A

VARIATIONAL FORMULATION

Often, obtaining an approximate solution to coupled boundary value problem relies on appropriate variational formulations. Following Sandhu's [1970, 1971, 1975, 1977] extension of Mikhlin's [1965] basic variational theorem to coupled linear boundary value problems including nonhomogenous boundary conditions, we present a summary of the basic concepts for setting up the variational formulation applicable to the problem of laminated plates.

A.1 PRELIMINARIES

A.1.1 Boundary Value Problem

Consider the boundary value problem

$$A\,u = f \qquad \text{on } R \tag{A.1}$$

$$C\,u = g \qquad \text{on } \partial R \tag{A.2}$$

where ∂R is the boundary of the open connected region R in an euclidean space. $\bar{R}$ is the closure of R. A and C are the linear bounded operators. Let V_R and $V_{\partial R}$ be linear vector spaces defined on the regions indicated by the subscripts, and $W_R, W_{\partial R}$ be dense subsets in V_R and $V_{\partial R}$, respectively. Then the differential A and C can be regarded as the transformations

$$A: W_R \rightarrow V_R \tag{A.3}$$

$$C: W_{\partial R} \rightarrow V_{\partial R}$$

A.1.2 Bilinear Mapping

Let V and S be linear vector spaces. A bilinear mapping $B: V \times V \rightarrow S$ assigns to each ordered pair of vectors $u, v \in V$ an element in S. Furthermore, bilinearity is satisfied for $u_1, u_2, v_1, v_2, u, v \in V$, if

$$B(\alpha u_1 + u_2, v) = \alpha B(u_1, v) + B(u_2, v) \tag{A.4}$$

$$B(u, \alpha v_1 + v_2) = \alpha B(u, v_1) + B(u, v_2) \tag{A.5}$$

where α is scalar. For convenience, we shall use the notation

$$B_R(u, v) = \langle u, v \rangle_R \tag{A.6}$$

To set up a variational formulation, symmetric, nondegenerate bilinear mappings are used, i.e.,

$$\langle u, v \rangle_R = \langle v, u \rangle_R \tag{A.7}$$

and

$$\langle u, v \rangle_R = 0 \quad \text{for all } v \text{ if and only if } u = 0 \tag{A.8}$$

A.1.3 Self-Adjoint Operator

An operator A^* on V is said to be the adjoint of A with respect to symmetric bilinear mapping $B_R : V \times V \rightarrow S$, where S is a linear vector space, if

$$\langle u, Av \rangle_R = \langle v, A^* u \rangle_R + D_{\partial R}(v, u) \tag{A.9}$$

for all u and $v \in V$ and where $D_{\partial R}(u, v)$ represents quantities associated with boundary ∂R of R. If $A = A^*$, then A is said to be self-adjoint. If A is a self-adjoint operator, then $D_{\partial R}(v, u)$ is antisymmetric, i.e.,

$$D_{\partial R}(v, u) = -D_{\partial R}(u, v) \tag{A.10}$$

Furthermore, A is said to be symmetric with respect to the bilinear mapping if,

$$\langle u, Av \rangle_R = \langle v, Au \rangle_R \tag{A.11}$$

The boundary operator C is said to be consistent with the self-adjoint operator A if

$$D_{\partial R}(v,u) = <u, Cv>_{\partial R} - <v, Cu>_{\partial R} \tag{A.12}$$

A.1.4 Gateaux Differential

If $\Omega: V \to S$, where V is such that if u, $\bar{u} \in V$, $u + \lambda \bar{u} \in V$ for scalar λ, the Gateaux differential of a function Ω along the path $\bar{u}$ is defined by

$$\delta_u \Omega(u) = \lim_{\lambda \to 0} \frac{\Omega(u + \lambda\bar{u}) - \Omega(u)}{\lambda} \tag{A.13}$$

where $\bar{u}$ is referred to as the path.

A.2 THEOREM

For the field equations (A.1) we define

$$\Omega(u) = <u, Au>_R - 2<u, f>_R \tag{A.14}$$

The Gateaux differential of Ω is:

$$\delta_u \Omega(u) = \lim_{\lambda \to 0} \frac{<u + \lambda\bar{u}, A(u + \lambda\bar{u})> - 2<u + \lambda\bar{u}, f> - <u, Au> + 2<u, f>}{\lambda} \tag{A.15}$$

$$= <u, A\bar{u}> + <\bar{u}, Au> - 2<\bar{u}, f>$$

$$= 2<\bar{u}, Au - f>$$

The Gateaux differential vanishes at the solution $u = u_0$ where $Au_0 - f = 0$. Conversely, if $\delta_u \Omega(u)$ vanishes for all $\bar{u}$, nondegeneracy of $< , >$ implies $A u_0 - f = 0$. If the range of the bilinear mapping is the real line, vanishing of the function Ω would imply its minimum, maximum, or stationary value, depending upon the operator A being positive, negative or semi-definite.

A.3 LINEAR COUPLED PROBLEMS

The above discussion for a single-valued function u can be extended to the case of several variables. If there are n variables, V is defined as the direct sum

$$V = V_1 + V_2 + \dots\dots\dots + V_n \tag{A.16}$$

and an element $u \in V$ is an n-tuple $(u_1, u_2, \dots, u_n)$ with $u_i \in V_i$ for i=1, 2, ..., n. A bilinear mapping on V is defined as

$$< u, v > \; = \; <u_1, v_1>_{R^1} + <u_2, v_2>_{R^2} + \dots\dots + <u_n, v_n>_{R^n} \tag{A.17}$$

where $< \, , \, >_{R^i}$ is defined for components u_i, v_i of $\{u_i\}, \{v_i\}$ respectively.

If the field equations and the boundary conditions for a linear coupled boundary value problem are:

$$\sum_{j=1}^{n} A_{ij} u_j = f_i \qquad \text{on } R \tag{A.18}$$

$$\sum_{j=1}^{n} C_{ij} u_j = g_i \qquad \text{on } \partial R \qquad i=1,2,\dots,n \tag{A.19}$$

the governing functional based on Eqs. (A.18, A.19) is

$$\Omega(u) = \sum_{i=1}^{n} < u_i, \sum_{j=1}^{n} A_{ij} u_j - 2f_i >_R + \sum_{i=1}^{n} < u_i, \sum_{j=1}^{n} C_{ij} u_j - 2g_i >_{\partial R} \tag{A.20}$$

The set of operators A_{ij} is said to be self-adjoint with respect to the bilinear mapping, if

$$< v_i, \sum_{j=1}^{n} A_{ij} u_j >_R \; = \; \sum_{j=1}^{n} < u_j, A_{ji} v_i >_R + D_{\partial R}(u_j, v_i) \tag{A.21}$$

where $D_{\partial R}(u_j, v_i)$ represents quantities associated with boundary ∂R of R. The boundary operator C_{ij} are said to be consistent with the set of field operators A_{ij} if

$$D_{\partial R}(u_j, v_i) = \sum_{j=1}^{n} < u_j, C_{ji} v_i >_{\partial R} - < v_i, \sum_{j=1}^{n} C_{ij} u_j >_{\partial R} \tag{A.22}$$

Appendix B

PUBLICATIONS AND PRESENTATIONS

B.1 RESEARCH REPORTS

1. M. Moazzami, R.S. Sandhu, and W.E. Wolfe, "A Finite Element Procedure for Analysis of Laminated Composite Plates", Wright Laboratory Technical Report WL-TR-91-3023, Flight Dynamics Directorate, Wright Laboratory, Wright-Patterson Air Force Base, Ohio, 1991.

2. R.S. Sandhu, W.E. Wolfe, S.J. Hong, and H.S. Chohan, "A Consistent Shear Deformable Theory for Laminated Plates," Wright Laboratory Technical Report WL-TR-91-3021, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

3. R.S. Sandhu, W.E. Wolfe, S.J. Hong, and H.S. Chohan, "A Computer Program for Analysis of Laminated Plates Based on a Consistent Shear Deformable Theory," Wright Laboratory Technical Report WL-TR-91-3018, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

4. R.S. Sandhu, W.E. Wolfe, C.C. Chang, and H.R. Chu, "Computer Program CTQ23 for Finite Element Analysis of Free-Edge Delamination Specimens", Wright Laboratory Technical Report WL-TR-91-3017, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

5. R.S. Sandhu, W.E. Wolfe, R.L. Sierakowski, C.C. Chang, and H.R. Chu, "Finite Element Analysis of Free-Edge Delamination in Laminated Composite Specimens," Wright Laboratory Technical Report WL-TR-91-3022, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

6. R.S. Sandhu, R.L. Sierakowski, W.E. Wolfe, and R.A. Dandan, "Finite Element Analysis of Laminated Composite Axisymmetric Solids, Vol. 1 - Theory and Applications," Wright Laboratory Technical Report WL-TR-91-3020, Vol. 1, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

7. R.S. Sandhu, W.E. Wolfe, and R.A. Dandan, "A Computer Program for Finite Element Analysis of Laminated Composite Axisymmetric Solids," Wright Laboratory Technical Report WL-TR-91-3020, Vol. 2, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

8. R.S. Sandhu, W.E. Wolfe, and H.H. Chyou, "Variational Formulation and Finite Element Implementation of Pagano's Theory of Laminated Plates," Wright Laboratory Technical Report WL-TR-91-3016, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

9. R.S. Sandhu, W.E. Wolfe, and H.H. Chyou, "A Finite Element Computer Program for Analysis of Laminated Plates by Pagano's Theory," Wright Laboratory Technical Report WL-TR-91-3019, Flight Dynamics Directorate, Wright-Patterson Air Force Base, Ohio, 1991.

B.2 CONFERENCE PROCEEDINGS.

1. C.C. Chang and R.S. Sandhu, "Finite Element Analysis of Free Edge Effect in Composite Laminates," 11th Canadian Congress for Applied Mechanics, Edmonton, Alberta, June 1987.

2. C.C. Chang, R.S. Sandhu, R.L. Sierakowski, and W.E. Wolfe, "Continuous Strain Finite Element Analysis of Free Edge Delamination Specimens," ASME Pressue Vessel and Piping Conference, San Diego, California, June 1987.

3. S.J. Hong and R.S. Sandhu, "A Consistent Shear Deformation Theory for Laminated Composite Plates," 13th Mechanics of Composites Review, Florida, November 1988.

4. S.J. Hong, R.S. Sandhu, and H.S. Chohan, "Consistent Shear Constitutive Relations for Laminated Plates", 12th Canadian Congree for Applied Mechanics, Ottawa, Ontario, June 1989.

5. R.S. Sandhu, "Some Recent Developments in Mechanics of Laminated Plates," International Conference on Mechanics, Physics, and Structure of Materials: A Celebration of Aristotle's 23 Centuries, Thessalonika, Greece, August 1990.

6. R.S. Sandhu and M. Moazzami, "A Variational Approach for Consistent Development of Constitutive Relations for Laminated Plates", 15th Mechanics of Composites Review, Dayton, October 24- 25, 1990.

7. R.S. Sandhu, and M. Moazzami, "Constitutive Relations for Force Resultants in Laminated Plates," Third International Conference on Constitutive Laws for Engineering Materials: Theory and Applications, Tucson, Arizona, January 1991.

B.3 REFEREED JOURNAL ARTICLES.

1. Chern-Chi Chang, Ranbir S. Sandhu, Robert L. Sierakowski, and William E. Wolfe, "Continuous Strain Finite Element Analysis of Free-Edge Effect in Laminated Composite Specimens," American Society for Testing and Materials, Journal of Composite Technology and Research, Vol. 10, No. 2, 54-64, Summer 1988.

2. C.C. Chang, R.S. Sandhu, R.L. Sierakowski, and W.E. Wolfe, "Continuous Stress Finite Element Analysis of A Free-Edge Delamination Specimen," Computers and Structures, Vol. 29, No. 5, 783-793, 1988.

3. Soon Jo Hong, Ranbir S. Sandhu, and Harpal S. Chohan, "Consistent Shear Constitutive Relations for a Laminated Plate", under review for possible publication in ASTM Journal of Composite Technology and Research.

B.4 DISSERTATIONS AND THESES.

1. Chern-Chi Chang, "Finite Element Analysis of Laminated Composite Free-Edge Delamination Specimens", Ph.D. Dissertation, The Ohio State University, Columbus, Ohio, 1987.

2. Soon Jo Hong, "A Consistent Shear Deformable Theory for the Vibration of Laminated Plates", Ph.D. Dissertation, The Ohio State University, Columbus, Ohio, 1988.

3. Hui-Huang Abel Chyou, "Variational Formulation and Finite Element Implementation of Pagano's Theory of Laminated Plates", Ph.D. Dissertation, The Ohio State University, Columbus, Ohio, 1989.

4. Razan A. Dandan, "Finite Element Analysis of Laminated Composite Axisymmetric Solids", M.S. Thesis, The Ohio State University, Columbus, Ohio, 1988.

5. Mehdi Moazzami, "A Higher-Order Layerwise Theory of Laminated Plates", Ph.D. Dissertation, The Ohio State University, Columbus, Ohio, 1991.

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