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MINIMAX LINEAR SPLINES

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An algorithm is derived for obtaining a mesh that minimizes the maximum local interpolatory error for a linear spline, where the error is measured in any classical Banach norm. This algorithm is based on the standard method of approximate error equidistribution advocated by C. de Boor.

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INTRODUCTION

In order to enable an industrial machine with primitive computational ability to use complicated or difficult to compute functional relationships repeatedly, efficiently, and accurately, it is necessary to supply the machine with these functional relationships as sets of data in tabular form. It is assumed that the machine can deal with continuous, piecewise linear functions (linear splines). A graphics tube is a good example. Such a tube can draw only straight lines, but drawing many short, connected line segments can represent an arbitrary curve well. In order to represent these functions most accurately, a nonuniform mesh must be used. Finding such a mesh is, in principle, a very difficult nonlinear optimization problem, but C. de Boor (refs 1-3) advocated a general method by which the mesh can be found quickly, easily, robustly (and approximately) without any recourse to optimization methods! We present herein a robust addition to de Boor's standard method which improves its accuracy without increasing the essential complexity of his algorithm.

INTERPOLATORY ERROR

Let ℓ be the linear interpolant of function f on a subinterval of length h. The error is given by

$$f(t) = \ell(t) + e(t) \quad (\mu - \frac{h}{2} \leq t \leq \mu + \frac{h}{2})$$

Expand e in a Taylor series around the midpoint (μ) of the subinterval

$$e(t) = \sum_{i=0}^{\infty} \frac{e^{(i)}(\mu)}{i!} (t-\mu)^i$$

Applying the two boundary conditions

$$e(\mu - \frac{h}{2}) = 0 = e(\mu + \frac{h}{2})$$

ultimately yields

$$e\left(\frac{ht}{2}+\mu\right) = \sum_{i=1}^{\infty} \frac{f^{(2i)}(\mu)}{(2i)!} \left(\frac{h}{2}\right)^{2i} (t^{2i}-1)$$

$$+ \sum_{i=1}^{\infty} \frac{f^{(2i+1)}(\mu)}{(2i+1)!} \left(\frac{h}{2}\right)^{2i+1} t(t^{2i}-1)$$

Taking the first two terms of each sum

$$e\left(\frac{ht}{2}+\mu\right) = \frac{f''(\mu)}{2^3} h^2(t^2-1)$$

$$+ \frac{f^{(3)}(\mu)}{2^4 \cdot 3} h^3 t(t^2-1)$$

$$+ \frac{f^{(4)}(\mu)}{2^7 \cdot 3} h^4(t^4-1)$$

$$+ \frac{f^{(5)}(\mu)}{2^8 \cdot 3 \cdot 5} h^5 t(t^4-1) + O(h^6)$$

Letting

$$\rho_i = f^{(2+i)}(\mu) / f''(\mu)$$

one has

$$e\left(\frac{ht}{2}+\mu\right) = \frac{f''(\mu)}{2^3} h^2(t^2-1)\{1 + \frac{\rho_1}{2 \cdot 3} ht + \frac{\rho_2}{2^4 \cdot 3} h^2(t^2+1)$$

$$+ \frac{\rho_3}{2^5 \cdot 3 \cdot 5} h^3 t(t^2+1) + O(h^4)\} = \frac{f''(\mu)}{2^3} h^2(t^2-1)(1+S)$$

where

$$S = a_1 ht + a_2 h^2(t^2+1) + a_3 h^3 t(t^2+1) + O(h^4)$$

and

$$a_1 = \frac{\rho_1}{2 \cdot 3}, \quad a_2 = \frac{\rho_2}{2^4 \cdot 3}, \quad a_3 = \frac{\rho_3}{2^5 \cdot 3 \cdot 5}$$

INTERPOLATORY ERROR NORM

The local L^n norm of the error on a subinterval of length h is defined by

$$\|e\|_{n,h} = \left(\int_{\mu-h/2}^{\mu+h/2} |e(t)|^n dt\right)^{1/n}$$

where $1 \leq n < \infty$ and n is an integer. For $n = \infty$, we have the maximum error.

Now,

$$\|e\|_{n,h}^{n} = \frac{h}{2} \int_{-1}^{1} \left| e\left(\frac{ht}{2} + \mu\right) \right|^{n} dt$$

but

$$e\left(\frac{ht}{2} + \mu\right) = \frac{f''(\mu)}{2^3} h^2 (t^2 - 1)(1+S)$$

So if we let h be sufficiently small so that $|S| < 1$ on $(-1,1)$, we have

$$\|e\|_{n,h}^{n} = \frac{|f''(\mu)|^n h^{2n+1}}{2^{3n+1}} \int_{-1}^{1} (1-t^2)^n (1+S)^n dt$$

Since only the even terms of $(1+S)^n$ contribute to the integral, we have

$$\|e\|_{n,h}^{n} = \frac{|f''(\mu)|^n h^{2n+1}}{2^{3n}} \int_{0}^{1} (1-t^2)^n Ev(1+S)^n dt$$

where $Ev(1+S)^n$ denotes the even terms of $(1+S)^n$.

Hence,

$$Ev(1+S)^n = 1 + \binom{n}{1} a_2 h^2 (1+t^2) + \binom{n}{2} a_1^2 h^2 t^2 + O(h^4)$$

Letting

$$I_{n,i} = \int_0^1 (1-t^2)^n t^{2i} dt$$

we therefore have

$$\int_0^1 (1-t^2)^n Ev(1+S)^n dt$$

$$= \int_0^1 (1-t^2)^n \left(1 + h^2\left(na_2(1+t^2) + \frac{n(n-1)}{2} a_1^2 t^2\right) + O(h^4)\right) dt$$

$$= I_{n,o} + nh^2\left(a_2(I_{n,o} + I_{n,1}) + \frac{n-1}{2} a_1^2 I_{n,1}\right) + O(h^4)$$

$$= I_{n,o}\left(1 + nh^2\left(a_2\left(1 + \frac{I_{n,1}}{I_{n,o}}\right) + \frac{n-1}{2} a_1^2 \frac{I_{n,1}}{I_{n,o}}\right) + O(h^4)\right)$$

Using integration-by-parts on $I_{n,i}$ and solving the resulting recursion ultimately yields

$$I_{n,i} = \frac{2^{2n} n! (2i)! (i+n)!}{i! (2i+2n+1)!}$$

from which we conclude that

$$I_{n,o} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

$$I_{n,1} = \frac{2^{2n+1}n!(n+1)!}{(2n+3)!}$$

and

$$\frac{I_{n,1}}{I_{n,o}} = \frac{1}{2n+3}$$

Hence,

$$\int_0^1 (1-t^2)^n Ev(1+S)^n dt$$

$$= \frac{2^{2n}(n!)^2}{(2n+1)!} \left(1+nh^2\left(\frac{2(n+1)}{2n+3} a_2 + \frac{n-1}{2(2n+3)} a_1^2\right) + O(h^4)\right)$$

and

$$\|e\|_{n,h}^n = \frac{\lfloor f''(\mu) \rfloor^n h^{2n+1}(n!)^2}{2^n(2n+1)!} \left(1+nh^2\left(\frac{2(n+2)}{2n+3} a_2 + \frac{n-1}{2(2n+3)} a_1^2\right) + O(h^4)\right)$$

or

$$\|e\|_{n,h} = k \mid f''(\mu) \mid h^{2+1/n}\left(1+h^2\left(\frac{2(n+2)}{2n+3} a_2 + \frac{n-1}{2(2n+3)} a_1^2\right) + O(h^4)\right)$$

where

$$k = \frac{1}{2} \left(\frac{(n!)^2}{(2n+1)!}\right)^{1/n}$$

Using Stirling's approximation to the factorial, it is easy to show that

$$\lim_{n\to\infty} k = \frac{1}{8}$$

Recalling that

$$a_1 = \frac{\rho_1}{2\cdot 3} \quad \text{and} \quad a_2 = \frac{\rho_2}{2^4\cdot 3}$$

we finally have

$$\|e\|_{n,h} = k \mid f''(\mu) \mid h^{2+1/n}\left(1 + \frac{h^2}{24} \left(\frac{n+2}{2n+3} \rho_2 + \frac{n-1}{3(2n+3)} \rho_1^2\right) + O(h^4)\right)$$

as $h\to 0$, where

$$k = \frac{1}{2}\left(\frac{(n!)^2}{(2n+1)!}\right)^{1/n}$$

and

$$\rho_i = f^{(2+i)}(\mu)/f''(\mu)$$

NORM OF ARBITRARY FUNCTION

The local L^p norm of arbitrary function ϕ over a subinterval of length h is defined as

$$\|\phi\|_{p,h} = \left(\int_{\mu-h/2}^{\mu+h/2} |\phi(t)|^p dt\right)^{1/p}$$

where $p > 0$, finite and real. In this context, we allow $p < 1$ even though Minkowski's triangle inequality holds only for $p \geqslant 1$.

Expand ϕ in a Taylor series around the midpoint of the subinterval

$$\phi(t) = \phi(\mu) \sum_{i=0}^{\infty} \frac{\rho_i}{i!} (t-\mu)^i$$

where

$$\rho_i = \phi^{(i)}(\mu)/\phi(\mu)$$

Now,

$$\|\phi\|_{p,h}^p = \frac{h}{2} \int_{-1}^{1} \left|\phi\left(\frac{ht}{2} + \mu\right)\right|^p dt$$

but

$$\phi\left(\frac{ht}{2} + \mu\right) = \phi(\mu)(1+S)$$

where

$$S = \sum_{i=1}^{\infty} a_i t^i$$

and

$$a_i = \frac{\rho_i}{i!} \left(\frac{h}{2}\right)^i$$

Hence, letting h be sufficiently small so that $|S| < 1$ on $(-1,1)$, we have

$$\|\phi\|_{p,h}^p = \frac{h}{2} |\phi(\mu)|^p \int_{-1}^{1} (1+S)^p dt = \frac{h}{2} |\phi(\mu)|^p \int_{-1}^{1} Ev(1+S)^p dt$$

but

$$S = a_1t + a_2t^2 + a_3t^3 + O(h^4)$$

hence

$$Ev(1+S)^p = 1 + \binom{p}{1}a_2t^2 + \binom{p}{2}a_1^2t^2 + O(h^4)$$

We therefore have

$$\|\phi\|_{p,h}^p = h\,|\,\phi(\mu)\,|^p\int_0^1 1 + pt^2(a_2 + \frac{p-1}{2}\,a_1^2) + O(h^4)dt$$

$$= h\,|\,\phi(\mu)\,|^p(1 + \frac{ph^2}{24}\,(\rho_2 + (p-1)\rho_1^2) + O(h^4))$$

or

$$\|\phi\|_{p,h} = h^{1/p}\,|\,\phi(\mu)\,|\,(1 + \frac{h^2}{24}\,(\rho_2 + (p-1)\rho_1^2) + O(h^4))$$

as $h \to 0$.

STANDARD APPROXIMATION TO $\|e\|_{n,h}$

Recalling that

$$h^{-1/p}\|f''\|_{p,h} = |\,f''(\mu)\,|\,(1 + \frac{h^2}{24}(\rho_2 + (p-1)\rho_1^2) + O(h^4))$$

and

$$\|e\|_{n,h} = kh^{2+1/n}\,|\,f''(\mu)\,|\,(1 + \frac{h^2}{24}(\frac{n+2}{2n+3}\,\rho_2 + \frac{n-1}{3(2n+3)}\,\rho_1^2) + O(h^4))$$

we multiply the first equation by $kh^{2+1/n}$ and subtract from the second, getting

$$\|e\|_{n,h} = kh^{2+1/n-1/p}\|f''\|_{p,h}$$

$$+ kh^{2+1/n}\,|\,f''(\mu)\,|\,(\frac{h^2}{24}(-\,\frac{n+1}{2n+3}\,\rho_2 + (\frac{7n+8}{6n+9} - p)\rho_1^2) + O(h^4))$$

If we now let $p = \frac{n}{2n+1}$, we have

$$\|e\|_{n,h} = k\|f''\|_{n/(2n+1),h}$$

$$+ kh^{4+1/n}\,|\,f''(\mu)\,|\,(\frac{1}{24}(-\,\frac{n+1}{2n+3}\,\rho_2 + \frac{8n^2+14n+8}{12n^2+24n+9}\,\rho_1^2) + O(h^2))$$

$$= k\|f''\|_{p,h} + kh^{4+1/n}\,|\,f''(\mu)\,|\,(\frac{1}{24}(-\,a\rho_2 + b\rho_1^2) + O(h^2))$$

$$= k\|f''\|_{n/(2n+1),h} + O(h^{4+1/n})$$

For $n = 1, 2$, and ∞, respectively, we have

$$\|e\|_{1,h} = \frac{1}{12} \|f''\|_{1/3,h} + O(h^5)$$

$$\|e\|_{2,h} = \frac{1}{2\sqrt{30}} \|f''\|_{2/5,h} + O(h^{9/2})$$

$$\|e\|_{\infty,h} = \frac{1}{8} \|f''\|_{1/2,h} + O(h^4)$$

STANDARD ERROR EQUIDISTRIBUTION FOR ANY BANACH NORM

In this section, we justify the standard method of error equidistribution with respect to any Banach norm. The global L^n norm of the error over interval (a,b) is

$$\|e\|_n = \left(\int_a^b |e(t)|^n dt\right)^{1/n}$$

Hence, for a mesh $a = x_1 < x_2 < \ldots < x_N = b$

$$\|e\|_n^n = \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} |e_i(t)|^n dt = \sum_{i=1}^{N-1} \|e\|_{n,h_i}^n$$

Let single bars around the error denote the standard approximation to the error norm and analogously define

$$|e|_n^n = \sum_{j=1}^{N-1} |e|_{n,h_j}^n$$

but

$$|e|_{n,h_j} = k\|f''\|_{p,h_j} = k\left(\int_{x_j}^{x_{j+1}} |f''(t)|^p dt\right)^{1/p}$$

where

$$p = \frac{n}{2n+1}$$

Hence, letting

$$I_{p,h_j} = \int_{x_j}^{x_{j+1}} |f''(t)|^p dt$$

we have

$$|e|_n^n = k^n \sum_{j=1}^{N-1} I_{p,h_j}^{2n+1}$$

We will refer to the integrals $I_{p,h}$ as the standard or de Boor integrals.

It follows trivially, using Leibnitz's rule, that

$$\frac{\partial}{\partial x_i} | e |_n^n = 0 \qquad 1 < i < N$$

implies that

$$I_{p,h_{i-1}} = I_{p,h_i} \qquad 1 < i < N$$

Hence, the condition $I_{p,h}$ = constant determines the mesh which minimizes the standard global approximation to $\|e\|_n$.

For a linear spline approximation to f", it is a fairly simple (see COMPUTATION) matter to find the mesh for which the de Boor integrals are constant.

CONVERGENCE OF STANDARD METHOD

Recall that

$$\|e\|_{n,h} = k\|f''\|_{p,h} + \frac{k}{24} h^{4+1/n} | f''(\mu) | (-a\rho_2+b\rho_1^2) + O(h^{6+1/n})$$

Letting

$$F = -af''(\mu)f^{(4)}(\mu) + bf^{(3)}(\mu)^2$$

we have the following one term approximation to the difference between $\|e\|_{n,h}$ and $| e |_{n,h}$:

$$\|e\|_{n,h} - | e |_{n,h} \simeq \frac{kFh^{4+1/n}}{24| f''(\mu) |}$$

but

$$\|e\|_{n,h} \simeq kh^{2+1/n} | f''(\mu) |$$

Therefore, we also have

$$\frac{\|e\|_{n,h} - | e |_{n,h}}{\|e\|_{n,h}} \simeq \frac{Fh^2}{24| f''(\mu) |^2}$$

but also

$$\|f''\|_{p,h} \simeq h^{1/p} | f''(\mu) |$$

hence,

$$h \simeq \left(\frac{\|f''\|_{p,h}}{|f''(\mu)|}\right)^p = \left(\frac{I_{p,h}^{1/p}}{|f''(\mu)|}\right)^p = \frac{I_{p,h}}{|f''(\mu)|^p}$$

In addition, for the correct mesh

$$I_{p,h} = \frac{I_p}{N-1}$$

hence,

$$h \simeq \frac{I_p}{(N-1)|f''(\mu)|^p}$$

and therefore,

$$\frac{\|e\|_{n,h} - |e|_{n,h}}{\|e\|_{n,h}} \simeq \frac{FI_p^2}{24|f''(\mu)|^{2+2p}(N-1)^2}$$

This tells us that the relative difference between $\|e\|_{n,h}$ and $|e|_{n,h}$ is $O(\frac{1}{N^2})$ as $N\to\infty$, which means that the standard method works better and better ($\|e\|_{n,h}$ will be more nearly constant) as N gets larger and larger. This is all true, however, with the proviso that

$$\frac{F}{|f''(\mu)|^{2+2p}}$$

is bounded throughout the region of interest. It stands to reason, therefore, that the standard method will perform worst where f" is not bounded away from zero.

IMPROVED APPROXIMATION TO $\|e\|_{n,h}$

Recall that

$$\|e\|_{n,h} = k|f''(\mu)|h^{2+1/n}(1 + \frac{h^2}{24}(\frac{n+2}{2n+3}\rho_2 + \frac{n-1}{3(2n+3)}\rho_1^2) + O(h^4))$$

and

$$\|f''\|_{q,h} = h^{1/q}|f''(\mu)|(1 + \frac{h^2}{24}(\rho_2 + (q-1)\rho_1^2) + O(h^4))$$

Multiplying h by r in the second equation, we have

$$r^{-1/q}\|f''\|_{q,rh} = h^{1/q}|f''(\mu)|(1 + \frac{h^2}{24}(r^2\rho_2 + r^2(q-1)\rho_1^2) + O(h^4))$$

Multiplying this equation by kh^Q gives us

$$kr^{-1/q}h^Q\|f''\|_{q,rh} = kh^{Q+1/q}\, f''(\mu)\,|\,(1 + \frac{h^2}{24}(r^2\rho_2 + r^2(q-1)\rho_1^2) + O(h^4))$$

Now, in order to make this equation look as much like the very first one as possible, we set

$$r^2 = \frac{n+2}{2n+3} \quad , \quad r^2(q-1) = \frac{n-1}{3(2n+3)}$$

and

$$Q + \frac{1}{q} = 2 + \frac{1}{n}$$

Solving for r, q, and Q, we have

$$r = (\frac{n+2}{2n+3})^{1/2}$$

$$q = \frac{4n+5}{3n+6}$$

and

$$Q = \frac{5n^2+8n+5}{4n^2+5n}$$

A simple subtraction then gives us an improved approximation to $\|e\|_{n,h}$

$$\|e\|_{n,h} = kr^{-1/q}h^Q\|f''\|_{q,rh} + O(h^{6+1/n})$$

where before, we had

$$\|e\|_{n,h} = k\|f''\|_{p,h} + O(h^{4+1/n}) = |\,e\,|_{n,h} + O(h^{4+1/n})$$

It must be mentioned however, that although this improved approximation is asymptotically more efficient, no such approximation can be uniformly superior in all cases. Bearing this in mind, we dispense with approximations on all subintervals not having f" bounded away from zero and instead use the exact error

$$e_i(x) = \int_{x_i}^{x}\int_{x_i}^{t} f''(u)dudt - \frac{x-x_i}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\int_{x_i}^{t} f''(u)dudt$$

COMPUTATION

In actual computation, we assume the existence of a piecewise linear approximation to $|f''|$. The mesh over which this function is defined is referred to as the "original" mesh. In order to deal with the standard and improved asymptotic integral approximations to the local error norm, we will need to deal with integrals of the form

$$L = \int_c^{c+\ell} \lambda(t)^{m/n} dt$$

where λ is a nonnegative linear function with slope s

$$\lambda(t) = \lambda(c) + s(t-c)$$

with

$$\lambda(t) \geqslant 0 \quad \text{for} \quad c \leqslant t \leqslant c + \ell$$

and where m and n are arbitrary positive integers.

In the following, let

$$\alpha = \lambda(c)^{1/n}$$

and

$$S_k = \sum_{i=0}^{k} \alpha^i \beta^{k-i} = \frac{\beta^{k+1} - \alpha^{k+1}}{\beta - \alpha}$$

First, we need to compute L as a function of ℓ

$$L = \frac{\ell n S_{m+n-1}}{(m+n) S_{n-1}} = A(\ell)$$

where

$$\beta = (\lambda(c) + s\ell)^{1/n}$$

Second, we need to compute ℓ as a function of L

$$\ell = \frac{L(m+n) S_{n-1}}{n S_{m+n-1}} = B(L)$$

where

$$\beta = (\lambda(c)^{m/n+1} + (\frac{m}{n} + 1) sL)^{1/(m+n)}$$

A and B are therefore inverse functions, i.e.,

$$A(B(x)) = x = B(A(x))$$

or

$$A^{-1} = B \quad \text{and} \quad B^{-1} = A$$

Now let values of u denote the original mesh and let g be the piecewise linear interpolant to the $(u_i, |f_i''|)$ data.

Define the integral

$$G(x) = \int_{u_1}^{x} g(t)^{m/n} dt$$

Now if $u_i \leqslant x \leqslant u_{i+1}$,

$$G(x) = \int_{u_1}^{u_i} g(t)^{m/n} dt + \int_{u_i}^{u_i+x-u_i} g_i(t)^{m/n} dt$$

$$= G(u_i) + L$$

where $\lambda = g_i$, $c = u_i$, and $\ell = x-u_i$. Hence,

$$G(x) = G(u_i) + A(x-u_i)$$

explicitly defines G for all x in the domain of interest.

In order to get the standard mesh, we will also have to compute the inverse of G (only for $m/n = p$).

$$B(G(x) - G(u_i)) = B(A(x-u_i)) = x-u_i$$

Hence,

$$x = u_i + B(G(x) - G(u_i))$$

but if $G(x) = \gamma$, then $x = G^{-1}(\gamma)$. Therefore,

$$G^{-1}(\gamma) = u_i + B(\gamma - G(u_i))$$

for

$$G(u_i) \leqslant \gamma \leqslant G(u_{i+1})$$

and provided

$$G(u_i) \neq G(u_{i+1})$$

Define

$$I_i = G(x_{i+1}) - G(x_i) = \int_{x_i}^{x_{i+1}} g(t)^p dt$$

ere x is the standard or improved mesh, obtained by prescribing values for the I's. The standard method prescribes

$$I_i = \text{const} = \frac{G(x_N)}{N-1} \qquad (1 \leqslant i \leqslant N)$$

For the improved mesh, the I's will vary, but the mesh is still obtained in the standard way. Since

$$G(x_{i+1}) = G(x_i) + I_i$$

we have immediately that

$$x_{i+1} = G^{-1}(G(x_i) + I_i) \qquad i = 1,2,\ldots,N-2$$

ALGORITHM

Let * denote a standard or improved mesh and ** denote the succeeding improved mesh. We have seen that the main contributor to the ratios $\|e\|_{n,h^{**}}/\|e\|_{n,h^{*}}$ and $|e|_{n,h^{**}}/|e|_{n,h^{*}}$ is

$$\left(\frac{h^{**}}{h^{*}}\right)^{2+1/n} \left| \frac{f''(\mu^{**})}{f''(\mu^{*})} \right|$$

We therefore have the approximate asymptotic relation

$$\frac{|e|_{n,h^{**}}}{|e|_{n,h^{*}}} \approx \frac{\|e\|_{n,h^{**}}}{\|e\|_{n,h^{*}}}$$

But we would like $\|e\|_{n,h^{**}}$ to be constant, hence we have the proportionality

$$|e|_{n,h^{**}} \propto \frac{|e|_{n,h^{*}}}{\|e\|_{n,h^{*}}}$$

or

$$I_{p,h^{**}} \propto \left(\frac{|e|_{n,h^{*}}}{\|e\|_{n,h^{*}}}\right)^p$$

We calculate the I's accordingly and multiply them by the appropriate constant to get

$$\sum_{i=1}^{N-1} I_{p,h_i^{**}} = \frac{G(x_N)}{N-1}$$

The quantities $\|e\|_{n,h_i^*}$ are computed either from the improved asymptotic approximation or exactly (relative to the original data) depending on whether or not f" is bounded away from zero on the subinterval in question. It is important to note that this <u>approximate</u> relation between the * and ** meshes can lead to <u>exact</u> convergence (rapidly) to the minimax mesh. If the * mesh <u>is</u> the minimax mesh ($\|e\|_{n,h^*}$ = constant), then the de Boor integrals ($I_{p,h}$) on the ** mesh will be no different from those on the * mesh.

The practical convergence properties of this algorithm are as follows. If f" is well bounded away from zero, the standard de Boor method gives impeccable results without any iteration. If f" is not bounded away from zero, convergence to a virtually perfect minimax mesh can easily occur in only two iterations. A few iterations may be needed in the presence of multiple inflection points.

In any case, even the very first iteration improves the mesh markedly.

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