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STURM-LIOUVILLE EIGENFUNCTIONS
EXPRESSED IN
DETERMINANT FORM

by

Michael D. Phillips

June, 1991

Thesis Advisor: G. E. Latta

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Sturm-Liouville Eigenfunctions Expressed in Determinant Form

by

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ABSTRACT

The purpose of this thesis is to investigate and establish Sturm-Liouville properties for special eigenfunctions which are expressed in determinant form. In particular, a special case is presented where the elements of the determinant are Legendre polynomials. This type of determinant has a probability background dealing in birth and death processes. The method of analysis used in this thesis is a new approach to solving this specific example. This investigation involves systems of differential equations and Prufer's analysis in the phase plane. The following are new results obtained in addition to solving the special case mentioned above.

Special determinants of hypergeometric functions also possess Sturm-Liouville properties. As a special case, a different proof of Turan's Inequality is provided. Finally, several theorems are presented for Sturm-Liouville systems of differential equations with polynomial coefficients.

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I. INTRODUCTION

A. THE PROBLEM

The main purpose of this thesis is to investigate and establish the essential Sturm-Liouville properties for special eigenfunctions which are expressed in determinant form. In particular, a special case is presented where the elements of the determinant are Legendre polynomials such that

$$\Phi_{n,k}(x) = \begin{vmatrix} P_k(x) & P_{k+1}(x) \\ P_n(x) & P_{n+1}(x) \end{vmatrix} \quad \begin{matrix} (n>k) & k=0,1,2,\ldots \\ (-1<x<1) & n=1,2,3,\ldots \end{matrix} \quad . \tag{1.1}$$

This type of determinant has a probability background dealing in birth and death processes and was studied by Dr. S. Karlin and Dr. J. L. McGregor. In their researches, Karlin and McGregor encountered determinants of classical polynomials as examples. In order to complete their results, they needed to know such properties as: number of zeros, interlacing zeros, and completeness. Their obvious starting point was to construct an ordinary differential equation such that (1.1) was the solution. However, this differential equation was not in a form that suggested how to answer any of these questions. [Ref. 1]

The method of analysis used in this thesis is a new approach to solving this specific example. The investigation that follows provides a constructive answer, not only to this example, but, to a whole class of such problems. The following are new results obtained in addition to solving Karlin's problem.

Special determinants of other hypergeometric functions also possess the essential Sturm-Liouville properties.

As a special case of (1.1), a different proof of Turan's Inequality is provided in Chapter III. Turan's Inequality is of the form

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} < 0 \qquad \begin{matrix} -1<x<1 \\ n=0,1,2,\ldots \end{matrix} \tag{1.2}$$

where the elements of the determinant are Legendre polynomials.

Finally, several theorems are presented for Sturm-Liouville systems of differential equations with polynomial coefficients. In general, boundary value problems of systems are not well-posed. Until now, very little has been established concerning these systems with polynomial coefficients.

B. STRATEGY FOR SOLVING THE PROBLEM

The basic strategy developed here is to construct a second order differential equation such that (1.1) is a solution. Next, this differential equation is reduced to a special system of first order differential equations such that the coefficients are polynomials. Then, the solution trajectories are analyzed in the corresponding phase plane by a method known as the "Prüfer analysis". From this analysis, several theorems of Sturm-Liouville type are developed for general systems of differential equations with polynomial coefficients. Finally, completeness and orthogonality are established with respect to a positive definite matrix weight function.

II. CONSTRUCTION OF A STURM-LIOUVILLE SYSTEM OF DIFFERENTIAL EQUATIONS

A. DEVELOPMENT OF A NONSTANDARD STURM-LIOUVILLE DIFFERENTIAL EQUATION WITH SPECIAL DETERMINANT EIGENFUNCTIONS

The objective of this section is to construct a second order linear differential equation, with appropriate boundary conditions, such that its solution is (1.1). For the general case, with arbitrary n and k, this method is somewhat lengthy and requires extensive manipulation of equations. However, considering a special case of k = 1, this method is not as complicated yet still illustrates all of the important steps. For completeness, the outline of the solution for the general case the same procedure and is provided in the Appendix.

In the general case, the eigenfunction $\Phi_{n,k}(x)$ has the following form

$$\Phi_{n,k}(x) = \begin{vmatrix} P_k(x) & P_{k+1}(x) \\ P_n(x) & P_{n+1}(x) \end{vmatrix} \quad \begin{matrix} (n>k) & k=0,1,2,\ldots \\ (-1<x<1) & n=1,2,3,\ldots \end{matrix} \quad . \qquad (2.1)$$

Setting k = 1 results in

$$\Phi_{n,1}(x) = \begin{vmatrix} x & \tfrac{1}{2}(3x^2-1) \\ P_n(x) & P_{n+1}(x) \end{vmatrix} \quad .$$

The strategy is to use recursion formulas and to develop three equations for $\Phi_{n,1}(x)$, $\Phi'_{n,1}(x)$, and $\Phi''_{n,1}(x)$ as functions of $P_n(x)$ and $P'_n(x)$. Then, from these equations, a system of equations in the matrix form AX = 0 is constructed where X $\neq$ 0. Finally, the determinant of A is calculated and set equal to zero. This results in a differential equation having (2.1) as a solution.

First, the equations for $\Phi_{n,1}(x)$, $\Phi'_{n,1}(x)$, and $\Phi''_{n,1}(x)$ are developed. Substituting the recursion formula

$$P_{n+1}(x) = xP_n(x) - \frac{(1-x^2)}{(n+1)} P_n'(x)$$

into (2.1) for $P_{k+1}(x)$ and $P_{n+1}(x)$ results in

$$\Phi_{n,k}(x) = \begin{vmatrix} P_k(x) & xP_k(x) - \frac{(1-x^2)}{k+1} P_k'(x) \\ P_n(x) & xP_n(x) - \frac{(1-x^2)}{n+1} P_n'(x) \end{vmatrix} .$$

Since k = 1, $P_k(x)$ and $P'_k(x)$ are replaced by x and 1, respectively. In addition, by computing the determinant and collecting like terms, we obtain the following equation for $\Phi_{n,1}(x)$.

$$\Phi_{n,1} = \frac{(1-x^2)}{2} P_n(x) - \frac{x(1-x^2)}{n+1} P_n'(x) \qquad (2.2)$$

Next, we differentiate both sides of (2.2). Thence,

$$\Phi'_{n,1}(x) = \frac{(1-x^2)}{2}P'_n(x) - xP_n(x) - \frac{(1-x^2)}{n+1}P'_n(x)$$
$$-x\frac{d}{dx}\left[\frac{(1-x^2)}{n+1}P'_n(x)\right] . \tag{2.3}$$

Then, we substitute the Legendre differential equation

$$\frac{d}{dx}\left[(1-x^2)P'_n(x)\right] = -n(n+1)P_n(x)$$

into (2.3) and collect like terms. Then, (2.3) becomes

$$\Phi'_{n,1}(x) = (n-1)xP_n(x) + \frac{(n-1)(1-x^2)}{2(n+1)}P'_n(x) \quad . \tag{2.4}$$

Repeating the same procedure on (2.4) results in the following relation for $\Phi''_{n,1}(x)$.

$$\Phi''_{n,1}(x) = \left[\frac{-n(n-1)}{2} + (n-1)\right]P_n(x) + (n-1)xP'_n(x) \tag{2.5}$$

Now, a system of equations in matrix form is constructed from equations (2.2), (2.4), and (2.5). The system takes the form AX = 0, i.e.,

$$AX=\begin{bmatrix} \Phi''_{n,1}(x) & \frac{-n(n-1)}{2}+(n-1) & (n-1)x \\ \Phi'_{n,1}(x) & (n-1)x & \frac{(n-1)(1-x^2)}{2(n+1)} \\ \Phi_{n,1}(x) & \frac{1-x^2}{2} & \frac{-x(1-x^2)}{n+1} \end{bmatrix}\begin{bmatrix} -1 \\ P_n(x) \\ P'_n(x) \end{bmatrix}=O. \quad (2.6)$$

Since the trivial solution can not be a solution of (2.6), A is not invertible. Therefore, the determinant of A must equal zero. Hence, by setting the determinant of A equal to zero we obtain a second order differential equation where $\Phi_{n,1}(x)$ is a solution.

$$|A| = (1-x^2)(1+3x^2)\Phi''_{n,1}(x) - 6x(1-x^2)\Phi'_{n,1}(x) + (n-1)\Big[(3n+6)x^2+n-2\Big]\Phi_{n,1}(x) = 0 \quad (2.7)$$

By setting $\lambda = n-1$, then (2.7) takes the form

$$(1-x^2)(1+3x^2)\Phi''_{n,1}(x) - 6x(1-x^2)\Phi'_{n,1}(x) + \lambda\Big[3(\lambda+3)x^2+\lambda-1\Big]\Phi_{n,1}(x) = 0 \quad . \quad (2.8)$$

Note that, for each Legendre polynomial solution $P_n(x)$ to the Legendre differential equation, there is a corresponding logarithmic solution, $Q_n(x)$. This logarithmic solution obeys the same recursion formulas as $P_n(x)$. Hence, by the same procedure, the second solution to (2.8) is

$$\begin{vmatrix} P_k(x) & P_{k+1}(x) \\ Q_n(x) & Q_{n+1}(x) \end{vmatrix} .$$

Therefore, the general solution is

$$C_1 \begin{vmatrix} x & \tfrac{1}{2}(3x^2-1) \\ P_n(x) & P_{n+1}(x) \end{vmatrix} + C_2 \begin{vmatrix} x & \tfrac{1}{2}(3x^2-1) \\ Q_n(x) & Q_{n+1}(x) \end{vmatrix} . \qquad (2.9)$$

Finally, translated boundary conditions on (2.9) are imposed so the second logarithmic solution is eliminated.

Note that (2.8) is not in the standard Sturm-Liouville form. However, $\Phi_{n,1}(x)$ still possesses all of the essential properties.

B. REDUCTION TO A SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS

The objective in this section is to reduce (2.8) to a system of differential equations of the special form

$$\begin{aligned} (1-x^2)\Phi' &= \alpha\Phi + (1+3x^2)\overline{\Psi} \qquad -1<x<1 \\ (1-x^2)\overline{\Psi}' &= \beta\Phi + \gamma\overline{\Psi} \end{aligned} \qquad (2.10)$$

where α, β, and γ are polynomials in x, $\Phi = \Phi_{n,1}(x)$, and $\overline{\Psi}$ is an arbitrary function in x. There are an infinite number of ways to transform (2.8) into a system of differential equations. However, by requiring polynomial coefficients we remove the singularities at $\pm(-3)^{-\frac{1}{2}}$. Hence, all solutions are well behaved on the interval $-1<x<1$.

The strategy is to convert (2.10) into a second order differential equation and then equate the polynomial coefficients of Φ'', Φ' and Φ.

The system (2.10) has the following equivalent form

$$(1-x^2)(3x^2+1)\Phi''-\Big[(\alpha+\gamma+2x)(3x^2+1)+6x(1-x^2)\Big]\Phi' -$$

$$\Big[\alpha'(3x^2+1)-\alpha 6x+\beta\frac{(3x^2+1)^2}{1-x^2} - \frac{\gamma\alpha(3x^2+1)}{1-x^2}\Big]\Phi = 0 \quad .$$

First, note that the coefficients of Φ'' are the same. Next, equating the coefficients of Φ' and simplifying results in

$$\gamma = -2x-\alpha \quad . \tag{2.11}$$

Equating the coefficients of Φ and substituting (2.11) for γ leads to

$$\lambda\Big[3(\lambda+3)x^2+\lambda-1\Big] = -\alpha'(3x^2+1)+\alpha 6x-\beta\frac{(3x^2+1)^2}{1-x^2} + \frac{\alpha(-2x-\alpha)(3x^2+1)}{1-x^2} \quad . \tag{2.12}$$

Dividing both sides by $3x^2+1$ and simplifying yields

$$\frac{\lambda[3(\lambda+3)x^2+\lambda-1]}{3x^2+1} = -\alpha' + \frac{\alpha 6x}{3x^2+1}$$
$$- \beta\frac{(3x^2+1)}{1-x^2} + \frac{\alpha(-2x-\alpha)}{1-x^2} \quad . \tag{2.13}$$

Next, we consider components of (2.13) that have $3x^2+1$ in the denominator. The idea is to select α so that

$$\lambda[3(\lambda+3)x^2+\lambda-1]-\alpha 6x = 0 \bmod(3x^2+1) \quad .$$

This condition on α is essential in removing the singularities at $\pm(-3)^{-\frac{1}{2}}$. For now, set α and β so that

$$\alpha=px \quad , \; \beta=q$$

where p and q are constants. Substituting px, p, and q into (2.12) for α, α' and β, respectively, and simplifying results in

$$(-3\lambda^2-9\lambda)x^4+(2\lambda^2+10\lambda)x^2+(\lambda^2-\lambda) =$$
$$(-3p^2-9p-9b)x^4+(-p^2+2p-6b)x^2+(-p-b) \quad .$$

Next, we equate the coefficients. Thus, $p = 2\lambda$ and $q = -\lambda(\lambda+1)$ which implies that $\alpha = 2\lambda x$ and $\beta = -\lambda(\lambda+1)$. Substituting back into (2.11), it follows that $\gamma = (\lambda+1)(-2x)$. Therefore, the system (2.10) takes the form

$$
\begin{aligned}
(1-x^2)\Phi' &= \quad 2\lambda x\Phi + \quad (1+3x^2)\bar{\Psi} \qquad -1<x<1 \quad . \\
(1-x^2)\bar{\Psi}' &= -\lambda(\lambda+1)\Phi + (\lambda+1)(-2x)\bar{\Psi}
\end{aligned}
$$

A simple substitution of $(\lambda+1)\Psi = \bar{\Psi}$ implies the following equivalent form

$$
\begin{aligned}
(1-x^2)\Phi' &= 2\lambda x\Phi + (\lambda+1)(1+3x^2)\Psi \qquad -1<x<1 \quad . \\
(1-x^2)\Psi' &= \quad -\lambda\Phi + \quad (\lambda+1)(-2x)\Psi
\end{aligned}
\tag{2.14}
$$

Hence, (2.14) is a system of first order differential equations that is equivalent to (2.8).

C. PRÜFER ANALYSIS

The Prüfer analysis is used to study the oscillatory behavior of trajectory solutions in the corresponding phase plane [Ref. 2:p. 312]. It is a very powerful method that is extremely useful. The objective of this section is to use this method to establish the theorems listed below for general systems of differential equations, and then apply these theorems to (2.14). These theorems for systems are adapted from the analogous theorems for Sturm-Liouville differential equations [Ref. 2:pp. 312-353].

<u>Theorem 1.</u> (STURM SEPARATION THEOREM)

If $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ are linearly independent solutions of the system

$$(1-x^2)\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix} = \begin{pmatrix}\alpha & Q\\ B & \gamma\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \quad -1<x<1$$

and

$$4B(-Q)-[\gamma-\alpha]^2>0$$

where α, B, γ, and Q are polynomials in x, then Φ_1 must vanish at one point between any two successive zeros of Φ_2. In other words, the zeros of Φ_1 and Φ_2 interlace.

<u>Theorem 2.</u> (STURM COMPARISON THEOREM)

Let $(\Phi\ \Psi)$ and $(\xi\ \eta)$ be nontrivial solutions of the systems

$$(1-x^2)\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix} = \begin{pmatrix}\alpha & Q\\ B & \gamma\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \text{ and } (1-x^2)\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \begin{pmatrix}\overline{\alpha} & \overline{Q}\\ \overline{B} & \overline{\gamma}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \quad -1<x<1$$

where α, B, γ, Q. $\overline{\alpha}$, $\overline{B}$, $\overline{\gamma}$, and $\overline{Q}$ are polynomials in x, and

$$4B(-Q)-[\gamma-\alpha]^2>0$$

$$0<\overline{B}<B \qquad 4\overline{B}(-\overline{Q})-[\overline{\gamma}-\overline{\alpha}]^2>0 \quad .$$

$$4(B-\overline{B})(-(Q-\overline{Q}))-[\gamma-\overline{\gamma}-(\alpha-\overline{\alpha})]^2>0$$

Then, ξ vanishes at least once between any two zeros of Φ, unless $(\xi\ \eta)$ is a constant multiple of $(\Phi\ \Psi)$.

<u>Theorem 3.</u> (OSCILLATION THEOREM)

Consider the associated Prüfer differential equation

$$(1-x^2)\frac{d\theta}{dx}=\lambda\beta\cos^2\theta+[(\lambda+1)\gamma-\lambda\alpha]\sin\theta\cos\theta-(\lambda+1)Q\sin^2\theta$$

obtained from the system of differential equations

$$8(1-x^2)\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix}=\begin{pmatrix}\lambda\alpha & (\lambda+1)Q\\ \lambda\beta & (\lambda+1)\gamma\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}\quad -1<x<1 \tag{2.15}$$

where α, β, γ, and Q are polynomials in x. If

$$4\beta(-Q)-[\gamma-\alpha]^2>0$$

and $\theta(-1)=\tau$, such that $-\pi/2<\tau<\pi/2$ for each λ, then the solution θ is a continuous and strictly decreasing function of λ for fixed x on $-1<x<1$. Moreover

$$\lim_{\lambda\to\infty}\theta(x)=-\infty \quad and \quad \lim_{\lambda\to-\infty}\theta(x)=\infty$$

for $-1<x<1$.

<u>Theorem 4.</u> Consider the system of differential equations (2.15) which satisfies the Oscillation Theorem. If

$$\theta(-1)=\tau,\quad -\frac{\pi}{2}<\tau<\frac{\pi}{2}\quad \text{for all} -\infty<\lambda<\infty$$
$$\theta(1)=\sigma-n\pi,\quad \frac{-\pi}{2}<\sigma<\frac{\pi}{2}\quad n=0,\pm1,\pm2,\ldots$$

then (2.15) has an infinite sequence of real eigenvalues

$$\ldots<\lambda_{-2}<\lambda_{-1}<\lambda_0<\lambda_1<\lambda_2<\ldots$$

with

$$\lim_{n\to\infty}\lambda_n=\infty \quad \textit{and} \quad \lim_{n\to-\infty}\lambda_n=-\infty \quad .$$

The eigenfunction Φ_n belonging to the eigenvalue λ_n has exactly n zeros in the interval -1<x<1, and is uniquely determined up to a constant factor.

1. **Phase Plane**

The objective of this section is to convert to polar coordinates the following general system of differential equations

$$\begin{aligned}(1-x^2)\Phi' &= \alpha\Phi + \rho\Psi \\ (1-x^2)\Psi' &= \beta\Phi + \gamma\Psi\end{aligned} \qquad -1<x<1 \qquad (2.16)$$

where α, β, γ, and Q are polynomials in x. By introducing polar coordinates

$$\Phi=r(x)\cos\theta(x), \quad \Psi=r(x)\sin\theta(x)$$

where θ is positive in the counterclockwise direction, the new dependent variables become r and θ.

Expressing r and θ explicitly as functions of Φ and Ψ leads to

$$\theta = \arctan\left(\frac{\Psi}{\Phi}\right) \quad \textit{and} \tag{2.17}$$

$$r^2 = \Phi^2 + \Psi^2 \quad . \tag{2.18}$$

Then, by differentiating both sides of (2.17) and (2.18), an equivalent system of differential equations for (2.16) is presented as functions of r and θ. Differentiating both sides of (2.17) yields

$$\sec^2\theta \, \frac{d\theta}{dx} = \frac{\Psi'\Phi - \Phi'\Psi}{\Phi^2} \quad . \tag{2.19}$$

Since

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{\Psi}{\Phi}\right)^2 = \frac{\Phi^2 + \Psi^2}{\Phi^2}$$

then (2.19) becomes

$$\frac{d\theta}{dx} = \frac{\Psi'\Phi - \Phi'\Psi}{\Phi^2 + \Psi^2} \quad . \tag{2.20}$$

By substituting from the general system (2.16), equation (2.20) takes the form

$$(1-x^2)\frac{d\theta}{dx} = \frac{[\beta\Phi + \gamma\Psi]\Phi - [\alpha\Phi + \varrho\Psi]\Psi}{\Phi^2 + \Psi^2} \quad .$$

Substituting in polar coordinates, the differential equation for θ becomes

$$(1-x^2)\frac{d\theta}{dx}=\beta\cos^2\theta+(\gamma-\alpha)\sin\theta\cos\theta-\rho\sin^2\theta \quad . \tag{2.21}$$

Finally, we differentiate both sides of (2.18) and follow similar steps. The differential equation for r then becomes

$$(1-x^2)\frac{dr}{dx}=r[\alpha\cos^2\theta+(\rho+\beta)\sin\theta\cos\theta+\gamma\sin^2\theta] \quad . \tag{2.22}$$

Therefore, (2.21) and (2.22) represent an equivalent system to (2.16).

The advantage of this system over (2.16) is that (2.21) is a first order differential equation of only θ. This allows the study of the oscillatory behavior of the solution trajectories independent of r. To find r for any solution θ, one solves (2.22) directly.

We also note that r is always positive for nontrivial solutions. This can be seen by examining the solution of (2.22) [Ref. 2:p. 312].

Next, we recall the following result from elementary algebra. A function

$$F(x,y) = a\ x^2 + b\ xy + c\ y^2$$

is negative definite if and only if

$$a<0 \quad and \quad 4ac - b^2>0$$

and positive definite if and only if

$$a>0 \quad and \quad 4ac - b^2>0 \quad .$$

Next, we apply this result to (2.21). If we can show that

$$4\beta(-\Omega)-[\gamma-\alpha]^2>0$$

then equation (2.21) is either negative definite or positive definite, depending on the sign of β. If equation (2.21) is positive definite, then the change in the angle Θ is always greater than zero. This means that the angle Θ is always increasing. Since our polar coordinate system is set up so that the positive direction of rotation is counterclockwise, then the solution trajectory of Θ always rotates counterclockwise. If equation (2.21) is negative definite, then the solution trajectory of Θ always rotates clockwise. In addition, since the change in the angle Θ is either positive definite or negative definite, then the direction of the solution trajectory Θ can never change. Therefore, we can conclude that once a solution trajectory crosses an axis it can not cross back.

Also, since $\Phi = r\cos\Theta$, then the zeros of Φ occur when $\Theta = \pi/2+n\pi$ where $n = 0, \pm1, \pm2, \ldots$.

Consider the system of differential equations (2.14). The associated Prüfer equation is negative definite for $\lambda>0$, and positive definite for $\lambda<0$. However, for (2.14), it is easy to determine the direction of rotation across the Φ-axis and Ψ-axis without the Prüfer equation. First, we examine the case when $\lambda>0$ by directly substituting into (2.14).

By setting $\Phi = 0$ we can determine the direction of rotation along the Ψ-axis. Also, by setting $\Psi = 0$ we can determine the direction of rotation along the Φ-axis. Hence, if we substitute these values back into (2.14), then the corresponding phase plane for $\lambda>0$ takes the form in Figure 1.

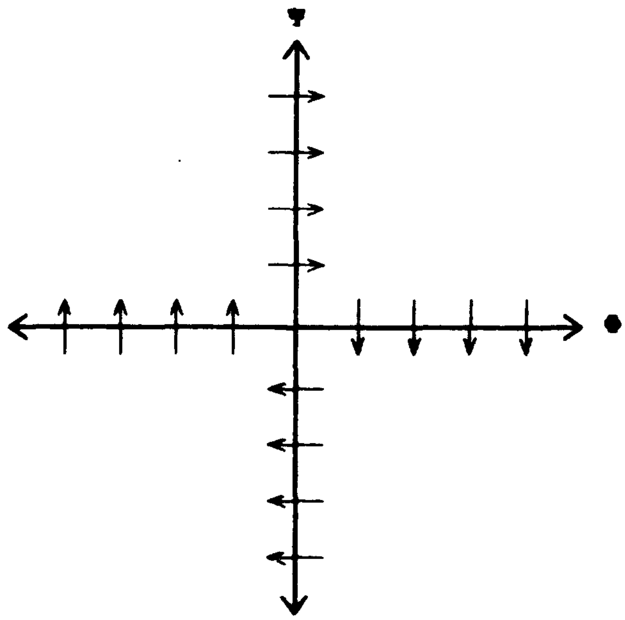

Figure 1. Phase Plane Analysis Along Ψ-axis and Φ-axis for $\lambda>0$

Clearly, the solution trajectories can cross the Φ-axis and Ψ-axis only in a clockwise direction. In the case when $\lambda<0$, the trajectories rotate in the counterclockwise direction.

2. **Sturm Separation Theorem**

Suppose $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ are two linearly independent solutions of the system

$$(1-x^2)\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix} = \begin{pmatrix}\alpha & Q\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \quad -1<x<1 \tag{2.23}$$

and

$$4\beta(-Q)-[\gamma-\alpha]^2>0$$

where α, β, γ, and Q are polynomials in x.

There are two cases to consider, one being when $\beta<0$ and the other when $\beta>0$. First, we consider the case when $\beta<0$.

We examine the area "A" of the triangle created by joining the points $(\Phi\ \Psi)_1$, $(\Phi\ \Psi)_2$, and the origin in the (Φ,Ψ) plane, Figure 2.

By inspection, the area "A" is equal to zero only when $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ are on the same straight line through the origin. This can occur with $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ both in the same quadrant, or in opposite quadrants. However, this can never occur since $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ are linearly independent, so "A" has positive area.

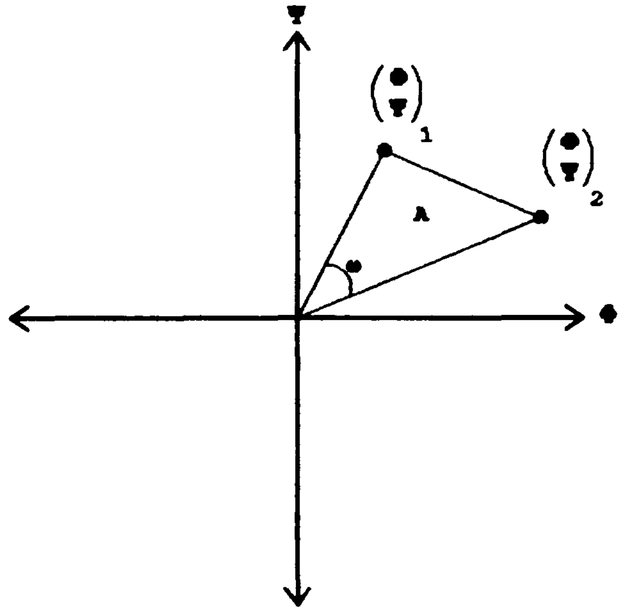

Figure 2. Area Principle Analysis

Recall that two nontrivial solutions of (2.23) are linearly dependent if and only if they are located on the same straight line through the origin in the (Φ,Ψ) plane. Hence, the angle, ω, between the linearly independent solutions, $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$, is always $0<\omega<\pi$. In other words, $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ are always in the same rotating half plane and never cross each other.

Analytically, we can show that the area "A" is never zero by the following method. Using the area principle, we calculate the area "A" using the following formula

$$A=\tfrac{1}{2}\begin{vmatrix}1 & \Phi_1 & \Psi_1\\ 1 & \Phi_2 & \Psi_2\\ 1 & 0 & 0\end{vmatrix} = \tfrac{1}{2}\begin{vmatrix}\Phi_1 & \Psi_1\\ \Phi_2 & \Psi_2\end{vmatrix} = \tfrac{1}{2}(\Phi_1\Psi_2-\Phi_2\Psi_1) \quad . \qquad (2.24)$$

The change in the area is

$$\frac{dA}{dx} = \tfrac{1}{2}[\Phi_1'\Psi_2 + \Phi_1\Psi_2' - \Phi_2'\Psi_1 - \Phi_2\Psi_1'] \quad .$$

Substituting in Φ'_1, Φ'_2, Ψ'_1, and Ψ'_2 from (2.16) and simplifying gives

$$\frac{dA}{dx}=\tfrac{1}{2}(\alpha+\gamma)(\Phi_1\Psi_2 - \Phi_2\Psi_1) \quad .$$

Substituting into this equation from (2.24) results in

$$\frac{dA}{dx}=(\alpha+\gamma)A \quad .$$

Finally, by solving for "A", the following formula is derived to determine the area "A" at any point $-1<x<1$;

$$A=k\ \exp\left\{\int_{-1}^{x}(\alpha+\gamma)\,dt\right\}$$

where $k = A(-1)$ is the initial area.

Since α and γ are polynomials and $-1<x<1$, the area is finite. Moreover, $A(-1)$ can not equal zero since the two solutions are linearly independent. Hence, the area "A" can never be zero.

Next, we recall that $\beta<0$ and

$$4\beta(-Q)-[\gamma-\alpha]^2>0 \quad .$$

Therefore, $d\Theta/dx$ is negative definite which implies that all solutions rotate in a clockwise direction. Also, once a solution crosses the Ψ-axis, it can never cross back. Finally, since the area can not equal zero, the angle ω between $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ is always satisfies $0<\omega<\pi$. Therefore, as the solutions rotate clockwise in the (Φ,Ψ) plane, $(\Phi\ \Psi)_1$ and $(\Phi\ \Psi)_2$ always cross the Ψ-axis in the same order. Therefore, the zeros of Φ_1 and Φ_2 interlace. The analysis for the case when $\beta>0$ proceeds in exactly the same way except that the solutions rotate counterclockwise. q.e.d.

3. **Sturm Comparison Theorem**

Let $(\Phi\ \Psi)$ and $(\xi\ \eta)$ be nontrivial solutions of the systems

$$(1-x^2)\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix} = \begin{pmatrix}\alpha & Q\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \text{ and}$$

$$(1-x^2)\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \begin{pmatrix}\overline{\alpha} & \overline{Q}\\ \overline{\beta} & \overline{\gamma}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \quad -1<x<1 \qquad (2.25)$$

respectively, where α, β, γ, Q, $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, and $\overline{Q}$ are polynomials in x, and

$$4\mathcal{B}(-\mathcal{C})-[\gamma-\alpha]^2>0$$

$$o<\bar{\mathcal{B}}<\mathcal{B} \qquad 4\bar{\mathcal{B}}(-\bar{\mathcal{C}})-[\bar{\gamma}-\bar{\alpha}]^2>0$$

$$4(\mathcal{B}-\bar{\mathcal{B}})(-(\mathcal{C}-\bar{\mathcal{C}}))-[\gamma-\bar{\gamma}-(\alpha-\bar{\alpha})]^2>0$$

Again, there are two main cases to consider. One case is when $\mathcal{B}-\bar{\mathcal{B}} > 0$ and the other when $\mathcal{B}-\bar{\mathcal{B}} < 0$. The second case is the same as the first except that the solutions rotate in the counterclockwise direction. Hence, it suffices to consider only the case when $\mathcal{B}-\bar{\mathcal{B}} > 0$ and that all solutions rotate clockwise. The objective is to analyze the two solutions in the (Φ,Ψ) plane through the use of the Prüfer equation.

The associated Prüfer equations for (2.25) are listed below, respectively.

$$(1-x^2)\frac{d\theta_1}{dx}=\mathcal{B}\cos^2\theta_1+(\gamma-\alpha)\sin\theta_1\cos\theta_1-\mathcal{C}\sin^2\theta_1 \qquad (2.26)$$

$$(1-x^2)\frac{d\theta_2}{dx}=\bar{\mathcal{B}}\cos^2\theta_2+(\bar{\gamma}-\bar{\alpha})\sin\theta_2\cos\theta_2-\bar{\mathcal{C}}\sin^2\theta_2 \qquad (2.27)$$

By subtracting (2.27) from (2.26), the resulting equation indicates the rate of change of the angle formed between $(\Phi\ \Psi)$ and $(\xi\ \eta)$. In addition, by superimposing one solution on top of the other, with $\theta_1 = \theta_2$ (Figure 3), then the equation becomes

$$(1-x^2)\frac{d}{dx}(\theta_1-\theta_2) = (\beta-\overline{\beta})\cos^2\theta_1 + [\gamma-\overline{\gamma}-(\alpha-\overline{\alpha})]\sin\theta_1\cos\theta_1 - (\rho-\overline{\rho})\sin^2\theta_1 \quad .$$

The resulting equation is a Prüfer equation. This equation facilitates the analysis of the solution vectors $(\Phi\ \Psi)$ and $(\xi\ \eta)$ by examining the behavior of the two angles, θ_1 and θ_2.

Since $\beta-\overline{\beta} > 0$, and

$$4(\overline{\beta}-\beta)(-(\rho-\overline{\rho}))-[\gamma-\overline{\gamma}-(\alpha-\overline{\alpha})]^2>0$$

then, $d/dx(\theta_1-\theta_2) > 0$. This means the rate of change of the angle between $(\Phi\ \Psi)$ and $(\xi\ \eta)$ is always increasing, and $(\xi\ \eta)$ "runs away" from $(\Phi\ \Psi)$, Figure 4.

Note that $(\Phi\ \Psi)$ and $(\xi\ \eta)$ are solutions of two different systems of differential equations. The two solution may be linearly independent or dependent. Hence, one solution could "lap" the other solution one or more times.

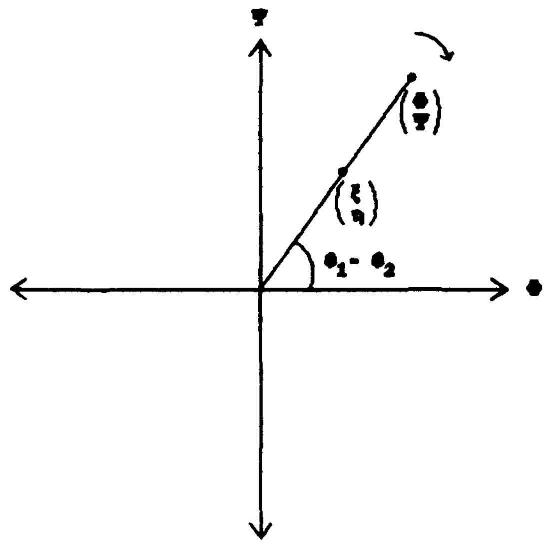

Figure 3. Superimposition of Solutions

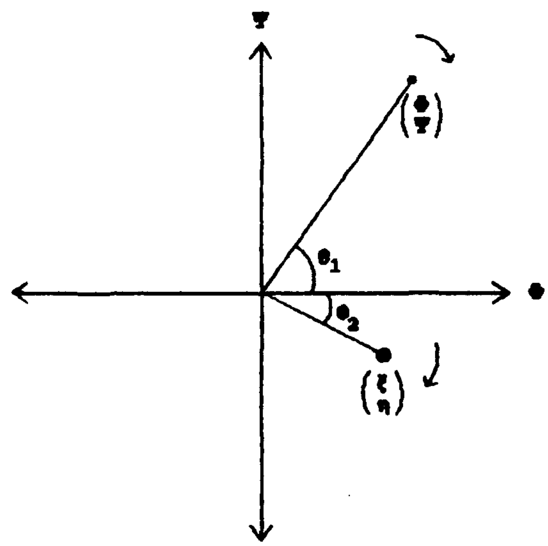

Figure 4. One Solution "Runs Away" from the Other

Recall that the zeros of Φ and ξ occur when $(\Phi\ \Psi)$ and $(\xi\ \eta)$ respectively cross the Ψ-axis. Therefore, as the solutions continue to rotate, between any two zeros of Φ there is at least one zero of ξ. Note that, if $(\Phi\ \Psi) = c(\xi\ \eta)$, for all $-1<x<1$, then the zeros occur simultaneously. Hence, ξ vanishes at least once between any two zeros of Φ, unless $(\xi\ \eta)$ is a constant multiple of $(\Phi\ \Psi)$. q.e.d.

4. Oscillation Theorem

We consider the system of differential equations of the form

$$(1-x^2)\begin{pmatrix}\Phi' \\ \Psi'\end{pmatrix} = \begin{pmatrix}\lambda\alpha & (\lambda+1)Q \\ \lambda\beta & (\lambda+1)\gamma\end{pmatrix}\begin{pmatrix}\Phi \\ \Psi\end{pmatrix} \quad -1<x<1 \tag{2.28}$$

where α, β, γ, and Q are polynomials in x, and

$$4\beta(-Q)-[\gamma-\alpha]^2>0 \quad .$$

The system (2.28) is basically the same as (2.23) except now we have introduced an eigenvalue, λ. The objective now is to vary λ thereby to study the zeros of Φ. The Prüfer equation for (2.28) is

$$(1-x^2)\frac{d\theta_1}{dx}=\lambda\beta\cos^2\theta_1+[(\lambda+1)\gamma-\lambda\alpha]\sin\theta_1\cos\theta_1 \\ -(\lambda+1)Q\sin^2\theta_1 \quad . \tag{2.29}$$

As before, the zeros occur only on the Ψ-axis which is the same as saying the zeros occur at $\theta = \pi/2+n\pi$ for n = 0, ±1, ±2,.... Next, we impose endpoint conditions on θ such that

$$
\begin{aligned}
&\theta(-1)=\tau, \quad -\frac{\pi}{2}<\tau<\frac{\pi}{2} \quad \textit{for all} -\infty<\lambda<\infty \\
&\theta(1)=\sigma-n\pi, \quad \frac{-\pi}{2}<\sigma<\frac{\pi}{2} \quad n=0,\pm1,\pm2,\ldots
\end{aligned}
\tag{2.30}
$$

so that the angle τ is as close to $\pi/2$ as possible. Note that the endpoint conditions are computed from $\tan\theta = \Psi/\Phi$.

Now, we consider a second system

$$
(1-x^2)\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \begin{pmatrix}\mu\alpha & (\mu+1)\Omega\\ \mu\beta & (\mu+1)\gamma\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \quad -1<x<1
$$

where μ is an eignevalue. The Prüfer equation for this system is

$$
\begin{aligned}
(1-x^2)\frac{d\theta_2}{dx}=\mu\beta\cos^2\theta_2+[(\mu+1)\gamma-\mu\alpha]\sin\theta_2\cos\theta_2 \\
-(\mu+1)\Omega\sin^2\theta_2 \quad .
\end{aligned}
\tag{2.31}
$$

Suppose that x is fixed and the differential equation (2.31) satisfies the following endpoint condition

$$
\theta(-1)=\tau, \quad \frac{-\pi}{2}<\tau<\frac{\pi}{2} \quad \textit{for} \quad -\infty<\mu<\infty \quad .
$$

Following the same basic proof as in the Sturm Comparison Theorem, we subtract (2.29) from (2.31), and equate θ_1 to θ_2 at $\theta_1(-1) = \theta_2(-1) = \tau$. Hence,

$$(1-x^2)\frac{d}{dx}(\theta_2-\theta_1) = (\mu-\lambda)[\beta\cos^2\theta_1+(\gamma-\alpha)\sin\theta_1\cos\theta_1-\rho\sin^2\theta_1] \quad . \tag{2.32}$$

Since

$$4\beta(-\rho)-[\gamma-\alpha]^2>0$$

then the discriminant of (2.32) is

$$(\mu-\lambda)^2[4\beta(-\rho)-(\gamma-\alpha)^2]>0 \quad .$$

Therefore, $d/dx(\theta_2-\theta_1)$ is either positive definite or negative definite. Accordingly, there are several cases to consider. However, by examining one case carefully, all other cases follow directly, with only the direction of rotation varying. Thus, it suffices to show only one case.

Let x be fixed. Suppose that $0<\mu<\lambda$ which implies that $\mu-\lambda < 0$. Now, consider the case when $\beta<0$.

Since $\beta<0$ and $\mu-\lambda<0$, then $d/dx(\theta_2-\theta_1) > 0$. Hence, by considering the conditions that $0<\mu$ and $0<\lambda$, it is implied that all solutions rotate in the clockwise direction, and $(\Xi\ \Upsilon)$ "runs away" from $(\xi\ \eta)$ as λ goes to positive infinity, Figure 5.

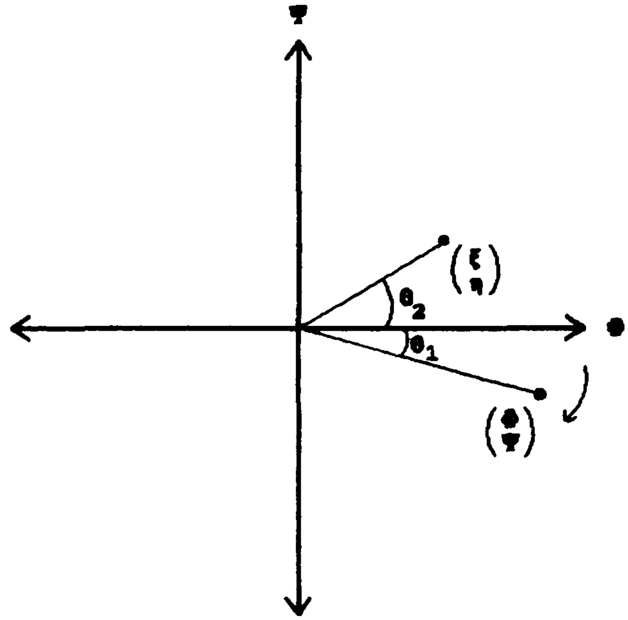

Figure 5. $(\Phi\ \Psi)$ "Runs Away" from $(\xi\ \eta)$, $\lambda>0$

If $\lambda<0$, then $\mu-\lambda > 0$ which implies that $d/dx(\theta_2-\theta_1) < 0$. Therefore, when $\lambda<0$, all solutions rotate in the counterclockwise direction and $(\Phi\ \Psi)$ "runs away" from $(\xi\ \eta)$ as λ goes to negative infinity, Figure 6.

In summary, as λ goes from $-\infty$ to ∞, the angle θ_1 goes from ∞ to $-\infty$ in a strictly decreasing manner. Hence, for fixed x, θ is a strictly decreasing function of the variable λ. In addition,

$$\lim_{\lambda\to\infty} \theta=-\infty \quad \textit{and} \quad \lim_{\lambda\to-\infty} \theta=\infty \quad .$$

q.e.d.

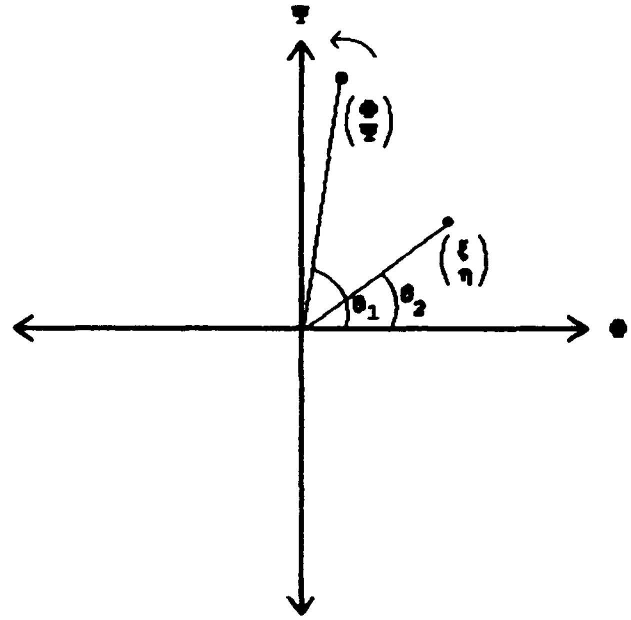

Figure 6. $(\Phi\ \Psi)$ "Runs Away" from $(\xi\ \eta)$, $\lambda<0$

5. **Sequence of Eigenvalues and Interlacing Zeros**

Consider the system of differential equations (2.15) which satisfies the Oscillation Theorem. Now we continue with our analysis and maintain the conditions of the previous proof. Namely, let $0<\mu<\lambda$ and

$$\theta(-1)=\tau,\ \frac{-\pi}{2}<\tau<\frac{\pi}{2},\ \textit{for all}\ -\infty<\lambda<\infty$$

$$\theta(1)=\sigma-n\pi,\ \frac{-\pi}{2}<\sigma<\frac{\pi}{2},\ n=0,\pm1,\pm2,\ldots$$

where τ is as close to $\pi/2$ as possible.

Now, we let μ be fixed and $\beta<0$. Note that a value for λ which satisfies both endpoint conditions is an eigenvalue for the system of differential equations (2.28). Let $\theta_1(-1) = \tau$. Then, as λ increases, the solution trajectory of θ_1 rotates in a clockwise direction. Since $d\theta_1/dx < 0$, the trajectory of θ_1 eventually crosses $\theta_1(1) = \sigma - n\pi$ ($n = 0$). Hence, the first time that the trajectory of θ_1 crosses the line $\theta_1(1) = \sigma - n\pi$ is when $n = 0$. We call this eigenvalue λ_0. Therefore, when $\lambda = \lambda_0$, the solution trajectory of θ_1 does not cross the Ψ-axis. Hence, there are no interior zeros for the corresponding eigenfunction Φ_0 of λ_0, Figure 7.

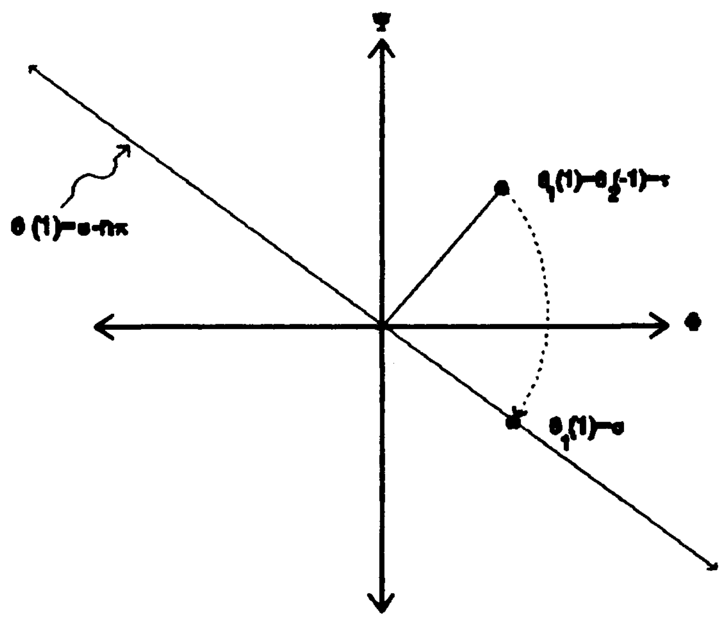

Figure 7. Interior Zeros of Φ_0

Recall that for each λ_n, the corresponding eigenfunction is determined by the formula

$$\Phi_n = r\cos\theta \quad .$$

Now, as λ continues to increase, the solution trajectory of Θ_1 continues to rotate in the clockwise direction. Eventually, the trajectory of Θ_1 again crosses the line $\Theta_1(1) = \sigma - n\pi$ when $n = 1$, or $\Theta_1(1) = \sigma - \pi$. We call this eigenvalue λ_1. Therefore, when $\lambda = \lambda_1$, the solution trajectory of Θ_1 crosses the Ψ-axis one time. Therefore, the eigenfunction Φ_1 of λ_1 has one interior zero. This process continues as λ increases to ∞. Similarly, for $\lambda<0$, the solution trajectory of Θ_1 rotates counterclockwise and this process continues as λ decreases to $-\infty$.

In summary, as λ goes to $\pm\infty$, each time that the solution trajectory of Θ_1 crosses the line $\Theta_1(1) = \sigma + n\pi$, the corresponding value for λ becomes an eigenvalue of the system (2.28). Hence, there is an infinite sequence of λ_n for $n = 0, \pm 1, \pm 2, \ldots$ for which the second endpoint condition is satisfied. Therefore, there are eigenvalues, λ_n, such that $\ldots<\lambda_{-1}<\lambda_0<\lambda_1<\ldots$, and the associated eigenfunction belonging to λ_n has exactly n interior zeros in the interval $-1<x<1$. Moreover, it follows that

$$\lim_{n\to\infty}\lambda_n = \infty \quad \textit{and} \quad \lim_{n\to-\infty}\lambda_n = -\infty \quad .$$

In addition, any two solutions which satisfy the same initial conditions are linearly dependent. Therefore, each eigenvalue uniquely determines an eigenfunction up to a constant factor. This completes the proof of Theorem 5.

For the system of differential equations (2.14), the discriminant of the associated Prüfer equation is positive. Therefore, all of the previous theorems apply.

D. ORTHOGONALITY AND COMPLETENESS

The objective of this section is to show that in a Hilbert space the eigenvectors of (2.14) are complete and orthogonal with respect to a weight function. [Ref. 3:pp. 344-353]

1. Orthogonality.

The system of differential equations (2.14) is equivalent to the following form

$$(1-x^2)\begin{pmatrix}\Phi'\\ \Psi'\end{pmatrix} = \lambda\begin{pmatrix}2x & 1+3x^2\\ -1 & -2x\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} + \begin{pmatrix}0 & 1+3x^2\\ 0 & -2x\end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}. \quad (2.34)$$

For convenience, represent (2.34) with the notation

$$(1-x^2)\,\mathbf{Z}_\lambda' = \lambda\mathbf{A}\mathbf{Z}_\lambda + \mathbf{B}\mathbf{Z}_\lambda \quad (2.35)$$

where Z_λ is the eigenvector $(\Phi\ \Psi)$ associated with the eigenvalue λ. A and B are the 2×2 matrices given below

$$\mathbf{A} = \begin{pmatrix} 2x & 1+3x^2 \\ -1 & -2x \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 1+3x^2 \\ 0 & -2x \end{pmatrix} \quad .$$

In addition, we let L stand for the linear differential operator, such that, when L operates on Z then LZ is defined by

$$L\mathbf{Z} = (1-x^2)\,\mathbf{Z}' - \lambda \mathbf{A}\mathbf{Z} - \mathbf{B}\mathbf{Z} = 0 \quad .$$

The general strategy is to find the adjoint operator, $L_\dagger$, such that $W^T(LZ) - (L^\dagger W^T)Z$ is an exact derivative. Then, we integrate both sides to establish orthogonality. Finally, matrix algebra can be used to find the corresponding weight function.

Let W be a 2 x 1 column vector such that

$$\begin{aligned} W^T(L\mathbf{Z}) &= (1-x^2)\,W^T\mathbf{Z}' - \lambda W^T\mathbf{A}\mathbf{Z} - W^T\mathbf{B}\mathbf{Z} \\ &= [(1-x^2)\,W^T\mathbf{Z}]' - [(1-x^2)\,W^T]'\mathbf{Z} - \lambda W^T\mathbf{A}\mathbf{Z} - W^T\mathbf{B}\mathbf{Z} \quad . \end{aligned}$$

Then, by requiring the exact derivative

$$W^T(L\mathbf{Z}) - (L^* W^T)\,\mathbf{Z} = \frac{d}{dx}[(1-x^2)\,W^T\mathbf{Z}] \quad , \qquad (2.36)$$

it follows that

$$(L^* W^T)\,\mathbf{Z} = -\{[(1-x^2)\,W^T]'\,\mathbf{Z} + \lambda W^T\mathbf{A}\mathbf{Z} + W^T\mathbf{B}\mathbf{Z}\} \quad .$$

Hence,

$$L^{*}W=-\{[(1-x^2)W]'+\lambda A^T W+B^T W\}=0 \quad .$$

For general μ, the adjoint differential equation is

$$\begin{aligned}&[(1-x^2)W^T]' = -\mu W^T A - W^T B \qquad or \\ &[(1-x^2)W]' = -\mu A^T W - B^T W \quad .\end{aligned} \tag{2.37}$$

Next, we return to (2.36). Substituting $W^T(LZ)$ and $(L^*W^T)Z$ into the equation and simplifying yields

$$W^T(LZ)-(L^*W^T)Z=\frac{d}{dx}[W^T(1-x^2)Z]$$

which simplifies to

$$W^T(1-x^2)Z'+[W^T(1-x^2)]'Z=\frac{d}{dx}[W^T(1-x^2)Z] \quad .$$

Substituting in (2.35) and (2.37) results in

$$(\lambda-\mu)W^T AZ=\frac{d}{dx}[W^T(1-x^2)Z] \quad .$$

Now, we integrate both sides. This leads to the following orthogonality relationship

$$(\lambda-\mu)\int_{-1}^{1}W^T AZ=(W^T(1-x^2)Z)\Big|_{-1}^{1}=0 \quad . \tag{2.38}$$

If the integrand is well defined, then the eigenvectors W are orthogonal to those of Z, with respect to the matrix weight function A. Next, we determine the nature of W.

Expanding (2.37) leads to

$$(1-x^2)\,W' = 2xW - \mu\mathbf{A}^T W - \mathbf{B}^T W \quad . \qquad (2.39)$$

We write W in the form W = JΣ where Σ is a 2 x 1 column vector, and J is a 2 x 2 matrix of the form

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad .$$

Then, (2.39) becomes

$$(1-x^2)\,J\Sigma' = 2xJ\Sigma - \mu\mathbf{A}^T J\Sigma - \mathbf{B}^T J\Sigma \quad .$$

Then we, premultiply both sides by J^{-1}. Note that $J^{-1} = -J$. Thus,

$$(1-x^2)\,\Sigma' = 2x\Sigma + \mu J\mathbf{A}^T J\Sigma + J\mathbf{B}^T J\Sigma \quad . \qquad (2.40)$$

Since

$$J\mathbf{A}^T J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2x & 3x^2+1 \\ -1 & -2x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{A}$$

and

$$JB^TJ = \begin{pmatrix} 2x & 3x^2+1 \\ 0 & 0 \end{pmatrix}$$

then (2.40) simplifies to

$$(1-x^2)\,\Sigma' = \mu A\Sigma + (B+4xI)\,\Sigma \qquad (2.41)$$

where I is the identity matrix. Next, letting

$$\Sigma = (1-x^2)^{-2}\Omega \qquad \textit{and}$$

$$\Sigma = 4x(1-x^2)^{-3}\Omega + (1-x^2)^{-2}\Omega$$

where Ω is a 2 x 1 column vector and substituting into (2.41) gives

$$(1-x^2)\,\Omega' = \mu A\Omega + B\Omega \quad . \qquad (2.42)$$

So, $\Omega = Z_\mu$ where Z_μ is the eigenvector corresponding to the eigenvalue μ. Back substituting results in

$$W = J\Sigma = J(1-x^2)^{-2}\Omega = J(1-x^2)^{-2}Z_\mu \quad .$$

If $\lambda \neq \mu$, then the orthogonality relationship (2.38) becomes

$$(\lambda-\mu)\int_{-1}^{1} Z_\mu^T \, \frac{J^T A}{(1-x^2)^2} \, Z_\lambda \, dx = 0 \quad .$$

Since $\lambda \neq \mu$, then $\lambda-\mu \neq 0$ which implies

$$\int_{-1}^{1} \mathbf{Z}_\mu^T \frac{J^T\mathbf{A}}{(1-x^2)^2} \mathbf{Z}_\lambda \, dx = 0 \quad .$$

When $\lambda = \mu$, then

$$\int_{-1}^{1} \mathbf{Z}_\lambda^T \frac{J^T\mathbf{A}}{(1-x^2)^2} \mathbf{Z}_\lambda \, dx \neq 0 \qquad (2.43)$$

since the integrand is always positive. In addition, since the integrand is a polynomial, then it is finite. This implies Z_λ is square-integrable with respect to the matrix weight function

$$\frac{J^T\mathbf{A}}{(1-x^2)^2} \quad .$$

Initially, the $(1-x^2)^{-2}$ term may seem to imply that the integrand is not well defined at ±1. However, $(1-x^2)^{-2}$ cancels in (2.43), and all the remaining entries are polynomials.

Next, we determine the sequence of eigenvectors. Clearly, λ = 1, 2, 3,... all yield solutions. This is based on the definition of λ such that λ = n-1 where n = 2, 3, 4,.... Now, we consider λ = -1, -2, -3.... Since

$$P_{-(n+1)} = P_n$$

for the Legendre polynomials, the following relationships hold.

$$P_{\lambda+1} = P_{-[(\lambda+1)+1]} = P_{-\lambda-2} \tag{2.44}$$

$$P_{\lambda+2} = P_{-[(\lambda+2)+1]} \; P_{-\lambda-3} \tag{2.45}$$

Substituting (2.44) and (2.45) into the original definition of $\Phi_{n,1}$ yields

$$\Phi_{n,1} = \begin{vmatrix} P_1 & P_2 \\ P_{\lambda+1} & P_{\lambda+2} \end{vmatrix} \rightarrow \Phi_{n,1} = \begin{vmatrix} P_1 & P_2 \\ P_{-\lambda-2} & P_{-\lambda-3} \end{vmatrix}$$

for $\lambda \geq 1$. Equivalently, for $\lambda \leq -1$,

$$\Phi_{n,1} = \begin{vmatrix} P_1 & P_2 \\ P_{\lambda-2} & P_{\lambda-3} \end{vmatrix} \quad .$$

This form provides the solution to the original differential equation (2.8). Therefore, there exists a sequence of eigenvectors

$$\ldots, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{-m}, \ldots, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{-2}, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{-1}, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{0}, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{1}, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{2}, \ldots, \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{m}, \ldots$$

where $(\Phi \; \Psi)$ has polynomial entries of degree $\leq m+3$.

2. Completeness

Finally, we determine that the sequence of continuous eigenvectors $(\Phi \; \Psi)_m$ for $m = 0, \pm 1, \pm 2, \pm 3, \ldots,$ which are orthogonal and square-integrable with respect to a matrix weight function, is complete. That is, we show that every square-integrable function $F = (1-x^2)\,(f\ g)^T$ can be expanded into an infinite series such that

$$(1-x^2)\begin{pmatrix} f \\ g \end{pmatrix} = \sum_{m=-\infty}^{\infty} c_m \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_m \quad .$$

Recall that there is a factor $(1-x^2)$ which cancels from both sides of the equation.

Consider the column vector $(p\ q)^T$ where p and q are polynomials of degree $\le m+3$. There exist constants c_{-m} and c_m such that

$$\begin{pmatrix} p \\ q \end{pmatrix} - \left[\; c_{-m} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{-m} + c_m \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_m \;\right]$$

is a column vector with polynomial entries of degree $\le m+2$. This follows by equating the coefficients of X^{m+3} and solving the resulting system of equations for the coefficients c_{-m} and c_m. Since $(\Phi\ \Psi)_{-m}$ and $(\Phi\ \Psi)_m$ are linearly independent, then c_{-m} and c_m exist. By mathematical induction, one can find constants c_k for $k = -m, \ldots, -1, 0, 1, \ldots, m$ such that

$$\begin{pmatrix} p \\ g \end{pmatrix} = c_{-m} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{-m} + \ldots + c_{-1} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_{-1} + c_0 \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_0 + c_1 \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_1 + \ldots + c_m \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_m$$

Finally, we let $F = (1-x^2)\ (f\ g)^T$ where f and g are any continuous functions on a finite closed interval $-1 \le x \le 1$, and let ϵ_1, $\epsilon_2 > 0$. Then, by the Weierstrass Approximation Theorem, there exist polynomials such that

$$\left|(1-x^2)\,f-P\right| < \epsilon_1 \quad \textit{and}$$

$$\left|(1-x^2)\,g-q\right| < \epsilon_2$$

for all x on $[-1,1]$. Hence,

$$\left|(1-x^2)\ \begin{pmatrix} f \\ g \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix}\ \right| = \left|\begin{pmatrix} (1-x^2)\,f-P \\ (1-x^2)\,g-q \end{pmatrix}\ \right| \le \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \le \max\,\{\epsilon_1, \epsilon_2\}\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which implies

$$\left|(1-x^2)\begin{pmatrix} f \\ g \end{pmatrix} - \sum_{k=-m}^{m} C_k \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_k\ \right| \le \max\,\{\epsilon_1, \epsilon_2\}\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad .$$

Therefore, F is approximated uniformly, within an arbitrarily small distance $\epsilon > 0$, by a linear combination of the $(\Phi\ \Psi)$. Hence, the sequence $\{(\Phi\Psi)_m\}$ for m = ... , -1, 0, 1, ... is a uniformly convergent sequence, which implies the sequence is mean square convergent on $-1 \le x \le 1$ with respect to an integrable positive weight function. Recall that $(1-x^2)^{-2}$ cancels in the integrand, leaving a positive definite matrix. Therefore, the sequence is complete. [Ref. 2:p. 353]

In addition, the constants c_k are determined in the following way

$$C_k = \frac{\int_{-1}^{1} \mathbf{Z}_k^T \frac{J^T \mathbf{A}}{(1-x^2)^2} (1-x^2) F \, dx}{\int_{-1}^{1} \mathbf{Z}_k^T \frac{J^T \mathbf{A}}{(1-x^2)^2} \mathbf{Z}_k \, dx} .$$

III. RESULTS AND CONCLUSIONS

A. RESULTS

We have shown in this thesis that there is a special function which takes the form

$$\Phi_{n,k}(x) = \begin{vmatrix} P_k(x) & P_{k+1}(x) \\ P_n(x) & P_{n+1}(x) \end{vmatrix} \quad \begin{matrix} (n>k) & k=0,1,2,\ldots \\ (-1<x<1) & n=1,2,3,\ldots, \end{matrix} \tag{3.1}$$

where the elements of the determinant are Legendre polynomials, and possesses the essential Sturm-Liouville properties. In addition, several interesting results are obtained and listed below.

1. Hypergeometric Functions

The Legendre differential equation is a special case of the hypergeometric differential equation. This suggests that a determinant of the form (3.1), with other hypergeometric functions as elements, might possess similar properties.

Recall that the recursion formula

$$P_{n+1}(x) = xP_n(x) - \frac{(1-x^2)}{(n+1)} P_n'(x)$$

and the Legendre differential equation are used extensively in the initial development of (3.1). In fact, it can be shown that since the hypergeometric function can be expressed in a recursion formula of the form

$$F(r+1\,;\,x) = A(r\,;\,x)F(r\,;\,x) + B(r\,;\,x)F'(r\,;\,x)\ ,$$

then the determinant (3.1), with other hypergeometric functions as elements and appropriate boundary conditions, possesses the same Sturm-Liouville properties.

2. Turan's Inequality

The proof of Turan's inequality is a straight forward application of the general case (3.1). Setting n = k+1 and substituting back into (3.1) results in a determinant that is of Turan's form. Recall, from the previous analysis of (3.1), that there exists a λ such that (3.1) has no zeros in the interval $-1<x<1$. Hence, Turan's inequality is established.

3. General Systems of Differential Equations

In general, boundary value problems of systems of differential equations are not well-posed. Until now, very little has been established concerning systems with polynomial coefficients. In Chapter II, several theorems are developed and proved for such systems. Hence, if the conditions of the theorems are satisfied, then the system is said to be of Sturm-Liouville form.

B. CONCLUSION

The analysis of this one problem presents several possible avenues of further research. Only the first step was taken in a direction which could result in establishing several new and substantial results. For example, the next step would be to establish the case when (3.1) has other hypergeometric functions as elements. Then, one could continue the research on Sturm-Liouville systems of differential equations by examining the possibility of expanding the necessary and sufficient conditions for such problems. Research on this subject is ongoing.

APPENDIX

I. PROOF FOR GENERAL CASE - $\Phi_{n,k}(x)$

A. PURPOSE

In this appendix we briefly outline the procedure for establishing the Sturm-Liouville properties for the general case, $\Phi_{n,k}(x)$. The basic procedure is very similar to that illustrated in Chapter II. Thus, only the important steps are highlighted.

B. DEVELOP DIFFERENTIAL EQUATION

By following the procedure in Chapter II, it is easy to establish the following relations for $\Phi_{n,k}(x)$, $\Phi'_{n,k}(x)$, and $\Phi''_{n,k}(x)$.

$$\Phi = -(1-x^2)\left[\frac{P_k P'_n}{n+1} - \frac{P_n P'_k}{k+1}\right] \tag{1}$$

$$\Phi' = \frac{(n-k)(1-x^2)}{(n+1)(k+1)} P'_k P'_n + (n-k) P_n P_k \tag{2}$$

$$\Phi'' = (n-k)\left[\frac{(n-k+1)}{n+1} P'_n P_k + \frac{2x}{(n+1)(k+1)} P'_n P'_k + \frac{(k-n+1)}{k+1} P_n P'_k\right] \tag{3}$$

Equations (1), (2), and (3) lead to the differential equation

$$(1-x^2)\left[P_k^2+\frac{(1-x^2)}{(k+1)^2}(P_k')^2\right]\Phi''-(1-x^2)\left[\frac{2}{k+1}P_kP_k'+\frac{2x}{(k+1)^2}(P_k')^2\right]\Phi'+ \tag{4}$$

$$(n-k)\left[(n-k+1)P_k^2+\frac{2x}{k+1}P_kP_k'-\frac{(k-n+1)(1-x^2)}{(k+1)^2}(P_k')^2\right]\Phi=0$$

where the general solution is

$$C_1\begin{vmatrix}P_k & P_{k+1}\\ P_n & P_{n+1}\end{vmatrix}+C_2\begin{vmatrix}P_k & P_{k+1}\\ Q_n & Q_{n+1}\end{vmatrix} .$$

Applying appropriately translated boundary conditions eliminates all logarithms, Q_n , in the general solution. Now, setting λ = n-k-1 and letting

$$Q=\frac{1-x^2}{(k+1)^2}(P_k')^2+P_k^2 \quad \textit{and} \tag{5}$$

$$Q'=\frac{2P_k'}{k+1}\left[\frac{x}{k+1}P_k'+P_k\right] \tag{6}$$

then (4) takes the form

$$(1-x^2)\Phi''-(1-x^2)\frac{Q'}{Q}\Phi'+(\lambda+1)\left[\lambda+\frac{2P_k(P_k+\frac{x}{k+1}P_k')}{Q}\right]\Phi=0\ . \tag{7}$$

C. REDUCTION TO A SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS

Next, we reduce (7) to a system of first order differential equations of the form

$$\begin{aligned}(1-x^2)\Phi' &= \alpha\Phi + Q\Psi \qquad -1<x<1 \\ (1-x^2)\Psi' &= \beta\Phi + \gamma\Psi\end{aligned} \tag{8}$$

where α, β, γ, and Q are polynomials in x.

First, note that Q and Q′ have no zeros in common. This implies that all polynomial coefficients of Φ, Φ', and Φ'' are reduced to lowest terms and only simple zeros occur in the denominator. The proof is as follows.

If $Q'=0$, then either $P_k'=0$ or

$$\frac{x}{k+1}P_k' + P_k = 0 \quad . \tag{9}$$

If $P_k'=0$, then Q=0 only when $P_k=0$. However, it is easy to show that P_k and P_k' are relatively prime. Hence, when $P_k'=0$, then $Q \neq 0$.

If (9) holds, then by solving for P_k and substituting back into Q it follows that Q=0 only when $P_k'=0$. But, by the first case, this can not occur. Therefore, Q and Q′ have no zeros in common.

By transforming (8) into a second order differential equation and equating coefficients of Φ'' and Φ' it is easy to establish that

$$\gamma = -2x-\alpha \quad . \tag{10}$$

Equating the coefficients of Φ and substituting (10) for γ, results in the following equation.

$$\alpha' + \frac{Q\beta}{1-x^2} - \frac{\alpha(-2x-\alpha)}{1-x^2} - \frac{\alpha Q'}{Q} = -\lambda(\lambda+1) - \frac{\lambda+1}{Q}\left[2P_k\left(P_k + \frac{x}{k+1}P_k'\right)\right] \tag{11}$$

Now, we isolate and examine the components of the equation that have Q in the denominator. The idea is to select α such that

$$\alpha Q' - (\lambda+1)\left[2P_k\left(P_k + \frac{x}{k+1}P_k'\right)\right] \equiv 0 \bmod(Q) \quad .$$

Substituting (6) for Q' implies

$$\alpha\left[\frac{2}{k+1}P_k'\left(\frac{x}{k+1}P_k' + P_k\right)\right] - (\lambda+1)\left[2P_k\left(P_k + \frac{x}{k+1}P_k'\right)\right] \equiv 0 \bmod(Q) \quad .$$

However,

$$\frac{x}{k+1}P_k' + P_k \tag{12}$$

and Q are relatively prime. This is shown by the following argument.

Set (12) equal to zero and then solve for P_k. By substituting back into Q it follows that (12) and Q have zeros in common when P_k and P_k' have zeros in common. However, P_k and P_k' are relatively prime. Hence, (12) and Q are relatively prime.

Therefore, since (12) and Q are relatively prime, then there exists a polynomial M such that

$$\frac{\alpha}{k+1}P_k' - (\lambda+1)P_k = M\left[\frac{(1-x^2)}{(k+1)^2}(P_k')^2 + P_k^2\right] \quad . \tag{13}$$

By examining (13) it follows that

$$(\lambda+1)P_k + MP_k^2 \equiv 0 \bmod (P_k') \quad .$$

Moreover, since P_k and P_k' are relatively prime, then there exists a polynomial N such that

$$(\lambda+1) + MP_k = NP_k' \quad . \tag{14}$$

However, by the Euclidean Algorithm for Polynomials, there exist unique polynomials M_1 and N_1 such that

$$M_1P_k + N_1P_k' = 1 \quad .$$

From (14), this implies that

$$M = -(\lambda+1)M, \quad \textit{and} \quad N = (\lambda+1)N_1 \quad .$$

Hence, there are unique polynomials M and N that determine α such that all the zeros of Q divide out.

Finally, choose β to satisfy (11).

D. STURM-LIOUVILLE PROPERTIES

Although it is somewhat tedious, it can be shown that

$$4\beta(-Q)-[\gamma-\alpha]^2>0 \quad .$$

Therefore, all of the theorems previously established apply. In addition, the properties of orthogonality and completeness follow in the same manner.

LIST OF REFERENCES

1. Karlin, S. and McGregor, J. L., "The differential equations of birth and death processes, and the Stieltjes moment problem," *Transactions of the American Mathematical Society*, v. 85, pp. 489-546, 1957.

2. Birkhoff, G., and Rota, G., *Ordinary Differential Equations*, John Wiley & Sons, Inc., 1989.

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