OFFICE OF NAVAL RESEARCH

Grant N00014-90-J-1193

TECHNICAL REPORT No. 77

Exact Solutions for a Mode of the Electromagetic Field in a Resonator with Time-Dependent Characteristics of the Internal Medium

by

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Prepared for publication

in

Journal of Soviet Laser Research

Departments of Chemistry and Physics
Washington State University
Pullman, WA 99164-1046

January 1992

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) WSU/DC/92/TR-77	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Depts. Chemistry & Physics Washington State University 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) 428 French Administration Building Pullman, WA 99164-1046	7b. ADDRESS (City, State, and ZIP Code) Chemistry Program 800 N. Quincy Street Arlington, Virginia 22217
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Office of Naval Research 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER Grant N00014-90-J-1193
8c. ADDRESS (City, State, and ZIP Code) Chemistry Program 800 N. Quincy Street Arlington, Virginia 22217	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
Exact Solutions for a Mode of the Electromagnetic Field in a Resonator with Time-Dependent Characteristics of the Internal Medium

12. PERSONAL AUTHOR(S)
V. V. Dodonov, T. F. George, O. V. Man'ko, C. I. Um and K. H. Yeon

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
	FROM ______ TO ______	January 1992	17

16. SUPPLEMENTARY NOTATION
Prepared for publication in Journal of Soviet Laser Research

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	ELECTROMAGNETIC FIELD; RESONATOR; TIME-DEPENDENT CHARACTERISTICS; INTERNAL MEDIUM; EXACT SOLUTIONS; COHERENT STATES

19. ABSTRACT (Continue on reverse if necessary and identify by block number)
The problem of quantization of the electromagnetic field inside a resonator filled with a dielectric medium with time-dependent characteristics in the presence of the external alternating current is studied. The exact propagator, coherent and Fock's states are obtained for a separate quantized mode of the field represented by a quantum oscillator with time-dependent frequency linearly coupled via momentum with the external current. Delta-pulse excitation of the medium is considered as an example.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr. John C. Pazik	22b. TELEPHONE (Include Area Code) (202) 696-4410 22c. OFFICE SYMBOL

DD Form 1473, JUN 86 Previous editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

Journal of Soviet Laser Research. in press

Exact solutions for a mode of the electromagnetic field in a resonator with time-dependent characteristics of the internal medium

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Abstract.

The problem of quantization of the electromagnetic field inside a resonator filled with a dielectric medium with time-dependent characteristics in the presence of ~~the~~ an external alternating current is studied. The exact propagator. coherent and Fock's states are obtained for a separate quantized mode of the field represented by a quantum oscillator with time-dependent frequency linearly coupled via momentum with the external current. Delta-pulse excitation of the medium is considered as an example.

1. Introduction.

Time-dependent systems are observed in various physics experiments. Two general types of such systems are: that which is formed through its own environmental conditions, and that which is formed when external forces are added. In regard to the second type, various experiments are being carried out to see how an applied time-dependent electric, magnetic or other field can alter the physical properties of materials such as semiconductors and superconductors. Experiments show that a system becomes time-dependent when a time-dependent electric or magnetic field (such as a.c.) is applied.

The aim of the present paper is to investigate the behaviour of a quantized mode of the electromagnetic field inside a resonator filled with a dielectric medium acted on by some external "pumping" electromagnetic field and external alternating current. Due to this "pumping" the properties of the internal medium become time-dependent.

First we study the problem of quantization of the electromagnetic field in the case of a time-dependent medium and external current. We show that the dynamics of every mode is the same as the dynamics of a time-dependent forced harmonic oscillator. Therefore, the rest of the paper is devoted to the problem of finding exact solutions for the time-dependent forced harmonic oscillator. This last problem was considered by many authors: see, e.g., refs. /1-13/ and especially /14-19/ where quantum systems with the most general quadratic multidimensional Hamiltonians were studied in detail. The difference of our study from the previous ones consists in the unusual coupling with the external force: not through coordinate but via momentum. Besides, we represent the known formulas for the propagators and eigenfunctions in another parametrization than in the previous studies and calculate some products of matrix elements which were not given earlier.

The special case of time-dependences of the frequency and external force in the form of delta-pulses is considered to illustrate general formulas.

2. Field quantization in a time-dependent medium in the presence of an external current.

The well-known usual scheme of quantization of an electromagnetic field in a uniform time-independent medium (see, e.g. /20/) results in the replacement of the field variables by a countable set of generalized canonically conjugated coordinate and momentum operators obeying harmonic oscillator equation of motion. Here we consider a more general case when dielectric and magnetic permeabilities depend on time (but not depend on space variables), and some time-dependent external current is present. The same problem, but without external current, was considered in /21/, and the most general case of nonuniform and time-dependent medium (also without current) was studied in /22/.

For the sake of simplicity we confine ourselves to the simplest case of the so-called "unidimensional electrodynamics", when linearly-polarized mutually perpendicular electric and magnetic fields depend only on the single space variable ξ and time t. Then Maxwell's equations assume the following form:

$$\frac{\partial E}{\partial \xi} = \frac{1}{c}\frac{\partial B}{\partial t}, \quad \frac{1}{\mu(t)}\frac{\partial B}{\partial \xi} = \frac{1}{c}\frac{\partial D}{\partial t} + \frac{4\pi}{c} j, \quad D=\varepsilon(t)E. \tag{2.1}$$

Introducing the vector potential as usual,

$$E = -\frac{1}{c}\frac{\partial A}{\partial t}, \quad B = -\frac{\partial A}{\partial \xi}, \tag{2.2}$$

we get the second order equation

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left[\varepsilon(t)\frac{\partial A}{\partial t}\right] - \frac{1}{\mu(t)}\frac{\partial^2 A}{\partial \xi^2} = \frac{4\pi}{c} j(\xi,t), \tag{2.3}$$

which coincides with Euler's equation

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial A/\partial t)} + \frac{\partial}{\partial \xi}\frac{\partial L}{\partial(\partial A/\partial \xi)} - \frac{\partial L}{\partial A} = 0 \tag{2.4}$$

for the Lagrangian density

$$L = \frac{1}{8\pi}\left\{\frac{\varepsilon(t)}{c^2}\left[\frac{\partial A}{\partial t}\right]^2 - \frac{1}{\mu(t)}\left[\frac{\partial A}{\partial \xi}\right]^2\right\} + \frac{1}{c} jA. \tag{2.5}$$

The cannonically conjugated density is

$$\Pi = \frac{\partial L}{\partial(\partial A/\partial t)} = \frac{\varepsilon(t)}{4\pi c^2}\frac{\partial A}{\partial t} = -\frac{1}{4\pi c}D. \qquad (2.6)$$

so that the Hamiltonian density is given by

$$W = \Pi\frac{\partial A}{\partial t} - L = \frac{1}{8\pi}\left\{\frac{D^2}{\varepsilon(t)} + \frac{1}{\mu(t)}\left[\frac{\partial A}{\partial \xi}\right]^2\right\} - \frac{1}{c}jA. \qquad (2.7)$$

We assume the field to occupy an ideal resonator with walls at the points $\xi=0$ and $\xi=L$. Then the following mode decomposition is natural:

$$D(\xi,t) = c\left[\frac{2\pi}{L}\right]^{3/2}\sum_n nx_n(t)\sin\left[\frac{n\pi}{L}\xi\right]. \qquad (2.8)$$

$$A(t,\xi) = \left[\frac{8L}{\pi}\right]^{1/2}\sum_n \frac{p_n(t)}{n}\sin\left[\frac{n\pi}{L}\xi\right]. \qquad (2.9)$$

After this the field Hamiltonian turns into the Hamiltonian of a countable set of noninteracting harmonic time-dependent forced oscillators:

$$H(t) = \int_0^L W(t,\xi)d\xi = \sum_n\left\{\frac{1}{2}\left[\frac{p_n^2}{\mu(t)} + \left[\frac{c\pi n}{L}\right]^2\frac{x_n^2}{\varepsilon(t)}\right] + f_n(t)p_n\right\}. \qquad (2.10)$$

where

$$f_n(t) = -\int_0^L\left[\frac{8L}{\pi c^2 n^2}\right]^{1/2} j(\xi,t)\sin\left[\frac{n\pi}{L}\xi\right]d\xi. \qquad (2.11)$$

To quantize the Hamiltonian (2.10), we have to treat the variables x_n and p_n as operators satisfying canonical commutation relations. At least two variants are possible. First, we may consider p_n as the generalized coordinate, and x_n as a generalized momentum. Secondly, we may make the opposite choice: p_n as a generalized momentum, and x_n as a generalized coordinate. In both cases we have the harmonic oscillator Hamiltonians with, generally speaking, variable masses and frequencies. Now let us take into account that really the magnetic permeability differs from unity to a very small value in all cases when the linear equations (2.1) hold. Thus from the physical point of view it is sufficient to consider the function $\mu(t)$=const=1. Then we may study either the oscillator with a constant frequency but with a time-dependent mass, or the oscillator with a constant mass and with a time-dependent frequency. The second situation seems much more familiar and convenient. Therefore our choice is to treat x_n as the generalized coordinate operator, and p_n as the generalized momentum operator satisfying the commutation relations

$$\left[\hat{x}_n , \hat{p}_m \right] = i\hbar\delta_{nm} . \tag{2.12}$$

The physical significance of such a choice is that the electric displacement vector plays the role of the generalized coordinate while the generalized momentum is related to the vector potential or the magnetic induction vector (see also refs./23-26/ in this connection).

Since all modes in the Hamiltonian (2.10) are uncoupled, we shall omit hereafter the number of the mode and consider the following one-mode Hamiltonian:

$$\hat{H} = \frac{1}{2} \left[\hat{p}^2 + \omega^2(t)\hat{x}^2 \right] + f(t)\hat{p} , \tag{2.13}$$

$$\omega^2(t) = \frac{(c\pi n)^2}{L^2\varepsilon(t)} . \tag{2.14}$$

This Hamiltonian possesses ~~the~~ a curious peculiarity: it contains coupling with an external time-dependent current (represented by the function f(t)) not through the coordinate, but through the momentum. Therefore, obtaining explicit solutions to this not very usual problem seems a rather interesting task.

3. Propagator.

We begin with calculating the exact propagator for the Schrödinger equation, i.e. the integral kernel relating the values of the wave function at the instants of time t' and t

$$\psi(x,t) = \int dx' K(x,t;x't')\psi(x',t'), \quad t>t'. \tag{3.1}$$

Since the Hamiltonian (2.13) is a nonuniform quadratic form of the coordinate and momentum operator, the propagator also is an exponential of some nonuniform quadratic form /12-19/

$$K(x,t;x',t') = \exp\Big[a(t,t')x^2 + b(t,t')xx' + c(t,t')x'^2 + g(t,t')x + h(t,t')x' + d(t,t')\Big]. \qquad (3.2)$$

Substituting this expression into the Schrödinger equation, one can obtain a system of coupled nonlinear ordinary differential equations for time-dependent coefficients of the quadratic form, e.g.

$$\frac{da}{dt} = 2i\hbar a^2 + \omega^2(t)/2i\hbar \,, \qquad (3.3)$$

$$\frac{db}{dt} = 2i\hbar ab, \qquad (3.4)$$

$$\frac{dc}{dt} = \frac{i\hbar}{2} b^2, \qquad (3.5)$$

$$\frac{dc}{dt'} = -\,2i\hbar c^2 + \frac{i}{2\hbar}\,\omega^2(t'), \qquad (3.6)$$

$$\frac{db}{dt'} = -\,2i\hbar cb, \qquad (3.7)$$

$$\frac{da}{dt'} = -\,\frac{i\hbar}{2}\,b^2 \,, \qquad (3.8)$$

and similar equations for the coefficients of linear form in (3.2). Eqs. (3.3)-(3.8) can be solved with the aid of the anzatz

$$a(t) = \frac{i}{2\hbar q(t)}\frac{dq}{dt}\,, \quad c(t') = -\,\frac{i}{2\hbar q(t')}\frac{dq}{dt'}\,. \qquad (3.9)$$

where the new function $q(t)$ obeys the classical equation for the oscillator with a time-dependent frequency

$$\frac{d^2q}{dt^2} + \omega^2(t)q(t) = 0. \qquad (3.10)$$

However, certain care is required, since eq.(3.10) possesses **two independent solutions**, so that the functions $q(t)$ and $q(t')$ in (3.9) are different in the general case.

There exists a more direct and convenient method of calculating the propagator proposed in /16/ and developed in /17,18/. It is based on the concept of time-dependent integrals of motion (whose role for obtaining exact solutions of quantum mechanical problems was stressed in refs. /1,2/). An integral of motion is an operator satisfying the equation

$$i\hbar \frac{\partial \hat{I}}{\partial t} = \left[\hat{H}, \hat{I} \right]. \tag{3.11}$$

Suppose two operators $\hat{X}(t,t')$ and $\hat{P}(t,t')$ satisfying the conditions $\hat{X}(t,t')=\hat{x}$, $\hat{P}(t,t')=\hat{p}$ are known. Then the kernel $K(x,t;x't')$ can be found from the equations /16-18/

$$\hat{X}(t,t')K(x,t;x't') = x'K(x,t;x't'), \tag{3.12}$$

$$\hat{P}(t,t')K(x,t;x't') = i\hbar\frac{\partial}{\partial x'} K(x,t;x't'). \tag{3.13}$$

The ~~For~~ quadratic Hamiltonians operators $\hat{X}$ and $\hat{P}$ are evidently **linear** combinations of operators $\hat{x}$ and $\hat{p}$:

$$\hat{P}(t,t') = \lambda_1(t,t')\hat{p} + \lambda_2(t,t')\hat{x} + \delta_1(t), \tag{3.14}$$

$$\hat{X}(t,t') = \lambda_3(t,t')\hat{p} + \lambda_4(t,t')\hat{x} + \delta_2(t). \tag{3.15}$$

Substituting these expression into (3.11) we get a set of linear ordinary differential equations

$$\frac{d\lambda_1}{dt} = -\lambda_2, \quad \frac{d\lambda_2}{dt} = \omega^2(t)\lambda_1, \quad \frac{d\delta_1}{dt} = -f(t)\lambda_2, \tag{3.16}$$

$$\frac{d\lambda_3}{dt} = -\lambda_4, \quad \frac{d\lambda_4}{dt} = \omega^2(t)\lambda_3, \quad \frac{d\delta_2}{dt} = -f(t)\lambda_4, \tag{3.17}$$

with the initial conditions

$$\lambda_1(t,t) = \lambda_4(t,t) = 1, \quad \lambda_2(t,t) = \lambda_3(t,t) = 0,$$

$$\delta_1(t,t) = \delta_2(t,t) = 0.$$

The solutions of eqs. (3.16)-(3.18) are as follows:

$$\lambda_1(t,t') = q_1(t,t'), \quad \lambda_2(t,t') = -\frac{dq_1}{dt}, \quad \delta_1(t,t') = \int_{t'}^{t} f(\tau)\frac{dq_1}{d\tau}\, d\tau, \tag{3.19}$$

$$\lambda_3(t,t') = -q_2(t,t'),\ \lambda_4(t,t') = \frac{dq_2}{dt},\ \delta_2(t,t') = -\int_{t'}^{t} f(\tau)\frac{dq_2}{d\tau}\,d\tau, \tag{3.20}$$

where both functions q_1 and q_2 satisfy the same equation (3.10) but with different initial conditions

$$q_1(t,t) = 1,\ \frac{dq_1}{dt}(t,t) = 0,\ q_2(t,t) = 0,\ \frac{dq_2}{dt}(t,t) = 1. \tag{3.21}$$

After this, equations (3.12) and (3.13) immediately determine all coefficients of the quadratic form (3.2) except the free term $d(t,t')$ which can be found from the Schrödinger equation

$$i\hbar\frac{dK}{dt} = \hat{H}K. \tag{3.22}$$

The final result is as follows:

$$K(x,t;x',t') = \left[2\pi i\hbar q_2(t,t')\right]^{-1/2}\exp\left\{\frac{i}{2\hbar q_2}\left[\dot{q}_2x^2 + q_1x'^2 - 2xx'\right] - \right. \tag{3.23}$$

$$\left. - \frac{i}{\hbar q_2}\left[xF_2 + x'\left[F_1q_2 - F_2q_1\right]\right] + \frac{i\ q_1}{2\hbar\ q_2}F_2^2 - \frac{i}{\hbar}\int_{t'}^{t} f(\tau)\dot{q}_1(\tau)F_2(\tau)\right\},$$

where

$$F_j(t,t') = \int_{t'}^{t} f(\tau)\dot{q}_j(\tau)d\tau,\ j=1,2.$$

Now let us discuss the structure of the functions q_1 and q_2 -- solutions of eq.(3.10). Suppose we look for a complex solution of this equation in the form $q(t)=\eta(t)\exp[i\gamma(t)]$, η and γ being real functions. Then we get two equations

$$\ddot{\eta} + \omega^2(t)\eta = \eta\dot{\gamma}^2, \tag{3.25}$$

$$\eta\ddot{\gamma} + 2\dot{\eta}\dot{\gamma}=0. \tag{3.26}$$

From the last equation we get

$$\eta^2\dot{\gamma} = \text{const} = \Omega. \tag{3.27}$$

Then the function $\eta(t)$ obeys the nonlinear equation

$$\ddot{\eta} + \omega^2(t)\eta = \Omega^2/\eta^3. \qquad (3.28)$$

Suppose $\eta(t)$ is an arbitrary solution of eq.(3.28). Then functions $q_1(t,t')$ and $q_2(t,t')$ can be expressed in terms of $\eta(t)$ and

$$\gamma(t,t') = \int_{t'}^{t} \frac{\Omega d\tau}{\eta^2(\tau)} \qquad (3.29)$$

as follows:

$$q_1(t,t') = \frac{\eta(t)\cos\gamma(t,t')}{\eta(t')} - \frac{\dot{\eta}(t')\eta(t)\sin\gamma}{\Omega}, \qquad (3.30)$$

$$q_2(t,t') = \frac{\eta(t')\eta(t)}{\Omega}\sin\gamma(t,t') = \frac{\sin\gamma(t,t')}{[\dot{\gamma}(t)\dot{\gamma}(t')]^{1/2}}. \qquad (3.31)$$

Using these expressions, we can rewrite (3.23) in a more symmetric form:

$$K(x,t;x',t') = \left\{\frac{[\dot{\gamma}(t)\dot{\gamma}(t')]^{1/2}}{2\pi\hbar i\sin\gamma(t,t')}\right\}^{1/2} \exp\left\{\frac{i}{2\hbar}\left[\frac{\dot{\eta}(t)}{\eta(t)}x^2 - \right.\right.$$
$$\left. - \frac{\dot{\eta}(t')}{\eta(t')}x'^2\right] + \frac{i}{2\hbar\sin\gamma(t,t')}\left[\cos\gamma(t,t')\left[\dot{\gamma}(t)x^2 + \dot{\gamma}(t')x'^2\right] - \right.$$
$$- 2\left[\dot{\gamma}(t)\dot{\gamma}(t')\right]^{1/2}xx' - 2x\int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left\{\left[\frac{\dot{\gamma}(t)}{\dot{\gamma}(\tau)}\right]^{1/2}\sin\gamma(\tau,t')\right\} -$$
$$- 2x'\int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left\{\left[\frac{\dot{\gamma}(t')}{\dot{\gamma}(\tau)}\right]^{1/2}\sin\gamma(t,\tau)\right\} - \qquad (3.32)$$
$$\left.\left. - 2\int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left\{\frac{\sin\gamma(t,\tau)}{(\dot{\gamma}(\tau))^{1/2}}\right\}\int_{t'}^{\tau} ds f(s)\frac{d}{ds}\left\{\frac{\sin\gamma(s,t')}{(\dot{\gamma}(s))^{1/2}}\right\}\right]\right\}.$$

4. Coherent states.

If we choose the complex solution of eq.(3.10) satisfying an additional condition

$$\dot{q}q^* - \dot{q}^*q = 2i, \qquad (4.1)$$

which is equivalent to the choice Ω=1 in relation (3.27), then we can construct a nonhermitian operator --linear integral of motion /14,18/

$$\hat{A}(t) = \iota(2\hbar)^{-1/2}\left[q(t)\hat{p} - \dot{q}(t)\hat{x} + \int d\tau f(\tau)\dot{q}(\tau)\right] = \qquad (4.2)$$

$$= \iota(2\hbar)^{-1/2} e^{\iota\gamma(t)}\left[\eta(t)\hat{p} - (\dot{\eta} + \iota\dot{\gamma}\eta)\hat{x} + \int_{t'}^{t} d\tau f(\tau)(\dot{\eta} + \right.$$

$$\left. - \iota\dot{\gamma}\eta) e^{\iota\gamma(\tau)-\iota\gamma(t)} \right] \qquad (4.3)$$

(t' is arbitrary) satisfying the bosonic annihilation-creation operators commutation relation

$$\left[\hat{A}(t), \hat{A}^{+}(t)\right] = 1. \qquad (4.4)$$

Thus we can construct an overcomplete set of coherent states $\psi_{\alpha}(x,t)$ satisfying both the Schrödinger equation and the equation

$$\hat{A}(t)\psi_{\alpha}(x,t) = \alpha\psi_{\alpha}(x,t), \qquad (4.5)$$

α being an arbitrary complex function. The explicit expression for these states can be written as a special case of general formulas for coherent states of multidimensional quadratic systems found in /15,18/

$$\psi_{\alpha}(x,t) = \left(\frac{\dot{\gamma}}{\pi\hbar}\right)^{1/4} \exp\left\{ -\frac{\dot{\gamma}}{2\hbar}\left[1 - \frac{\iota\dot{\eta}}{\eta\dot{\gamma}}\right]x^2 - \frac{1}{2}\iota\gamma(t) - \right.$$

$$- \frac{1}{2}\alpha^2 e^{-2\iota\gamma} - \frac{1}{2}|\alpha|^2 + \left[\frac{\dot{\gamma}}{\hbar^2}\right]^{1/2} e^{-\iota\gamma(t)} x \left[\sqrt{2\hbar}\,\alpha - \right. \qquad (4.6)$$

$$\left. - \iota\int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left[\frac{e^{\iota\gamma(t)}}{\sqrt{\dot{\gamma}(\tau)}}\right]\right] + \left[\frac{2}{\hbar}\right]^{1/2} e^{-\iota\gamma(t)}\alpha\int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left[\frac{\sin\gamma(t,\tau)}{\sqrt{\dot{\gamma}(\tau)}}\right] -$$

$$\left. - \frac{\iota}{\hbar}\int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left[\frac{\sin\gamma(t,\tau)}{\sqrt{\dot{\gamma}(\tau)}}\right]\int_{t'}^{\tau} ds f(s)\frac{d}{ds}\left[\frac{e^{-\iota\gamma(t,s)}}{\sqrt{\dot{\gamma}(s)}}\right]\right\}.$$

The expectation values of x and p in these states are

$$\langle\alpha|x|\alpha\rangle = \left[\frac{2\hbar}{\dot{\gamma}}\right]^{1/2} \mathrm{Re}\left[\alpha e^{-\iota\gamma(t)}\right] - \int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left\{\frac{\sin\gamma(t,\tau)}{[\dot{\gamma}(t)\dot{\gamma}(\tau)]^{1/2}}\right\}. \qquad (4.7)$$

$$\langle\alpha|p|\alpha\rangle = \frac{d}{dt}\langle\alpha|x|\alpha\rangle. \tag{4.8}$$

The variances

$$\sigma_p = \langle\alpha|p^2|\alpha\rangle - \left[\langle\alpha|p|\alpha\rangle\right]^2, \quad \sigma_x = \langle\alpha|x^2|\alpha\rangle - \left[\langle\alpha|x|\alpha\rangle\right]^2$$

depend neither on α nor on $f(t)$, but are completely determined by the function $q(t)$ /18/:

$$\sigma_x = \frac{\hbar}{2}|q(t)|^2 = \frac{\hbar}{2\dot{\gamma}}, \tag{4.9}$$

$$\sigma_p = \frac{\hbar}{2}|\dot{q}(t)|^2 = \frac{\hbar}{2}\dot{\gamma}\left[1 + \frac{\dot{\eta}^2}{\dot{\gamma}}\right]. \tag{4.10}$$

Their product

$$\sigma_x\sigma_p = \frac{\hbar^2}{4}\left[1 + \dot{\eta}^2/\dot{\gamma}\right] \tag{4.11}$$

varies in time; moreover, it is greater than the minimal possible value $\hbar^2/4$ provided $\dot{\eta}\neq0$. This is explained by the fact that the state (4.6) is really **correlated coherent states** /18,27/ with nonzero **correlated coefficient**

$$r = \frac{\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle/2 - \langle\hat{x}\rangle\langle\hat{p}\rangle}{(\sigma_x\sigma_p)^{1/2}} = \frac{\dot{\eta}}{[\dot{\eta}^2 + \dot{\gamma}]^{1/2}} \tag{4.12}$$

Correlated coherent states minimize the Schrödinger-Robertson generalized uncertainty relation /18,27/

$$\sigma_x\sigma_p\left[1 - r^2\right] \geq \hbar^2/4, \tag{4.13}$$

since the substitution of (4.11) and (4.12) into (4.13) transforms this inequality into the identity.

5. Fock's states.

It is well known /28/ that coherent states are generating functions for Fock's states:

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n\rangle, \tag{5.1}$$

$$\hat{A}^+\hat{A}|n\rangle = n|n\rangle. \tag{5.2}$$

On the other hand, on exponential function of a quadratic form is the generating function for Hermite's polynomials /29/. Thus one can obtain the following formula:

$$\langle x|n\rangle = \phi_n(x,t) = \left[2^n n!\right]^{-1/2} e^{-in\gamma(t)}\psi_0(x,t)\circ$$

$$\circ\, H_n\left(\left[\frac{\dot{\gamma}}{\hbar}\right]^{1/2}\left\{x + \int_{t'}^{t} d\tau f(\tau)\frac{d}{d\tau}\left[\frac{\sin\gamma(t,\tau)}{\{\dot{\gamma}(t)\dot{\gamma}(\tau)\}^{1/2}}\right]\right\}\right), \tag{5.3}$$

$\psi_0(x,t)$ being given by eq.(4.6) with $\alpha=0$. Functions (5.3) are also eigenstates of the operator [illegible] (considered first by Lewis /1,2/)

$$\hat{E}(t) = \hbar\Omega\left[\hat{A}^+\hat{A} + 1/2\right] = \frac{1}{2}\left\{\eta^2\hat{p}^2 + \left[\dot{\eta}^2 + \eta^2\dot{\gamma}^2\right]\hat{x}^2 - \dot{\eta}\eta\left[\hat{x}\hat{p} + \hat{p}\hat{x}\right] + \right.$$

$$\left. + 2\mathrm{Re}\left[F(t)e^{-i\gamma(t)}\left[\eta\left[\hat{p} - i\dot{\gamma}\hat{x}\right] + \dot{\eta}\hat{x}\right]\right] + |F(t)|^2\right\}, \tag{5.4}$$

where

$$F(t) = \int_{t'}^{t} d\tau f(\tau)\left[\dot{\eta} + i\dot{\gamma}\eta\right]e^{i\gamma(\tau)}, \tag{5.5}$$

and the functions $\eta(t)$, $\gamma(t)$ satisfy eqs. (3.25)-(3.29) with the given positive constant Ω.

$$\hat{E}(t)\phi_n(x,t) = \hbar\Omega\left[n + 1/2\right]\phi_n, \quad n=0,1,2,\ldots \tag{5.6}$$

The operator $\hat{E}(t)$ is an integral of the motion coinciding with the usual energy operator in the case of time-independent functions Ω and f (provided the choice $\Omega=\omega$ is made). Therefore, it can be called a "generalized energy operator".

If we define an "uncertainty product" as

$$\Delta x \Delta p = \left\{ \left[\left[\langle m|x^2|n\rangle - \langle m|x|n\rangle^2 \right]^* \left[\langle m|x^2|n\rangle - \langle m|x|n\rangle^2 \right] \right]^{1/2} \circ$$

$$\circ \left[\left[\langle m|p^2|n\rangle - \langle m|p|n\rangle^2 \right]^* \left[\langle m|p^2|n\rangle - \langle m|p|n\rangle^2 \right] \right]^{1/2} \right\}^{1/2}, \quad (5.7)$$

then the following formulas can be obtained:

$$\left[\Delta x \Delta p\right]_{n,n} = \hbar \left[1 + \frac{\dot{\eta}^2}{\dot{\gamma}^2 \eta^2} \right]^{1/2} \left[n + 1/2\right]. \quad (5.8)$$

$$\left[\Delta x \Delta p\right]_{n+2,n} = \hbar \left[1 + \frac{\dot{\eta}^2}{\dot{\gamma}^2 \eta^2} \right]^{1/2} \left[(n+2)(n+1)\right]^{1/2}. \quad (5.9)$$

$$\left[\Delta x \Delta p\right]_{n,n+2} = \hbar \left[1 + \frac{\dot{\eta}^2}{\dot{\gamma}^2 \eta^2} \right]^{1/2} \left[(n-1)n\right]^{1/2}. \quad (5.10)$$

All other products are equal to zero.

6. Mode excitation by delta-impulse.

As an example, we consider the simplest time dependences of the effective frequency and "force" in the form of delta-impulses:

$$\omega^2(t) = \Omega^2 + W\delta(t), \quad W>0. \quad (6.1)$$

$$f(t) = \begin{cases} f_o, & |t| < a/2 \\ 0, & |t| > a/2 \end{cases}, \quad a\to 0, \ f\to\infty, \ af\to F=\mathrm{const}. \quad (6.2)$$

The dependence (6.1) can be realized approximately in the case of very fast ionization of the medium when the dielectric permeability $\varepsilon = 1 - 4\pi ne^2/m\omega^2$ turns into zero due to an instant growth of the electron concentration (here ω is the resonance frequency of the mode) and then quickly restores ~~the~~ its initial value due to recombination. Some physical effects in such a "plasma window" were discussed in ref. /30/.

Further calculations are based in part on results of ref. /31/.

The Hamiltonian of this system is of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\hat{x}^2}{2}\left[\Omega^2 + W\delta(t)\right] + f(t)\hat{p}. \tag{6.3}$$

Following the usual procedure /18/ and taking into consideration the results obtained in /31/ one can construct the linear integral of the motion for the Hamiltonian (6.3),

$$\hat{A}(t) = \frac{i}{2^{1/2}}\left\{\frac{\hat{p}}{p_o}\left[e^{i\Omega t} - \frac{W}{\Omega}\sin\Omega t\right] - \frac{\hat{x}}{x_o}\left[i\Omega e^{i\Omega t} - W\cos\Omega t\right]\right\} + \delta(t),$$

$$t>0, \tag{6.4}$$

where

$$p_o = \left[\hbar m\Omega\right]^{1/2},\quad x_o = \left[\hbar/m\Omega\right]^{1/2},\quad \delta(t) = \left[\frac{m}{\hbar\Omega}\right]^{1/2}(i\Omega F - FW/2),\quad F = \lim_{\substack{a\to 0\\ f_o\to\infty}} f_o a.$$

One can check that the integral of the motion (6.4) and its hermitian conjugate operator satisfy the commutation relations of boson creation and annihilation operators,

$$[\hat{A},\hat{A}^{+}] = 1.$$

The ground state of a quantum oscillator transforms to a correlated coherent state (4.6) after δ-kicking of the frequency (it is necessary to put $q(t) = e^{i\Omega t} - (W/\Omega)\sin\Omega t$ in the formula (4.2)), with the variances (4.9), (4.10) equal to

$$\sigma_x = \frac{\hbar}{2m\Omega}\left[1 + \frac{W^2}{\Omega^2}\sin^2\Omega t - \frac{W}{\Omega}\sin 2\Omega t\right],$$

$$\sigma_p = \frac{1}{2m\Omega}\left[1 + \frac{W^2}{\Omega^2}\cos^2\Omega t + \frac{W}{\Omega}\sin 2\Omega t\right].$$

The correlation coefficient (4.12) is nonzero after δ-kicking and is equal to

$$r = 1 - \left\{1 + \left[W^4/4\Omega^4\right]\sin^2 2\Omega t + \left[W^2/\Omega^2\right]\cos^2\Omega t + \left[W^3/2\Omega^3\right]\sin 4\Omega t\right\}^{-1}.$$

The squeezing coefficient $k = m\Omega\left[\sigma_x/\sigma_p\right]^{1/2}$ is not equal to unity and is given by the formula

$$k = \left[\frac{1 + \dfrac{W^2}{\Omega^2}\sin^2\Omega t - \dfrac{W}{\Omega}\sin 2\Omega t}{1 + \dfrac{W^2}{\Omega^2}\cos^2\Omega t + \dfrac{W}{\Omega}\sin 2\Omega t} \right]^{1/2} .$$

Acknowledgements.

This research was supported in part by Center for Thermal and Statistical Physics, and the National Science Foundation [redacted] under grant CHE-9016789 and the U.S.Office of Naval Research.

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