**Title and Subtitle:**
Comparison of One-way Wave Propagation Algorithms in Underwater Acoustics - Error Estimates and Sensitivity

**Authors:**
Dr. Louis Fishman

**Performing Organization:**
Dept. of Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401

**Sponsoring Organization:**
Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217-5000

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13. Abstract

The long-range objective is to develop and apply “microscopic” phase space methods and global path integral constructions to gain a deeper theoretical and computational understanding of acoustic, electromagnetic, and seismic direct and inverse wave propagation problems. This seems to be an appropriate approach for ocean seismo-acoustic modeling, which is characterized by rapidly changing, multidimensional environments extending over many wavelengths. Much of the mathematical development can indeed be motivated by the well-known parabolic (paraxial) approximation. Combining wave field splitting, invariant imbedding, and phase space (pseudo-differential and Fourier integral operator) methods has led to the development of both one- and two-way direct Helmholtz solvers, in addition to providing the framework for multidimensional profile reconstruction algorithms based on exact solution methods.
OBJECTIVE

The long-range objective is to develop and apply “microscopic” phase space methods and global path integral constructions to gain a deeper theoretical and computational understanding of electromagnetic, seismic, and acoustic direct and inverse wave propagation problems. Specifically, this project focuses on the development of new, multidimensional algorithms for direct (forward) propagation and generalized tomography and inversion. Special emphasis is placed on applying these methods to ultimately derive meaningfully sharp global error estimates for a wide range of underwater acoustic propagation algorithms. While most of the analysis addresses the scalar (acoustic) formulation, current work involves vector (electromagnetic, elastic) extensions.

BACKGROUND

The analysis, understanding, and fast, accurate numerical computation of the wave equation are quite difficult for rapidly changing, multidimensional environments extending over many wavelengths. This is particularly so for environments characterized by a refractive index field with a compact region of arbitrary variability superimposed upon a transversely inhomogeneous background profile. For both the forward and inverse problems, the entire domain is in the scattering regime. There is, generally, no accessible asymptotically homogeneous regime where simplified wave fields can be constructed or appropriate data collected in these model ocean environments.

APPROACH

The introduction and widespread application of the parabolic (paraxial) approximation marked a significant advance in wave propagation modeling for ocean seismo-acoustic and other strong channeling environments. In many respects, this is a most natural motivation for the application of phase space factorization and path integral methods to the Helmholtz equation. The parabolic is, after all, an approximation to the full square root (pseudo-differential) operator, while the associated split-step FFT computational algorithm follows from a direct integration of the phase space Feynman path integral. It is then quite natural, in attempting to extend the ordinary parabolic approximation...
beyond its limited region of validity, to examine more closely the full square root operator and the more general path integral representation associated with its corresponding propagator (fundamental solution). For the one-way theory, both the square root Helmholtz operator construction and the phase space path integral representation and marching algorithm depend crucially upon a detailed analysis of the operator symbol. The construction of exact, numerical, and uniform perturbation solutions to the (Helmholtz) Weyl composition equation for the operator symbol and their application to the development of the marching algorithm are considered. This analysis can be extended to the operator symbol matrices associated with Maxwell’s equations and the theory of elasticity. The phase space analysis can be combined with wave field splitting and invariant imbedding methods to address the full two-way Helmholtz wave theory. The result is a set of first-order in range, nonlinear, nonlocal, well-posed equations for the appropriate reflection and transmission operators (symbols) associated with the environment. Further, for the two-way theory, the evaluation of the Feynman/Garrod approximate path integral and the construction of a rigorous path integral for the homogeneous limit are considered. It is in addressing the full two-way problem and attempting to construct true path functionals (sum over paths) that a more physically transparent underlying structure is introduced (even at the level of the one-way problem). For the inverse analysis, the generalized Fourier integral operator structure of both the macroscopic (finite) and microscopic (infinitesimal) propagators, the semigroup property of the one-way equation, and the (Helmholtz) Weyl composition equation are considered to exactly reconstruct the refractive index profile for transversely inhomogeneous environments. This is intended to extend current ocean tomographic analysis in both frequency domain and inhomogeneity strength. The phase space/path integral analysis can be combined with the previously mentioned wave field splitting/invariant imbedding methods to provide the basis for a computational algorithm for two- and three-dimensional profile reconstruction. This approach is based on an exact mathematical solution, thus treating the full nonlinear nature of the inverse problem and is applicable to the seabed inversion problem.

RESULTS

During this project period, the model problem $K^2(q) = K_0^2 + A \tanh pq + B \text{sech}^2 pq$ has been studied in detail in the phase space/path integral formulation. This soluble profile allows for modeling of (1) symmetric and asymmetric profiles with trapped modes and (2) sharp gradient effects. The model provides the initial benchmark case for evaluating wave propagation algorithms within the phase space framework. In conjunction with previous studies on generalized quadratic profiles, certain uniform symbol approximations can be constructed and evaluated. This is key for the one-way, numerical, phase space marching algorithm. Also, in work with Ron Brent (Univ. of Mass. at Lowell), the phase space and path integral techniques have been applied to vector wave equations (Maxwell’s and elasticity), resulting in new first-order Weyl pseudo-differential equations, which are recognized as exact one-way wave equations for transversely inhomogeneous environments. Perturbation treatments of the appropriate Weyl composition equations
for the operator symbol matrix yield high-frequency and other asymptotic wave theories. This is an exact formulation at the level of the wave field—no special symmetry and/or far-field assumptions are made. Finally, wave field splitting, invariant imbedding, and phase space methods have been combined to construct a direct numerical propagation algorithm for the two-way Helmholtz equation models of seismo-acoustic environments. In joint work with Suhrit Dey (Eastern Illinois), this computational algorithm is being numerically implemented and tested. The same analysis has also been used to construct theoretical inversion schemes based on exact methods for the generalized acoustic tomography problem.

**DISCUSSION OF RESULTS**

The analysis and fast, accurate numerical computation of both direct and inverse wave propagation problems are often quite difficult for rapidly changing, multidimensional environments extending over many wavelengths. This is particularly so for environments characterized by a refractive index field with a compact region of arbitrary (n-dimensional) variability superimposed upon a transversely inhomogenous ((n - 1)-dimensional) background profile. For such environments, the entire domain is in the scattering regime, with the subsequent absence of an “asymptotically free,” or homogeneous, region. For the most part, classical, “macroscopic” methods have resulted in direct wave field approximations (perturbation theory, ray-theory asymptotics, field splitting/invariant imbedding, transmutation theory, modal analysis, hybrid ray-mode methods, Gaussian beams), derivations of approximate wave equations (scaling analysis, field-splitting techniques, formal operator expansions, approximation theory), and discrete numerical approximations (finite differences, finite elements, spectral methods, FFP). In the last several decades, however, developments in Fourier analysis, partial differential equations, mathematical physics, among others have been synthesized into what is now called harmonic analysis in phase space. This analysis on the configuration space $R^n$ done by working in the phase space $R^n \times R^n$ has produced sharp, “microscopic” tools (pseudo-differential and Fourier integral operators, wave packets) appropriate for attacking wave propagation problems in extended environments. In conjunction with the global functional integral techniques pioneered by Wiener (Brownian motion) and Feynman (quantum mechanics), and so successfully applied today in quantum field theory and statistical physics, the $n$-dimensional wave field propagators can be both represented explicitly and computed directly. The phase space, or “microscopic,” method and path (functional) integral representations provide the appropriate framework to extend homogeneous Fourier methods to extended inhomogeneous environments, in addition to suggesting the basis for the formulation and solution of corresponding arbitrary-dimensional nonlinear inverse problems. These pseudo-differential and generalized Fourier integral operator methods are largely based upon the properties and interrelationships of the corresponding operator symbols.

So, why bring all of the machinery of path integrals and pseudo-differential and generalized Fourier integral operators to bear on the problems of direct and inverse wave
propagation? The answer lies in the difficult nature of the physical environments often encountered. The ocean/bottom environment provides an excellent example. This environment is inhomogeneous everywhere - there may be no accessible asymptotically homogeneous regime where simplified wave fields can be constructed and utilized or appropriate data collected. This impacts on both the direct and inverse problems. Methods based on approximations to the wave field, such as the Born, Rytov, and supereikonal, will generally fail due to the extended nature of the inhomogeneities. Even for frequency regimes where ray theory is appropriate, the resulting ray structure can be quite complex, exhibiting numerous caustic structures. Moreover, in many applications, the frequency is sufficiently low that the geometrical-acoustics limit is in serious quantitative error. Further, coupled-mode methods and purely numerical techniques can tend to obscure the underlying physics. These problems are typical for strong channeling (or focusing) media.

It is clear from an outline of the phase space and path integral analysis that the square root Helmholtz operator construction, the phase space path integral representation and algorithm, and the inverse analysis all depend crucially upon a detailed symbol analysis. There are also other areas of direct application. Weston's extension of wave field splitting and invariant imbedding methods to the direct and inverse analysis of the time-domain wave equation in $\mathbb{R}^3$ ultimately relies on computing (in one manner or another) the application of the square root of the second-order hyperbolic wave operator to an arbitrary field. Further, the phase space analysis is appropriate for extending backpropagation algorithms to transversely inhomogeneous background environments. Indeed, DeFacio and Brander have even applied coherent-state path integral constructions to the multidimensional ($n \geq 3$) inverse scattering problem. Thus, the (Helmholtz) Weyl composition equation is central to the overall scope of the work. From this perspective, the construction of exactly soluble cases of this equation is of great value. Such nontrivial constructions provide for detailed specific illustrations of the mathematical theory, possible insights into a general inversion approach to the composition equation, reference examples to check both numerical and approximate analytical symbol constructions, and a means to numerically assess the discretization effects in the marching algorithm. Of course, the explicit construction of nontrivial symbols corresponding to the square root of the indefinite Helmholtz operator is of mathematical interest in its own right.

Since the number of exact solutions to the composition equation is limited, approximate analytical solutions are extremely important, providing the core of the marching algorithm. The approximate symbol constructions are initially based on the well-known pseudo-differential operator asymptotic (in smoothness) treatment of the (Helmholtz) Weyl composition equation. Moreover, formal operator rational approximations to the square root Helmholtz operator have also played a historical and influential role in this wave propagation modeling. These approximations can be reformulated in terms of operator symbols, allowing for the resulting high-angle-parabolic-equation (HAPE) implicit-finite-difference (IFD) algorithms to be analyzed and computed by phase space and path integral methods. This is an important point since, for example, the naive application of such rational operator approximations in the elastic case can lead to inherently unstable
algorithms due to the improper treatment of complex wavenumber modes.

The pseudo-differential operator and functional integral analysis of the factored scalar Helmholtz equation can be carried over to vector wave equations (EM and elasticity). This results in new first-order Weyl pseudo-differential equations, which are recognized as exact one-way wave equations for transversely inhomogeneous environments. Perturbation treatments of the appropriate Weyl composition equations for the operator symbol matrix yield high-frequency and other asymptotic wave theories. Unlike the scalar Helmholtz equation case, the one-way vector equations (and a scalar analogue provided by the Klein-Gordon equation of relativistic physics) require the solution of generalized quadratic operator equations. While these operator solutions do not have a simple formal representation as in the straightforward (acoustic) square root case, they are conveniently constructed in the Weyl pseudo-differential operator calculus.

The one-way analysis leads naturally into the full two-way problem. Due to the extremely large size of realistic three-dimensional ocean propagation problems, there is an understandable desire to incorporate marching (one-way) methods into the solution algorithm for the inherently elliptic, frequency-domain formulation. This has provided the impetus for the development of the so-called “marching elliptic methods.” While not quite an oxymoron, “marching elliptic methods” generally require (1) the backpropagation of a backward wave field component and (2) the knowledge of both the wave field and its normal derivative on an initial plane. The first requirement leads to an ill-posed problem, requiring a regularization which effectively filters out the evanescent portion of the spectrum or a stability requirement which ultimately limits the spatial resolution. For strongly backscattering environments, for example, such filtering can result in spurious oscillations in the wave field, reflecting the important role played by the high-frequency portion of the spectrum under such conditions. The second requirement is even more problematical. For the elliptic problem, both the wave field and its normal derivative cannot be independently specified on the initial plane. Given the initial wave field, the determination of the corresponding normal derivative requires scattering data from the entire domain. Constructing this scattering information, however, effectively solves the original problem.

The above-mentioned problems notwithstanding, the original motivation is still sound. The goal only needs to be restated. It is now desirable to develop an algorithm for the elliptic propagation problem which can exploit marching methods as much as possible in a well-posed manner. The geometry of the problem provides the key insight. For the purpose of illustration, imagine a two-dimensional ocean waveguide with flat top and bottom surfaces divided into three distinct regions: left- and right-hand half-spaces separated by a transition region of arbitrary length. The two half-spaces are taken to be transversely inhomogeneous, while the transition region is both depth and range dependent. The vertical lines defining the transition region may or may not correspond to physical interfaces. If the vertical lines are imaginary, then the slab should be thought of as being smoothly imbedded in the larger medium. Now, the ocean propagation problem is not in the form of the classical, well-posed elliptic boundary-value problem, where appropriate total wave field (or derivative) values are prescribed on the entire boundary.
In the ocean problem, boundary data is given on the top and bottom boundaries; however, no total wave field (or derivative) values are known on the vertical transition-region boundaries. Only sources in the left- and right-hand half-spaces are prescribed. The problem is a scattering problem. The problems of interest include the reflection and transmission from the transition region and (possibly) the wave field within the transition region itself. While radiation boundary conditions could be constructed beyond the sources in the two half-spaces to create a well-posed boundary-value problem (which could be treated by, for example, finite-element methods), this leads to an excessively large problem and obscures the physical picture.

Recognizing that typical ocean propagation problems are essentially scattering problems in terms of a transition region and transversely inhomogeneous half-spaces, wave field splitting, invariant imbedding, and phase space methods reformulate the problem in terms of an operator scattering matrix characteristic of the transition region. The wave field splitting incorporates the kinematically correct physics and provides a convenient representation for the weak range-dependent limit. Invariant imbedding techniques enable the derivation of the scattering (reflection and transmission) operator equations. The phase space analysis transforms the operator equations to equations on well-behaved functions (symbols), in addition to providing the one-way propagation algorithms. The resulting equations for the reflection and transmission operators (symbols) are first-order in range, nonlinear (Riccati-like), and, in general, nonlocal. The system is well-posed, but stiff. The reflected and transmitted wave fields can be computed in a very efficient manner, while the wave field in the transition region (if desired) can be computed by essentially a layer-stripping algorithm. In principle, the transition region can be divided into subregions, allowing for parallel computations and subsequent recombination. This, of course, is not the first approach to treat the ocean propagation problem as a scattering problem. For example, the boundary integral equation method in conjunction with modal analysis has exploited this recognition. Invariant imbedding methods, however, focus directly on the reflection and transmission operators (symbols) in the scattering matrix, resulting in a minimal and physically useful formalism and an efficient use of marching (one-way) methods in the overall (two-way) algorithm.

The reflection and transmission operator equations provide the framework for constructing inverse algorithms based on, in principle, exact solution methods. This is directly applicable to the acoustic tomography problem where the ocean features to be reconstructed are confined to the transition region and the sources and receivers are located in the half-spaces. Rotating the formulation by 90 degrees makes it appropriate for the seabed parameter reconstruction (reflection seismology) problem. The principle idea is to determine the reflection operator (symbol), subsequently construct the Dirichlet-to-Neumann map, and finally recover the refractive index field from the Dirichlet-to-Neumann map. Algorithms based on this approach have already been developed and successfully implemented in the time-domain formulation of the acoustic problem in one spatial dimension. The development of robust, numerical algorithms for multidimensional inversion based on exact solution methods is clearly a desirable goal since the computational algorithms currently being employed in these areas primarily in-
volve either perturbative methods (an initial linearization of the problem in some form) or massive optimization (search) algorithms.

PUBLICATIONS (Books)


PUBLICATIONS (Papers, Book Chapters)


**SUBMITTED PAPERS**


**PAPERS OR TALKS PRESENTED**


L. Fishman, “A Comprehensive Atmospheric Wave Propagation Modeling Program,”

L. Fishman, “Wave Field Splitting, Invariant Imbedding, and Phase Space Analysis
in Direct and Inverse Scattering,” Acoustical Society of America Spring Meeting,

L. Fishman, “Phase Space and Path Integral Methods in Scalar and Vector Wave Propagation,” Inverse Problems – Computational Algorithms, Texas A & M University,
College Station, TX, March 10-14, 1991.

L. Fishman, “Atmospheric Wave Propagation Modeling and Computational Algorithms,”
Joint Electronic Warfare Center, Kelley Air Force Base, San Antonio, TX, December 17, 1990.


L. Fishman, “Phase Space and Path Integral Methods for Direct and Inverse Scattering with Applications to Ocean Acoustics,” Joint Mathematics and Physics Colloquium,
Montana State University, September 21, 1990.


L Fishman, “Phase Space and Functional Integral Methods in Direct and Inverse Wave Propagation,” Ames DOE Laboratory, Iowa State University, Ames, IA, June 27, 1990.

L. Fishman, “Phase Space and Path Integral Methods in Computational Acoustics,”


HONORS

Invited Presentations:


L. Fishman, "Phase Space and Functional Integral Methods in Direct and Inverse Scattering," Northeastern University, Boston, Massachusetts, March 14, 1990.


Other Honors:

Elected to membership in The Electromagnetics Academy.

Appointed to Editorial Board of the Journal of Computational Acoustics.

One month paid visiting position at Applied Mathematical Sciences, Ames DOE Laboratory, Iowa State University, June, 1990.


Requested by Joint Electronic Warfare Center of the Joint Chiefs of Staff to conduct long-range atmospheric propagation modeling studies.

Offered complimentary membership in SPIE.


COLLABORATIONS

The following scientific collaborations have been maintained or started.

**Stephen C. Wales** (NRL, Washington, D.C.) – continued work on numerical aspects of project; work on error estimates for model profiles.

**Mike Porter** (NJIT, Newark, NJ) – continued work on numerical symbol construction; work on error estimates for model profiles.

**Mike Fiddy** (University of Massachusetts at Lowell, Lowell, MA) – continued work on inverse problems associated with method; initiation of work on phase retrieval problem.

**David J. Thomson and Gary Brooke** (DREP Victoria, B.C.) – work on error estimates for model profiles.

**James Corones** (Ames Laboratory, Iowa State University, Ames, Iowa) and **Vaughan Weston** (Purdue University, W. Lafayette, IN) – book on field-splitting methods in direct and inverse scattering; with **Curt Vogel** (Montana State University, Bozeman, MT) and **John Gustafson** (Ames) – multidimensional inverse scattering algorithms.
John A. DeSanto (CSM)  daily conversations and joint work with Thomson and Brooke. Work with former post-doc Wombell is in progress.

Steve Pruess and Frank Hagin (CSM) -- development of a second (distinct) method for numerical symbol construction.

Ron Brent (University of Massachusetts at Lowell, Lowell, MA) -- application of phase space/path integral methods to EM wave propagation problems.

Ding Lee (NUSC and Yale) -- development of two-way direct wave propagation algorithms for underwater acoustics.

Suhrit Dey (Eastern Illinois) -- numerical testing of two-way direct wave propagation algorithm developed with Ding Lee.

Joe Ormsby (JEWC, San Antonio, TX) -- long-range atmospheric wave propagation modeling and numerical algorithm development.