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REFLECTION AND TRANSMISSION OF WAVES FROM AN INTERFACE WITH A PHASE-TRANSFORMING SOLID

by

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ABSTRACT

This paper is concerned with waves in a composite elastic bar, the left half of which is composed of a linearly elastic material, while the nonlinearly elastic material of the right half can undergo a phase transition. We assume that a wave in the left portion of the bar is incident upon the interface between the two materials, and we investigate the question of whether the phase transition can be exploited to augment or diminish the strength of the reflected or transmitted wave.

1. Introduction. We consider a composite tensile specimen consisting of two dissimilar elastic bars joined end-to-end. One of the two elastic materials ("material 2") is capable of undergoing a stress-induced transition to a second phase, while the other ("material 1") is not. When an incident wave traveling in the single-phase material strikes the bimaterial interface, it may nucleate a phase transition in material 2. The reflection and transmission characteristics of the product phase in such a transition will in general differ from those of the parent phase. We designate as "material 3" the single-phase elastic material whose properties are identical with those of the parent phase of material 2. Our interest lies in comparing the strengths, relative to the incident wave, of the reflected and transmitted waves generated in the material 1/material 2 composite bar with the relative strengths of those that would occur in a bar composed of material 1 and material 3. In particular, we investigate the extent to which the phase-transforming capability of material 2 can be exploited to control the strength of the reflected or transmitted wave.
The composite bar is viewed as consisting of two perfectly bonded, semi-infinite one-dimensional elastic continua. Material 1 is taken to be linearly elastic, while material 2 is treated on the basis of a nonlinear continuum model of the kind currently receiving much attention in discussions of the macroscopic aspects of phase transformations of martensitic type; see, for example the references cited in [1]. This model is based on a "two-well" potential energy for material 2, and it includes a nucleation criterion and a kinetic relation governing the initiation and evolution, respectively, of the phase transition. Thermal effects are neglected here, so that the model is a purely mechanical one. It has been applied to quasi-static phase transformations in [2], where it was shown to lead to results in qualitative agreement with some experimental observations on materials of the shape-memory type. The dynamics of the model have been studied in [3].

There are many recent papers devoted to the general issue of the continuum-mechanical modeling of the macroscopic effects of phase transformations; examples may be found in the references cited in [1]. The only work of which we are aware that is related to the specific problem under discussion here is that of Pence [4, 5]. In [4], Pence studies the reflection and transmission of an acoustic shear wave from an initially stationary phase boundary in an elastic solid. The analysis in [5] is concerned with the structure of the fields in two elastic bars, one of which (the impactor) is composed of a single-phase material, while the other (the target) is made of a material capable of sustaining a phase transformation. Neither of these papers addresses the issues of primary interest here.

We describe the basic model to be used in the following section. Section 3 contains the formulation of the underlying wave propagation problem, and Section 4 is devoted to its solution. In Section 5, we show that the ratio of the relative strength of the reflected wave in the presence of the phase transition to its relative strength in the absence of the transition is governed by two material parameters: one is the ratio of the mechanical impedances of material 1 and the
parent phase of material 2, while the other parameter is inherently related to the phase-transition properties of material 2. In terms of these two parameters, we derive in Section 5 conditions under which (i) the reflectivity is always increased by the phase transition, independently of the kinetic relation and the nucleation criterion, (ii) the reflectivity is always decreased, and (iii) the effect of the transition on reflectivity depends on the details of kinetics and nucleation. Some analogous results for the transmitted wave are stated without proof in Section 6.

2. The model. In a reference configuration, the composite bar occupies the entire x-axis, with material 1 in x < 0, material 2 in x > 0; the referential cross-sectional area is A. We treat longitudinal motions in which a particle at x in the reference state is carried to the point x + u(x, t) at time t. The displacement u is to be continuous with piecewise continuous first and second derivatives. The strain and particle velocity are defined by \( \gamma(x, t) = u_x(x, t) \) and \( v(x, t) = u_t(x, t) \), respectively, where the derivatives exist; in order to assure that the mapping \( x \rightarrow x + u \) is one-to-one, we require that \( \gamma > -1 \) everywhere. At points in the x,t-plane where the fields are smooth, balance of linear momentum in the absence of body force requires that

\[
\sigma_x = \rho(x) v_t, \tag{2.1}
\]

where \( \sigma(x, t) \) is the stress in the bar, and

\[
\rho(x) = \begin{cases} 
\rho_1, & x < 0, \\
\rho_2, & x > 0; 
\end{cases} \tag{2.2}
\]

the constants \( \rho_1 \) and \( \rho_2 \) are the mass densities of materials 1 and 2 in the reference state, respectively. From the definitions of \( \gamma \) and \( v \), one also has
\[ \nu_x - \gamma_t = 0 \]  

(2.3)

where the fields are smooth.

The stress-strain relation for the composite material 1/material 2 bar is

\[
\sigma = \tilde{\sigma}(\gamma, x) \equiv \begin{cases} 
\tilde{\sigma}_1(\gamma), & x < 0, \gamma > -1, \\
\tilde{\sigma}_2(\gamma), & x > 0, \gamma > -1,
\end{cases}
\]

(2.4)

where \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \) are the respective stress response functions of materials 1 and 2. For material 1, we assume that

\[
\tilde{\sigma}_1(\gamma) = \mu_1 \gamma, \quad -1 < \gamma < \infty,
\]

(2.5)

where \( \mu_1 \) is Young's modulus. The stress response function for material 2 is taken to have the following "trilinear" form:

\[
\tilde{\sigma}_2(\gamma) = \begin{cases} 
\mu_2 \gamma, & -1 < \gamma \leq \gamma_M, \\
\sigma_M - (\sigma_M - \sigma_m)(\gamma - \gamma_M)/(\gamma_M - \gamma_M), & \gamma_M \leq \gamma \leq \gamma_m, \\
\mu_2 (\gamma - \gamma_0), & \gamma \geq \gamma_m.
\end{cases}
\]

(2.6)

The graph of \( \tilde{\sigma}_2(\gamma) \) is shown schematically in Figure 1; the significance of the constants \( \mu_2, \gamma_0, \gamma_M, \gamma_m, \sigma_M \) and \( \sigma_m \) may be read from the graph. Each of the three branches of the stress-strain curve in the figure is identified with a phase of material 2: the two rising branches are the low-strain and high-strain phases, and the declining branch is the unstable phase. For simplicity, we have assumed that the Young's modulus \( \mu_2 \) is the same in both the low-strain and high-strain
phases of material 2, but this is by no means necessary. The transformation strain $\gamma_0$ is the distance between the low-strain and high-strain branches of the stress-strain curve at any given value of stress; because $\sigma_M > \sigma_m$, one has $\gamma_0 > \gamma_m - \gamma_M$. If $\gamma_0 < \gamma_m$, the stress $\sigma_m$ at the local minimum is positive, as in the figure. If $\gamma_0 \geq \gamma_m$, this minimum stress is non-positive, and the right half of the bar possesses more than one unstressed configuration. The stress $\sigma_0$ shown in Figure 1 is such that the shaded areas are equal; $\sigma_0$ is called the Maxwell stress. It is given by

$$\sigma_0 = \frac{1}{2} (\sigma_M + \sigma_m) = \frac{\mu_2}{2} (\gamma_M + \gamma_m - \gamma_0);$$

for simplicity, we assume that $\sigma_0 > 0$.

Equations (2.1)-(2.6) yield a system of two differential equations for $\gamma$ and $v$ for $x < 0$ corresponding to material 1, and a second pair of differential equations for $\gamma$ and $v$ in the interval $x > 0$ for the part of the bar made of material 2.

Suppose that there is a discontinuity in strain or particle velocity along the curve $x = s(t)$ in the $x,t$-plane. Balance of linear momentum and the assumed smoothness of the motion require that the following jump conditions hold:

$$[[\sigma]] = - \dot{s} [[p v]],$$

$$\dot{s} [[\gamma]] = - [[v]],$$

where for any $g(x, t)$, we write $[[g]] = [[g(x, t)]] = g(s(t)^+, t) - g(s(t)^-, t)$.

The strain energy per unit reference volume for the composite bar is
\[ W(x, x) = \begin{cases} W_1(\gamma), & x < 0, \gamma > -1, \\ W_2(\gamma), & x > 0, \gamma > -1, \end{cases} \quad (2.10) \]

where the separate energy densities for materials 1 and 2 are given by

\[ W_k(\gamma) = \int_0^\gamma \frac{\gamma'}{\sigma_k(\gamma')} \, d\gamma', \quad \gamma > -1, \ k = 1, 2. \quad (2.11) \]

The potential energies for the two materials are defined by \( G_k(\gamma, \sigma) = W_k(\gamma) - \sigma \gamma \). For every fixed \( \sigma \), \( G_1(\gamma, \sigma) \) as a function of \( \gamma \) has a single minimum at \( \gamma = \sigma / \mu \). In contrast, the potential \( G_2 \) for material 2 is such that, at a fixed value of \( \sigma \) between \( \sigma_m \) and \( \sigma_M \), \( G_2(\cdot, \sigma) \) has two local minima separated by a local maximum. When \( \sigma \) is outside this range, \( G_2(\cdot, \sigma) \) has only one minimum. Thus when the stress is such that material 2 can exist in either the low- or the high-strain phase, \( G_2 \) is a "two-well" potential typical of two-phase materials.

Suppose that in a piece of the bar corresponding to \( x_1 < x < x_2 \), the strain and particle velocity are discontinuous at \( x = s(t) \) but are otherwise smooth. Writing

\[ E(t) = \int_{x_1}^{x_2} \left[ W(\gamma(x, t), x) + \frac{1}{2} \rho(x) v^2(x, t) \right] \, dx \quad (2.12) \]

for the total energy in this portion of the bar at time \( t \), one can show by a direct calculation that

\[ \sigma(x_2, t) \dot{A}v(x_2, t) - \sigma(x_1, t) \dot{A}v(x_1, t) = f(t)A \dot{s}(t), \quad (2.13) \]

where
\[ f(t) = [W(\gamma(x,t), x)] - <\sigma(x,t)> [[\gamma(x,t)]] . \quad (2.14) \]

and \(<\sigma(x,t)> = (1/2)\{ \sigma(s(t)+, t) + \sigma(s(t)-, t) \}. \) We call \( f(t) \) the driving traction acting on the discontinuity at \( x = s(t) \). In view of (2.13), one may think of \( f(t)A \dot{s}(t) \) as the instantaneous dissipation rate associated with the discontinuity; if either \( f(t) = 0 \) or the discontinuity is stationary (in the Lagrangian sense) so that \( \dot{s}(t) = 0 \), this dissipation rate vanishes. At any discontinuity, it is required that the dissipation rate be non-negative:

\[ f(t) \dot{s}(t) \geq 0 . \quad (2.15) \]

One can show that, if the motion of the bar is viewed as occurring isothermally, (2.15) is a consequence of the second law of thermodynamics; we shall therefore refer to (2.15) as the entropy inequality.

In the problem to be treated here, we shall be concerned with three different types of strain discontinuity. The first of these occurs at the bimaterial interface at \( x = 0 \); because this discontinuity is stationary, (2.15) is trivially satisfied. The second type of strain discontinuity to be encountered involves a strain jump either in material 1 or between strains \( \gamma(s(t)\pm, t) \), both of which belong to the same phase in material 2. Such a discontinuity is a sound wave. Because the stress-strain relation is linear between the strains \( \gamma(s(t)\pm, t) \) in either of these circumstances, the definition (2.14) yields \( f(t) = 0 \), so that (2.15) is automatically satisfied at a sound wave as well. Finally, we shall need to deal with strain jumps in material 2 for which \( \gamma(s(t)-, t) \) is in the high-strain phase while \( \gamma(s(t)+, t) \) is in the low-strain phase. Such a discontinuity is an example of a phase boundary; for the problem to be considered here, phase boundaries in material 2 will always move to the right, so that \( \dot{s}(t) > 0 \). It then follows from (2.15) that \( f(t) \) must be non-negative at the phase boundaries arising in the present problem. Moreover, it is easy to show with the help of (2.14), (2.10), (2.11) for \( k=2 \), (2.6) and (2.7) that, at
any phase boundary with high strain on the left, low strain on the right, the driving traction is
given by

\[ f(t) = \frac{\mu_2}{2} \gamma_0 \left[ \dot{\gamma}(t) + \gamma(t) - \gamma_M - \gamma_m \right]. \] (2.16)

As in the analyses in [2,3], a kinetic relation is to be prescribed at a phase boundary; we
take it to have the form of a relation between driving traction \( f \) and phase boundary velocity \( \dot{s} \):

\[ f(t) = \varphi(\dot{s}(t)), \] (2.17)

where \( \varphi \) is a function determined by material 2. It is assumed that \( \varphi(\dot{s}) \) is a continuous function
that increases monotonically with \( \dot{s} \). The entropy inequality (2.15) imposes the restriction

\[ \varphi(\dot{s}) \dot{s} \geq 0 \] (2.18)

on the kinetic response function \( \varphi \); this and the continuity of \( \varphi \) imply in particular that \( \varphi(0)=0 \).

Again following the arguments in [2,3], we impose a nucleation criterion for the
initiation of a phase transformation from the low-strain phase to the high-strain phase in material
2: such a transformation takes place through the emergence at \( x = 0 \) of a phase boundary
whenever the associated driving traction \( f \) would be at least as great as a given critical value \( f_* \)
that is also determined by the nature of material 2. After entering the bar, the phase boundary
moves to the right in accordance with the kinetic relation (2.17). We assume that the critical
value of driving traction satisfies

\[ 0 \leq f_* \leq \frac{\gamma_0}{2} (\sigma_M - \sigma_m). \] (2.19)
The lower bound in (2.19) is a necessary consequence of the entropy inequality (2.15); the right inequality guarantees that material 2 will support slowly propagating phase boundaries as well as the fast ones permitted here; see the discussion of quasi-static phase transitions in [1-3]. Imposing the upper bound on \( f_\ast \) in (2.19) also simplifies the details of nucleation in the problem to be treated here, so we adopt it even though it is not strictly necessary to do so.

Wave propagation properties of the material 1/material 2 bar modeled above are ultimately to be compared with corresponding properties of a material 1/material 3 bar, in which material 3 is a linearly elastic material whose density and Young’s modulus coincide with their counterparts in the low-strain phase of material 2: \( \rho_3 = \rho_2, \mu_3 = \mu_2 \). Thus to treat the material 1/material 3 bar, one must modify the basic field equations (2.1)-(2.6) and jump conditions (2.8), (2.9) only to the extent of replacing \( \sigma_2(\gamma) \) by \( \sigma_3(\gamma) \) in (2.4), and thereupon replacing (2.6) by

\[
\sigma_3(\gamma) = \mu_2 \gamma, \quad -1 < \gamma < \infty.
\]

Equations (2.10)-(2.15) pertaining to the energetics of the bar remain valid but trivial, since for discontinuities in either material 1 or material 3, (2.14) always yields \( f = 0 \) in place of (2.16). Since phase transitions cannot occur in either material 1 or material 3, the kinetic relation relation (2.17), the attendant restriction (2.18) and the nucleation criterion all become irrelevant and are discarded.

3. The wave propagation problem. We now formulate the wave propagation problem to be considered. We suppose that, at time \( t = -\infty \), an incident wave bearing a given tensile strain \( \gamma_l > 0 \) and a given particle velocity \( \nu_l \) is initiated at \( x = -\infty \) in material 1, traveling to the right and striking the bimaterial interface \( x = 0 \) at time \( t = 0^- \). At \( t = 0^+ \), a sound wave will be
reflected from the interface back into material 1, and a second sound wave will be transmitted into the right half of the bar, which heretofore was at rest and unstrained, and therefore in the low-strain phase of material 2. If the incident wave fails to be strong enough to nucleate a phase transition in material 2, only these two sound waves will be generated. On the other hand, if nucleation does occur, there will also be a phase boundary that emerges from the interface $x = 0$ and travels to the right, leaving the particles of material 2 that are behind it in the high-strain phase.

Thus we are given an incident wave of the form

$$\gamma(x, t) = \begin{cases} \gamma_1, & x < c_1 t, \\ 0, & x > c_1 t, \end{cases}$$

$$v(x, t) = \begin{cases} v_1, & x < c_1 t, \\ 0, & x > c_1 t, \end{cases}$$

for $t < 0$, where $\gamma_1$ and $v_1$ are given constants, and $c_1$ is the sound speed in material 1. Observe that, because $\gamma$ and $v$ are piecewise constant, the field equations (2.1) - (2.5) are trivially satisfied away from $x = c_1 t$ for $t < 0$. When one applies the jump conditions (2.8), (2.9), specialized appropriately for material 1, to the discontinuity at $x = c_1 t$ for $t < 0$, one finds that

$$c_1 = \left(\mu_1/\rho_1\right)^{1/2}, \quad v_1 = -c_1 \gamma_1,$$

thus determining the sound speed in material 1 and imposing a restriction on the strain and particle velocity in the incident wave.

Given the incident wave and assuming first that nucleation does occur, we seek functions $\gamma(x, t)$ and $v(x, t)$ of the following form on the upper half of the $x, t$-plane (see Figure 2):
\[
\gamma(x, t) = \begin{cases} 
\gamma_1, & -\infty < x < -c_1 t, \\
\gamma_R, & -c_1 t < x < 0, \\
\gamma_T, & 0 < x < c_2 t, \\
0, & c_2 t < x < \infty,
\end{cases}
\quad v(x, t) = \begin{cases} 
v_1, & x < -c_1 t, \\
v_R, & -c_1 t < x < 0, \\
v_T, & 0 < x < c_2 t, \\
0, & c_2 t < x < \infty,
\end{cases}
\]

here the two constants \(\gamma_R, v_R\) associated with the reflected wave and the four constants \(\gamma_T, v_T\) of the transmitted wave are to be determined, as are the respective constant speeds \(s\) and \(c_2\) of the phase boundary and the sound wave in the low-strain phase of material 2. It is assumed in (3.3) that \(s\) is less than \(c_2\), so that the phase boundary moves subsonically in material 2; one can show that this is in fact necessary.

Since \(\gamma\) and \(v\) of (3.3) are piecewise constant, the field equations (2.1)-(2.6) are trivially satisfied away from discontinuities. We shall speak of the problem of determining the unknown constants in (3.3) from the jump conditions (2.8), (2.9) and the entropy inequality (2.15) at the two sound waves, the bimaterial interface and the phase boundary \(xy\) as the wave propagation problem. We shall find that this problem has a 1-parameter family of solutions, parameter \(s\); \(s\) is then determined by enforcing the kinetic relation at the phase boundary \(x = st\).

If the incident wave fails to nucleate a phase transition in material 2, the appropriate form for the solution corresponding to the incident wave (3.1) is that obtained by deleting from (3.3) the wedge \(0 < x < st\) associated with the transformed material:

\[
\begin{align*}
\gamma(x, t) &= \begin{cases} 
\gamma_1, & -\infty < x < -c_1 t, \\
\gamma_R, & -c_1 t < x < 0, \\
\gamma_T, & 0 < x < c_2 t, \\
0, & c_2 t < x < \infty,
\end{cases} \\
v(x, t) &= \begin{cases} 
v_1, & x < -c_1 t, \\
v_R, & -c_1 t < x < 0, \\
v_T, & 0 < x < c_2 t, \\
0, & c_2 t < x < \infty,
\end{cases} 
\end{align*}
\]
The form (3.4) is also appropriate for the composite bar composed of materials 1 and 3. Formally, one can infer properties of the solution whose form is (3.4) from that of the form (3.3) by setting $s = 0$ prior to enforcing the kinetic relation, thus avoiding the need to consider (3.4) separately from (3.3).

4. Determination of $\gamma$ and $v$.

Case 1. Nucleation occurs. We first assume that the incident wave causes a phase transition to occur in material 2, and we construct the corresponding reflected and transmitted waves.

When one imposes the jump conditions (2.8), (2.9) at the reflected sound wave $x = -c_1t$ in material 1, at the bimaterial interface $x = 0$, at the phase boundary $x = st$ and at the transmitted sound wave $x = c_2t$ in material 2, one obtains a system of eight algebraic equations for the eight quantities $\gamma_R$, $v_R$, $T^R$, $V_T$, $c_2$ and $s$. Of these equations, the two arising from the reflected sound wave are not independent because of (3.2), and the two arising from the transmitted sound wave determine $c_2$ as

$$c_2 = (\mu_2/\rho)^{1/2}, \quad (4.1)$$

and then reduce to a redundant pair. This leaves six independent equations for the seven unknowns $\gamma_R$, $v_R$, $T^R$, $V_T$ and $s$. These equations are readily solved for the $\gamma$'s and $v$'s in terms of $s$, furnishing

$$\gamma_R = \frac{2}{1+\beta} \gamma_T - \frac{1}{\alpha(1+\beta)} \frac{s}{c_2 + s} \gamma_0, \quad (4.2)$$
\[ \dot{\gamma}_T = \frac{2\alpha\beta}{1 + \beta} \gamma_I + \frac{\dot{s} + (1 + \beta)c_2}{(1 + \beta)(c_2 + \dot{s})} \gamma_0, \]  
(4.3)

\[ \dot{\gamma}^+_T = \frac{2\alpha\beta}{1 + \beta} \gamma_I - \frac{\dot{s}(\beta c_2 + \dot{s})}{(1 + \beta)(c_2^2 - s^2)} \gamma_0, \]  
(4.4)

\[ \nu_R = \nu_T = -\frac{2\beta}{1 + \beta} c_1 \gamma_I - \frac{1}{\alpha(1 + \beta)} \frac{\dot{s}}{c_2 + \dot{s}} c_1 \gamma_0, \]  
(4.5)

\[ \nu_T^+ = -\frac{2\beta}{1 + \beta} c_1 \gamma_I + \frac{(\beta c_2 + \dot{s})s}{(1 + \beta)(c_2^2 - s^2)} c_2 \gamma_0, \]  
(4.6)

where we have introduced the symbols

\[ \alpha = \frac{c_1}{c_2}, \quad \beta = \frac{\rho_1 c_1}{\rho_2 c_2}, \]  
(4.7)

for the respective ratios of the sound speeds and the mechanical impedances of materials 1 and 2, and we have also eliminated the particle velocity \( \nu_I \) of the incident wave from the results by using (3.2)_2.

The representations (4.2)-(4.6) for the strains and particle velocities involve the as yet unknown value of the phase boundary speed \( \dot{s} \), which will ultimately be determined by the kinetic relation.

To assure that \( \dot{\gamma}_T \) and \( \dot{\gamma}^+_T \) are respectively in the high- and low-strain phases of material 2, we must necessarily enforce the phase segregation inequalities \( \dot{\gamma}_T \geq \gamma_m, \dot{\gamma}^+_T \leq \gamma_M \). By (4.3) and (4.4), these inequalities are equivalent to

\[ \gamma_I \geq G_m(s) = \frac{1 + \beta}{2\alpha\beta} \gamma_m - \frac{1}{2\alpha\beta} \frac{(1 + \beta)c_2 + \dot{s}}{c_2 + \dot{s}} \gamma_0, \quad 0 < \dot{s} < c_2, \]  
(4.8)
The functions $G_M$ and $G_m$ defined in (4.8), (4.9) are both monotonically increasing with $\dot{s}$; moreover, $G_M(0) > 0$, and $G_M(s)$ tends to $+\infty$ as $\dot{s}$ tends to $c_2$, while $G_m(c_2)$ is finite. The inequalities in (4.8), (4.9) are restrictions on the datum $\gamma_I$ and the phase boundary speed $\dot{s}$. These restrictions are illustrated in the $s$, $\gamma_I$-plane of Figure 3, where the curves $\Gamma_m: \gamma_I = G_m(\dot{s})$ and $\Gamma_M: \gamma_I = G_M(\dot{s})$ are shown schematically; in the figure, it has been assumed for definiteness that $G_m(0) > 0$ (or equivalently $\sigma_m > 0$), though this need not be the case. Only the pairs $(\dot{s}, \gamma_I)$ that correspond to points on or between these two curves are permitted by the phase segregation requirement.

To find the driving traction $f$ at the phase boundary, one substitutes for $\dot{\gamma}_T$ and $\dot{T}_T$ from (4.3), (4.4) into (2.16); the result is

$$f = \frac{1}{2} \mu_2 \gamma_0 \left\{ \frac{4\alpha \beta}{1 + \beta} \gamma_1 + \frac{(1 + \beta) c_2^2 - 2\beta c_2 \dot{s} - 2\dot{s}^2}{(1 + \beta)(c_2^2 - \dot{s}^2)} \gamma_0 - \gamma_M - \gamma_m \right\}. \quad (4.10)$$

The curve $\Gamma_0$ in the $\dot{s}$, $\gamma_I$-plane along which $f = 0$ is therefore given by

$$\Gamma_0: \gamma_I = G_0(\dot{s}) \equiv \frac{1}{2} \left[ G_m(\dot{s}) + G_M(\dot{s}) \right]; \quad (4.11)$$

we call $\Gamma_0$ the Maxwell curve. For $0 < \dot{s} < c_2$, the right side of (4.11) is a monotonically increasing function of $\dot{s}$ that tends to $+\infty$ as $\dot{s}$ tends to $c_2$. Also, from (4.11), (4.8), (4.9), (2.6) and the assumption that the Maxwell stress $\sigma_0$ is positive, one has
\[ G_0(0) = \frac{1 + \beta}{4\alpha \beta} (\gamma_M + \gamma_m - \gamma_0) = \frac{1 + \beta}{2\alpha \beta} \frac{\sigma_0}{\mu_2} > 0. \quad (4.12) \]

The curve $\Gamma_0$ is also shown schematically in Figure 3. The requirement $f \geq 0$, imposed by the entropy inequality (2.15), holds only for points $(\hat{s}, \gamma_t)$ in the closed curvilinear strip between $\Gamma_0$ and $\Gamma_M$.

Finally, we must assure that $\gamma(x, t) > -1$ everywhere in the $x,t$-plane. One can show that, if $(\hat{s}, \gamma_t)$ lies in the strip between $\Gamma_0$ and $\Gamma_M$, the strains $\gamma_R$ and $\gamma_T$ defined in (4.2), (4.3) automatically fulfill this requirement. On the other hand, it turns out that $\frac{\dot{\gamma}_T}{\gamma_T} > -1$ if and only if $(\hat{s}, \gamma_t)$ lies above the curve $\Gamma_1$ defined by

\[ F_1 G_1 \equiv \hat{s} > \frac{\gamma_0}{2\alpha \beta} + \frac{\gamma_0}{\gamma_M + \gamma_m + 2} \quad 0 < \hat{s} < c_2. \quad (4.13) \]

From (4.13) and (4.9), it can be seen that $\Gamma_M$ always lies above $\Gamma_1$. After some algebra, one shows that $\Gamma_1$ intersects $\Gamma_0$ exactly once; the intersection occurs at a value $\hat{s} = \hat{s}_*$ given by

\[ \hat{s}_* = \left(1 - \frac{\gamma_0}{\gamma_M + \gamma_m + 2}\right)^{1/2} c_2 < c_2. \quad (4.14) \]

For values of $\hat{s}$ to the left of $\hat{s}_*$, $\Gamma_1$ lies below $\Gamma_0$, while for $\hat{s}_* < \hat{s} < c_2$, the reverse is true.

Let $S$ stand for the closed curvilinear strip bounded above by $\Gamma_M$ and below in part by $\Gamma_0$ and in part by $\Gamma_1$; $S$ is shown hatched in Figure 3. It then follows that for each value of the
incident strain $\gamma_1$ that lies in the interval $(G_0(0), +\infty)$, there is a 1-parameter family (parameter $\dot{s}$) of admissible fields $\gamma(x, t), \nu(x, t)$ of the form (4.2)-(4.6) that fulfill all of the conditions of the wave propagation problem. For each given $\gamma_1$ in this interval, the permissible range of the parameter $\dot{s}$ is that corresponding to the end-points of the associated horizontal line segment in the $\dot{s}, \gamma_1$-plane connecting either $\Gamma_M$ or the vertical axis on the left to either $\Gamma_0$ or $\Gamma_{-1}$ on the right, as appropriate. Each of these fields that corresponds to a positive value of $\dot{s}$ involves a phase transition. For each value of $\gamma_1$ outside $(G_0(0), +\infty)$, there is no solution to the wave propagation problem that involves a phase transition; as we shall see in Case 2 below, the only solution available for such an initial datum $\gamma_1$ is one in which no phase transition occurs.

Case 2. Nucleation does not occur. In this case, we can find the solutions of the form (3.4) arising from the incident wave (3.1) by formally setting $\dot{s} = 0$ in the expressions (4.2)-(4.6). This yields

$$\gamma(x, t) = \begin{cases} 
\frac{2}{1+\beta} \gamma_1, & -c_1t < x < 0, \\
\frac{2\alpha\beta}{1+\beta} \gamma_1, & 0 < x < c_2t,
\end{cases}$$

$$\nu(x, t) = -\frac{2\alpha\beta}{1+\beta} c_2 \gamma_1, & -c_1t < x < c_2t, \quad t > 0,$$  

(4.15)

for the reflected and transmitted waves in the absence of the phase transformation. Since the solution (4.15) involves only sound waves, the entropy inequality (2.15) holds automatically. The phase segregation requirement demands that $\gamma(x, t) \leq \gamma_M$ for $0 < x < c_2t$; from (4.15) and (4.9), this leads to

$$\gamma_1 \leq G_M(0).$$  

(4.16)

For each positive initial datum $\gamma_1$ satisfying (4.16), the fields (4.15) comprise a solution of the
wave propagation problem that is uniquely determined by the data and in which no phase
transition takes place. Since the incident strain $\gamma_I$ has been assumed positive (tensile) throughout,
the requirement $\gamma(x, t) > -1$ is automatically satisfied by the strain field of $(4.15)_1$.

It is important to note that $(4.15)$ also represents the solution to the wave propagation
problem in which the left half of the bar is composed of material 1, while the right half is made
of material 3. In this case, however, $(4.15)$ represents a solution for all tensile values of the
initial datum $\gamma_I$, and not merely for those satisfying $(4.16)$.

Combining the conclusions reached in Cases 1 and 2 establishes the following results.
(i) For each value of the initial datum $\gamma_I$ in the interval $(0, G_0(0))$, there is exactly one solution to
the wave propagation problem; it is given by $(4.15)$ and does not involve a phase transition. (ii)
For each $\gamma_I$ in $(G_0(0), G_M(0)]$, there is a 1-parameter family of solutions $(4.2)$-$(4.6)$ to the wave
propagation problem that involve a phase transition, and a single solution $(4.15)$ that does not.
(iii) For values of $\gamma_I > G_M(0)$, there is only the 1-parameter family of solutions $(4.2)$-$(4.6)$, each
of which involves a phase transition.

The ambiguity remaining when $\gamma_I$ is in $(G_0(0), G_M(0)]$ (case (ii) above) is resolved by
first invoking the nucleation criterion, then the kinetic relation. Setting $f$ in $(4.10)$ equal to the
nucleation value $f_*$ defines a curve $\Gamma_*: \gamma_I = G_*(s)$ in the $s, \gamma_I$-plane; we omit the formula for $G_*$,
and we do not show $\Gamma_*$ in Figure 3. The constitutive inequality $(2.19)$ can be shown to guarantee
that $\Gamma_*$ always lies in the between $\Gamma_0$ and $\Gamma_M$, and that $G_0(0) \leq G_*(0) \leq G_M(0)$. We assume
further that $f_*$ is such that $\Gamma_*$ lies in the slightly smaller strip $S$ in Figure 3. At points on or
between $\Gamma_*$ and $\Gamma_M$ (4.10) yields values of $f$ at least as great as $f_*$. It then follows that
nucleation will occur if the incident strain $\gamma_I$ is at least as great as the critical value $\gamma_I^*$ given by

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When the incident wave carries a strain \( \gamma^* \), that is less than \( \gamma^*_1 \), no phase transition occurs, and the appropriate solution to the wave propagation problem is that given by (4.15). On the other hand, if \( \gamma^*_1 \) is at least as great as \( \gamma^*_1 \), nucleation will take place in material 2, a phase boundary will emerge at \( x = 0 \), and the appropriate solution must be selected from the 1-parameter family of admissible fields (4.2)-(4.6) involving a phase transition. The kinetic relation (2.17) provides the mechanism for this selection. Substituting for \( f \) from (4.10) into (2.17) yields the equation for \( \dot{s} \):

\[
\gamma^*_1 = G_k(s) \equiv G_0(s) + \frac{1 + \beta}{2\alpha\beta\mu_2\gamma_0} \phi(s), \quad \gamma \geq \gamma^*_1,
\]  

(4.18)

where \( \phi \) is the kinetic response function of the material. Because \( \phi(\dot{s}) \) and \( G_0(\dot{s}) \) both increase monotonically with \( \dot{s} \), the same is true of the right side of (4.18). Since \( \phi(0) = 0 \), one has \( G_k(0) = G_0(0) \); moreover, \( G_k(\dot{s}) \) tends to \( +\infty \) as \( \dot{s} \) tends to \( c_2 \), and \( 0 < G_0(\dot{s}) < G_k(\dot{s}) < G_M(\dot{s}) \) for \( 0 < \dot{s} < c_2 \). Thus the curve \( \Gamma_k \) in the \( \dot{s}, \gamma \)-plane represented by (4.18) lies between \( \Gamma_0 \) and \( \Gamma_M \).

We further assume that \( \phi \) is such that \( \Gamma_k \) lies in the strip \( S \) (see Figure 3). Clearly (4.18) determines exactly one value of \( \dot{s} \) for each given \( \gamma_1 \) in \((G_0(0), +\infty)\). When inserted into (4.2)-(4.6), this value of \( \dot{s} \) completes the determination of the response to the incident wave when a phase transition is nucleated.

5. Reflectivity. For a given incident wave in material 1, we wish to compare the strengths of the reflected waves when (i) the right half of the bar consists of material 2, and (ii) the right half is composed of material 3. To motivate our notion of the strength of the transmitted or reflected wave, it is helpful to reconsider the energetics of the composite bar. Consider first the case of the material 1/material 2 bar. By extending the energy considerations
of Section 2 to a portion of the bar long enough to include the four discontinuities at $x = -c_1 t$, $x = 0$, $x = \dot{s}t$ and $x = c_2 t$, one can establish the following identity among energy rates for fields of the form (3.3):

$$e_I = e_R + e_T + fA\dot{s}, \quad (5.1)$$

where

$$e_1 = \left\{ \frac{1}{2} c_1^2 \gamma_T^2 + \frac{1}{2} v_1^2 \right\} \rho_1 c_1 A, \quad (5.2)$$

$$e_R = \left\{ \frac{1}{2} c_1^2 (\gamma_R - \gamma_T)^2 + \frac{1}{2} (v_R - v_1)^2 \right\} \rho_1 c_1 A, \quad (5.3)$$

$$e_T = \left\{ \frac{W_2(\dot{\gamma}_T)}{\rho_2} + \frac{1}{2} \dot{v}_T^2 \right\} \rho_2 \dot{s} A + \left\{ \frac{1}{2} c_2^2 \gamma_T^2 + \frac{1}{2} \dot{v}_T^2 \right\} \rho_2 (c_2 - \dot{s}) A, \quad (5.4)$$

$f$ is the driving traction at the phase boundary, $A$ is the cross-sectional area of the bar and $\dot{s}$ is the phase boundary speed. In (5.3), (5.4), $\gamma_R$, $\gamma_T$, $v_R$ and $v_T$ are given in terms of $\dot{s}$ by (4.2)-(4.6), and $W_2(\dot{\gamma}_T)$ refers to the stored energy density $W_2$ of material 2. We call $e_I$, $e_R$ and $e_T$ the incident, reflected and transmitted energy rates; note that all have the units of energy per unit time, and all are non-negative. By (2.15), the dissipation rate $fA\dot{s}$ is non-negative as well. Thus neither $e_R$ nor $e_T$ is greater than $e_I$.

Setting $\dot{s} = 0$ in (5.1)-(5.4) provides the corresponding identity among energy rates for the material 1/material 3 bar in which no phase transformation can occur.

The reflectivity and transmissivity are defined by
respectively; they are measures of the strengths of the reflected and transmitted waves relative to the incident wave. We are interested in

\[
q_R = \frac{e_R}{e_1} \leq 1, \quad q_T = \frac{e_T}{e_1} \leq 1,
\]

which are the ratios of reflectivity and transmissivity in the material 1/material 2 bar to their respective counterparts in the material 1/material 3 composite. We study \(Q_R\) in the present section, \(Q_T\) in the next. When \(\beta = 1\), the reflectivity \(q_{R|s=0}\) of the material 1/material 3 bar vanishes, and (5.6) fails to define \(Q_R\); we exclude this possibility for simplicity by assuming that \(\beta \neq 1\).

The reflectivity ratio \(Q_R\) is a function on the admissible strip \(S\) shown hatched in Figure 3. For a given kinetic relation, only the values of \(Q_R\) at points on the associated kinetic curve \(\Gamma_k\) are relevant; for a given value of the strain \(\gamma_1\) in the incident wave, only the value of \(Q_R\) at the point on \(\Gamma_k\) whose ordinate is \(\gamma_1\) is relevant. If, for example, one has \(Q_R(s, \gamma_1) < 1\) at this point, then the reflectivity of the material 1/material 2 bar is less than that of the material 1/material 3 bar, so that reflection has been diminished by the occurrence of the phase transition. We study the properties of \(Q_R\) as a function on \(S\).

From (5.6)_1, (5.5)_1, (5.3), (4.2) and (4.5), one finds
it may be noted from Figure 3 that $\gamma_1 \geq G_0(0) > 0$ at all points on $S$; this fact, together with the assumption that $\beta \neq 1$, guarantees that $Q_R$ is well defined on $S$. Of course, $Q_R(0, \gamma_1) = 1$; we now determine the locus of points $(\hat{s}, \gamma_1)$ in the $\hat{s}, \gamma_1$-plane at which $Q_R = 1$ and $\hat{s} \neq 0$; such points correspond to situations in which the reflectivity is unaltered by the phase transition. This locus consists of the interior points on the curve $\Gamma_R^1$ defined by

$$
\Gamma_R^1: \quad \gamma_1 = \frac{1}{2\alpha(1-\beta)} \frac{\hat{s}}{c_2 + \hat{s}} \gamma_0, \quad 0 \leq \hat{s} \leq c_2.
$$

Only those points on $\Gamma_R^1$, if any, that lie in the admissible strip $S$ are relevant. If a portion of $\Gamma_R^1$ lies in the interior of $S$, the strip is divided into two parts: on one, $Q_R < 1$, while on the other, $Q_R > 1$. Whether $\Gamma_R^1$ has points in common with the interior of $S$ is determined by two material parameters $\beta$ and $\lambda$, where $\beta = \rho_1 c_1 / \rho_2 c_2$ is the (positive) impedance ratio introduced earlier, and $\lambda$ is defined by

$$
\lambda = \frac{\gamma_M + \gamma_m}{\gamma_0}.
$$

By (2.7) and our assumption that the Maxwell stress $\sigma_0$ is positive, we have $\lambda > 1$. We first show that $\Gamma_R^1$ fails to intersect the interior of $S$ if either of the following conditions holds:

(i) $\beta > 1$ or (ii) $0 < \beta < 1$ and $\lambda > \Lambda(\beta^2)$.

$$
(i) \beta > 1 \text{ or } (ii) 0 < \beta < 1 \text{ and } \lambda > \Lambda(\beta^2).
$$
where

\[
\Lambda(z) = 1 + \frac{1}{2} \left\{ \frac{1 + z - \left[(1 - z)(1 + 3z)\right]^{1/2}}{1 - z} \right\}, \quad 0 < z < 1. \tag{5.11}
\]

One can show that \(\Lambda(z)\) increases monotonically from the value 1 at \(z = 0\), tending to \(+\infty\) as \(z\) tends to 1.

To establish (5.10), we first note that by (4.11), (4.12) and (5.8), the curve \(\Gamma_R^1\) lies below the Maxwell curve \(\Gamma_0\) near both endpoints of the interval \(0 < \hat{s} < c_2\), so that \(\Gamma_R^1\) cannot have points in common with the interior of the admissible strip \(S\) unless \(\Gamma_R^1\) intersects \(\Gamma_0\) at least twice. To investigate such intersections, one equates the right sides of (4.11) and (5.8), obtaining a quadratic equation for \(\hat{s}\). It is readily shown that (5.10) describes precisely the conditions under which this equation has no real roots in the interval \(0, c_2\). Thus (5.10) is indeed sufficient to assure that \(\Gamma_R^1\) fails to intersect the interior of \(S\). Conversely, when

\[
\beta < 1 \text{ and } \lambda < \Lambda(\beta^2), \tag{5.12}
\]

the quadratic equation mentioned above has two real roots in \((0, c_2)\), corresponding to two points at which \(\Gamma_R^1\) and \(\Gamma_0\) intersect. By a calculation too lengthy to be included here, one can show that the smaller of these two roots lies in \((0, \hat{s}_x)\) (Figure 3), so that at the left point of intersection, \(\Gamma_R^1\) indeed enters \(S\).

We are now in a position to delineate conditions under which the occurrence of the phase transition always increases the reflectivity, always decreases it or might do either. Figure 4 shows three open regions marked I, II and III in the quadrant \(\beta > 0, \lambda > 1\) in the \(\beta, \lambda\)-plane.
Region III corresponds precisely to the inequalities (5.12) and hence to the case in which $F_R$ enters $S$. It follows that the reflectivity ratio $Q_R$ is less than one at some points $(\beta, \lambda)$ in this region, greater than one at others; the actual effect obtained depends on the particular kinetic relation and possibly on the particular value of the strain $\gamma_1$ in the incident wave as well. The region marked I corresponds to values of $\beta$ and $\lambda$ for which, on the interior of the admissible strip $S$ of Figure 3, reflectivity at the interface is always less in the material 1/material 2 bar than it is in the material 1/material 3 bar. Thus assuming that nucleation has occurred, the phase transformation induced in material 2 by the incident wave always acts to reduce reflection if $(\beta, \lambda)$ corresponds to a point in region I, regardless of the particular kinetic relation involved. In region II, precisely the reverse is true: reflectivity is always increased at points in the interior of $S$ by the phase transformation. For values of $(\beta, \lambda)$ belonging to the various regions I, II and III, Figure 5 schematically shows the curve $\Gamma_R^1$ in the $s, \gamma_1$-plane.

If the impedance of material 1 is greater than that of the parent phase of material 2, then $\beta > 1$, and it follows from (5.7) that $Q_R(s, \gamma_1)$ increases with $s$ for each fixed $\gamma_1$. Thus at each given $\gamma_1$, the maximum value of the reflectivity ratio $Q_R$ occurs on the right boundary of $S$, while the minimum value occurs on the left boundary. Thus when $\beta > 1$, fast kinetics promote the increase of reflectivity, while slow kinetics are best for reducing it.

If the impedance ratio $\beta < 1$, the behavior of $Q_R$ as a function of phase boundary speed $s$ at fixed incident strain $\gamma_1$ is more complicated; under certain circumstances, $Q_R$ may vanish at some points in $S$, corresponding through (5.7) to a minimum in $Q_R$. 
6. Transmissivity. We state here without proof some results pertaining to the transmissivity ratio $Q_T$, whose behavior is more complicated than that of the reflectivity ratio $Q_R$. Using $(5.6)_2$, $(5.5)_2$, $(5.4)$, and $(4.3)$ - $(4.6)$, one can show that

$$Q_T(s, \gamma) = 1 - \frac{1}{2\alpha\beta} \frac{\gamma_0}{\gamma_1} T_1(s) + \frac{1}{8\alpha^2\beta^2} \left( \frac{\gamma_0}{\gamma_1} \right)^2 T_2(s),$$  \hspace{1cm} (6.1)

where

$$T_1(s) = \frac{[2\beta + (1 + \beta)s/c_2]}{c_2 + \dot{s}},$$  \hspace{1cm} (6.2)

$$T_2(s) = (\lambda - 1)(1 + \beta)^2s/c_2 + \frac{c_2 s^2}{(c_2^2 - s^2)(c_2 + \dot{s})} T_3(s),$$  \hspace{1cm} (6.3)

$$T_3(s) = 2\beta^2 + (1 + 4\beta + \beta^2) \frac{\dot{s}}{c_2} + (1 - \beta^2) \frac{s^2}{c_2^2},$$  \hspace{1cm} (6.4)

and $\lambda$ is defined in $(5.9)$. It is possible to show that, for $\beta > 1$, one has $Q_T < 1$ at all interior points of the admissible strip $S$; on the other hand if $\beta < 1$, one finds that $Q_T$ takes values greater than 1 at some points in $S$, less than 1 at others.
REFERENCES


FIGURE 1. STRESS-STRAIN CURVE FOR MATERIAL 2.
FIGURE 2. INCIDENT, REFLECTED AND TRANSMITTED WAVES.
FIGURE 3. THE ADMISSIBLE STRIP $S$ IN THE $\dot{s}, \gamma_1$-PLANE.
FIGURE 4. REGION I: $Q_R < 1$;
REGION II: $Q_R > 1$;
REGION III: $Q_R - 1$ MAY HAVE EITHER SIGN.
FIGURE 5. THE CURVE $\Gamma_{R}^{-1}$ FOR (a) $(\beta, \lambda) \in I$, (b) $(\beta, \lambda) \in II$, (c) $(\beta, \lambda) \in III$. 
This paper is concerned with waves in a composite elastic bar, the left half of which is composed of a linearly elastic material, while the nonlinearly elastic material of the right half can undergo a phase transition. We assume that a wave in the left portion of the bar is incident upon the interface between the two materials, and we investigate the question of whether the phase transition can be exploited to augment or diminish the strength of the reflected or transmitted wave.