ON SINGULAR SEMILINEAR ELLIPTIC EQUATIONS

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ON SINGULAR SEMILINEAR ELLIPTIC EQUATIONS

Abstract: For the semilinear elliptic equation \( \Delta u + p(x)u^{-\gamma} = 0, \ x \in \mathbb{R}^n, \)

\( n \geq 3, \gamma > 0, \) we show via the barrier method the existence of a positive

entire solution behaving like \( |x|^{2-n} \) near \( \infty. \)
Abstract: - For the semilinear elliptic equation \( \Delta u + p(x)u^{-\gamma} = 0 \), \( x \in \mathbb{R}^n \), \( n \geq 3 \), \( \gamma > 0 \), we show via the barrier method the existence of a positive entire solution behaving like \( |x|^{2-n} \) near \( \infty \).

1 Introduction

We study the singular semilinear elliptic equation
\[
\Delta u + p(x)u^{-\gamma} = 0
\]
in \( \mathbb{R}^n \). This type of equation arises in the boundary layer theory of viscous fluids [3,4]. From the results of Fulks and Maybee [8], Crandall, Rabinowith, and Tartar [5], Gomes [9], and recently Lazer and McKenna [14], it follows that (1) has a unique classical solution within a bounded domain \( \Omega \), where \( p(x) \) is a sufficiently regular function which is positive on \( \Omega \). Kusano and Swanson [12] gave the existence proof on exterior domains. As for the existence of entire solutions, not much is known. Edelson [7], Kusano and Swanson [13] have been able to show the existence of entire solutions of (1) with \( \gamma \in (0,1) \), and \( p(x) \) sufficiently regular. In this paper we show via the upper and lower solution method, which is also referred to as the barrier method, that (1) has a bounded positive entire solution vanishing at \( \infty \) in \( \mathbb{R}^n \) for \( n \geq 3 \) and all \( \gamma > 0 \).

The author learned after this paper was finished that a similar result was given earlier by R. Dalmasso [6], but by a different approach.
2 Preliminaries

We first state the theorem by Kusano and Swanson [13] for the case $0 < \gamma < 1$.

**Lemma 1.** Equation (1) has an entire bounded positive solution $u(z)$ in $\mathbb{R}^n$ for $n \geq 3$, and $|x|^{n-2}u(x)$ is bounded and bounded away from zero near $\infty$ if $p(x)$ satisfies the following conditions:

1. $p(x) \in C_{\text{loc}}(\mathbb{R}^n)$, $n \geq 3$, $p(x) > 0$, $x \in \mathbb{R}^n \setminus \{0\}$,
2. $\exists C > 0$, such that $C \phi(|x|) \leq p(x) \leq \phi(|x|)$, $\phi(x) = \max_{|x|=t} p(x)$, $0 \leq t < \infty$,
3. $\int_1^{\infty} t^{n-1+\gamma(n-2)} \phi(t) dt < \infty$.

The term "entire" has often been used for solutions of equation (1) in $\mathbb{R}^n$. To avoid confusion with the traditional definition for entire functions, we use the term "$C^{2+\alpha}$-entire". A $C^{2+\alpha}$-entire solution of (1) is defined to be a function $u(x) \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^n)$ that satisfies (1) pointwise in $\mathbb{R}^n$.

The method that we shall be using heavily in our proof is the so-called barrier method, or upper-lower solution method.

We consider the elliptic boundary value problem

$$
\begin{cases}
Lu + f(x,u) = 0 & \text{in } D \\
Bu = a \frac{\partial u}{\partial \nu} + bu = g & \text{on } \partial D
\end{cases}
$$

where $D$ is a smoothly bounded domain in $\mathbb{R}^N$ and $\nu = (\nu_1, \cdots, \nu_n)$ is a smoothly varying outward normal vector field on $\partial D$ which is of class $C^{2+\alpha}$, while $a$ and $b$ are positive constants. We also assume that $f \in C^\alpha$ and that $g$ has an extension $\hat{g}$ to the interior of $D$ such that $\hat{g} \in C^{2+\alpha}$.

An upper solution to the above problem is a function $\phi$ satisfying

$$
\begin{cases}
L\phi + f(x,\phi) \leq 0 & \text{in } D \\
B\phi \geq g & \text{on } \partial D.
\end{cases}
$$

A lower solution to the above problem is a function $\psi$ satisfying

$$
\begin{cases}
L\psi + f(x,\psi) \geq 0 & \text{in } D \\
B\psi \leq g & \text{on } \partial D.
\end{cases}
$$

We assume that $\partial D, f, g$, and the coefficients of $L$ are smooth in what follows.
Lemma 2. (Theorem 2.3.1 of [16]) Let $\phi$ be an upper solution and $\psi$ a lower solution with $\psi \leq \phi$ on $D$. Then there exists a solution $u$ to the above boundary value problem with $\psi \leq u \leq \phi$.

We consider the following example:

$$
\begin{cases}
  u'' + \lambda u - u^3 = 0 & x \in (0, \pi) \\
  u = 0 & x = 0, \pi
\end{cases}
$$

By the above theorem, if $\lambda > 1$, then the problem has at least three solutions.

Actually, $u = \epsilon \sin x$ with $\epsilon$ small is a lower solution, and $\bar{u} = Rx^{1/2}$ with $R$ large is an upper solution. Therefore there exists a solution $u$ such that $u \leq u \leq \bar{u}$ in $(0, \pi)$. Clearly $-u$ and 0 are also solutions to this problem.

The following lemma on the barrier method for $D = \mathbb{R}^n$ is due to Ni [15] in 1982. A special case was proved earlier by Ako and Kusano [1] in 1964. The proof is standard. Using the well known result on the upper-lower solution approach in bounded regions (see Sattinger [16]), we first solve the equation

$$Lu + F(x, u) = 0$$

on $B_R$. Then by letting, $R \to \infty$, we obtain a solution on $\mathbb{R}^n$ by a diagonal process.

Lemma 3. Let $u_1 \geq u_2$ in $\mathbb{R}^n$ be such that

where $f$ is locally Hölder continuous in $(x, u)$ and locally Lipschitz in $u$, and $L$ is an elliptic operator of second order. Then there exists a solution $u$ of $Lu + f(x, u) = 0$ with $u_1 \geq u \geq u_2$.

3 Main Result

Theorem 1. Under the same conditions as given in Lemma 1, the equation (1) has a $C^{2+\alpha}$-entire positive solution in $\mathbb{R}^N$, $N \geq 3$, vanishing at $\infty$ at the rate of at least $|x|^{q(N-2)}$ with some $q \in (0,1)$ for any $\gamma > 0$. 
The difficulty in constructing the proof is to find an appropriate upper solution to equation (1). In order to use the barrier method we first study the nonsingular equation

\[ \Delta u + p(x)[\delta + u]^{-\gamma} = 0. \]

For each fixed \( \gamma \) there corresponds a solution \( u_\gamma(x) \). Letting \( \gamma \to \infty \), we show that the limiting function is the desired solution.

**Proof:** By Lemma 1, for \( \gamma = \gamma_1 \in (0, 1) \), equation (1) has a \( C^{2+\alpha} \)-entire positive solution \( u_1(x) \) in \( R^n, n \geq 3 \), vanishing at \( \infty \) at the rate \( r^{2-n} \). We claim that \( \bar{u} = cu_1^\gamma \) is an upper solution of the equation (1) for \( \gamma \geq 1 \), where

\[ q < \frac{1 + \gamma_1}{1 + \gamma} < 1, \]

\[ c > \left( \frac{M^{1+\gamma_1-q(1+\gamma)}}{q} \right)^{1+\gamma_1}, \quad M = \max_{x \in R^n} |u(x)|. \]

In fact:

\[ \Delta \bar{u} + \frac{p(x)}{\bar{u}^\gamma} \]

\[ = cg(q - 1)u^{q-2}|\nabla u|^2 - cgu^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma} \]

\[ \leq -cgu^{q-1}p(x)u^{-\gamma_1} + p(x)c^{-\gamma}u^{-\gamma} \]

\[ = \frac{p(x)}{c^\gamma u^{\gamma}} (1 - \frac{c^{1+\gamma} q}{u^{1+\gamma_1-q(1+\gamma)}}) \]

\[ \leq \frac{p(x)}{c^\gamma u^{\gamma}} (1 - \left( \frac{M^{1+\gamma_1-q(1+\gamma)}}{q} \frac{q}{u^{1+\gamma_1-q(1+\gamma)}} \right)) \]

\[ \leq \frac{p(x)}{c^\gamma u^{\gamma}} (1 - 1) = 0. \]

Let \( \delta \) be a fixed positive number. We then observe that \( \bar{u} \) is an upper solution of the equation

\[ \Delta u(x) + p(x)[u(x) + \delta]^{-\gamma} = 0, \quad x \in R^n. \]

\( \underline{u} = 0 \) is a lower solution of (4). Since \( \bar{u} = cu_1^\gamma > 0, \bar{u} \geq \underline{u} \) in \( R^n \). By Lemma 2, (4) has a solution \( u \) such that \( \underline{u} \leq u \leq \bar{u} \).
For \( \delta < \delta \), \( u \) is a lower solution of (4) with \( \delta = \delta \). Lemma 2 then implies that (4) has a solution \( \hat{u} \) for \( \delta = \delta \) such that \( u \leq \hat{u} \leq \bar{u} \).

Let \( \{\delta_n\}_1^\infty \) be a sequence of strictly decreasing positive numbers, and let \( u_n(x) \) be a smooth positive solution of (4) when \( \delta = \delta_n \). From the construction of our lower solutions, it is clear that \( u_n(x) \geq u_{n-1}(x) \) for all \( n \). So \( \lim_{n \to \infty} u_n(x) = u(x) \) exists for all \( x \in \mathbb{R}^n \) and

\begin{equation}
\tag{5}
u \leq u \leq \bar{u}
\end{equation}

for \( x \in \mathbb{R}^n \).

We can now assert that \( u \in C^{2+\alpha}(\mathbb{R}^n) \) and that

\begin{equation}
\tag{6}\Delta u + p(x)u^{-\gamma} = 0
\end{equation}

for \( x \in \mathbb{R}^n \). This follows from more or less standard arguments.

Let \( x_0 \in \mathbb{R}^n \) and \( r > 0 \). We consider the ball of radius \( r \) centered at \( x_0 \), \( B(x_0, r) \) in \( \mathbb{R}^n \). Let \( \Psi \) be a \( C^\infty \) function which is equal to 1 on \( B(x_0, r/2) \) and equal to 0 off \( B(x_0, r) \). We have

\[ \Delta(\Psi u_n) = 2\nabla\Psi \cdot \nabla u_n + p_n \]

for \( n \geq 1 \), where \( p_n \) is a term whose \( L^\infty \) norm is bounded independently of \( n \). Therefore for \( n \geq 1 \) we have

\[ \Psi u_n \Delta(\Psi u_n) = \sum_{j=1}^N b_{nj} \frac{\partial(\Psi u_n)}{\partial x_j} + q_n, \]

where \( b_{nj}, j = 1, \ldots, n \) and \( q_n \) are terms bounded independently of \( n \) for \( n \geq 1 \). Integrating the above equation, we have that there exist constants \( c_1 > 0 \) and \( c_2 > 0 \) independent of \( n \) such that

\[ \int_{B(x_0, r)} |\nabla u_n|^2 dx \leq c_1 \left( \int_{B(x_0, r)} |\nabla u_n|^2 dx \right)^{1/2} + c_2. \]

From this, it follows that the \( L^2(B(x_0, r)) \)-norm of \( |\nabla \Psi u_n| \) is bounded independently of \( n \). Hence, the \( L^2(B(x_0, r/2)) \)-norm of \( |\nabla u_n| \) is bounded independently of \( n \). Let \( \Psi_1 \) be a \( C^\infty \) function which is equal to 1 on \( B(x_0, r/4) \) and equal to 0 off \( B(x_0, r/2) \). We have for \( n \geq 1, \)

\[ \Delta(\Psi_1 u_n) = 2\nabla \Psi_1 \cdot \nabla u_n + p_{1n}. \]
where $p_{2n}$ is a term whose $L^\infty(B(x_o, r/4))$-norm is bounded independently of $n$. From standard elliptic theory, the $W^{2,2}(B(x_o, r/2))$-norm of $\Psi_1u_n$ is also bounded independently of $n$ and hence, the $W^{2,2}(B(x_o, r/4))$-norm of $u_n$ is bounded independently of $n$. Since the $W^{1,2}(B(x_o, r/4))$-norm of the components of $\nabla u_n$ are bounded independently of $n$, it follows from the Sobolev embedding theorem that if $q = 2n/(n-2) > 2$ for $n > 2$ and in addition if $q > 2$ is arbitrary for $n \leq 2$, then the $L^q(B(x_o, r/4))$-norm of $|u_n|$ is bounded independently of $n$. Let $\Psi_2$ be a $C^\infty$ function which is equal to 1 on $B(x_o, r/8)$ and equal to 0 iff $B(x_o, r/4)$. We have for $n \geq 1$,

$$\Delta(\Psi_2u_n) = 2\nabla\Psi_2 \cdot \nabla u_n + p_{2n},$$

where $p_{2n}$ is a term whose $L^\infty(B(x_o, r/4))$-norm is bounded independently of $n$. Since the right hand side of the above equation is bounded in $L^q(B(x_o, r/4))$ independently of $n$, the $W^{2,q}(B(x_o, r/4))$-norm of $\Psi_2u_n$ is also bounded independently of $n$. Hence, the $W^{2,q}(B(x_o, r/8))$-norm of $u_n$ is bounded independently of $n$. Continuing this line of reasoning, after a finite number of steps, we find a number $r_1 > 0$ and $q_1 > n/(1-\alpha)$ such that the $W^{2,q_1}(B(x_o, r_1))$-norm of $u_n$ is bounded independently of $n$. Hence, there is a subsequence of $\{u_n\}^\infty$, which we may assume is the sequence itself, which converges in $C^{1+\alpha}(B(x_o, r_1))$. If $\theta$ is a $C^\infty$ function which is equal to 1 on $(B(x_o, r_1/2))$ and 0 off $B(x_o, r_1)$, then

$$\Delta(\theta u_n) = 2\nabla\theta \cdot \nabla u_n + \tilde{p}_n,$$

where $\tilde{p}_n = \theta \Delta u_n + u_n \Delta \theta$.

The right-hand side of the above equation converges in $C^\alpha(B(x_o, r_1))$. Hence by Schauder theory, $\{\theta u_n\}^\infty$ converges in $C^{2+\alpha}(B(x_o, r_1))$ and thus $\{u_n\}^\infty$ converges in $C^{2+\alpha}(B(x_o, r_1/2))$. Since $x_o$ was arbitrary, this shows that $u \in C^{2+\alpha}(R^n)$. Clearly (6) holds.

4 Some remarks

Remark 1:

For $n = 1$, the properties of positive solutions of equation (1) have been studied by Taliaferro [17], and Gatica [10]. For $n = 2$, no entire positive solution of equation (1) exists regardless of its asymptotic behavior at $\infty$ (see [13]).

Remark 2:
It is observed by Callegari, Friedman and Nachman [2], [3,4] that if the partial differential equations describing the boundary layer behind a rarefaction or shock wave (with viscosity proportional to the temperature) traveling down, and perpendicular to, a flat plate are written in terms of a stream function and a similarity variable the following Blasius-type equation emerges [18].

\[ f'''(\eta) + f(\eta)f''(\eta) = 0, \]

where

\[ f(0) = 0, \quad f'(0) = K, \quad f'(<\infty) = 1. \]

Here, \( 0 < K < 1 \), for rarefaction waves and, \( 1 < K < 6 \), for shock waves. (\( K = 0 \) corresponds to the classical Blasius problem.) Adopting the Crocco variables

\[ x = f'(\eta), \quad g = f''(\eta) \]

results in the system

\[ gg'' + x = 0, \]
\[ g'(K) = 0, \quad g(1) = 0, \]

which falls into the class of equation discussed in this paper.

References


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