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Perturbations, Untruncated in Eccentricity, For an Orbit in an Axi-Symmetric Gravitational Field

by

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SUMMARY

By recourse to a particular definition of a satellite's mean orbital elements and to a particular system of spherical polar coordinates based on the mean orbital plane, an orbital theory has been developed that leads to extremely compact first-order perturbation formulae associated with the general zonal harmonic, J_ℓ. The formulae are complete (untruncated in eccentricity) and generalize, via recurrence relations, the author's earlier results for the effects of J_2 (analysed to second order) and J_3. To illustrate the compact nature of individual expressions, the (untruncated) perturbation formulae due to J_4 are given.

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R. H. Gooding

By recourse to a particular definition of a satellite's mean orbital elements and to a particular system of spherical polar coordinates based on the mean orbital plane, an orbital theory has been developed that leads to extremely compact first-order perturbation formulae associated with the general zonal harmonic, J_ℓ . The formulae are complete (untruncated in eccentricity) and generalize, via recurrence relations, the author's earlier results for the effects of J_2 (analysed to second order) and J_3 . To illustrate the compact nature of individual expressions, the (untruncated) perturbation formulae due to J_4 are given.

INTRODUCTION

In an earlier paper[1], the author summarized a theory for satellite perturbations due to the harmonics J_2 and J_3 of the Earth's gravitational field, presenting formulae that are complete (untruncated in eccentricity) to second order in J_2 and first order in J_3 . The novelty of the theory arises from the way in which short-period perturbations in the osculating element are amalgamated into perturbations in a set of spherical-polar coordinates (r, b, w), based on a mean orbital plane: r is the geocentric distance (radial direction), whilst b and w are quasi-latitude (cross-track direction) and quasi-longitude (along-track direction). Particular definitions of the mean orbital elements were adopted, to make the coordinate-perturbation expressions as compact as possible, these expressions being complemented by formulae for the rates of change of the mean elements, to take care of the secular and long-period behaviour.

Full details of the J_2/J_3 theory are given in a recent RAE report[2], and a subsequent report[3] gives the details of the theory's extension (to first order) to the general zonal harmonic, J_ℓ . The present paper is essentially a précis of Ref. 3, which will often be referred to simply as 'the Report'.

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The starting point for the theory is the decomposition of U_ℓ, the potential due to J_ℓ, into a finite sum of component terms, U_ℓ^k, each of which is associated with a particular function of the orbital inclination. Then, to integrate Lagrange's planetary equations, $(p/r)^{\ell-1}$ (where p is the semi-latus rectum of the orbit) is expanded as a finite sum of terms, each of which is associated with an eccentricity function. The two sets of functions can be generated by the use of recurrence relations, and certain of these relations are also vital in the development of the coordinate-perturbation formulae (expressions for δr, δb and δw, associated with U_ℓ^k) that constitute the principal results of the paper.

For each of δr, δb and δw, the general formula (associated with U_ℓ^k) involves a sum of terms over a third index, j. Certain values of j lead to zero denominators in particular terms, but for δr and δb these terms can be eliminated altogether, by the definition of the mean elements. For δw, on the other hand, most of the potentially infinite terms do not just disappear; instead, they are replaced by particular finite terms that are induced by the element definitions. The formulae for these 'particular' terms supplement the general formulae. A further complication originates from the necessity to change the integration variable, in the planetary equations, from t (time) to v (true anomaly). Since the mean elements evolve with t, rather than v, an additional short-period perturbation is induced by every secular or long-period component of the rate of change of a mean element; the perturbations induced by the long-period components (but not the secular components) are best treated like the pure short-period perturbations, by amalgamating them into perturbations in the coordinates.

To exemplify the general theory, specific results are given for the harmonic J_4. The secular and long-period expressions for the element rates of change are well-known, but the expressions for δr, δb and δw have not been given before.

The formulae associated with the general zonal harmonic do not immediately extend to the tesseral harmonics, because of the complication introduced by the Earth's rotation. For a zero rate of rotation, however, the extension is simple and straightforward, if of little practical value, and this is the final topic of the paper.

It is convenient to conclude the introduction with some remarks on notation. We assume the usual elliptic elements (a, e, i, Ω, ω, M), with n and σ defined such that $M = \sigma + \int n\, dt$; further, this integral will often be written just as $\int$. As in the author's earliest work[4] on orbital theory, we also find it convenient to utilize the quasi-elements ψ, ρ and L, defined such that $d\psi = d\omega + c\, d\Omega$, where $c = \cos i$ (and we also write $s = \sin i$), $d\rho = d\sigma + q\, d\psi$ (where $q^2 = 1 - e^2$) and $dL = dM + q\, d\psi$. Finally, and as a valuable shorthand, we define

$$C_j^k = \cos (jv + ku') \quad \text{and} \quad S_j^k = \sin (jv + ku') , \tag{1}$$

where $u' = \omega + v - \frac{1}{2}\pi$ (argument of latitude measured from the north apex of the orbit); when no ambiguity is possible, we will frequently omit the superfix k in this notation.

POTENTIAL DEVELOPED VIA THE INCLINATION FUNCTIONS

The standard expression for U_ℓ is

$$U_\ell = -\frac{\mu}{r} J_\ell (R/r)^\ell P_\ell(\sin\beta) . \tag{2}$$

As in Ref. 4, we expand $P_\ell(\sin\beta)$ via the addition theorem for zonal harmonics; thus

$$P_\ell(\sin\beta) = \sum_{k=0}^{\ell} u_k \frac{(\ell - k)!}{(\ell + k)!} P_\ell^k(0)\, P_\ell^k(c) \cos ku' . \tag{3}$$

Here $u_0 = 1$, $u_k = 2$ if $k > 0$, and the Legendre function P_ℓ^k is defined by

$$P_\ell^k(c) = s^k \frac{d^k P_\ell(c)}{dc^k} . \tag{4}$$

The second factor (the k'th derivative) in (4) is a polynomial in c , which (with $k \le \ell$) does not vanish when $c = 1$, its value then being $(\ell + k)!/\{2^k\, k!\, (\ell - k)!\}$. Hence this factor may be normalized, in a certain useful sense, and we write

$$\frac{d^k P_\ell(c)}{dc^k} = \frac{(\ell + k)!}{2^k\, k!\, (\ell - k)!} A_\ell^k(i) , \tag{5}$$

where $A_\ell^k(i)$ is a pure polynomial in s^2 if k has the same parity as ℓ , but has an additional factor c if k and ℓ are of opposite parity; in each case the constant term in the polynomial is unity, by the normalization. Explicit expressions for the $A_\ell^k(i)$ are given in Table 1 of the Report.

We can now rewrite (3) as

$$P_\ell(\sin\beta) = \sum_{k=0}^{\ell} \alpha_{\ell k}\, s^k A_\ell^k(i) \cos ku' , \tag{6}$$

where the constant, $\alpha_{\ell k}$, is given by

$$\alpha_{\ell k} = u_k P_\ell^k(0) / (2^k\, k!) . \tag{7}$$

It is clear, from the last paragraph, that $P_\ell^k(0)$ vanishes when k and ℓ are of opposite parity, and it may be shown that when the parity is the same,

$$P_\ell^k(0) = (-1)^{\frac{1}{2}(\ell-k)} \frac{(\ell + k)!}{2^\ell \{\frac{1}{2}(\ell + k)\}! \{\frac{1}{2}(\ell - k)\}!} . \qquad (8)$$

In substituting (6) into (2) it is of great benefit to introduce a new quantity, $A_{\ell k}$, defined by

$$A_{\ell k} = J_\ell (R/p)^\ell \alpha_{\ell k} s^k A_\ell^k(i) . \qquad (9)$$

This permits us to write

$$U_\ell^k = - \frac{\mu}{p} (p/r)^{\ell+1} A_{\ell k} C_0^k . \qquad (10)$$

It will be noted that, whereas $A_\ell^k(i)$ is defined and useful regardless of parity, $A_{\ell k}$ (and hence U_ℓ^k) is only non-zero when k and ℓ are of the same parity. However, a use will be found for quantities that behave in the opposite way from $\alpha_{\ell k}$ and $A_{\ell k}$, anticipating which we define (with bold letters to make the distinction)

$$\boldsymbol{\alpha}_{\ell \kappa} = u_\kappa (\ell - \kappa + 1) P_{\ell+1}^\kappa(0)/(2^\kappa \kappa! \ \ell) \qquad (11)$$

and

$$\mathbf{A}_{\ell \kappa} = J_\ell (R/p)^\ell \boldsymbol{\alpha}_{\ell \kappa} s^\kappa A_\ell^\kappa(i) , \qquad (12)$$

where k has been replaced by κ to signify that we now have quantities that are non-zero only when κ and ℓ are of <u>opposite</u> parity.

We will require derivatives of the inclination functions. It is evident from (5) that

$$\frac{d}{di}\{A_\ell^k(i)\} = - \frac{(\ell - k)(\ell + k + 1)}{2(k + 1)} s A_\ell^{k+1}(i) ; \qquad (13)$$

from this and (9) it follows that the (partial) derivative of $A_{\ell k}$ with respect to i is given by

$$A'_{\ell k} = J_\ell (R/p)^\ell \alpha_{\ell k} s^{k-1} \left\{kc A_\ell^k(i) - \frac{(\ell - k)(\ell + k + 1)}{2(k + 1)} s^2 A_\ell^{k+1}(i)\right\} . \qquad (14)$$

We will also require, finally, the particular combinations of $A_{\ell k}$ and $A'_{\ell k}$ denoted by $A^+_{\ell k}$ and $A^-_{\ell k}$, and given by

$$A^\pm_{\ell k} = ks^{-1} A_{\ell k} \pm c^{-1} A'_{\ell k} ; \qquad (15)$$

these definitions reduce[3] to

$$A^-_{\ell k} = J_\ell (R/p)^\ell \alpha_{\ell k} \frac{(\ell - k)(\ell + k + 1)}{2(k + 1)} c^{-1} s^{k+1} A_\ell^{k+1}(i) \qquad (16)$$

and

$$A^{+}_{\ell k} = 2k\, J_\ell (R/p)^\ell\, \alpha_{\ell k}\, c^{-1}\, s^{k-1}\, A^{k-1}_\ell(i)\ . \tag{17}$$

Various recurrence relations for the functions $A^k_\ell(i)$ were given in the Report, and these may be used (if any possibility of instability[5] is disregarded) to generate the $A_{\ell k}$ and $A_{\ell\kappa}$. It is also possible to connect $A_{\ell k}$ and $A'_{\ell k}$ with the appropriate $A_{\ell\kappa}$, and the following pair of formulae will be needed in the sequel:

$$\ell \left(\frac{A_{\ell,k+1}}{u_{k+1}} - \frac{A_{\ell,k-1}}{u_{k-1}} \right) = 2kcs^{-1}\, \frac{A_{\ell k}}{u_k} \tag{18}$$

and

$$\ell \left(\frac{A_{\ell,k+1}}{u_{k+1}} + \frac{A_{\ell,k-1}}{u_{k-1}} \right) = -\frac{2A'_{\ell k}}{u_k}\ . \tag{19}$$

ECCENTRICITY FUNCTIONS REQUIRED IN SUBSEQUENT ANALYSIS

The term U_ℓ of the potential, specified by (2), has now been decomposed into the U^k_ℓ defined by (10), the latitude (β) having been eliminated. The longitude was absent from U_ℓ from the beginning, because of axial symmetry, so it remains to eliminate the radius vector (r) . Since $p/r = 1 + e \cos v$, an expansion of the form

$$(p/r)^{\ell-1} = \sum_{j=0}^{\ell-1} u_j\, B_{\ell j} \cos jv \tag{20}$$

is possible, for $\ell \geq 1$, and we regard $B_{\ell j}$ as defined by this expansion; clearly, $B_{\ell j}$ is a polynomial in e . We shall find it useful, and entirely natural, to extend the definition of $B_{\ell j}$ to negative j , by defining $B_{\ell j} = B_{\ell,|j|}$, and to take $B_{\ell j} = 0$ when $|j| \geq \ell$. On this basis we can replace (20) by

$$(p/r)^{\ell-1} = \sum B_{\ell j}\, C^0_j\ , \tag{21}$$

where the summation effectively runs from $j = -\infty$ to $j = +\infty$. The $B_{\ell j}$ are directly related to the Hansen X functions of classical celestial mechanics, since

$$B_{\ell j} = q^{2\ell-1}\, X_0^{-\ell-1,j}\ . \tag{22}$$

We proceed, as in Ref. 4, to express the e-polynomial $B_{\ell j}$, when $\ell \geq 1$ and $0 \leq j < \ell$, in terms of a normalized polynomial, the connecting relation being

$$B_{\ell j} = \binom{\ell-1}{j} (e/2)^j\, B^j_\ell(e)\ ; \tag{23}$$

$B^j_\ell(e)$ in (23) is a polynomial in e^2 , with constant term unity by the normalization. Explicit expressions for the $B^j_\ell(e)$ are given in Table 3 of the Report.

In contrast with the $A_{\ell k}$, however, it is usually better to work with the $B_{\ell j}$ directly, rather than through $B_\ell^j(e)$ and (23). One reason for this is that only alternate values of the $A_{\ell k}$ are non-zero, whereas (for $|j| < \ell$ and $e \neq 0$) all the $B_{\ell j}$ are non-zero. Further, no difficulty arises with the $B_{\ell j}$ when $j < 0$, whereas $B_\ell^j(e)$ would then be infinite (if $|j| < \ell$). We can even allow ℓ to be negative (or zero) as well as j; the validity of this follows from the universality of (22).

In regard to derivatives of the eccentricity functions, it can be shown that, for $1 \leq j < \ell$,

$$\frac{d}{de}\{B_\ell^j(e)\} = 2je^{-1}\,\{B_{\ell-1}^{j-1}(e) - B_\ell^j(e)\}\,. \tag{24}$$

The universal formula (valid for all j) for the derivative of $B_{\ell j}$ can then be shown to be

$$B'_{\ell j} = (\ell - 1)\,B_{\ell-1,j-1} - j\,e^{-1}\,B_{\ell j}\,. \tag{25}$$

However, because we only introduce the $B_{\ell j}$ after each planetary equation has been set up, we effectively only use (25) in expressing the rates of change of the mean elements. Since this involves

$$\frac{\partial}{\partial e}(q\,A_{\ell k}\,B_{\ell k}) = q^{-1}\,A_{\ell k}\,\{q^2\,B'_{\ell k} + (2\ell - 1)\,e\,B_{\ell k}\}\,, \tag{26}$$

we define

$$E_{\ell k} = q^2\,B'_{\ell k} + (2\ell - 1)\,e\,B_{\ell k}\,; \tag{27}$$

then (25) leads (via a recurrence relation) to

$$E_{\ell k} = e^{-1}\,(\ell e^2 - k)\,B_{\ell k} + (\ell - k)\,B_{\ell,k-1}\,. \tag{28}$$

A number of recurrence relations for the $B_{\ell j}$ are given in the Report. However, these may all be generated from just two relations, and we use here the following pair, as being most useful in the sequel:

$$(j - \ell)\,e\,B_{\ell,j-1} + 2j\,B_{\ell j} + (j + \ell)\,e\,B_{\ell,j+1} = 0 \tag{29}$$

and

$$(\ell - j)\,e\,B_{\ell+1,j} + \ell q^2\,B_{\ell,j+1} - (\ell + j + 1)\,B_{\ell+1,j+1} = 0\,. \tag{30}$$

Alternatively, we can replace the unsymmetrical (30) by the more symmetrical (and simpler) equation

$$\ell e\,(B_{\ell,j-1} - B_{\ell,j+1}) = 2j\,B_{\ell+1,j}\,, \tag{31}$$

given by Zafiropoulos[6] in his orbital theory for an axi-symmetric field.

RATES OF CHANGE OF OSCULATING AND MEAN ELEMENTS

It is straightforward to derive finite expressions for the perturbations in the osculating elements, so long as we change the variable in Lagrange's planetary equations from t to v before integrating. The equation needed for the change of variable is

$$dv/dt = n\,q^{-3}\,(p/r)^2 \,. \tag{32}$$

We obtain the expression for δa first, however, because a direct derivation is available, which is simpler than using the planetary equation.

The starting point for the direct derivation of δa is, as in previous work[1-4], the existence of an energy-related absolute constant of the motion, which we denote by a' ; the relation

$$a = a'\,(1 + 2aU/\mu) \tag{33}$$

is exact for any time-independent disturbing function, U, and in particular for the axi-symmetric U_ℓ^k . It follows that there is no long-term variation in a, to whatever order of magnitude the perturbation analysis is conducted. Further, the short-period perturbation, δa, is given exactly, on substituting for U_ℓ^k from (10); thus

$$\delta a = -\,2a'q^{-2}\,A_{\ell k}\,(p/r)^{\ell+1}\,C_0^k \,. \tag{34}$$

This does not mean that an exact perturbation can be written down for semi-major axis, however, as the right-hand side of (34) is expressed in terms of osculating elements; as soon as mean elements are introduced, the result is no more than a first-order perturbation expression.

To present δa in the form appropriate for later use, we combine C_0^k with one of the factors p/r . We retain another p/r factor explicitly, and expand the remaining $(p/r)^{\ell-1}$ by (21). By this means, we effectively transform the term $2C_0$, for example, into $C_j + C_{-j}$. But each pair of terms (such as this) for positive j, in the infinite summation of (21), is matched by the same pair (in reverse order) for negative j, so we can express our desired result as

$$\delta a = -\,aq^{-2}\,A_{\ell k}\,(p/r)\,\textstyle\sum B_{\ell j}\,(eC_{j-1} + 2C_j + eC_{j+1}) \,. \tag{35}$$

For the remaining elements $(e, i, \Omega, \omega$ and $M)$, it is convenient to start with the time rates of change of p, pc^2, Ω, ψ and ρ, since the right-hand sides of the planetary equations for these rates are the single-term expressions $(2q/na)\,\partial U/\partial\omega$, $(2qc/na)\,\partial U/\partial\Omega$, $(1/na^2qs)\,\partial U/\partial i$, $(q/na^2e)\,\partial U/\partial e$ and $(-2/na)\,\partial U/\partial a$, respectively. We now use (32) to change the variable to v, and (21) to eliminate r, as a result of which we get (with only a finite number of non-zero terms from each infinite summation):

$$dp/dv = 2kp\,A_{\ell k}\,\textstyle\sum B_{\ell j}\,S_0 \,, \tag{36}$$

$$d(pc^2)/dv = 0 \,, \tag{37}$$

$$d\Omega/dv = -s^{-1} A'_{\ell k} \sum B_{\ell j} C_j \,, \tag{38}$$

$$d\psi/dv = -\tfrac{1}{4} e^{-1} A_{\ell k} \sum B_{\ell j} \{(\ell + 1 - k)e\, C_{j-2} + 2(\ell + 1 - 2k)C_{j-1} + 2(\ell + 1)e\, C_j + 2(\ell + 1 + 2k)C_{j+1} + (\ell + 1 + k)e\, C_{j+2}\} \tag{39}$$

and

$$d\rho/dv = -2(\ell + 1)q\, A_{\ell k} \sum B_{\ell j} C_j \,. \tag{40}$$

We also require a formula for the rate of change of the (shorthand) quantity $\int$; the appropriate formula is

$$d\int/dv = n'/\dot{\nabla} + 3q\, A_{\ell k} \sum B_{\ell j} C_j \,, \tag{41}$$

where n' is the exact constant defined by $n'^2 a'^3 = \mu$.

To obtain the rates of change of the mean elements, we select the terms involving C^k_{-k} or S^k_{-k} (in the equations other than (39) this implies just $j = -k$), since these are the terms that are independent of v . We abandon p and pc^2 in favour of e and i , but we retain ψ and ρ , since the secular and long-period rates of these (mean) 'elements' are useful in practice, and this is also true for L ; since $\bar{a}$ (i.e. a') does not vary, our full results may be written (after simplification where necessary):

$$\dot{\bar{e}}_{\ell k} = -kne^{-1}q^2 A_{\ell k} B_{\ell k} S^k_{-k} \,, \tag{42}$$

$$\dot{\bar{I}}_{\ell k} = kncs^{-1} A_{\ell k} B_{\ell k} S^k_{-k} \,, \tag{43}$$

$$\dot{\bar{\Omega}}_{\ell k} = -ns^{-1} A'_{\ell k} B_{\ell k} C^k_{-k} \,, \tag{44}$$

$$\dot{\bar{\psi}}_{\ell k} = -ne^{-1} A_{\ell k} E_{\ell k} C^k_{-k} \,, \tag{45}$$

$$\dot{\bar{\rho}}_{\ell k} = -2(\ell + 1)nq\, A_{\ell k} B_{\ell k} C^k_{-k} \tag{46}$$

and

$$\dot{\bar{L}}_{\ell k} = -(2\ell - 1)\, nq\, A_{\ell k} B_{\ell k} C^k_{-k} \,. \tag{47}$$

The remaining terms from (36) - (40), those that depend on v , lead to the pure short-period perturbations in the elements. For future reference we supplement this set of equations by giving those for e , i , M and L , which can be derived from the original set (with an appropriate formula for da/dv included); thus:

$$de/dv = \tfrac{1}{4} A_{\ell k} \sum B_{\ell j} \{(k - \ell - 1)e\, S_{j-2} + 2(2k - \ell - 1)S_{j-1} + 6ke\, S_j + 2(2k + \ell + 1)S_{j+1} + (k + \ell + 1)e\, S_{j+2}\} \,, \tag{48}$$

$$di/dv = kcs^{-1} A_{\ell k} \sum B_{\ell j} S_j \,, \tag{49}$$

$$dM/dv = \tfrac{1}{4} e^{-1} q\, A_{\ell k} \sum B_{\ell j} \{(\ell + 1 - k)e\, C_{j-2} + 2(\ell + 1 - 2k)C_{j-1} - 6(\ell - 1)e\, C_j + 2(\ell + 1 + 2k)C_{j+1} + (\ell + 1 + k)e\, C_{j+2}\} \quad (50)$$

and

$$dL/dv = -(2\ell - 1)q\, A_{\ell k} \sum B_{\ell j}\, C_j\ . \quad (51)$$

The integration of these equations is entirely straightforward, since the analysis is only being taken to first order; also, zero denominators cannot occur, because the terms that would produce them are precisely the ones that have been dealt with separately and are no longer present. To save space, therefore, these integrals are omitted.

In the rest of the paper we are concerned with the amalgamation of the integrals for short-period δa etc into formulae for the coordinate perturbations, δr, δb and δw. However, we conclude the present section with two remarks about the secular and long-period perturbations - they will be relevant in obtaining the formulae associated with J_4, to exemplify the general results. First, the long-period perturbations induce additional terms in δr, δb and δw, as noted in the Introduction. Second, when ℓ is even it is convenient to deal with the secular perturbation in M by use of a 'mean mean motion', $\bar{n}$, that is not the same as n' (except, as it happens, for $\ell = 2$); since $\bar{a}$ is identified with a', this means that Kepler's third law does not hold for mean n and a. (This is covered in detail in the Report, and Ref. 7 is almost entirely devoted to the matter.)

COORDINATE PERTURBATIONS (GENERAL CASE)

In this section we develop general expressions for the δr, δb and δw that can be derived from the (untruncated) first-order formulae

$$\delta r = (r/a)\, \delta a - (a \cos v)\, \delta e + (aeq^{-1} \sin v)\, \delta M\ , \quad (52)$$

$$\delta b = (\cos u')\, \delta i + (s \sin u')\, \delta\Omega \quad (53)$$

and

$$\delta w = \delta\psi + \{q^{-2} \sin v\, (1 + p/r)\}\, \delta e + q^{-3} (p/r)^2\, \delta M\ . \quad (54)$$

Special cases, associated with the particular choices of mean elements that will be derived, are reserved for the next section of the paper.

Generation of the expressions for δr and δw is essentially straightforward in that the analysis starts with the δa etc due to U_ℓ^k and finishes with $\delta r_{\ell k}$ and $\delta w_{\ell k}$. With δb, however, there is a complication, due to the appearance of u' in (53), as opposed to v in (52) and (54); we deal with the difficulty by deriving $\delta b_{\ell\kappa}$, rather than $\delta b_{\ell k}$, where κ has values of opposite parity to those of k.

We do not give expressions for $\delta\dot{r}$, $\delta\dot{b}$ and $\delta\dot{w}$, but they are immediately available[2,3] from the expressions for δr, δb and δw, just by replacing S_j and C_j by $(k + j)\, \bar{n}\, C_j$ and $-(k + j)\, \bar{n}\, S_j$.

We can do better than this if we allow for the (overall) rate of change of $\bar{\omega}$, replacing $(k + j)\bar{n}$ by $(k + j)\bar{n} + k\dot{\bar{\omega}}$, assuming C_j and S_j still to be shorthand for C_j^k and S_j^k.

The Perturbation δr

We have to apply (52) with δa, δe and δM given by (35) and (the integrals of) (48) and (50). The integrals combine in a very natural way, and we are able to write

$$\delta r = -\tfrac{1}{4} a A_{\ell k} \sum B_{\ell j} R_j , \tag{55}$$

where

$$R_j = e(j + \ell - 1)\left(\frac{1}{k + j - 2} + \frac{3}{k + j}\right)C_{j-1} + 2\left(\frac{2j + \ell - 1}{k + j - 1} + \frac{2j - \ell + 1}{k + j + 1}\right)C_j + e(j - \ell + 1)\left(\frac{3}{k + j} + \frac{1}{k + j + 2}\right)C_{j+1} . \tag{56}$$

It can be seen that (55) is a summation in which R_j, as given by (56), has three components, each component being expressed as the sum of two multiples of the same 'C quantity'. We separate the first multiple from the second (in each component of R_j), feeding them back separately into the summation of (55); this leads to two distinct summations that we can denote by Σ_- and Σ_+. Thus Σ_- involves $\Sigma B_{\ell j} R_{j-}$, where

$$R_{j-} = \frac{j + \ell - 1}{k + j - 2} eC_{j-1} + 2\,\frac{2j + \ell - 1}{k + j - 1} C_j + 3\,\frac{j - \ell + 1}{k + j} eC_{j+1} . \tag{57}$$

Since sums over $B_{\ell j}$ can be regarded as running from $-\infty$ to $+\infty$, it follows that we can rearrange the three sets of terms in $\Sigma B_{\ell j} R_{j-}$ so that (with j now used in a different way)

$$\sum B_{\ell j} R_{j-} = \sum\{(j + \ell)e\, B_{\ell,j+1} + 2(2j + \ell - 1)B_{\ell j} + 3(j - \ell)e\, B_{\ell,j-1}\}(k + j - 1)^{-1} C_j . \tag{58}$$

Using the recurrence relations (29) and (30), we can simplify this to

$$\sum B_{\ell j} R_{j-} = 2(\ell - 1)q^2 \sum (k + j - 1)^{-1} B_{\ell-1,j} C_j . \tag{59}$$

Similarly,

$$\sum B_{\ell j} R_{j+} = -2(\ell - 1)q^2 \sum (k + j + 1)^{-1} B_{\ell-1,j} C_j . \tag{60}$$

The final result we require now follows from (55), (59) and (60). Because of its importance, we write C_j in full. Thus

$$\delta r_{\ell k} = -(\ell - 1)\, p\, A_{\ell k} \sum_j \frac{1}{(k + j + 1)(k + j - 1)} B_{\ell-1,j} \cos(ku' + jv) . \qquad \ldots\ldots (61)$$

If $j = -k \pm 1$, there is a zero denominator in (61), and terms with these values of j must be excluded; in the next section we determine integration constants for $\delta e_{\ell k}$ and $\delta M_{\ell k}$ so that the terms with these two values of j are forced to zero. It will be noted that all the cosine terms occurring in (61), for a given J_ℓ and all possible k, are distinct, except that if $k = 0$ (ℓ even) then equal and opposite values of j lead to identical terms in $\cos jv$. On this basis we can derive the total number of terms required to express δr for a given value of ℓ. This is done in detail in the Report, the result being $\ell^2 - \frac{1}{2}(3\ell - 1)$ when ℓ is odd and $\ell^2 - \frac{1}{2}(3\ell - 2)$ when ℓ is even.

The perturbation δb

We get δb from (53), where δi and $\delta\Omega$ are given by the integrals of (49) and (38). This is on the assumption that δb ($= \delta b_{\ell k}$) is associated with U_ℓ^k, following the decomposition of U_ℓ. We shall find, however, that it is much more convenient to decompose the total δb (associated with U_ℓ) as $\sum \delta b_{\ell\kappa}$, where the summation is for values of κ that are of opposite parity to ℓ and we no longer associate the individual δb ($= \delta b_{\ell\kappa}$) with specific components of U_ℓ.

In relation to U_ℓ^k, we get

$$\delta b_{\ell k} = -\sum B_{\ell j} \left\{ \frac{kcs^{-1}}{k + j} A_{\ell k} C_j^k \cos u' + \frac{1}{k + j} A'_{\ell k} S_j^k \sin u' \right\} . \tag{62}$$

The trigonometrical products are replaced by sums, in the usual way, and we can then invoke the notation of (15) to write

$$\delta b_{\ell k} = -\tfrac{1}{2}c \sum B_{\ell j} (k + j)^{-1}(A^+_{\ell k} C_j^{k-1} + A^-_{\ell k} C_j^{k+1}) . \tag{63}$$

This expression may be contrasted with (55) and (56) for δr. In view of the difference in superfix (whereas the suffix alone varied in the terms of R_j), in the two C terms of (63), we would now like to combine a pair of terms with different k indices, before the summation over the j index operates.

With the philosophy just indicated, we make the new decomposition

$$\delta b_\ell = \sum \delta b_{\ell\kappa} , \tag{64}$$

where each δb ($= \delta b_{\ell\kappa}$) is of the form

$$\delta b = \sum T_j B_{\ell j} C_j^\kappa \tag{65}$$

and we require an expression for T_j. We note first that since (for non-trivial results) k runs from 0 or 1 to ℓ (taking alternate values), it follows that, in principle, κ runs from -1 or 0 to $\ell + 1$ (again alternate values, but of opposite parity to k): for the minimum value of κ, only the term in $A^+_{\ell k}$, in (63), contributes to T_j, whilst for the maximum value of κ, only the term in $A^-_{\ell k}$ contributes; for intermediate values (if any), both terms contribute. But we can straight away dismiss the 'maximum value' ($\kappa = \ell + 1$), because $A_{\ell\ell}$ is just a multiple of s^{ℓ}; from this it follows that $A^{-}_{\ell\ell}$, defined by (15), is zero. (Also $B_{\ell\ell} = 0$ anyway!) We find that we do not require the 'minimum value' ($\kappa = -1$) either. Then, using (16) and (17) for $A^{-}_{\ell k}$ and $A^{+}_{\ell k}$, respectively, we get

$$T_j = -J_\ell \left\{ \frac{\kappa + 1}{\kappa + j + 1} \alpha_{\ell,\kappa+1} + \frac{(\ell - \kappa + 1)(\ell + \kappa)}{4\kappa(\kappa + j - 1)} \alpha_{\ell,\kappa-1} \right\} \times$$
$$\times (R/p)^{\ell} s^{\kappa} A^{\kappa}_{\ell}(i) \,. \qquad (66)$$

The quantity in curly brackets in (66) is a pure constant, in which the $\alpha_{\ell k}$ are given by (7): thus the first α involves $P^{\kappa+1}_{\ell}(0)$ and the second involves $P^{\kappa-1}_{\ell}(0)$, these being given by (8). By relating these to $P^{\kappa}_{\ell+1}(0)$, and hence, via $\alpha_{\ell\kappa}$ given by (11), to $A_{\ell\kappa}$ given by (12), we can rewrite (66) as

$$T_j = \frac{\ell A_{\ell\kappa}}{2u_\kappa} \left\{ \frac{u_{\kappa+1}}{\kappa + j + 1} - \frac{u_{\kappa-1}}{\kappa + j - 1} \right\} . \qquad (67)$$

The preceding argument is the general one, for $1 \le \kappa \le \ell - 1$, and for $\kappa \ge 2$ we can obviously cancel u_κ out with $u_{\kappa+1}$ and $u_{\kappa-1}$. Very conveniently (with the full argument given in the Report), (67) is also correct without the occurrences of u for $\kappa = 0$ (ℓ odd), and even for $\kappa = 1$ (ℓ even) on the basis that the case $\kappa = -1$ is then automatically covered.

We can now write down the final result we require, on substituting (67) into (65) and expressing C^{κ}_{ℓ} in full. Thus

$$\delta b_{\ell\kappa} = -\ell A_{\ell\kappa} \sum_j \frac{1}{(\kappa + j + 1)(\kappa + j - 1)} B_{\ell j} \cos(\kappa u' + jv) \,. \qquad (68)$$

As already indicated, this formula is unlike (61), the corresponding one for δr, in that it cannot be taken in isolation as relating to a sub-component of U_ℓ. It is like (61) in one respect, however, in that terms of $\delta b_{\ell\kappa}$ with $j = -\kappa \pm 1$ are excluded. In the next section we determine constants for $\delta i_{\ell k}$ and $\delta\Omega_{\ell k}$ (k, not κ, now being the appropriate symbol) such that these terms are forced to zero. On this basis we can derive the total number of terms (without duplication of C_j) required to express δb for a given value of ℓ. The result (derived in the Report) is $\ell^2 - \frac{3}{2}(\ell - 1)$ when ℓ is odd and $\ell^2 - \frac{1}{2}(3\ell - 2)$ when ℓ is even (the latter being the same as for δr).

The perturbation δw

The analysis for δw is much more like the δr analysis than the δb analysis, because each U^{k}_{ℓ} can again be treated separately throughout. There are two complications, however. First, (54) effectively involves $\cos 2v$ and $\sin 2v$, not just $\cos v$ and $\sin v$ (we see this at equation (69), following), and this means that the values $j = -k \pm 2$ are special as well as $j = -k \pm 1$. Second, we cannot take δw to be zero for any of these special cases, since the constants in δe and δM must now be assumed to have been already assigned; formulae for the four special δw will be obtained in the next section. Actually, a fifth special case emerges, corresponding to $j = -k$ and a zero denominator $k + j$; δw for this case can be set to zero, since we still have (for each k) the unassigned constant in $\delta\omega$ available for the purpose.

We start by rewriting (54) as

$$\delta w = 2q^{-2}\,(\delta e \sin v + eq^{-1}\,\delta M \cos v) + \tfrac{1}{2}\,eq^{-2}\,(\delta e \sin 2v + eq^{-1}\,\delta M \cos 2v) + \tfrac{3}{2}\,e^2q^{-3}\,\delta M + q^{-1}\,\delta L\,, \tag{69}$$

where δe , δM and δL are available from the integrals of (48), (50) and (51). The integrals for δe and δM combine in a very natural way and we eventually get, changing the interpretation of j (as in the analysis for δr) so that we use the same S_j in each term:

$$\begin{aligned}\delta w = \tfrac{1}{8}q^{-2}\,A_{\ell k}\sum\Bigg\{&3e^2\left(\frac{1-\ell+k}{k+j+2} + \frac{1+\ell-k}{k+j}\right)B_{\ell,j+2} + 2e\left(\frac{1+\ell+2k}{k+j+2}\right.\\
&\left. + 6\frac{1-\ell+k}{k+j+1} + 3\frac{1+\ell-2k}{k+j} + 2\frac{1+\ell-k}{k+j-1}\right)B_{\ell,j+1} + \left(e^2\frac{1+\ell+k}{k+j+2}\right.\\
&+ 8\frac{1+\ell+2k}{k+j+1} + 2\frac{4(1-2\ell) + e^2(5-\ell)}{k+j} + 8\frac{1+\ell-2k}{k+j-1}\\
&\left. + e^2\frac{1+\ell-k}{k+j-2}\right)B_{\ell j} + 2e\left(2\frac{1+\ell+k}{k+j+1} + 3\frac{1+\ell+2k}{k+j} + 6\frac{1-\ell-k}{k+j-1}\right.\\
&\left. + \frac{1+\ell-2k}{k+j-2}\right)B_{\ell,j-1} + 3e^2\left(\frac{1+\ell+k}{k+j} + \frac{1-\ell-k}{k+j-2}\right)B_{\ell,j-2}\Bigg\}\,S_j\,.\end{aligned} \tag{70}$$

Though the algebra is tedious, we can eliminate $B_{\ell,j+2}$ and $B_{\ell,j-2}$ via appropriate versions of (29). If we express the result as

$$\delta w = \tfrac{1}{8}q^{-2}\,A_{\ell k}\sum\,(V_{j,1}\,B_{\ell,j+1} + V_{j,0}\,B_{\ell j} + V_{j,-1}\,B_{\ell,j-1})\,S_j\,, \tag{71}$$

the formulae for $V_{j,1}$, $V_{j,0}$ and $V_{j,-1}$ are initially very complicated. They can be greatly simplified, however; for $V_{j,0}$ this was done by a technique akin to partial fractions. The resulting expressions are

$$V_{j,1} = 2e(\ell + j)\left(\frac{1}{k+j+2} - \frac{6}{k+j+1} + \frac{3}{k+j} + \frac{2}{k+j-1}\right), \tag{72}$$

$$\begin{aligned}V_{j,0} = {}& 8\left(\frac{\ell+2k+1}{k+j+1} - \frac{2\ell-1}{k+j} + \frac{\ell-2k+1}{k+j-1}\right)\\
&- 2e^2\left(\frac{\ell+k+1}{k+j+2} - \frac{2(\ell+1)}{k+j} + \frac{\ell-k+1}{k+j-2}\right)\end{aligned} \tag{73}$$

and

$$V_{j,-1} = 2e(\ell - j)\left(\frac{2}{k+j+1} + \frac{3}{k+j} - \frac{6}{k+j-1} + \frac{1}{k+j-2}\right). \tag{74}$$

As a result of this considerable simplification, $V_{j,1} B_{\ell,j+1}$ and $V_{j,-1} B_{\ell,j-1}$ are now in a form suitable for the elimination of $B_{\ell,j+1}$ and $B_{\ell,j-1}$ in favour of $B_{\ell j}$ and $B_{\ell-1,j}$, via an application of two recurrence relations, (30) and a similar one. Thus, if we now write

$$\delta w = \tfrac{1}{8}A_{\ell k} \sum \left(W_{\ell,0}\, B_{\ell j} + W_{\ell,-1}\, B_{\ell-1,j}\right)S_j , \tag{75}$$

we get

$$W_{\ell,0} = 2\left(\frac{\ell+k+1}{k+j+2} - 2\,\frac{\ell+1}{k+j} + \frac{\ell-k+1}{k+j-2}\right) \tag{76}$$

and

$$W_{\ell,-1} = -2(\ell-1)\left(\frac{1}{k+j+2} - \frac{4}{k+j+1} + \frac{6}{k+j} - \frac{4}{k+j-1} + \frac{1}{k+j-2}\right). \tag{77}$$

The final result we require follows from the substitution of (76) and (77) into (75). Writing S_j in full, we get

$$\delta w_{\ell k} = A_{\ell k} \sum_j \frac{1}{(k+j+2)(k+j)(k+j-2)}\left\{[2(\ell+1) - k(k+j)]\, B_{\ell j} - \frac{6(\ell-1)}{(k+j+1)(k+j-1)}\, B_{\ell-1,j}\right\} \sin(ku' + jv) . \tag{78}$$

Equation (78) is the general formula for δw due to U_ℓ^k. As with (61) and (68), for δr and δb respectively, it applies for all $\ell \geq 1$; like (68) but unlike (61), on the other hand, values of $|j|$ up to $\ell - 1$ (not just $\ell - 2$) are required to cover all the non-zero terms. For each k, zero denominators exist for five different values of j: for four of these values ($j = -k \pm 1$ and $j = -k \pm 2$), special formulae are required, in place of (78), as already noted; only for the fifth value ($j = -k$) can a term (for each k) be actually excluded. It should be noted that one specific null term arises for each even value of ℓ. Thus, for $k = 2$ and $j = \ell - 1$, we see from (78) that the coefficient of $B_{\ell j}$ is identically zero (independently of ℓ), but $B_{\ell-1,j}$ is itself zero when $j = \ell - 1$, so this specific term of $\delta w_{\ell,2}$ always vanishes. On this basis we can derive the total number of terms required to express δw for a given value of ℓ, counting in the terms derived in the next section. The result (derived in the Report) is ℓ^2 when ℓ is odd and $\ell^2 - 1$ when ℓ is even.

THE SPECIAL CASES, AND INTEGRATION CONSTANTS

The main results in this section are the formulae required to supplement (78), the general formula for δw . These formulae, covering the cases $j = -k \pm 1$ and $-k \pm 2$, are forced by the 'constants' for δe and δM , which are determined so that the terms for $j = -k \pm 1$ can be excluded from δr . We also give the formulae for the constants associated with the main elements, i.e. for the quantities independent of v that are deliberately introduced when the v-dependent components of $d\Omega/dv$, for example, are integrated; by giving these formulae we effectively define the mean elements underlying the theory. (The constants associated with J_2^2 perturbations are given in Ref. 2).

Mandatory Constants for δa

We go back to (34), the original expression for δa due to U_ℓ^k . We can expand the complete factor $(p/r)^{\ell+1}$ in terms of the $B_{\ell+2,j}$ (cf the expansion via the $B_{\ell j}$ in (35)). On taking just the term of the expansion with $j = -k$, we isolate the constant term that (for each k , and a given J_ℓ) is mandated by taking $\bar{a} = a'$.

The result can be written in the form (for the 'constant' component of $\delta a_{\ell k}$)

$$\delta a_{\ell k(c)} = -2aq^{-2} A_{\ell k} B_{\ell+2,k} \cos k\omega' . \tag{79}$$

Constants for δe and δM

We have to derive the formulae for $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$ that will legitimize our taking the terms in $\delta r_{\ell k}$ for $j = -k + 1$ and $-k - 1$ to be zero. These 'constants' will complete the formulae, for δe and δM , given by the integrals of (48) and (50) respectively.

We start by noting that (61), the general formula for $\delta r_{\ell k}$, was obtained by combining the two different denominators from (59) and (60). If we do not combine the denominators, we can rewrite the formula as

$$\delta r_{\ell k} = -\tfrac{1}{2}(\ell - 1)p\, A_{\ell k} \sum \left(\frac{1}{k + j - 1} - \frac{1}{k + j + 1}\right) B_{\ell-1,j}\, C_j . \tag{80}$$

The first denominator here is associated with the Σ_- summation of (57), and if this summation still applied for $j = -k + 1$, the result would be an infinite coefficient of $B_{\ell-1,-k+1}\, C_{-k+1}$. We actually want this coefficient to be $-\tfrac{1}{2}(\ell - 1)p\, A_{\ell k}$, since it will then neutralize the coefficient that arises without difficulty from the second denominator in (80). The situation is reversed when $j = -k - 1$ and we want the coefficient of $B_{\ell-1,-k-1}\, C_{-k-1}$, from the second term of (80), to be $\tfrac{1}{2}(\ell - 1)p\, A_{\ell k}$ (and not infinity) to neutralize the first term. What we do, therefore, is to obtain the coefficients of C_{-k+1} and C_{-k-1} that would apply in the absence of the constants $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$; we can then derive the appropriate values of these constants to cancel these putative coefficients.

The details of this analysis (given in the Report) are omitted here, and we proceed directly to the results, which can be written as

$$\delta e_{\ell k(c)} = -\tfrac{1}{4} A_{\ell k} \{eB_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} + 2(\ell + 2)eB_{\ell k} - (\ell - k - 4)B_{\ell,k-1} + eB_{\ell,k-2}\} \cos k\omega' \quad (81)$$

and

$$\delta M_{\ell k(c)} = \tfrac{1}{4} e^{-1} q A_{\ell k} \{eB_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} - 2keB_{\ell k} + (\ell - k - 4)B_{\ell,k-1} - eB_{\ell,k-2}\} \sin k\omega' . \quad (82)$$

Constants for δi and $\delta\Omega$

We have to derive formulae for $\delta i_{\ell k(c)}$ and $\delta\Omega_{\ell k(c)}$ to legitimize taking the terms for $j = -\kappa + 1$ and $-\kappa - 1$ in (68), the general expression for $\delta b_{\ell\kappa}$, to be zero. The analysis is somewhat simpler than that in the preceding section, in spite of the complexity entailed by the need to work with both k and κ.

As with $\delta r_{\ell k}$, we start by observing that (68) was obtained by combining two denominators, which appeared separately in equation (67). When $j = -\kappa + 1$, the second denominator becomes zero and no longer operates; from the first alone we get, as the effective term in (68), $\tfrac{1}{4}\ell A_{\ell\kappa} B_{\ell,-\kappa+1} C^{\kappa}_{-\kappa+1}$. When $j = -\kappa - 1$, similarly, the first denominator in (67) does not operate, and (68) effectively reduces to $\tfrac{1}{4}\ell A_{\ell\kappa} B_{\ell,-\kappa-1} C^{\kappa}_{-\kappa-1}$. These terms have to be cancelled by the use of $\delta i_{\ell k(c)}$ and $\delta\Omega_{\ell k(c)}$, with appropriate k. Again we leave the details to the Report, proceeding directly to the results, which are

$$\delta i_{\ell k(c)} = \tfrac{1}{2}A'_{\ell k} B_{\ell k} \cos k\omega' \quad (83)$$

and

$$\delta\Omega_{\ell k(c)} = \tfrac{1}{2}kcs^{-2} A_{\ell k} B_{\ell k} \sin k\omega' . \quad (84)$$

Forced Terms in δw

We now have, for each U^k_ℓ, only one 'constant' at our disposal; denoted by $\delta\omega_{\ell k(c)}$, we shall determine it (in the next sub-section) so as to validate the nulling of the term for $j = -k$ in the formula, (78), for $\delta w_{\ell k}$. For $j = -k \pm 1$ and $-k \pm 2$, on the other hand, we are forced to accept non-null terms that arise, via (54), from (81) and (82); we now have to derive the formulae for these terms. For each of the four special values of j, in principle we embark on a procedure that is similar to that employed in the derivation of $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$, though more direct. In practice, however, instead of developing our four special formulae more or less ab initio, we start four times from the (final) general formula for δw, (78), and modify it each time in the appropriate manner, replacing 'general' terms that would be infinite by special terms based on $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$.

As usual, we omit the details (supplied in the Report) and simply quote the results. Our four special-case formulae can be expressed as

$$\delta w_{\ell k,-k+1} = -\tfrac{1}{3} A_{\ell k} \{(2\ell - k + 2) B_{\ell,k-1} + (\ell - 1) B_{\ell-1,k-1}\} \sin (v + k\omega') , \tag{85}$$

$$\delta w_{\ell k,-k-1} = -\tfrac{1}{3} A_{\ell k} \{(2\ell + k + 2) B_{\ell,k+1} + (\ell - 1) B_{\ell-1,k+1}\} \sin (v - k\omega') , \tag{86}$$

$$\delta w_{\ell k,-k+2} = -\tfrac{1}{48} A_{\ell k} \{3(\ell + k + 5)B_{\ell,k-2} - 19(\ell - 1)B_{\ell-1,k-2}\} \sin (2v + k\omega') \tag{87}$$

and

$$\delta w_{\ell k,-k-2} = -\tfrac{1}{48} A_{\ell k} \{3(\ell - k + 5)B_{\ell,k+2} - 19(\ell - 1)B_{\ell-1,k+2}\} \sin (2v - k\omega') . \tag{88}$$

Constants for $\delta\omega$

It remains to determine the constant, $\delta\omega_{\ell k(c)}$, that legitimizes our taking the term for $j = -k$ in (78) to be zero. We already have $\delta M_{\ell k(c)}$, given by (82), so we only need to determine $\delta L_{\ell k(c)}$, the constant in $\delta L_{\ell k}$, for $\delta\omega_{\ell k(c)}$ to be known at once.

It turns out (with the details in the Report) that

$$\delta L_{\ell k(c)} = - kq A_{\ell k} B_{\ell k} \sin k\omega' . \tag{89}$$

From (82) and the definitions of L and ψ , it follows that

$$\delta\psi_{\ell k(c)} = -\tfrac{1}{4} e^{-1} A_{\ell k} \{e B_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} + 2keB_{\ell k} + (\ell - k - 4)B_{\ell,k-1} - e B_{\ell,k-2}\} \sin k\omega' . \tag{90}$$

Finally, $\delta\Omega_{\ell k(c)}$ is given by (84), so the formula for $\delta\omega_{\ell k(c)}$ is

$$\delta\omega_{\ell k(c)} = -\tfrac{1}{4} e^{-1} A_{\ell k} \{e B_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} + 2kes^{-2} B_{\ell k} + (\ell - k - 4)B_{\ell,k-1} - e B_{\ell,k-2}\} \sin k\omega' . \tag{91}$$

RESULTS FOR J_4

The formulae of the two preceding sections are valid for $\ell \geq 1$, the case $\ell = 1$ being trivial. To exemplify the straightforward use of these fomulae, and the complementary formulae for the rates of change of the mean elements, results for $\ell = 1$, 2, 3 and 4 (together with an analysis for the exceptional case, also trivial, $\ell = 0$) were given in the Report. Only the results for $\ell = 4$ were new, however, so here we confine ourselves to these. For convenience in expressing the formulae, we define $G = \frac{1}{1024} J_4 (R/p)^4$ and $f = s^2$.

We start with the secular rates of change for $\bar{\Omega}$ and $\bar{\omega}$. From (44) and (45) we get, with $\ell = 4$ and $k = 0$,

$$\dot{\bar{\Omega}} = 480\ Gnc(4 - 7f)(2 + 3e^2) \tag{92}$$

and

$$\dot{\bar{\Psi}} = -120\ Gn(8 - 40f + 35f^2)(4 + 3e^2)\ , \tag{93}$$

from which it follows that

$$\dot{\bar{\omega}} = -120\ Gn\{4(16 - 62f + 49f^2) + 9e^2(8 - 28f + 21f^2)\}\ . \tag{94}$$

We avoid an explicit secular perturbation in $\bar{M}$ by modifying Kepler's third law. From equation (15) of Ref. 7 (or the formulae in the Report) we require (for J_4 the only non-zero harmonic)

$$\bar{n}^2\ \bar{a}^3 = \mu\{1 + 288Gq^3(8 - 40f + 35f^2)\}\ . \tag{95}$$

Next, we cover the long-period rates for all the elements (except a). From (42) - (47), we get, with $\ell = 4$ and $k = 2$,

$$\dot{\bar{e}} = -480\ Gneq^2f(6 - 7f)\sin 2\omega\ , \tag{96}$$

$$\dot{\bar{I}} = 480\ Gne^2cs(6 - 7f)\sin 2\omega\ , \tag{97}$$

$$\dot{\bar{\Omega}} = -960\ Gne^2c(3 - 7f)\cos 2\omega\ , \tag{98}$$

$$\dot{\bar{\Psi}} = -240\ Gnf(6 - 7f)(2 + 5e^2)\cos 2\omega\ , \tag{99}$$

$$\dot{\bar{p}} = -2400\ Gne^2qf(6 - 7f)\cos 2\omega \tag{100}$$

and

$$\dot{\bar{L}} = -1680\ Gne^2qf(6 - 7f)\cos 2\omega\ . \tag{101}$$

Turning to the perturbations in the coordinates, we expect the number of terms in δr, δb and δw to be 11, 11 and 15, respectively, from the formulae given earlier. Starting with δr, we note that there are terms for $k = 4$, $k = 2$ and $k = 0$, with values of j, a priori, satisfying $|j| \leq 2$; but for $k = 2$ we exclude $j = -1$; and for $k = 0$ we exclude $j = \pm 1$, whilst the terms for $j = \pm 2$ are identical. Then (61) gives, corresponding to the three values of k,

$$\delta r = -2Gpf^2\{6e^2\cos 2(2u + v) + 35e\cos(4u + v) + 28(2 + e^2)\cos 4u + 105e\cos(3u + \omega) + 70e^2\cos 2(u + \omega)\}, \tag{102}$$

$$\delta r = -8Gpf(6 - 7f)\{2e^2\cos 2(u + v) + 15e\cos(2u + v) + 20(2 + e^2)\cos 2u - 30e^2\cos 2\omega\} \tag{103}$$

and

$$\delta r = -24Gp(8 - 40f + 35f^2)\{e^2 \cos 2v - 3(2 + e^2)\} . \qquad (104)$$

For δb, the effects are for $\kappa = 3$ and $\kappa = 1$, the a-priori values of j being the seven with $|j| \leq 3$. For $\kappa = 3$ we exclude $j = -2$, and for $\kappa = 1$ we exclude $j = 0$ and $j = -2$. Then (68) gives, corresponding to the two values of κ,

$$\begin{aligned}\delta b = -4Gcsf\,\{&4e^3 \sin 3(u + v) + 35e^2 \sin (3u + 2v) \\ &+ 28e(4 + e^2) \sin (3u + v) + 70(2 + 3e^2) \sin 3u \\ &+ 140e(4 + e^2) \sin (2u + \omega) - 140\, e^3 \sin 3\omega\}\end{aligned} \qquad (105)$$

and

$$\begin{aligned}\delta b = -4Gcs(4 - 7f)\{&4e^3 \sin (u + 3v) + 45e^2 \sin (u + 2v) \\ &+ 60e(4 + e^2) \sin (u + v) - 180e(4 + e^2) \sin \omega \\ &- 20e^3 \sin (2v - \omega)\} .\end{aligned} \qquad (106)$$

For δw, the effects are again for $k = 4$, 2 and 0, with the same a-priori j values as for δb. For $k = 4$, all seven j values yield terms, of which five come from the general (78); for $j = -2$ we use (87) and for $j = -3$ we use (85). For $k = 2$, the term with $j = -2$ is excluded, whilst for $j = 3$, (78) gives an example of a 'specifically null' term; there are non-null general terms for $j = 2$ and $j = 1$; and the terms for $j = 0$, -1 and -3 come from (87), (85) and (86) respectively. Finally, for $k = 0$ the term with $j = 0$ is excluded; the other terms come in pairs, being 'general' for $j = \pm 3$, from (87) and (88) for $j = \pm 2$, and from (85) and (86) for $j = \pm 1$. Corresponding to the three values of k, we get

$$\begin{aligned}\delta w = -Gf^2\{&4e^3 \sin (4u + 3v) + 31e^2 \sin 2(2u + v) + 4e(21 + 5e^2) \times \\ &\times \sin (4u + v) + 28(3 + 4e^2) \sin 4u + 28e(7 + e^2) \sin (3u + \omega) \\ &+ 175e^2 \sin 2(u + \omega) + 140e^3 \sin (u + 3\omega)\} ,\end{aligned} \qquad (107)$$

$$\begin{aligned}\delta w = 4Gf(6 - 7f)\,\{&2e^2 \sin 2(u + v) + 4e(5 + 2e^2) \sin (2u + v) \\ &+ 5(8 - 7e^2) \sin 2u - 80e(5 + e^2) \sin (u + \omega) \\ &- 40e^3 \sin (v - 2\omega)\}\end{aligned} \qquad (108)$$

and

$$\begin{aligned}\delta w = 4Ge(8 - 40f + 35f^2)\{&2e^2 \sin 3v - 3e \sin 2v \\ &- 6(24 + 5e^2) \sin v\} .\end{aligned} \qquad (109)$$

It remains to cover the induced short-period terms from the secular and long-period rates of change. The effects induced by the secular variation are included by adding $(\dot{\bar{\Omega}}/n)(v - M)$, $(\dot{\bar{\omega}}/n)(v - M)$ and $(\dot{\bar{M}}/n)(v - M)$ to $\bar{\Omega}$, $\bar{\omega}$ and $\bar{M}$, respectively. Here $\dot{\bar{\Omega}}$ and $\dot{\bar{\omega}}$ are given by (92) and (94), whilst $\dot{\bar{M}}$ is the rate implicit in (95), so that

$$\dot{\bar{M}} = 144\, Gnq^3(8 - 40f + 35f^2) ; \qquad (110)$$

The effects induced by the long-period variation, on the other hand, are allowed for via additional terms in the expressions for δr, δb and δw. Using (52) - (54), we find that these additional terms are given by

$$\delta r = 480\, Gpe(v - M)f(6 - 7f) \sin (u + \omega) , \tag{111}$$

$$\delta b = 240\, Ge^2(v - M)cs \{3(4 - 7f) \cos (v - \omega) - 7f \cos (u + 2\omega)\} \tag{112}$$

and

$$\delta w = 240\, Ge(v - M)f(6 - 7f)\{e \cos 2u + 4 \cos (u + \omega) - 4e \cos 2\omega\} . \quad \ldots. \tag{113}$$

EXTENSION TO TESSERAL HARMONICS

If the Earth's rotation rate is neglected, so that a' is still an absolute constant of the motion, the formulae that have been derived require surprisingly little change. The main change is, as one might expect, the replacement of all occurrences of $A_{\ell k}$ and $A_{\ell\kappa}$ by quantities $A_{\ell mk}$ and $A_{\ell m\kappa}$, to reflect the generalization of J_ℓ to $J_{\ell m}$, with the arguments of C_j^k and S_j^k, in equations (1), replaced by $jv + ku' + m(\Omega' - \lambda_{\ell m})$, where $\Omega' = \Omega - \nu - \frac{1}{2}\pi$, the sidereal angle ($\nu$) being supposed fixed; further, we now have to allow for negative values of k, which takes alternate values from $-\ell$ to ℓ. (For amplification of these remarks, with definitions of $A_{\ell mk_k}$ and $A_{\ell m\kappa}$, and a discussion of two possible definitions of the $A_{\ell m}^k(i)$ functions that generalize the $A_\ell^k(i)$, see Appendix A of the Report).

By making use of the quantities $A_{\ell mk}$ and $A_{\ell m\kappa}$, we find no difficulty in extending the theory, largely because the treatment of $(p/r)^{\ell+1}$, via the $B_{\ell j}$, goes through unchanged. Thus, equations (35), (48), (39) and (50), for δa, de/dv, $d\psi/dv$ and dM/dv, respectively, are unchanged apart from the appearance of $A_{\ell mk}$ in place of $A_{\ell m}$. Equation (38), for $d\Omega/dv$, requires a corresponding change, such that the derivative $A'_{\ell mk}$ replaces $A'_{\ell m}$. This just leaves (49), for di/dv, for which a slightly more complicated expression is now required, to reflect the fact that pc^2 is no longer an invariant. We have, in fact,

$$\frac{d(pc^2)}{dv} = 2mpc\, A_{\ell mk} \sum B_{\ell j}\, S_j , \tag{114}$$

from which we derive

$$\frac{di}{dv} = s^{-1}(kc - m)\, A_{\ell mk} \sum B_{\ell j}\, S_j ; \tag{115}$$

in comparison with equation (49), we see that the only additional change is the replacement of kc by $kc - m$.

Six of the seven formulae that define δr, δb and δw completely, for the zonal harmonics, apply immediately to the zonal harmonics, so long as $A_{\ell m k}$ replaces $A_{\ell m}$ and the trigonometric argument includes the term $m(\Omega' - \lambda_{\ell m})$. These six are (61) and (78), for the general δr and δw, and (85) - (88), the four special formulae for δw. In the seventh formula, (68) for δb, $\ell \mathbf{A}_{\ell\kappa}$ must be replaced by $(\ell + m)\mathbf{A}_{\ell m\kappa}$, in addition to the inclusion of the new term in the trigonometric argument.

The numbers of terms in δr, δb and δw, for a given $J_{\ell m}$, are greater for $m > 0$ than for $m = 0$, to reflect the distinction between positive and negative k. These numbers are otherwise independent of m, however, in consequence of which we find that there are $2\ell^2 - 3\ell + 1$ terms for δr, $2\ell^2 - 3\ell + 2$ terms for δb, and $2\ell^2$ (for odd ℓ) or $2\ell^2 - 1$ (for even ℓ) terms for δw.

CONCLUSION

The original idea of Kozai[8] - that the short-period perturbations due to J_2 can be more compactly expressed by amalgamating δa, δe, $\delta\omega$ and δM into δr and δu - has been carried to its logical conclusion by eliminating the short-period perturbations in all the elements in favour of effects in spherical coordinates. The general formulae for these effects are given by the summations in (61), (68) and (78), for the perturbations in r, b and w, respectively. Terms that would have a zero denominator are excluded from these summations, as a consequence of the optimal definition of mean elements, except that replacement terms are needed for the perturbations in w; the formulae for these are (85) - (88).

The formulae for coordinate perturbations are complemented by (42) - (47), which are the formulae for the rates of change of the mean elements. Relatively speaking, these formulae, which lead to the secular and long-period perturbations, contain very few terms, but over periods of time longer than an orbital revolution the effects are much greater than those from the short-period perturbations. Formulae to a further order of approximation, i.e. to J_2^3 and $J_2 J_\ell$ $(\ell > 2)$, have been given by Berger and Walch[9], and Kinoshita[10], but expressions appropriate to the mean elements used in the present paper have not yet been derived.

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17. Abstract

By recourse to a particular definition of a satellite's mean orbital elements and to a particular system of spherical polar coordinates based on the mean orbital plane, an orbital theory has been developed that leads to extremely compact first-order perturbation formulae associated with the general zonal harmonic, J_ℓ. The formulae are complete (untruncated in eccentricity) and generalize, via recurrence relations, the author's earlier results for the effects of J_2 (analysed to second order) and J_3. To illustrate the compact nature of individual expressions, the (untruncated) perturbation formulae due to J_4 are given.

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