A TUTORIAL ON THE
ANGULAR POSITIONS AND
VELOCITIES OF GROUND OBJECTS
VIEWED FROM AIRCRAFT (U)

Herschel C. Self

ARMSTRONG AEROSPACE MEDICAL RESEARCH LABORATORY

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FOR THE COMMANDER

CHARLES BATES, JR.
Director, Human Engineering Division
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This report is a primer or tutorial on the angular motion of ground objects relative to level-flying aircraft. Both qualitative and quantitative description is used. Equations are derived in detail for angular relationships and for angular velocities in alpha, azimuth, and declination. Terms are defined and included are tables, graphs and worked-out examples and problems. Problems in low-altitude flight, dynamic visual acuity, apparent motion, and angular velocities at aircraft windows are discussed.
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SUMMARY

In high-speed low-altitude flight, objects on the ground are viewable for only a short time and can have high angular velocities. This impairs navigation, flight safety, and the detection and tracking of targets. The present paper is a tutorial on angular velocity. The apparent motion of ground objects and texture streaming is discussed. Detailed derivations of equations for the angular velocities of ground objects are presented in textbook fashion. Universal tables of angular velocity based on the equations are provided. Example problems are presented and solutions are provided. The construction of flow charts is described. How to plot angular velocity at locations on aircraft windows is illustrated with figures of the windows of a B-52 aircraft for different-sized pilots and seat adjustments. The variation in angular velocity with aircraft height is described by examples and graphs. The influence of angular velocity on dynamic visual acuity is reviewed.

Previous literature used ground references to calculate angular velocity. The present paper uses azimuth and declination, which are observer references. Earlier papers concentrated on results or use of equations with little attention to how to derive and apply the equations. The present paper shows how.
PURPOSE

Viewed from an aircraft, ground objects have angular velocities and eventually exit from an observer's field-of-view. In high-speed flight at low altitudes, angular velocities can be high and available time short, making it difficult for aircrews to find ground objects and take effective action. System design, analyses of system performance, and mission planning all require data on angular relationships and angular velocities. Not all of the required equations are in the literature, and some equations are tedious to derive. There is also a shortage of tables and graphs for analysis purposes. To make up for this shortage, the present tutorial report was written as a primer on the angular positions and angular velocities of ground objects viewed from an aircraft in level flight. Both qualitative and quantitative descriptions of angular motion are presented.
1.0 INTRODUCTION - AIRCRAFT MOTION AND ANGULAR VELOCITIES

For an aircraft flying very low and very fast, the angular velocities of ground objects near the aircraft can be very high. Ground objects may be quite close to the aircraft when aircraft motion unmasks them from concealing vegetation, terrain, or man-made objects. Rapid angular motion in both azimuth and declination, both at different angular velocities, combined with little available viewing time, make it difficult for a crewmember to detect, identify and take action against ground objects. At low aircraft altitudes, vibration and turbulence, by degrading visual capabilities, increase the difficulty of carrying out these tasks. Systematic search is difficult or impossible, and aiming at and tracking ground objects are both difficult and inaccurate. Danger, stress and workload are all high in low-altitude high-speed flight. The problems in such flight regimes are indicated by the short list of problems of Appendix 4.

Since vehicle motion may degrade a crewmember's ability to observe, as well as his ability to take action, vehicle motion must be taken into account in the design of weapon systems, in analyses of system performance, and in the planning of missions.

Taking angular motion rates and available time into account requires equations for angular velocity as a function of aircraft speed and height, and the location of a ground object. Location may be specified relative to the aircraft or relative to the ground. For some purposes, such as aiming or tracking with a weapon or a sensor, or just where to look, the observer wants to know how many degrees clockwise from straight ahead, and how many degrees down from level or horizontal. These two angles are azimuth or azimuth angle $A$, and declination or declination angle (dip angle) $D$, respectively. (Note that azimuth angle (angle to the side) may also be defined as the angle between a straight-ahead horizontal line (zero azimuth) and a line from the aircraft passing directly over the ground point or object.) (Declination may also be defined as the angle between a horizontal line (zero declination) passing directly over a ground object and a line from the
aircraft to the object or ground point.) Figure 13 in Appendix I depicts the various angles and distances.

An equation containing distance ahead \( R \) and offset distance \( S \) from the flight path is not suitable for pointing or aiming. For straight and level flight, define alpha as the angle between a straight-ahead line along the flight path in the sky, i.e., along the aircraft velocity vector, and a line from the aircraft to a ground point or object. An equation for angular velocity in Alpha of ground objects relative to the observer is derived as Equation 16 in Appendix 1, where \( \frac{d\alpha}{dt} \) is the derivative of the angle with respect to time, i.e., is angular velocity:

\[
\frac{d\alpha}{dt} = 57.296(V/H)\sin D_i \cos D_i \sqrt{\tan^2 D_i + \sin^2 A_i} \text{ Degrees/Second}
\]

For some purposes, specifying object location relative to the terrain is useful. This type of specification was used by Erickson (1965), who used offset distance \( S \) of the ground object from the flight path and its distance \( R_i \) ahead of the aircraft. With this terrain reference, angular velocity in alpha is given by Equation 15 derived in detail in Appendix 1:

\[
\frac{d\alpha}{dt} = 57.296V \sqrt{H^2 + S^2} / (H^2 + S^2 + R_i^2)
\]

Havron (1962) used yet another specification of object position relative to the terrain. He used X and Y coordinates in an aircraft that is diving or landing.

There are three angular velocities of particular interest. Azimuth velocity or angular rate in azimuth, declination velocity or angular rate in declination, and alpha velocity or angular rate in alpha. Appendix 1 of the present paper derives equations for all three angular velocities, the relationships between them, and equations for ground range, slant range, distance ahead, and offset of objects from the aircraft's ground path. The various equations for angular rates and angular relationships, as well as the other quantities mentioned, are not difficult to derive. However, some of the equations are long and
tedious enough to invite errors in derivation. For readers of this report who want to peruse the derivations of the equations, Appendix 1 gives detailed derivations. The equations apply to straight level flight.

Tables are provided for quick determination of angular rates. Equations are given for calculating and graphing flow fields on azimuth-declination plots. The derived equations are applied to find the angular velocity rates in various areas of the windows of an aircraft from the pilot's eye position. Since this paper is tutorial, it uses a textbook approach and derivations of equations are presented in detail. Since low-altitude flight at high-speed provides conditions that pose angular rate problems for crewmembers, this flight regime is emphasized in the examples. However, the equations and tables are not limited to low altitude flight.

1.1 The Apparent Motion of Ground Objects

The motion of an aircraft over the ground changes the azimuth and the declination of ground objects: every object and every ground point is constantly changing position relative to the aircraft and observers on it. To an airborne observer, objects have angular velocities. In perception, the terrain, not the observer, appears to be moving. The apparent angular velocity of a point on the ground has the same magnitude as the angular velocity of the aircraft observed from the ground point. It is readily apparent to passengers in ground and airborne vehicles that nearby objects move rapidly, while distant objects, i.e., objects with low declinations, move slowly or not at all. Sometimes, distant objects appear to move with the observer. It is also readily noticed that the angular velocity of a ground object also varies with its position in the field of view, i.e., varies with its azimuth and declination. As distance decreases, the angle subtended at the eye by any dimension of an object increases, and it appears to grow in size. Angular motion becomes more apparent and, at close range, may be very high. As distance decreases, approach speed appears to increase, and motion downward and outward both increase. At close range, objects
appear to fly outward at high speed. Once past an object, it appears to recede with decreasing velocity, while inward and upward motion rates are decreasing and apparent size is shrinking at a decreasing rate.

Except for objects directly in front of the vehicle, the paths in azimuth and declination are curves. Each object follows its own curved path. This path does not vary with aircraft speed, although speed along the path is directly proportional to vehicle speed. Thus, if aircraft speed is multiplied by n, then angular velocity in azimuth and in declination are also multiplied by n. If an observer views the ground with a device that has a magnification M, angular motion rate is magnified by M.

When angular motion rates are high, an observer may notice that ground objects and ground texture appear to flow, forming a flow field. The apparent paths are flow lines or streamers, and they are useful in perception of motion. Gibson (1955) discusses the optical expansion pattern in aerial locomotion. He notes that the flow of optical stimulation due to the motion of an observer provides a continuous feedback of information used in controlling motion. Changing optical stimulation conforms to the principle of motion parallax or motion perspective. Retinal motion provides information about both space and the motion of the observer. A frontal surface perpendicular to the line of locomotion yields a radial pattern of velocities, a streamer pattern, while a longitudinal surface parallel to the line of locomotion yields a unidirectional pattern of velocities. The first is perceived as an expansion when moving toward the surface and a contraction when moving away from the surface. The second is perceived as a flow.
2.0 CONSTRUCTION OF FLOW CHARTS

As noted earlier, ground objects and ground texture appear to an airborne observer to move along paths that are called streamers or flow lines. A flow line may be plotted on an X-Y ground plot or on an azimuth-declination plot. A plot or graph with several flow lines is called a flow graph, flow diagram, or flow chart. It is a picture of streamer paths, the paths of apparent motion of ground points.

An azimuth-declination flow diagram depicts the path of ground objects in the field-of-view. The path's direction at any point in the field is the direction of motion of the object at that point. All flow lines on a graph may start at the same declination below horizontal. In constructing the graph, initial azimuth may be varied in steps, one flow line starting at each step and moving down the graph. Figure 1 is an example of a flow diagram. It contains several flow lines, each with a different initial or starting azimuth $A_0$, but with the same initial declination $D_0$ of 10°. In this figure, time ($t$) was varied in steps from zero until each line reached the edge of the graph. The graph is for a height of 200 feet and a ground speed of 440 feet/second (300 miles/hour). The paths of flow lines, although plotted for a given speed, are the same for any speed at the same height. However, speed along a flow line is proportional to aircraft speed. Position on a flow line at any time $t$ also depends on aircraft speed.

An example showing how to construct a flow line on a flow chart for straight and level flight at a height above ground of $H$ feet with a ground speed of $V$ feet/second illustrates the procedure for constructing flow charts. All lines of the chart are constructed by the same method. Equation 34 in Appendix 1 gives azimuth $A_i$ after $t$ seconds as $A_i = \text{ArcTan} \left( \frac{1}{\text{Cot}A_0 - Kt} \right)$, where $K = \frac{(V \text{Tan} D_0)}{H \text{Sin} A_0}$. In these equations, $A_0$ and $D_0$ are initial azimuth and declination, respectively, i.e., azimuth and declination when $t = 0$. In the present example, let initial declination $D_0$ be 10° below horizontal and initial azimuth $A_0$ be .5° for a height $H$ of 200 feet and a speed $V$ of 440 feet/second (300 miles/hour). For these conditions, the constant $K$ is $K = \frac{(440/200)}{\text{Tan}}$.
1^0/Sin .5^0 = 4.40050. With this value of K, Equation 34 is then \( A_i = \text{ArcTan} \frac{1}{(\cot 5^0 - 4.4005t)} = \text{ArcTan} \frac{1}{(114.589 - 4.4005t)} \). Here, \( t \) is in seconds. The corresponding declination \( D_i \) is given in Appendix 1 by Equation 35 as \( D_i = \text{ArcTan} (\tan D_0/Sin A_0) = \text{ArcTan} (\tan 1^0/Sin .5^0) Sin A_i \) = \( D_i = \text{ArcTan} (2.00023 Sin A_i) \). Figure 1 is the graph of a flow line based on these equations. The time in seconds since the initial condition is indicated at each plotted point. Using the same procedure, other flow lines starting at other initial locations may be constructed to obtain the flow chart of Fig 2. For example, it is easily shown, by the above equations, that the equations for a flow line starting at a declination \( D_0 \) of \( 1^0 \) and an azimuth \( A_0 \) of \( 10^0 \) are \( A_i = \text{ArcTan} 1/(5.67128 - .221143t) \) and \( D_i = \text{ArcTan} (.100520 Sin A_i) \).

When plotting a flow chart or when planning action of some kind, it may be necessary to determine how long it takes for an object at a position of azimuth \( A_0 \) and declination \( D_0 \) to reach an azimuth \( A_i \). The time required is obtained by solving the \( A_i \) equation in the above paragraph for \( t \). For the conditions of the example, \( \tan A_i = 1/(114.589 - 4.4005t) \), from which \( t = 26.0400 - 1/4.40050 \tan A_i \). The time to reach 45°, for example, is \( t = 26.0400 - 1/4.40050 \tan 45^0 = 25.813 \) seconds. As a second example, what time is required for \( A_i \) to reach 90°? For \( A_i = 90^0 \), \( \tan A_i = \infty \), hence \( t = 26.040 = 1/(\infty) = 26.040 - 0 = t = 26.040 \) seconds. Alternatively, for \( A_i = 90^0 \), from Equation 6 in Appendix 1, \( R_i = H \cos A_i/Tan D_i \), from which \( R_0 = H \cos A_0/Tan A_0 = (200) \cos .5^0/Tan 1^0 = 11,457.6 \) feet. At a speed of 440 feet/second, time for \( A_i \) to reach 90° is the time for \( R_i \) to reach zero, thus \( t = 11,457.6/440 = 26.040 \) seconds, as before.

When a flow chart with several lines is to be constructed, much labor will be saved by using a computer plotter. Figure 1 was constructed using a hand calculator and plotting by hand. The equations of Appendix 1 permit labeling selected points on the flow lines with ground range \( R_i^* \), slant range \( r_i \), distance ahead \( R_i \) to the ground point or object, elapsed time \( t \), and angular velocity in alpha \( d A_i/dt \), in azimuth \( dA_i/dt \), or in declination \( dD_i/dt \), or any combination of these. A picture of an aircraft window or the field-of-view of an optical
Fig 1. Flow diagram in azimuth and declination for an aircraft height of 200 feet. The curves are motion paths for ground objects and ground texture. Angular velocities are for a speed of 440 ft/sec. The dashed line indicates field locations having the same azimuth and declination velocities at all aircraft speeds.
Initial Conditions
\[ V = 440 \text{ ft/sec}, \ H = 200 \text{ ft}, \]
\[ A_o = 0.50 \text{ deg}, \ D_o = 1.0 \text{ deg} \]

Plotting Equations
\[ K = V \tan \frac{D_o}{H \sin A_o} \]
\[ = 4.40050 \]
\[ A_i = \arctan \left[ \frac{1}{\cot A_o - Kt} \right] \]
\[ = \arctan \left[ \frac{1}{114.589 - 4.0050t} \right] \]
\[ D_i = \arctan \left( \frac{\sin A_i \tan D_o}{\sin A_o} \right) \]
\[ = \arctan \left( 2.0003 \sin A_i \right) \]

Figure 2. An example flow line for an aircraft height of 200 feet. Time markers are for an aircraft speed of 440 feet/second.

Device or display may have a flow chart superimposed on it. For the flow line in Figure 1 that starts with an initial declination of 10 and azimuth of 250, selected points have been labeled with azimuth and declination velocity. Calculation of these velocities will be covered later on.

When display magnification M is not unity, angular rates at positions in the field-of-view will be magnified by M. Thus, if magnification is 2, rates are doubled. Flow charts for diving aircraft are discussed by Havron (1962) for X-Y ground plots, i.e., for flow lines plotted against ground coordinates.
3.0 ANGULAR VELOCITY IN ALPHA BY EQUATIONS

Erickson (1965), in his analysis and review of visual detection of ground targets, examines the angular rate in the angle alpha. Angular rate is also called angular velocity. Alpha is the angle between the flight path in the sky of an aircraft and a line from the aircraft to an object on the ground. This angle is also the angle between a line-of-sight that is level and straight ahead and a line to the ground object. For straight level flight at constant speed, the equation that Erickson provided for angular velocity in the angle alpha is

\[ \frac{d\alpha}{dt} = \frac{V}{\sqrt{H^2 + S^2}} \left( \frac{H^2 + S^2}{R_i^2} \right) \] radians/second.

Here, V is aircraft ground speed, H is aircraft height above the ground, S is offset distance of a ground object from the ground path or track of the aircraft, and \( R_i \) is the object's distance ahead of the aircraft measured parallel to the aircraft ground path. To obtain angular velocity in degrees/second, the above equation, which provides radians/second, must be multiplied by 57.296, converting it to Equation 15 in Appendix 1 of the present report. The equation is derived in detail in the appendix. In his report, Erickson (1965) includes a nomograph for angular velocity in alpha prepared by Roy Dule Cole. This nomograph yields angular velocity without having to do any calculation to an accuracy adequate for some purposes. Unfortunately, available copies of the report provide the nomograph in a small size and with poor reproduction quality. It is easier, quicker, and more accurate to use the angular velocity Equation 15.

From the equation, it is apparent that angular velocity in alpha is directly proportional to aircraft ground speed V. It is also clear that the angular velocity increases as distance ahead \( R_i \) decreases. The angular velocity situation for height changes is more complex, since increasing height also increases the declination of a ground object. Equation 15 also reveals that angular velocity in alpha is maximum when distance ahead \( R_i \) is zero. For a zero \( R_i \), a ground object is directly beneath the aircraft, or is on a line passing beneath the aircraft and perpendicular to the ground path. Here, Equation 15 becomes \( \frac{d\alpha}{dt} = 57.296 \frac{V}{\sqrt{H^2 + S^2}} \). When distance ahead \( R_i \) and offset S are both zero, Equation 15 becomes \( \frac{d\alpha}{dt} = 57.296 \frac{V}{H} \), the angular velocity for
objects directly beneath the aircraft. This position under the aircraft is the location where ground objects have the highest angular velocity.

An equation for angular velocity in alpha for ground objects on the ground path or ground track of the aircraft, where azimuth is zero, is obtained by setting offset distance \( S \) to zero, yielding \( \frac{d\alpha}{dt} = \frac{57.296VH}{(H^2 + R_i^2)} \). For objects straight ahead at a height \( H \), \( \tan D_i = \frac{H}{R_i} \), from which \( R_i = H/\tan D_i \). Substituting for \( R_i \) in the above equation, \( \frac{d\alpha}{dt} = \frac{57.296 \cdot VH}{(H^2 + R_i^2)} = \frac{57.296 \cdot VH}{(H^2 + H^2/\tan^2 D_i)} = \frac{57.296 \cdot V}{H} \cdot (\sec^2 D_i/\tan^2 D_i) = \frac{57.296 \cdot V}{H} \cdot (\sin^2 D_i/\cos^2 D_i) = \frac{d\alpha}{dt} = \frac{57.296 \cdot V}{H} \cdot \sin^2 D_i \) degrees/second for ground objects on the aircraft's ground path. Alpha velocity for zero azimuth is also derivable from Equation 16 in Appendix 1. \( \frac{d\alpha}{dt} = \frac{57.296 \cdot (V/H)}{\sin D_i \cos D_i} \cdot \frac{1}{\cos D_i} \). When \( A_i \) is zero, \( \sqrt{\tan^2 D_i + \sin^2 A_i} = \sqrt{\tan^2 D_i + 0} = \tan D_i \cdot \sin D_i/\cos D_i \), so that \( \frac{d\alpha}{dt} = \frac{57.296 \cdot V}{H} \cdot (\sin D_i \cos D_i) \cdot (\sin D_i/\cos D_i) = \frac{57.296 \cdot V}{H} \cdot \sin^2 D_i \), as above. From this equation, note that, when azimuth is zero, angular velocity in alpha increases as \( \sin^2 D_i \), approaching zero as declination \( D_i \) approaches zero. Also, for declination \( D_i = 90^\circ \) (straight down), angular velocity in alpha is \( 57.296 \cdot V/H \) degrees/second.

As shown earlier, when declination as well as azimuth is zero, the ground object is straight ahead on the level of the aircraft and \( \sin^2 D_i \) is zero, so that angular velocity in alpha, by the equation, is zero. The aircraft is on a collision course with the ground object and the object is at the center of an expanding pattern of flow lines. The vectors of all angular velocities point radially away from the impact point. Collision can occur in level flight with tall objects, such as buildings, radio and television towers, tall trees, power lines, hills and cliffs, or upward sloping terrain.

From Equation 16, when a ground object is at an azimuth \( A_i \) of either \( 90^\circ \) (straight right) or \( 270^\circ \) (straight left), \( \sqrt{\tan^2 D_i + \sin^2 A_i} = \sqrt{\tan^2 D_i + 1} = \sqrt{\sec^2 D_i} = 1/\cos D_i \). The equation then becomes \( \frac{d\alpha}{dt} = \frac{57.296 \cdot V}{H} \cdot \sin D_i \cdot \cos D_i \cdot (1/\cos D_i) = \frac{57.296 \cdot V}{H} \cdot \sin
Thus, for an azimuth of either 90° or 270°, angular velocity in alpha increases as the sine of the declination angle, reaching a maximum, as before, of 57.296 (V/H) for \( D_i = 90° \) (straight down). Also, for objects at either 90° or 270°, alpha velocity varies inversely with height.

When declination is constant, variation of azimuth from 0° to 360° defines a circle on the ground. From Equation 16 in Appendix 1, \( \frac{d\alpha_i}{dt} = 57.296 (V/H)\sin D_i\cos D_i\sqrt{\tan^2 D_i + \sin^2 A_i} \), it is clear that minimum alpha velocity occurs when azimuth is 90° and 270°, where \( \sin A_i \) has its maximum of 1. It is also clear that minimum alpha velocity occurs when azimuth is 0° and 180°, where \( \sin A_i = 0 \). The minimum in both cases, from the equation, is:

\[
\frac{d\alpha_i}{dt} = 57.296 (V/H)\sin D_i\cos D_i\sqrt{\tan^2 D_i + 0} = 57.296 (V/H)\sin D_i\cos D_i\tan D_i
\]

\[
\frac{d\alpha_i}{dt} = 57.296 (V/H)\sin D_i\cos D_i (\sin D_i/\cos D_i)
\]

\[
\frac{d\alpha_i}{dt} = 57.296 (V/H)\sin^2 D_i\cos D_i.
\]

Going back to Equation 16, with constant declination, alpha velocity on the ground circle defined by this constant declination increases from 0° to 90°, decreases from 90° to 180°, increases from 180° to 270°, and decreases from 270° to 360°. The minima at 0° and 180° are equal, and the maxima at 90° and 270° are equal. The decline from 90° to 180° mirrors the growth from 0° to 90°.

To determine how alpha velocity varies with change in offset distance \( S \), note that the derivative of angular velocity with respect to offset \( S \) is change in angular velocity with change in offset. Similarly, the derivative with respect to height is change in angular velocity with change in height. To calculate these derivatives, alpha velocity Equation 16 in Appendix 1 may be written as \( \frac{d\alpha_i}{dt} = 57.296V \sqrt{H^2 + S^2/(H^2 + S^2 + R_i^2)} = KV(H^2 + S^2)^{1/2}(H^2 + S^2 + R_i^2)^{-1} \). Note that, when \( W \) and \( Z \) are functions of \( S \), \( d(Wz)/ds = W(dZ/ds) + Z(dW/ds) \). Applying this differentiation rule to the above alpha velocity equation, and differentiating both size of the equation,
\[ \frac{d(\alpha_i/dt)}{dS} = KV(H^2 + S^2)^{1/2}(-1)(H^2 + S^2 + R_i^2)(2S) + KV(H^2 + S^2 + R_i^2)^{-1}(1/2)(H^2 + S^2)^{-1/2}(2S) = \]

\[ \frac{d(\alpha_i/dt)}{dS} = \frac{-2SV\sqrt{H^2 + S^2}}{(H^2 + S^2 + R_i^2)^2} + \frac{SV}{(\sqrt{H^2 + S^2})(H^2 + S^2 + R_i^2)} \]

Placing both fractions over a common denominator and combining them,

\[ \frac{d(\alpha_i/dt)}{dS} = \frac{\left[ -2SV(H^2 + S^2) + SV(H^2 + S^2 + R_i^2) \right]^1}{(\sqrt{H^2 + S^2})(H^2 + S^2 + R_i^2)^2} = \frac{SV \left[ R_i^2 - (H^2 + S^2) \right]}{(\sqrt{H^2 + S^2})(H^2 + S^2 + R_i^2)^2} \]

From this equation it may be seen that, when \( R_i^2 > (H^2 + S^2) \), an increase in offset distance \( S \) increases the angular rate, and when \( (H^2 + S^2) > R_i^2 \), an increase in \( S \) decreases the angular rate in \( \alpha \).

This result was shown in Erickson's paper (1965), with the steps not shown. Erickson noted, from examination of the equations, that substituting \( HV \) for \( SV \) in the numerator gives

\[ \frac{d(\alpha_i/dt)}{dH} = HV \left[ R_i^2 - (H^2 + S^2) \right] / (\sqrt{H^2 + S^2})(H^2 + S^2 + R_i^2) \]

for the rate of change of angular velocity in \( \alpha \) with change in aircraft height \( H \). Thus, his Figure 13 show a boundary on the ground where the sign of \( d(\alpha_i/dt) \) changes.

Some of the results obtained in this section from examination of the equations for \( \alpha \) velocity are as follows:

1. Angular velocity in \( \alpha \) is directly proportional to aircraft ground speed, but varies in a more complex way with height.

2. The closer an object on the ground is to an aircraft, the higher its angular velocity in \( \alpha \).
3. The maximum angular velocity in alpha of any ground object is 57.296 (V/H), and is for objects directly beneath the aircraft.

4. For objects on the ground path or track of the aircraft (zero azimuth), alpha velocity is \(\frac{d\alpha_i}{dt} = 57.296 \left(\frac{V}{H}\right) \sin^2 d_i\), which has a maximum of 57.296(V/H) when \(D_i\) is 90° (straight down), and a minimum of 0 for \(D_i = 0\) (horizontal). Also, an equation not containing \(D_i\) for objects on the flight path, which gives the same results, is \(\frac{d\alpha_i}{dt} = 57.296 VH \left(\frac{H^2 + R_i^2}{R_i^2}\right)\). This equation has a maximum of 57.296 (V/H) when \(R_i\) is zero, straight down in this case, and a minimum of 0 for infinite \(R_i\), which is for horizontal viewing.

5. For a given declination \(D_i\), maximum alpha velocity is for objects with an azimuth of either 90° or 270°. For objects at either of these two azimuths, \(\frac{d\alpha_i}{dt} = 57.296(V/H) \sin D_i\) or, equivalently, 57.296(V/H), which are maximum for \(D_i = 90°\) (straight down), or for offset \(S = 0\), respectively, and minimum of zero for \(D_i = 0\) (horizontal), or for infinite offset \(S\) (also, horizontal).

6. Angular velocity in alpha is zero for an object directly ahead on a level with the aircraft (H=0). In this case, the aircraft is in a collision course with the object, and the object is at the center of a radially expanding flow pattern.

7. For any given declination, variation of azimuth from 0° to 360° describes a circle of ground points that are all equally distant from the aircraft. For objects on this ground circle, alpha velocity is lowest at zero azimuth (straight ahead), increases as azimuth increases to 90° (directly right), then decreases as azimuth further increases to 180° (directly to the rear). The decrease from 90° to 180° mirrors the increase from 0° to 90°. Angular velocity in alpha at 0° and 180° are equal, but of opposite sign (approaching versus receding).

8. When \(R_i^2 > (H^2 + S_i^2)\), an increase in offset distance \(S\) increases alpha velocity, and when \(R_i^2 < (H^2 + S_i^2)\), an increase in \(S\) decreases angular rate.
There are three angular velocities that are of interest for objects on the ground: velocity in alpha, azimuth velocity, and declination velocity. In the following discussion, each will be defined and equations will be given for calculating angular velocity. Universal tables are provided in Appendix 2 for each of the three angular velocity types. The equations and tables are for straight level flight at a constant velocity. The tables are for an aircraft ground speed of 440 feet/second (300 miles/hour) at an altitude of 200 feet above ground, but are usable for any speed or altitude, as explained later. Alpha is the angle between a level line along the aircraft flight path in the sky and a line from the aircraft to the ground point or ground object being observed. An equation for angular velocity in the angle alpha, using aircraft velocity $V$, height above ground $H$, azimuth angle $A_i$ and declination angle $D_i$ to the object, is derived in Appendix 1. Equation 16. This equation is $\frac{d\alpha_i}{dt} = \frac{57.296(V/H)\sin D_i \cos D_i \sqrt{\tan^2 D_i + \sin^2 A_i}}{\sin 2A_i}$ degrees/second. Here, $\frac{d\alpha_i}{dt}$ is the derivative with respect to time of the angle alpha, i.e., rate of change of alpha, commonly called velocity in alpha or just alpha velocity. $D_i$ is declination below horizontal, and $A_i$ is azimuth or angle off from straight ahead.

As may be noted from inspection of the equation, it does require some calculation which can be considerable if angular velocity is required for several azimuths and declinations. In this case, a universal angular velocity table may be of value for avoiding much computation. Table A1 in Appendix 1 is such a table. The table values are based on an aircraft speed of 440 feet/second at a height of 200 feet. For different height $H$ in feet and speed $V$ in feet/second, multiply table entries by $(V/440)(200/H)$, i.e., by $(5/11)(V/H)$. This can be done because, in the equation for alpha velocity given above, $V$ and $H$ appear as the multiplier $V/H$.

The data of table A1 are plotted in Figure 3, with velocity in alpha on the vertical axis, declination on the horizontal axis, and families of curves for various values of azimuth, one curve for each
DECLINATION ANGLE, $D_i$

Figure 3. Velocity in alpha for various azimuths for an aircraft speed of 440 feet/second and at a height of 200 feet above ground. The graph is a plot of the data in Table Al.
azimuth. The same data could, of course, be plotted with azimuth on the horizontal axis and a series of curves, one for each declination. From the figure, note that the azimuth curves all start at zero declination and converge to a peak at 90° declination. Had declination been plotted out to 180°, a mirror image of the azimuth curves for 0° to 90° would have been produced. All curves fall to 0°/second at an azimuth of 180°. The peak of the curves is at a declination of 90° and, as explained elsewhere, is 57.296 (V/H) = 57.296(440/200) = 126.05 degrees/second.

When the distance ahead of the aircraft $R_i$ and offset $S$ from the aircraft ground path are given, azimuth $A_i$ and declination $D_i$ may be calculated from equations in Appendix 1: $A_i = \arctan(S/R_i)$, and $D_i = \arctan(H/\sqrt{H^2+S^2+R_i^2})$, respectively. These values may then be used in the alpha velocity equation given above. A more direct approach, when $S$ and $R_i$ are given, is to use Equation 15 from Appendix 1: $\frac{dA_i}{dt} = \frac{(57.296 \sqrt{H^2+S^2})}{(H^2+S^2+R_i^2)}$.

For some purposes, such as aiming, pointing, or tracking, the angular velocities in azimuth and declination are more applicable than velocity in alpha $d\alpha_i/dt$. Equations for angular velocities in azimuth and declination, using $A_i$ and $D_i$, rather than $R_i$ and $S$, are derived in Appendix 1.

The azimuth or azimuth angle of a ground object is an angular measure of how far to the side of straight ahead it is located. It is the angle between a straight-ahead line level with the aircraft and a level line at aircraft height passing over the ground object. Azimuth velocity $dA_i/dt$ is rapidity of change of azimuth. An equation for azimuth velocity derived in Appendix 1 is Equation 21: $dA_i/dt = \frac{57.296 (V/H) \sin A_i \tan D_i}{\pi}$ degrees/second. The constant 57.296 is the number of degrees in one radian, i.e., 180/π.

Equation 21, with $A = 440$ feet/second, and $H = 200$ feet, was used to calculate the entries in Table A2 of Appendix 2, a universal azimuth velocity table. For aircraft velocity $V$ other than 440 and height 200, multiply the table entries by $(5/11)(V/H)$. Had the tables not been intended to provide actual angular velocities for examples and graphs,
table entries could have been calculated using only 57.296 Sin $A_i$ Tan $D_i$, and, for velocity $V$ and height $H$, entries would be multiplied by $V/H$, rather than by $(5/11)(V/H)$.

The data of Table A2 may be plotted on a graph, with azimuth on the horizontal axis and azimuth velocity on the vertical axis, with one curve for each selected value of declination. This was done to generate Figure 4. The graph is for straight and level flight at a speed of 440 feet/second (300 miles/hours) at a height of 200 feet. Note, from the graph, that, for any declination (for any curve), as azimuth increases, azimuth velocity increases (curves have a positive slope), but at a decreasing rate (curves less steep). Also, note that as declination increases, azimuth velocity increases (higher curves), and increases at a higher rate (steeper slopes) with azimuth increase. Examination of the Sin $A_i$ Tan $D_i$ of the angular velocity equation leads to the same conclusions.

Inspection of either the graph or the table from which it was constructed reveals that azimuth velocity may be quite high when neither azimuth nor declination are very high. For example, suppose that, for an azimuth of 30°, it is required to determine the declination below which azimuth velocity exceeds 30°/second. On the graph, trace up along a vertical line at a 30° azimuth. Note that the vertical intersects the declination curve for 25° just short of 30°/second. Thus, for any declination greater than just a little more than 25° at a 30° azimuth, azimuth velocity exceeds 30°/second. The equation for the table yields an exact value. Here, $\frac{dA_i}{dt} = 126.05 \sin A_i \tan D_i = 30 = 126.05 \sin 30^\circ \tan D_i$, from which $\tan D_i = 30/126.05/\sin 30^\circ = .47600$, and $D_i = \arctan .47600 = 25.450$ declination.

Azimuth velocity may also be calculated from offset $S$ and distance ahead $R_i$, rather than from azimuth and declination. The equation for doing this is derived in Appendix 1 as Equation 20: $\frac{dA_i}{dt} = 57.296 V S/(S^2 + R_i^2)$. Offset $S$ is the distance off to the side of the aircraft ground path of the ground object, and distance ahead $R_i$ is the
AZIMUTH IN DEGREES, $a_i$

Figure 4. Azimuth velocity for various declinations for an aircraft speed of 440 feet/second at a height of 200 feet above ground. The graph is a plot of the data in Table A2.
distance of the ground object ahead of the aircraft measured along the
ground parallel to the ground path or ground track of the aircraft.

An equation for angular velocity or angular rate in declination is
derived in Appendix 1 as Equation 22: $\frac{d\varphi_i}{dt} = 57.296 \frac{V}{H} \cos \varphi_i$
$\sin^2 \varphi_i$ degrees/second. This equation was used to calculate the entries
in Table A3 in Appendix 2. The entries are for an aircraft ground speed
of 440 feet/second and height above ground of 200 feet. For other
values of V and H, multiply the table entries by $(5/11) \frac{V}{H}$. Angular
velocity in declination is plotted as a graph in Figure 5, with angular
velocity in declination on the vertical axis and declination in degrees
on the horizontal axis. Curves for azimuth values from $0^\circ$ to $90^\circ$ were
plotted. Note the flattening out of the angular velocity curves, as
would be expected from the $\cos \varphi_i$ term in the equation, reaching zero
for a $90^\circ$ azimuth (straight to the side). Also, note how declination
velocity increases as declination increases, as expected from the $\sin^2 \varphi_i$
term in the equation.

Declination velocity may also be calculated using an Equation
containing offset $S$ and distance ahead $R_i$, instead of azimuth and
deciliation. An equation for this is derived in Appendix 1 as equation
22a: $\frac{d\varphi_i}{dt} = 57.296VHR_i/(H^2 + S^2 + R_i^2) \sqrt{S^2 + R_i^2}$. 

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Figure 5. Declination velocity for various azimuths for an aircraft speed of 440 feet/second at a height of 200 feet above ground. The graph is a plot of the data in Table A3.
5.0 VARIATION IN VELOCITY IN ALPHA WITH AIRCRAFT HEIGHT

It is instructive to work out numerical examples in which angular velocity in alpha is compared at different aircraft heights above the terrain. For a first example, suppose that ground speed \( V \), offset distance \( S \) from the aircraft ground path, and distance \( R_i \) of the ground object ahead of the aircraft are the same for both aircraft. In this case, the same ground object is viewed by observers at different heights above ground. Angular velocity in alpha for aircraft flying straight and level at a constant ground speed \( V \) and height \( H \) above ground is given by Equation 16 in Appendix 1 as 

\[
\frac{d\alpha_i}{dt} = \frac{57.296 V \sqrt{H^2 + S^2}}{(H^2 + S^2 + R_i^2)}. 
\]

Let \( H_1 \) be the height of the lower aircraft, and \( H_2 \) be the height of the higher aircraft. To compare angular velocities, use the ratio \( M \) of the alpha velocity of the higher aircraft to that of the lower one. Using the above equation, this ratio is 

\[
M = \left( \frac{d\alpha_i}{dt} \right)_2 \left/ \frac{d\alpha_i}{dt} \right)_1 = \frac{\sqrt{(H_2^2 + S^2)/(H_1^2 + S^2)}}{[(H_1^2 + S^2 + R_i^2)/((H_2^2 + S^2 + R_i^2)]}. 
\]

To illustrate variation of the angular velocity ratio \( M \) with height differences, suppose that the lower aircraft is at a height of 200 feet, and that the upper aircraft is directly above it and ten times higher at 2,000 feet. With these heights, the ratio of the angular velocities is 

\[
M = \frac{\sqrt{(2,000^2 + S^2)/(200^2 + S^2)}}{[(200^2 + S^2 + R_i^2)/((2,000)^2 + S^2 + R_i^2)]. 
\]

When several values of distance ahead \( R_i \) are used with a given value of offset distance \( S \), a curve of the ratio \( M \) may be plotted for that offset distance. In Figure 6, offset distances of 0, 200, 400, 800, 1,200, 2,000, and 4,000 feet are used with distances ahead \( R_i \) varied from 0 to 12,000 feet. Computation with the \( M \) equation yields a family of curves, one for each offset distance \( S \). The family of curves produced this way is shown in Figure 6.

From examination of the curves in the figure, it is clear that, the shorter the standoff distance \( S \), the shorter the distance ahead \( R_i \) at which the angular rate for the upper aircraft surpasses that for the lower aircraft, and the more rapidly the velocity ratio \( M \) increases with distance ahead \( R_i \), i.e., the steeper the curves. Note that, for a zero distance ahead \( R_i \), the \( S \) curves are lower for lower offset distances \( S \), i.e., the lower the angular velocity ratio \( M \). It is also
Figure 6. Ratio of angular velocity in alpha at a 2000-foot height to velocity at a 200-foot height when both aircraft speeds are equal and distance ahead $R_i$ is the same for both. One aircraft in directly above the other.

$$M = \frac{\text{Ratio of velocities in alpha}}{\left(\frac{H_1^2 + S^2 + R_1^2}{H_2^2 + S^2 + R_2^2}\right)\sqrt{\frac{H_2^2 + S^2}{H_1^2 + S^2}}}$$

OFFSET DISTANCE $S$, FEET

DISTANCE AHEAD IN THOUSANDS OF FEET, $R_i$
clear that the higher aircraft is presented with lower angular velocities ($M < 1$) only when distance ahead $R_i$ is less than about 1,000 feet. Appreciably lower angular velocity for the upper aircraft is present only for $R_i$ less than about 500-1,000 feet. It is clear that, for many combinations of offset and distance ahead, observers in the upper aircraft see angular rates that are higher to much higher than observers in the lower aircraft. Note, from the graph, that the largest $S$ value curve plotted was for an offset $S$ of 4,000 feet, and that this curve fell close to the $M = 1$, or equal angular velocity line, for all distance ahead. Also note that, for offsets of 0-200 feet at distances of over 2,000 feet, upper aircraft angular velocities are 3.5 to almost 10 times greater than those for the lower aircraft.

In the examples above, both aircraft were above the same ground point, with observers viewing the same object at the same distance ahead. When an aircraft is flying at a higher altitude, objects on the ground may be detected and observed at greater distances, since there may be less masking by vegetation, buildings, and terrain. For a second example comparing angular velocities at different heights, let the lower aircraft be at a 200-foot height and the upper one at 2,000 feet, as before. Now, however, let the upper aircraft view the ground object from a distance ahead $R_i$ that is ten times as great as for the lower aircraft. The same ground object is viewed from both vehicles with the same offset $S$ and aircraft speed $V$. Before calculating the angular velocity ratio $M$, a look at the ratio of slant ranges may be of interest. Slant range is the straight line distance from aircraft to ground object. From Equation 2 in Appendix 1, $r_i = \sqrt{H^2 + S^2 + R_i^2}$. The slant range ratio for the two air vehicles in this example is

$$\frac{\sqrt{(10H)^2 + S^2 + (10R_i)^2}}{\sqrt{H^2 + S^2 + R_i^2}} = \sqrt{10H^2 + S^2 + 100R_i^2}$$

For ground objects on the ground path of the aircraft, where $S$ is zero, this ratio is $\sqrt{100 (H^2 + R_i^2)} / \sqrt{H^2 + R_i^2} = 10$. For ground objects, then, slant range for the upper vehicle is 10 times as great for objects directly ahead, and decreases from 10 as offset increases from zero.
In this second example, the angular velocity ratio \( M \) is
\[
M = \frac{(d\alpha_i/dt)_{2000}/(d\alpha_i/dt)_{200}}{\sqrt{(2000^2 + S^2)(200^2 + S^2)}} \times \left[ \frac{(2000^2 + S^2 + R_i^2)/(200^2 + S^2 + 100 R_i^2)}{2000^2 + S^2 + 100 R_i^2} \right].
\]
Variation of \( S \) and \( R_i \) in this equation yields a family of curves shown in Figure 7. From this figure it may be seen that, when observing from a height ten times as great and with the ground object's distance ahead \( R_i \) ten times as great, results are very different from those of the first example. For distances ahead of more than about 700 feet for the lower vehicle and 7,000 for the upper one, at offsets up to 1,600 feet, the largest shown in the figure, alpha velocity is less than .4 as high for the upper aircraft. For a zero offset, where slant range is ten times as great, the angular velocity ratio curve is flat at .1.

Flying higher and observing ground objects while they are at a longer distance ahead of the aircraft can, in some situations, be beneficial by considerably reducing angular velocity in alpha, allowing more observing time, and being able to view ground objects masked at lower altitudes by intervening terrain or vegetation and manmade objects. The reduced angular velocity and the smoother ride sometimes present at higher altitude may permit better vision, better aiming and more accurate tracking of ground objects.
Figure 7. Ratio of angular velocity at 2000 ft to angular velocity at 200 feet when distance ahead is 10 times as great for the higher aircraft. Angular velocity is the rate of change of the angle alpha.
6.0 ANGULAR VELOCITY AT LOCATIONS ON AIRCRAFT WINDOWS

Relative to the eye of an observer in an aircraft, every point and object on the ground has an azimuth angle, or angle to the side of straight ahead, and a declination angle, or angle down from horizontal. Every point on the ground may be regarded as lying on a straight line from the point to the eye. The location or point on the window through which the line from the point passes defines the window location, conveniently expressed as the window azimuth and window declination of the ground point. Both of these window angles will vary with changes of the eye's location relative to the window. An observer with a greater eye-to-seat height can look down at a steeper angle, and given ground object appears at a higher point on the window. When the seat is moved forward, a steeper down look is possible, objects farther to the side appear in the window, and ground objects appear lower on the window and closer to zero window azimuth. Since aircraft windows are close to airborne observers, small changes in eye location produce appreciable changes in the window location of ground objects. However, ground objects are distant, so that changes of eye position within an aircraft have negligible effect upon the azimuth and declination of ground objects. Observer height and seat adjustment, then, change the angular coordinates of points on a window. Also, as fuel is used, in level flight the tilt or pitch of the aircraft may change so that, relative to a horizontal line from the observer's eye, window declinations change.

The azimuth and declinations of points on an aircraft window may be obtained by taking a photograph of the window with a level camera located at the observer's eye position and pointed straight ahead. An azimuth-declination or azimuth-altitude coordinate grid in the camera can be superimposed on the picture or added later. The grid allows every point on the window to be assigned an azimuth and a declination. There are several sources of error in making and using coordinate grids. One error source is to use nominal (marked) camera lens focal length rather than actual focal length. The two may differ by as much as 5% or more. Camera lens image distortions, particularly with wide-angle lenses, can also produce measurement errors. Window distortions of the
outside scene can cause significant errors in azimuth and declination: ground objects are not quite where they appear to be.

Once angular coordinates have been determined for points on an aircraft window, angular velocities in alpha, in azimuth, and in declination can be worked out. One way to determine angular velocity is to use the angular velocity equations in Appendix I. However, this method can require considerable computation. For example, note that by Equation 16 in Appendix I, angular velocity in alpha is:

\[
\frac{d\alpha}{dt} = 57.296 \frac{V}{H} \sin D_i \cos D_i \sqrt{\tan^2 D_i + \sin^2 A_i}
\]

An alternate to equations is to use tables listing angular velocity at selected azimuths and declinations, provided that the steps in angle size for azimuth and declination are adequate. If the aircraft speed and height of interest are not 440 feet/second and 200 feet, respectively, table entries must be corrected. Because the tables contain V and H as a multiplier, tables can be corrected for the speed V and height H of interest by multiplying them by \((H_o/V_o)(V/H)\), i.e., by \((200/440)(V/H)=(5/11)(V/H)\). For example, the tables may be used for an aircraft speed of 1550 feet/second at an altitude of 250 feet by multiplying table entries by \((H_o/V_o)(V/H)=(200/440)(1550/250)=2.818\).

To demonstrate how angular velocities in aircraft windows may be determined, plots were obtained showing the forward windows of a U.S. Air Force B-52 bomber as seen from the pilot's eye position. The plots contained a grid of azimuth (horizontal) versus declination (vertical). The plots were obtained from the AAMRL COMBIMAN computer program, and are shown in Figures 8, 9, and 10. The first two are for a male with an eye-to-seat distance corresponding to a 95th percentile Air Force male. Figure 7 is with the seat full back, and Figure 9 is for the seat full forward. Both seats are full up, i.e., at the maximum seat height position. Figure 10 is for a 5th percentile Air Force male with the seat full back and full up. In the three figures, within the grid squares, are numbers representing angular velocity in degrees/second in the angle alpha, the angle between a level line straight ahead from the pilot's eye and a line to the window position. Each number is for angular velocity at the center of the grid square in which it appears.
The three figures indicate angular velocity in the angle alpha. By using Table A2 for azimuth velocity, the numbers would be for azimuth velocity, or, Table A3 could be used for declination velocity. Examination of the three figures is instructive about how the azimuth and declination of points on windows and the angular velocities in the angle alpha corresponding to these locations vary with pilot eye-to-seat height and seat adjustment. The figures could have a flow diagram superimposed on the picture instead of or with the numbers giving angular velocity.
Figure 8. Angular velocity in alpha for ground objects for straight level flight. Numbers apply to centers of squares for a 200-foot height at 440 feet/second. For other values of H and V, multiply values shown by 5H/11V. Number 6 window: 95% male, middle eye location, pilot seat full back, full up.
Figure 9. Angular velocity in alpha for ground objects for straight level flight. Numbers apply to centers of squares for a 200-foot height 440 feet/second. For other values of speed V and height H, multiply values by 5H/11V. No. 6 window, 95% male, middle eye location pilot seat full forward, full up.
Figure 10. Angular velocity in alpha for ground objects for straight level flight. Numbers apply to centers of squares for a 200-foot height at 440 feet/second. For other values of V and H, multiply values by 5H/11V. No. 2 window, 5% Hale, middle eye position, pilot seat full back, full up.
7.0 STUDIES ON DYNAMIC VISUAL ACUITY

The use of angular velocity data in system design and mission planning requires data on the effect of angular velocities on human vision and perception. Havron (1962) reviewed 16 studies from the literature on the ability of observers to perceive angular motion and evaluated the ability of pilots to use angular motion cues from ground objects and ground texture. Erickson (1965), in his extensive analysis and review of the visual detection of targets, examined angular velocity in the angle alpha. He also reviewed results from some studies on threshold contrast and target detection.

Rather than repeating Havron's literature review and Erickson's material on target detection, the present paper reviews several studies on dynamic visual acuity as influenced by angular velocity.

One of the effects on an observer of relative angular velocity is a reduction in the observer's visual capabilities. One visual ability that decreases with angular motion is visual acuity, the ability to discern or resolve fine details in objects. When there is no relative motion, visual acuity is called static visual acuity. When either the observer or the viewed object is moving, i.e., when relative angular motion is present, visual acuity is called dynamic visual acuity. Dynamic visual acuity, then, is the ability to discriminate minute detail in objects with relative motion. Dynamic visual acuity is influenced by all of the factors or variables that influence static visual acuity, including the luminance, contrast and polarity (light on dark or vice versa) of the viewed test object or acuity test pattern, the orientation of the test pattern, type of pattern, length-to-width ratio of bars in grid patterns and the number of bars, size of the field-of-view, viewing or exposure duration, test administrator's criterion of resolution or legibility, and observer confidence level that a pattern is resolved. Some of these factors or variables are discussed in detail by Olzak and Thomas (1986), and others are covered by Stevens (1951) and by Farrell and Booth (1984). Individual observers also often differ greatly in acuity, adding to variability in data. In
dynamic visual acuity an additional factor or variable is difference in acuity with different direction of motion of an observed object or pattern. Whether or not an object is tracked by the eyes also makes a difference. Different reports differ in test conditions, i.e., in the values of the variables or factors that influence reported results. Results for acuity as a function of angular velocity are thus expected to differ, often by an appreciable amount, from one report to another. However, acuity-velocity curves have similar shapes in different studies, as will appear in discussion of results. It is apparent, then, that applying numerical values of dynamic acuity from the literature must be done with care.

The technical and scientific literature on dynamic visual acuity and the detection of moving or relatively-moving objects, both in the laboratory and in flight tests is very extensive, and only a few such studies can be discussed in this paper. Foley (1957), using moving digits, found that increasing the separation between digits improved legibility, and that dynamic acuity was better for horizontal motion than for vertical motion and better for upward motion than downward motion. Also, symbol movement from right to left was better than left to right.

Miller (1958) studied visual acuity for a stationary object and a moving observer, finding slightly poorer visual acuity for a moving observer than for a moving object. In the angular velocity range of 20-120 degrees/second, Miller found an acuity loss on the order of one minute or arc or less.

Elkin (1961) examined dynamic visual acuity with monocular tracking of Landolt-C rings at target angular velocities of 30, 60, 90 and 120 degrees/second, anticipatory tracking times of .2 and 1 second, and exposure time of .2 and .5 seconds with 12 subjects. Dynamic visual acuity worsened with increase in angular velocity. Acuity decreased from 1.35 arc minutes to 2.42, and from .79 to 1.10 arc minutes at the shortest and longest exposure-pair conditions, respectively. Acuity was better with 1 second tracking time than with .2 second, and acuity
decreased less with increased angular velocity at the longer tracking time and at the longer exposure time. The equation, acuity \( Y = A + BX^3 \), adequately described acuity as a function of angular velocity.

Miller and Ludvigh (1962) used dark Landolt-C rings as resolution patterns or targets. They found that, for individuals, static and dynamic acuity can be markedly and significantly different. They showed that dynamic visual acuity for a stationary observer viewing a moving object decreases (worsens) with increase in the angular velocity of objects moving at a right angle to the line-of-sight. It was shown that the equation \( Y = A + BX^3 \) describes dynamic visual acuity \( Y \), where \( X \) is angular velocity and \( A \) is static visual acuity, i.e., acuity when there is no motion. \( A \) and \( B \) in the equation are constants whose values depend upon conditions, such as illumination, direction of motion, contrast, etc. The equations held for subject motion, target motion and different directions of motion. In the range of 0 to 40 degrees/second of angular velocity, a visual acuity of about 2 minutes of arc is a representative value for laboratory conditions. When angular velocity reaches 120 degrees/second, visual acuity has decreased to about 10 minutes of arc or worse, depending upon conditions. Similar results were obtained earlier by Ludvigh (1949) who first formulated the semi-empirical equation \( Y = A + BX^3 \).

Lippert and Lee (1965) used black alphanumeric symbols subtending 39 minutes of arc on a high luminance background, with vertical top-to-bottom motion. Visibility was compared for 7.5 and 39 arc minute spacings. Mean object velocities for both zero and 100%, legibility was about three times higher for the 7.5 degree symbol spacing. Performance was about twice as good with a 30-degree aperture as with a 3-degree aperture.

Snyder and Greening (1965) examined the effect of direction and velocity of relative motion upon dynamic visual acuity. Acuity was appreciably better (lower acuity threshold) in the horizontal plane (0 degrees) of motion than in the vertical plane (90 degrees). Miller (1958) had found only a slight difference, in the 30 and 60 degree
planes, proportional decreases in acuity (higher acuity thresholds) were found. As the plane of motion changes from horizontal to vertical, acuity decreased. As angular velocity perpendicular to the line-of-sight increased from zero to two radians/second (115°/sec), the visual acuity worsened from 1.00 to over 3.00 minutes of arc. Miller and Ludvigh (1962) had found a static acuity of 2 minutes of arc. Snyder and Greening found that visual acuity decreased (worsened) as rate of approach to the observer increased. They also found that the exponent \(n\) in the equation \(Y=A+Bx^n\) was 2.3 rather than the 3 of Miller (1958) and of Miller and Ludvigh (1962). Snyder and Greening noted that studies in the literature that compared static and dynamic visual acuity had used only observers with superior (20/20 or better) static visual acuity. This truncation of the range of static acuity made it statistically unlikely that a significant correlation would be obtained between the two types of acuity. Snyder and Greening's correlations between static and dynamic visual acuity were positive and very small, of the same magnitude as those of other investigators who also used only observers with good static visual acuity.

Levine and Jauer (1972) examined dynamic acuity with TV imagery on a 5-inch TV cockpit display with image motion rates of zero to 10.5 inches/second at an 86 inch viewing distance. Critical detail in the target symbol subtended angles of .7 to 4.4 minutes of arc. The task was to determine the orientation of the symbol. For target-critical detail smaller than 2.2 minutes or arc, visual acuity decreased (worsened) linearly with increasing motion rate. They concluded that, for general design purposes, angular rates of 1.67 and 5 degrees/second require 1.4 and 1.9 minutes or arc visual angle, respectively, to equal visual acuity with a stationary one-minute-of-arc target.

Levine and Youngling (1972) used a cockpit mock-up and a 9 inch 525 line Contrac monitor and a flying spot scanner to display spliced panoramic photographs. Image scale was 12,500 and images were on display for 1, 2, 3, 4, 5 and 6 seconds, simulating aircraft speeds of 675 down to 113 knots. There were 24 targets and 12 subjects. Difficult targets had an image size on the display of 1/5 inch, and easy
ones were about 3.4 inch. Each target was briefed along with immediately-adjacent territory. Performance decreased as speed increased for both easy and hard targets at motion rates greater than 1.7 inches/second. Performance did not increase for display time over 3 seconds. Viewing distance was not specified, but if about 28 inches in the cockpit mock-up is assumed, angular velocity corresponding to the 1.7 inches/second was about $\arctan(1.7/28)$ or 57.296 (1.7/28), which is about 3.5 degrees/second. If an 86 inch viewing distance is assumed, a long distance, but one used by Levine and Jauer (1972), the rate is $(57.296)(1.7/86)$, or about 1.13 degrees/second.

Reddy (1975) examined the effect of contrast and linear velocity upon dynamic visual acuity using randomly-oriented black Landolt rings of different sizes with nine different white and grey backgrounds. Illumination was 120 feet candles. An increase in linear velocity markedly decreased dynamic visual acuity, while an increase in viewing time in the range of 1/8 to 1/2 second increased acuity. Dynamic visual acuity, as expected, was appreciably higher with higher contrast. The effect of linear velocity upon acuity is dependent upon viewing distance, for linear velocity and viewing distance determine angular velocity. Reddy, unfortunately, did not specify viewing distance.

Target detection using a TV sensor and a TV monitor when angular motion is present is complicated by TV resolution and the direction of motion across the display of objects relative to the scan lines of the system. Erickson et al (1974) did a series of four studies using, as targets, Landolt-Cs, square-wave grids, and vehicles moving at various velocities. Detection of pattern detail was measured as a function of pattern velocity, direction of motion, pattern size and pattern velocity. Tables and graphs show the loss in resolution as a function of these variables. A section of the report applies experimental results to the design of electro-optical systems. Estimates are made of the number of raster lines, sensor field-of-view, and display size required for satisfactory operator performance. Results and recommendations from this report are too voluminous to report here.
Those readers interested in TV viewing of moving objects will find this report very useful.

7.1 Summary of Dynamic Visual Acuity

Factors that influence static acuity, such as scene illuminance, object contrast, object pattern, and viewing time, also influence dynamic visual acuity. For the same angular velocity dynamic visual acuity is about the same whether the observer or the object is moving. Dynamic visual acuity is better for horizontal motion than vertical motion, and better for upward motion than downward motion. Symbol legibility is better for symbol movement from right to left than from left to right. Visually tracking objects with relative motion results in much less loss of acuity than viewing without tracking.

The data from dynamic visual acuity studies are a good fit to the equation \( Y = A + BX^n \), where \( Y \) is dynamic acuity in minutes of arc, \( A \) is a measure of static visual acuity, and \( X \) is angular velocity in degrees/second. The value of the exponent \( n \) has varied from as low as 2.34 to 3.00, with 3 being the most often reported value. Since more minutes of arc represents poorer visual acuity, dynamic visual acuity worsens with increase in angular velocity. Incidentally, visual acuity is often specified as the reciprocal of acuity in minutes of arc, so that larger numbers represent better acuity. The correlation between static and dynamic visual acuity is low: an individual's dynamic acuity must be measured, as it can not be predicted with adequate accuracy from his static visual acuity. The low correlations may be partly an artifact caused by using only observers who have good static visual acuity.

Visual acuity has usually been examined and measured using high-contrast test targets. However, acuity varies with the contrast of the test object or target. Targets viewed from aircraft, particularly objects of military interest, are often of low to very low contrast. The dynamic visual acuity of an airborne observer is reduced by vibration, buffeting, lack of clarity of the atmosphere, and the optical defects of the windscreen and its lack of cleanliness.
As ordinarily measured, visual acuity is related to ability to perceive high and intermediate spatial frequencies. Owsley, Sekular and Siemsen (1983) found that acuity, as measured by high contrast test patterns, is not related to ability to perceive low-contrast spatial frequencies.

Visual acuity is only one measure of visual ability and more comprehensive measures of visual ability are required to predict observer vision from aircraft. Two such measures are modulation transfer function (MTF) and contrast sensitivity. For discussion and data on modulation transfer function, the reader is referred to Westheimer (1986). For contrast sensitivity, the reader is referred to Boff, Kaufman and Thomas (1986) and Boff and Lincoln (1986).
8.0 REFERENCES


APPENDIX 1

DERIVATION OF EQUATIONS

(A) Derivation

The equations derived in this report are based on the situation geometry depicted in figures 11 and 12. In fig. 11, an aircraft at point Pi is in straight level flight at a height H with a constant speed V. An object or point of interest on the ground is at point T which is offset from the aircraft ground path by a distance S. Angular direction to the point on the ground may be designated by the angle alpha or \( \alpha_i \), which is the angle between the flight path in the sky of the aircraft and a line from the aircraft to the ground point. A second angular designation which is sometimes more useful is to use two angles: an azimuth or azimuth angle, \( A_i \) which is angle to the side of straight ahead, and declination or declination angle, \( D_i \), which is angle down from horizontal. Point Q is an imaginary point directly above the ground point of interest and at the height of the aircraft. Point Gi is a point on the ground directly beneath the aircraft. At the bottom of fig. 11 are five right angle triangles taken from the situation picture above them. Triangles B and E have the same angles and their sides are of the same length.

The following equations are based on fig. 11.

In triangle A, \( \sin D_i = H/r_i \), from which

\[ r_i = \frac{H}{\sin D_i} \]  (1)

In triangle B, \( R_i^2 = S^2 + r_i^2 \).

In triangle A, \( R_i^2 + H^2 = r_i^2 \), from which \( R_i^2 = r_i^2 - H^2 \).

Equating the two values of \( R_i^2 \),

\[ r_i^2 - H^2 = S^2 + R_i^2, \]

from which \( r_i^2 = H^2 + S^2 + R_i^2 \). Taking square roots,

\[ r_i = \sqrt{H^2 + S^2 + R_i^2} \]  (2)

Equating \( r_i \) values from (1) and (2),

\[ \frac{H}{\sin D_i} = \sqrt{H^2 + S^2 + R_i^2} \], from which

\[ \sin D_i = \frac{H}{\sqrt{H^2 + S^2 + R_i^2}} \]  (3)

From (3), \( \sin D_i = \frac{\sqrt{1 - \cos^2 D_i}}{\sqrt{H^2 + S^2 + R_i^2}} = \frac{H}{\sqrt{H^2 + S^2 + R_i^2}} \).

Squaring both sides,

\[ 1 - \cos^2 D_i = \frac{H^2}{(H^2 + S^2 + R_i^2)} \]

\[ \cos^2 D_i = 1 - \frac{H^2}{(H^2 + S^2 + R_i^2)} = \frac{(H^2 + S^2 + R_i^2 - H^2)}{(H^2 + S^2 + R_i^2)} = \frac{(S^2 + R_i^2)}{(H^2 + S^2 + R_i^2)}. \]

Taking square roots,

\[ \cos D_i = \sqrt{\frac{S^2 + R_i^2}{H^2 + S^2 + R_i^2}} \]  (4)
Aircraft Flight Path or Sky Path

A point at a height H above the target

Aircraft ground track or path

S = Offset

Fig. 11. Geometry of the aircraft-terrain situation.
From triangle A, \( \tan D_i = H/R_i^* \), from which \( R_i^* = H/\tan D_i \).

From triangle B, \( \sin A_i = S/R_i^* \), from which \( R_i^* = S/\sin A_i \).

Equating the two values of \( R_i^* \), \( H/\tan D_i = S/\sin A_i \), from which

\[
(5) \quad S = H \sin A_i / \tan D_i
\]

From triangle B, \( \tan A_i = S/R_i^* \), from which \( R_i^* = S/\tan A_i \).

From (5), \( S = H \sin A_i / \tan D_i \), so that \( R_i = S/\tan A_i = (H \sin A_i / \tan D_i)(\cos A_i / \sin A_i) = \)

\[
(6) \quad R_i = H \cos A_i / \tan D_i
\]

From triangle A,

\[
(7) \quad R_i^* = H / \tan D_i
\]

From triangle B,

\[
(8) \quad R_i^* = \sqrt{S^2 + R_i^*^2}
\]

From triangle B, \( \sin A_i = S/R_i^* \), from which

\[
(9) \quad R_i^* = S / \sin A_i
\]

From triangle C, \( \cos \alpha_i = R_i / r_i \).

From (1), \( r_i = H / \sin D_i \). Substituting this value of \( r_i \),

\( \cos \alpha_i = R_i / r_i = R_i / (H / \sin D_i) = \)

\[
(10) \quad \cos \alpha_i = (R_i / H) \sin D_i
\]

From triangle C, \( \cos \alpha_i = R_i / r_i \).

From (2), \( r_i = \sqrt{H^2 + S^2 + R_i^*^2} \). Substituting this value of \( A_i \),

\( \cos \alpha_i = R_i / r_i = \)

\[
(11) \quad \cos \alpha_i = R_i / \sqrt{H^2 + S^2 + R_i^*^2}
\]

From triangle C, \( \sin \alpha_i = \sqrt{H^2 + S^2} / r_i \)

From (2) \( r_i = \sqrt{H^2 + S^2 + R_i^*^2} \), so that

\[
(12) \quad \sin \alpha_i = \sqrt{H^2 + S^2} / \sqrt{H^2 + S^2 + R_i^*^2}
\]
From (5), \( S = H \sin A_i / \tan D_i \), so that
\[
\sqrt{\frac{H^2 + S^2}{H^2 + (H^2 \sin^2 A_i / \tan^2 D_i)}} = \sqrt{\frac{H^2 + 2}{H^2 + (H^2 \sin^2 A_i / \tan^2 D_i)}} = \sqrt{\frac{H^2 + 2}{H^2 + (H^2 \sin^2 A_i / \tan^2 D_i)}}(\tan^2 D_i + \sin^2 A_i)
\]
\[
\sqrt{\frac{H^2 + S^2}{H^2 + \tan^2 D_i}} = \frac{\tan^2 D_i + \sin^2 A_i}{(H/\tan D_i)}.
\]
From (3), \( \sin D_i = H/\sqrt{H^2 + S^2 + R_i^2} \), from which
\[
\sqrt{H^2 + S^2 + R_i^2} = H/\sin D_i.
\]
Inserting these values of \( \sqrt{H^2 + S^2} \) and \( \sqrt{H^2 + S^2 + R_i^2} \) into (12),
\[
\sin \alpha_i = \frac{\sqrt{H^2 + S^2}}{\sqrt{H^2 + S^2 + R_i^2}}
\]
\[
\sin \alpha_i = (H/\tan D_i) \sqrt{\frac{\tan^2 D_i + \sin^2 A_i}{(H/\sin D_i)}}
\]
\[
\sin \alpha_i = (\cos D_i/\sin D_i) \sin D_i \sqrt{\frac{\tan^2 D_i + \sin^2 D_i}{(H/\sin D_i)}}
\]
(13)
\[
\sin \alpha_i = \cos D_i \sqrt{\tan^2 D_i + \sin^2 D_i}
\]
From (10), \( \cos \alpha_i = (R_i/H) \sin D_i \).
From triangle B, \( \tan A_i = S/R_i \), from which \( R_i = S/\tan A_i \).
Substituting this into the above equation,
\[
\cos \alpha_i = (R_i/H) \sin D_i = (S/\tan A_i)(1/H) \sin D_i
\]
\[
\cos \alpha_i = (S/H) \sin D_i / \tan A_i \). From (5), \( S = H \sin A_i / \tan D_i \), thus
\[
\cos \alpha_i = (H \sin A_i / \tan D_i)(1/H)(\sin D_i / \tan A_i) =
\[
\cos \alpha_i = (\sin A_i)(\cos D_i/\sin D_i)(\sin D_i)(\cos A_i / \sin A_i) =
\]
(14)
\[
\cos \alpha_i = \cos A_i \cos D_i
\]
Equation (12), \( \sin \alpha_i = \sqrt{H^2 + S^2} / \sqrt{H^2 + S^2 + R_i^2} \) may be written as
\[
\sin \alpha_i = (\sqrt{H^2 + S^2})(H^2 + S^2 + R_i^2)^{-1/2} \), Note that \( \sqrt{H^2 + S^2} \) is a constant. Differentiating both sides with respect to time,
\[
(Cos \alpha_i)(d \alpha_i/dt) = (\sqrt{H^2 + S^2} - 1/2)(2R_i)(dR_i/dt)(H^2 + S^2 + R_i^2)^{-3/2}
\]
\[
(Cos \alpha_i)(d \alpha_i/dt) = (dR_i/dt)R_i(\sqrt{H^2 + S^2}/(H^2 + S^2 + R_i^2))H^2 + S^2 + R_i^2
\]
Now \( dR_i/dt = -V \), relative aircraft velocity.
From (11), \( \cos \alpha_i = R_i / \sqrt{H^2 + S^2 + R_i^2} \). The equation for \( \frac{d \alpha_i}{dt} \) then becomes

\[
\left( \frac{R_i}{\sqrt{H^2 + S^2 + R_i^2}} \right) \left( \frac{d \alpha_i}{dt} \right) = \frac{V \sqrt{H^2 + S^2}}{(H^2 + S^2 + R_i^2)^{1/2}} \cdot \frac{1}{\sqrt{H^2 + S^2 + R_i^2}} \cdot \frac{1}{\sqrt{H^2 + S^2}}.
\]

\[
\frac{d \alpha_i}{dt} = \frac{V \sqrt{H^2 + S^2}}{(H^2 + S^2 + R_i^2)^{1/2}} \text{ radians/second}. \text{To convert to degrees/second, multiply by 57.296.}
\]

(15) \[
\frac{d \alpha_i}{dt} = 57.296 \frac{V \sqrt{H^2 + S^2}}{(H^2 + S^2 + R_i^2)^{1/2}} \text{ deg/sec.}
\]

From (5), \( S = H \sin A_1 / \tan D_1 \). Replacing the \( S \) in \( H^2 + S^2 \) with this \( S \),

\[
H^2 + S^2 = H^2 + (H^2 \sin^2 A_1 / \tan^2 D_1) = H^2(1 + \sin^2 A_1 / \tan^2 D_1) =
\]

\[
= \left( \frac{H^2}{\tan^2 D_1} \right) (\tan^2 D_1 + \sin^2 A_1).
\]

Using this value of \( H^2 + S^2 \), and, from (6), \( R_i = H \cos A_1 / \tan D_1 \),

\[
H^2 + S^2 + R_i^2 = \left( \frac{H^2}{\tan^2 D_1} \right) (\tan^2 D_1 + \sin^2 A_1) + H^2 \cos^2 A_1 / \tan^2 D_1 =
\]

\[
= \left( \frac{H^2}{\tan^2 D_1} \right) (\tan^2 D_1 + \sin^2 A_1 + \cos^2 A_1). \text{Now, in general,}
\]

\[
\sin^2 A_1 + \cos^2 A_1 = 1, \text{and,} \tan^2 D_1 + 1 = \sec^2 D_1. \text{Using these as substitutions,}
\]

\[
H^2 + S^2 + R_i^2 = \left( \frac{H^2}{\tan^2 D_1} \right) \sec^2 D_1 = \frac{H^2}{\cos^2 D_1 \tan^2 D_1} =
\]

\[
= \left( \frac{H^2}{\cos^2 D_1} \right) (\cos^2 D_1 / \sin^2 D_1) = \frac{H^2}{\sin^2 D_1}. \text{Substituting this value of} \ H^2 + S^2 + R_i^2 \text{and the value of} \ H^2 + S^2 \text{worked out above into equation (15),}
\]

\[
\frac{d \alpha_i}{dt} = 57.296 \frac{V \sqrt{H^2 + S^2}}{(H^2 + S^2 + R_i^2)^{1/2}} \]

\[
= 57.296V \left( \frac{\sqrt{H^2 + \tan^2 D_1} (\tan^2 D_1 + \sin^2 A_1)}{(H^2 + \sin^2 D_1)} \right)
\]

\[
= 57.296 V \left( \frac{H \tan D_1 (\sin^2 D_1 / H^2) \sqrt{\tan^2 D_1 + \sin^2 A_1}}{(H^2 + \sin^2 D_1)} \right)
\]

(16) \[
\frac{d \alpha_i}{dt} = 57.296(V/H) \tan D_1 \cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1}
\]

From (13), \( \cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} = \sin \alpha_i \). Substituting this value of \( \cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} \) into (16),

(17) \[
\frac{d \alpha_i}{dt} = 57.296(V/H) \sin D_1 \cos D_1
\]
From (14), $\cos \alpha_i = \cos A_i \cos D_i$. Using the differentiation rule
\[
d(UW)/dt = U(dW/dt) + W(dU/dt),
\] and differentiating both sides,
\[
(-\sin \alpha_i)(d\alpha_i/dt) = (\cos A_i)(-\sin D_i)(dD_i/dt) + \\
(Cos D_i)(-\sin A_i)(dA_i/dt)
\]
\[
(sin \alpha_i)(d\alpha_i/dt) = \cos A_i \sin D_i)(dD_i/dt) + (\sin A_i \cos D_i)(dA_i/dt),
\]
from which
\[
\begin{align*}
(18) \quad d\alpha_i/dt &= (1/\sin \alpha_i) \left[ (\cos A_i \sin D_i)(dD_i/dt) + \\
&\quad \quad \quad (\sin A_i \cos D_i)(dA_i/dt) \right].
\end{align*}
\]
From (13), $\sin \alpha_i = \cos D_i \sqrt{Tan^2 D_i + Sin^2 A_i}$, from which
\[
1/\sin \alpha_i = 1/\cos D_i \sqrt{Tan^2 D_i + Sin^2 A_i}.
\]
Substituting this value of $1/\sin \alpha_i$ into (18),
\[
\begin{align*}
(19) \quad d\alpha_i/dt &= (1/\sqrt{Tan^2 D_i + Sin^2 A_i}) \left[ (Cos A_i \sin D_i/Cos D_i)(dD_i/dt) + \\
&\quad \quad \quad (1/Cos D_i)(\sin A_i \cos D_i)(dA_i/dt) \right].
\end{align*}
\]
From triangle B, $Tan A_i = S/R_i = SR_i^{-1}$. Differentiating both sides with respect to time,
\[
\begin{align*}
\tan A_i/dt &= (Sec^2 A_i)(dA_i/dt) = -(S/R_i^2)(dR_i/dt). \text{ Now } dR_i/dt = -v, \text{ relative aircraft velocity, so that } Sec^2 A_i(dA_i/dt) = SV/R_i^2,
&\quad \text{from which } dA_i/dt = SV/R_i^2 Sec^2 A_i = (SV/R_i^2)Cos^2 A_i.
\end{align*}
\]
From Triangle B, $Cos A_i = R_s/R_i = R_s/\sqrt{S^2 + R_i^2}$, so that
\[
\begin{align*}
dA_i/dt &= (VS/R_i^2)\left[R_s^2/(S^2+R_i^2)\right] = VS/(S^2+R_i^2) \text{ radians/second. To convert to degrees/second, multiply by 57.296.}
\end{align*}
\]
\[
(20) \quad dA_i/dt = 57.296 VS/(S^2 + R_i^2)
\]
In triangle B, \( \sin A = \frac{S}{\sqrt{S^2 + R^2}} \), so that \( \sqrt{S^2 + R^2} = S \sin A \), and, squaring both sides, \( S^2 + R^2 = \frac{S^2}{\sin^2 A} \).

From (5), \( S = H \sin A_1 / \tan D_1 \), so that
\[
S^2 + R^2 = \frac{S^2}{\sin^2 A_1} = \frac{(H^2 \sin^2 A_1 / \tan^2 D_1)}{\sin^2 A_1} = S^2 + R^2 = H^2 / \tan^2 D_1. \]
Substituting in (20) for \( S \) and for \( S^2 + R^2 \),
\[
d\frac{dA_1}{dt} = 57.296 V S / (S^2 + R^2) =
\]
\[
d\frac{dA_1}{dt} = 57.296 (V/H)(H \sin A_1 / \tan D_1) / (H^2 / \tan^2 D_1) =
\]
(21) \[
d\frac{dA_1}{dt} = 57.296 (V/H) \sin A_1 \tan D_1.
\]
From (3), \( \sin D_1 = H / \sqrt{H^2 + S^2 + R^2} = H(H^2 + S^2 + R^2)^{-1/2} \).

Differentiating both sides with respect to time,
\[
(C \cos D_1) \frac{dD_1}{dt} = (H) (-1/2)(2R_i)(H^2 + S^2 + R^2)^{-3/2} \frac{dR_i}{dt}.
\]
From (4), \( \cos D_1 = \sqrt{H^2 + S^2} / \sqrt{H^2 + S^2 + R^2} \). Inserting this value of \( \cos D_1 \) into the above equation,
\[
(\sqrt{S^2 + R^2} / \sqrt{H^2 + S^2 + R^2})(dD_1/dt) = HR_i / (H^2 + S^2 + R^2) \sqrt{H^2 + S^2 + R^2} (dR_i/dt).
\]
\[dD_1/dt = - \frac{[HR_i / (H^2 + S^2 + R^2) \sqrt{H^2 + S^2 + R^2}] (dR_i/dt)}.\]
\(dR_i/dt\) is \(-V\), relative aircraft velocity, and, since the equation gives radians/second, it must be multiplied by 57.296 to provide degrees/second. The equation then becomes
(22) \[
d\frac{dD_1}{dt} = 57.296 HR_i V / (H^2 + S^2 + R^2) \sqrt{S^2 + R^2}.
\]
From (6), \( R_i = H \cos A_1 / \tan D_1 \).
From (8), \( R^*_i = \sqrt{S^2 + R^2} \).
From (7) \( R^*_i = H / \tan D_1 \). Equating \( R^*_i \) values,
\[
\sqrt{S^2 + R^2} = H / \tan D_1.
\]
From (2), \( r_i = \sqrt{H^2 + S^2 + R^2} \). Squaring both sides,
\[
H^2 + S^2 + R^2 = r^2_i. \]
However, from (1), \( r_i = H / \sin D_1 \), so that
\[
r^2_i = H^2 / \sin^2 D_1 \text{ and } H^2 + S^2 + R^2 = H^2 / \sin^2 D_1 \).

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Substituting the above values of \( R_1, \sqrt{S^2 + R_1^2} \) and \( H^2 + S^2 + R_1^2 \) into (22),
\[
d\frac{dD_1}{dt} = 57.296 \frac{HR_1}{(H^2 + S^2 + R_1^2)} \sqrt{S^2 + R_1^2}
\]
\[
= 57.296 \frac{HV(H \cos A_1/Tan D_1)}{(H^2/\sin^2 D_1)(H/Tan D_1)}
\]
\[
= 57.296(V/H)(\cos A_1/Tan D_1)(\sin^2 D_1/Tan D_1)
\]

(23) \( d\frac{A_1}{dt} = 57.296 (V/H)\cos A_1\sin^2 D_1 \)

From (16), \( d\frac{\alpha_1}{dt} = 57.296(V/H)\sin D_1\cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} \),
from which \( 57.296(V/H) = (d\alpha_1/dt)/\sin D_1\cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} \).

From (23), \( d\frac{A_1}{dt} = 57.296(V/H)\sin A_1\tan D_1 \), from which

\( 57.296(V/H) = (d\frac{A_1}{dt})/\sin A_1\tan D_1 \).

Equating the above two values of \( 57.296(V/H) \),
\[
(d\frac{\alpha_1}{dt})/\sin D_1\cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} = (d\frac{A_1}{dt})/\sin A_1\tan D_1
\]
\[
d\frac{\alpha_1}{dt} = \frac{((\sin D_1\cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1}))/\sin A_1\tan D_1)}{(d\frac{A_1}{dt})}
\]
\[
d\frac{\alpha_1}{dt} = (\sin D_1\cos D_1)(\cos D_1/\sin D_1\sin A_1)(\sqrt{\tan^2 D_1 + \sin^2 A_1})(d\frac{A_1}{dt})
\]

(24) \( d\frac{\alpha_1}{dt} = (\cos^2 D_1/\sin A_1) \sqrt{\tan^2 D_1 + \sin^2 A_1} (d\frac{A_1}{dt}) \)

From (13), \( \sin \alpha_1 = \cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} \), so that
\[
\sqrt{\tan^2 D_1 + \sin^2 A_1} = \sin \alpha_1 /\cos D_1 \), Substituting into (24)
for \( \sqrt{\tan^2 D_1 + \sin^2 A_1} \),
\[
d\frac{\alpha_1}{dt} = (\cos^2 D_1/\sin A_1)(\sin \alpha_1 /\cos D_1)(d\frac{\alpha_1}{dt})
\]

(25) \( d\frac{\alpha_1}{dt} = (\cos D_1\sin \alpha_1 /\sin A_1)(d\frac{A_1}{dt}) \)

Solving (24) for \( d\frac{A_1}{dt} \),
\[
(26) \quad d\frac{A_1}{dt} = (\sin A_1/\cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} )(d\frac{\alpha_1}{dt})
\]

Solving (25) for \( d\frac{A_1}{dt} \),
\[
(27) \quad d\frac{A_1}{dt} = (\sin A_1/\cos D_1 \sin \alpha_1)(d\frac{\alpha_1}{dt})
\]
From (21), \( \frac{dA_i}{dt} = 57.296(V/H) \sin A_i \tan D_i \), from which
\[ 57.296(V/H) = \frac{(dA_i/dt)}{\sin A_i \tan D_i}. \]

From (23), \( \frac{dD_i}{dt} = 57.296(V/H) \cos A_i \sin^2 D_i \), from which
\[ 57.296(V/H) = \frac{(dD_i/dt)}{\cos A_i \sin^2 D_i}. \]

Equating the above two expressions for 57.296(V/H),
\[ \frac{(dA_i/dt)}{\sin A_i \tan D_i} = \frac{(dD_i/dt)}{\cos A_i \sin^2 D_i}, \]
from which
\[ \frac{dA_i}{dt} = \frac{(Tan A_i/\sin D_i \cos D_i)(dD_i/dt)}{\cos D_i}. \]

To graph the motion of a ground object or ground point relative
to an airborne observer, a starting point or initial set of
conditions at \( t=0 \) is required. The initial conditions may be \( H \)
and \( V \) plus offset distance \( S \) and initial distance ahead \( R_0 \), or
they may be \( H \) and \( V \) plus initial azimuth \( A_i \) and initial declination
\( D_i \). Initial conditions could even be \( H \), \( V \), \( S \), and initial slant
range \( r_0 \) or initial ground range \( R_0 \). In the following derivations
only the first two sets of initial conditions will be addressed.

After \( t \) seconds of flight, the aircraft has traveled a distance
\( Vt \) from its initial position above point \( G \) of fig. 12, and is
then above point \( G_i \). Distance ahead of the ground object is then

\[ R_i = R_0 - Vt \]

From fig. 13, \( \tan A_i = S/R_i = S/(R_0 - Vt) \), from which

\[ A_i = \text{Arc Tan} \left( S/(R_0 - Vt) \right) \]

From (5), \( S = H \sin A_i / \tan D_i \), from which
\[ \tan D_i = (H/S) \sin A_i \]. Taking the inverse,

\[ D_i = \text{Arc Tan} \left( (H/S) \sin A_i \right) \]

Equation (31), contains \( \sin A_i \). To calculate \( D_i \) independently of
\( A_i \), note that, in general,
\[ \tan A_i = \sin A_i / \cos A_i = \sin A_i / \sqrt{1 - \sin^2 A_i}. \]

From (30), \( \tan A_i = S/(R_0 - Vt) \). Equating these two values of \( \tan A_i \),
\[ \sin A_i / \sqrt{1 - \sin^2 A_i} = S/(R_0 - Vt). \] Squaring both sides,
Fig. 12. The ground situation at time zero and after t seconds.
\[
\sin^2 A_i / (1 - \sin^2 A_i) = S^2 / (R_o - Vt)^2.
\]
Cross multiplying,
\[
\sin^2 A_i = [S^2 / (R_o - Vt)^2] (1 - \sin^2 A_i)
\]
\[
= S^2 / (R_o - Vt)^2 - S^2 \sin^2 A_i / (R_o - Vt)^2,
\]
and
\[
\sin^2 A_i + [S^2 / (R_o - Vt)^2] \sin^2 A_i = S^2 / (R_o - Vt)^2
\]
\[
\sin^2 A_i \left[ 1 + S^2 / (R_o - Vt)^2 \right] = S^2 / (R_o - Vt)^2
\]
\[
\sin^2 A_i \left[ (R_o - Vt)^2 + S^2 \right] / (R_o - Vt)^2 = S^2 / (R_o - Vt)^2
\]
\[
\sin^2 A_i \left[ (R_o - Vt)^2 + S^2 \right] = S^2
\]
\[
\sin A_i = S / \sqrt{(R_o - Vt)^2 + S^2}.
\]
Taking square roots of both sides,
\[
\sin A_i = S / \sqrt{(R_o - Vt)^2 + S^2}.
\]
\[
\text{Inserting this value of } \sin A_i \text{ into (31),}
\]
\[
D_i = \arctan \left( \frac{H}{S} \sin A_i \right) = \arctan \left( \frac{H}{S / \sqrt{S^2 + (R_o - Vt)^2}} \right) =
\]
\[
D_i = \arctan \left[ \frac{H / \sqrt{S^2 + (R_o - Vt)^2}}{\sin A_i} \right] =
\]
\[
(32) \quad D_i = \arctan \left[ \frac{H / \sqrt{S^2 + (R_o - Vt)^2}}{\sin A_i} \right] =
\]
\[
\text{When initial conditions are } H, V, S, \text{ and } R_o, \text{ equations (30), (31), and (32) provide azimuth and declination angles after } t \text{ seconds of flight. When initial conditions are } H, V, A_o, \text{ and } D_o, \text{ the } S \text{ and } R_o \text{ of these equations must be replaced.}
\]
\[
\text{From (30), } A_i = \arctan \left[ \frac{S}{(R_o - Vt)} \right], \text{from which } \tan A_i = S / (R_o - Vt).
\]
\[
\text{From (5), } S = H \sin A_i / \tan D_i, \text{ so that } S = H \sin A_o / \tan D_o.
\]
\[
\text{From (6), } R_i = H \cos A_i / \tan D_i, \text{ so that } R_i = H \cos A_o / \tan D_o.
\]
Replacing \( S \) and \( R_o \) with these values, \( \tan A_i = S / (R_o - Vt) =
\]
\[
\tan A_i = \left( H \sin A_o / \tan D_o \right) / \left[ (H \cos A_o / \tan D_o) - Vt \right] =
\]
\[
\tan A_i = \left( H \sin A_o / \tan D_o \right) / \left[ (H \cos A_o - Vt \tan D_o) / \tan D_o \right] =
\]
\[
\tan A_i = (H \sin A_o / H \cos A_o - Vt \tan D_o). \text{ Taking the inverse,}
\]
\[
(33) \quad A_i = \arctan \left[ H \sin A_o / (H \cos A_o - Vt \tan D_o) \right]
\]
Dividing both numerator and denominator by \( H \sin A_o,
\]
\[
A_i = \arctan \left[ 1 / (\cos A_o / \sin A_o) - Vt \tan D_o / H \sin A_o \right] =
\]
\[
A_i = \arctan \left[ 1 / (\cot A_o - Vt \tan D_o / H \sin A_o) \right]
\]
This may be written as

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\[
A_i = \text{Arc Tan} \left[ \frac{1}{(\cot A_o - C_t)} \right], \text{ where } C = V \tan D_o / H \sin A_o.
\]

From (31), \(D_i = \text{Arc Tan} \left[ (H/S) \sin A_i \right]\), and,

From (5), \(S = H \sin A_o / \tan D_o\), so that

\[
D_i = \text{Arc Tan} \left\{ \frac{H}{(H \sin A_o / \tan D_o)} \sin A_i \right\}
\]

(35) \(D_i = \text{Arc Tan} \left[ (\tan D_o / \sin A_o) \sin A_i \right]\)

From (5), \(S = H \sin A_i / \tan D_i\), so that \(S = H \sin A_o / \tan D_o\).

From (6), \(R_i = H \cos A_i / \tan D_i\), so that \(R_o = H \cos A_o / \tan D_o\).

Replacing the \(S\) and \(R_o\) of (32) with these values,

\[
D_i = \text{Arc Tan} \left[ \frac{H}{\sqrt{S^2 + (R_o - Vt)^2}} \right]
\]

\[
D_i = \text{Arc Tan} \left[ \frac{H}{\sqrt{(H^2 \sin^2 A_o / \tan^2 D_o) + ((H \cos A_o / \tan D_o) - Vt)^2}} \right]
\]

\[
D_i = \text{Arc Tan} \left[ \frac{H}{\sqrt{(H^2 \sin^2 A_o / \tan^2 D_o) + (H \cos A_o - Vt \tan D_o) / \tan^2 D_o}} \right]
\]

(36) \(D_i = \text{Arc Tan} \left[ \frac{H}{\tan D_o} \sqrt{H^2 \sin^2 A_o + (H \cos A_o - Vt \tan D_o)^2} \right]\)

When values of both \(A_i\) and \(D_i\) have to be calculated, it may be more convenient to use equation (35), rather than (36) to calculate \(D_i\).

Note. By varying time \(t\) in equations (33)-(36), azimuth and declination at any time \(t\) may be calculated, thus permitting flow lines to be plotted on an azimuth-declination plot. The calculated azimuth and declination permit calculation of the azimuth and declination velocities, velocity in alpha, ground range, slant range, distance ahead, and the angle alpha.
B. COLLECTED SYMBOL DEFINITIONS AND EQUATIONS

Alpha, or $\alpha_i$. The angle between the path of the aircraft in the sky and a line from the aircraft to the ground point or ground object.

\[
\cos \alpha_i = \cos A_i \cos D_i \\
\sin \alpha_i = \sqrt{H^2 + S^2}/\sqrt{H^2 + S^2 + R_i^2} \\
\cos \alpha_i = R_i/r_i \\
\sin \alpha_i = \sqrt{r_i^2 - R_i^2}/r_i \\
\cos \alpha_i = (S/H)\sin D_i/\tan D_i \\
\sin \alpha_i = (\cos D_i)\sqrt{\sin^2 A_i + \tan^2 D_i} \\
\cos \alpha_i = R_i/\sqrt{H^2 + S^2 + (R_o - Vt)^2} = (R_o - Vt)/\sqrt{H^2 + S^2 + (R_o - Vt)^2}
\]

Azimuth, or $A_i$. The angle measured horizontally from a level-straight-ahead line to a point at aircraft height directly above a ground point or object. Angular deviation from straight ahead.

\[
\cos A_i = R_i/\sqrt{S^2 + R_i^2} \\
\sin A_i = S/R_i \\
\tan A_i = S/(R_o - Vt) \\
\sin A_i = S/\sqrt{S^2 + R_i^2} \\
\tan A_i = 1/((\cot A_o - C)t), \text{ where} \\
C = V \tan D_o/H \sin A_o \\
\]

Secondary Equations

\[
\sin A_i = S/R_i \\
\cos A_i = R_i/r_i \\
\sin A_i = (S/H) \tan D_i \\
\cos A_i = (\sqrt{S^2 + R_i^2})/r_i \\
\tan A_i = H/R_i \\
\tan A_i = (\tan D_o/\sin A_o) \sin A_i \\
\tan A_i = H/\sqrt{S^2 + R_i^2} \\
\tan A_i = (H/\tan D_o)\sqrt{H^2 \sin^2 A_o - (H \cos A_o - Vt \tan D_o)^2} \\
\]

Declination, declination angle, or $D_i$. The angle measured down from horizontal to the ground point or ground object. Angle down.

\[
\sin D_i = H/\sqrt{H^2 + S^2} \\
\cos D_i = \sqrt{(S^2 + R_i^2)/(H^2 + S^2 + R_i^2)} \\
\tan D_i = H/R_i \\
\tan D_i = (\tan D_o/\sin A_o) \sin A_i \\
\tan D_i = H/\sqrt{S^2 + R_i^2} \\
\tan D_i = (H/\tan D_o)\sqrt{H^2 \sin^2 A_o - (H \cos A_o - Vt \tan D_o)^2} \\
\]

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Distance Ahead, or $R_i$. The distance ahead of the aircraft of a ground point or ground object measured along the ground parallel to the ground path of the aircraft.

$$R_i = \frac{H \cos A_i}{\tan D_i}$$

$$R_i = \sqrt{H^2 + S^2} / \tan \alpha_i$$

**Secondary equations**

$$R_i = \frac{S}{\tan A_i}$$

$$R_i = \frac{\sqrt{H^2 - S^2}}{\tan \alpha_i}$$

$$R_i = R_i \cos A_i$$

$$R_i = R_i \cos D_i$$

$$R_i = r_i \cos \alpha_i$$

$$R_i = R_i - vt$$

Ground Range, or $R_i$. The distance along the ground from a point directly beneath the aircraft to the ground point or ground object of interest.

$$R_i^* = \frac{H}{\tan D_i}$$

$$R_i^* = \sqrt{S^2 + R_i^2}$$

**Secondary equations**

$$R_i^* = R_i / \cos A_i$$

$$R_i^* = r_i \cos D_i$$

$$R_i^* = S / \sin A_i$$

$$R_i^* = \sqrt{H^2 - H^2}$$

Slant Range, or $r_i$. The distance from the viewpoint in the aircraft to the ground point or ground object. Object distance.

$$r_i = \frac{H}{\sin D_i}$$

$$r_i = \sqrt{H^2 + S^2 + R_i^2}$$

**Secondary equations**

$$r_i = \sqrt{H^2 + R_i^2}$$

$$r_i = \frac{R_i}{\cos \alpha_i}$$

$$r_i = R_i^* / \cos D_i$$

$$r_i = \sqrt{H^2 + S^2} / \sin \alpha_i$$

Offset, offset distance, or $S$. Also called target offset. The perpendicular distance of a ground point or ground object from the ground path of the aircraft measured along the ground. Distance to the side.

$$S = H \sin A_i / \tan D_i$$

**Secondary equations**

$$S = R_i \tan A_i$$

$$S = r_i \sin A_i \cos D_i$$

$$S = R_i^* \sin A_i$$

$$S = \sqrt{R_i^*^2 - R_i^2}$$
Alpha velocity, or $d \alpha_i/dt$. Also called velocity in alpha or angular velocity in alpha. Rate of change of the angle between the aircraft flight path in the sky and a line from the aircraft to the ground point or ground object of interest. The following equations yield degrees/second.

$$d \alpha_i/dt = 57.296 \sqrt{V/H} \sin D_i \cos D_i \tan^2 D_i + \sin^2 A_i.$$  
$$d \alpha_i/dt = 57.296 \sqrt{H^2 + S^2} / (H^2 + S^2 + R_i^2).$$

Secondary equations

$$d \alpha_i/dt = 57.296 (V/H) \sin D_i \sin \alpha_i$$  
$$d \alpha_i/dt = (\cos^2 D_i / \sin A_i) \left(\sqrt{\tan^2 D_i + \sin^2 A_i}\right) (dA_i/dt).$$  
$$d \alpha_i/dt = (1/\sqrt{\tan^2 D_i + \sin^2 A_i}) \left[\left(\cos A_i \tan D_i \right)(dD_i/dt) + \left(\sin A_i \right)(dA_i/dt)\right].$$

Azimuth velocity, or $dA_i/dt$. Also called angular velocity in azimuth. Rate of change of azimuth.

$$dA_i/dt = 57.296 (V/H) \sin A_i \tan D_i$$  
$$dA_i/dt = 57.296 V/(S^2 + R_i^2).$$

Secondary equations

$$dA_i/dt = (\sin A_i / \cos^2 D_i) \left(\sqrt{\tan^2 D_i + \sin^2 A_i}\right) (d \alpha_i/dt)$$  
$$dA_i/dt = (\tan A_i / \sin A_i \cos D_i) (dD_i/dt)$$

In the above two equations, $d \alpha_i/dt$ and $dD_i/dt$, respectively, are in degrees/second.

Declination velocity, or $dD_i/dt$. Also called angular velocity in declination. Angular rate of up or down motion of points or objects in the field of view.
APPENDIX 2

ANGULAR VELOCITY TABLES
## TABLE A1

**ANGULAR VELOCITY IN ALPHA, d\(\alpha_i/dt\)**

<table>
<thead>
<tr>
<th>(0^\circ)</th>
<th>(1^\circ)</th>
<th>(5^\circ)</th>
<th>(10^\circ)</th>
<th>(15^\circ)</th>
<th>(20^\circ)</th>
<th>(25^\circ)</th>
<th>(30^\circ)</th>
<th>(35^\circ)</th>
<th>(40^\circ)</th>
<th>(45^\circ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0^\circ)</td>
<td>0.038</td>
<td>0.054</td>
<td>0.196</td>
<td>0.384</td>
<td>0.571</td>
<td>0.753</td>
<td>0.930</td>
<td>1.122</td>
<td>1.364</td>
<td>1.614</td>
</tr>
<tr>
<td>(5^\circ)</td>
<td>0.957</td>
<td>0.976</td>
<td>1.352</td>
<td>2.128</td>
<td>2.980</td>
<td>3.864</td>
<td>4.723</td>
<td>5.555</td>
<td>6.360</td>
<td>7.078</td>
</tr>
<tr>
<td>(30^\circ)</td>
<td>31.513</td>
<td>31.527</td>
<td>31.870</td>
<td>32.097</td>
<td>32.334</td>
<td>32.561</td>
<td>32.790</td>
<td>33.020</td>
<td>33.250</td>
<td>33.480</td>
</tr>
<tr>
<td>(35^\circ)</td>
<td>35.114</td>
<td>35.128</td>
<td>35.442</td>
<td>35.726</td>
<td>35.974</td>
<td>36.219</td>
<td>36.464</td>
<td>36.711</td>
<td>36.963</td>
<td>37.218</td>
</tr>
<tr>
<td>(40^\circ)</td>
<td>39.441</td>
<td>39.456</td>
<td>39.750</td>
<td>40.034</td>
<td>40.320</td>
<td>40.607</td>
<td>40.893</td>
<td>41.180</td>
<td>41.467</td>
<td>41.754</td>
</tr>
<tr>
<td>(45^\circ)</td>
<td>41.469</td>
<td>41.482</td>
<td>41.789</td>
<td>42.076</td>
<td>42.362</td>
<td>42.649</td>
<td>42.936</td>
<td>43.223</td>
<td>43.510</td>
<td>43.798</td>
</tr>
</tbody>
</table>

---

\[ d\alpha_i/dt = (180/\pi)(V/H) \sqrt{\tan^2 D_i + \sin^2 A_i} \]

The table is for a height \(H\) of 200 feet and a speed \(V\) of 440 feet/second. Here, \(180V/\pi H\) is 126.0571. For other values of \(H\) and \(V\), multiply table entries by \(5V/11H\).

*Since \(d\alpha_i/dt = (180/\pi)(V/H) \sqrt{\sin^2 D_i + \sin^2 A_i \sin^2 D_i \cos^2 D_i}\), for \(D_i = 90^\circ\), \(\sin^2 D_i = 1\), and \(\cos^2 D_i = 0\), so that \(d\alpha_i/dt = (180/\pi)(V/H) = 126.051\) deg./sec.*
### TABLE A2
ANGULAR VELOCITY IN AZIMUTH, $dA_1/dt$

**Degrees/Second in Azimuth, $dA_1/dt$**

<table>
<thead>
<tr>
<th>Azimuth in Degrees, $A_1$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0384</td>
<td>0.1918</td>
<td>0.3821</td>
<td>0.5695</td>
<td>0.7525</td>
<td>0.9298</td>
<td>1.1001</td>
<td>1.2620</td>
<td>1.4143</td>
<td>1.5558</td>
<td>1.6855</td>
</tr>
<tr>
<td>5</td>
<td>0.1925</td>
<td>0.9611</td>
<td>1.9150</td>
<td>2.8542</td>
<td>3.7718</td>
<td>4.6606</td>
<td>5.5140</td>
<td>6.3254</td>
<td>7.0886</td>
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</table>

Angular velocity in azimuth is $dA_1/dt = (180/\pi)(V/H)\sin A_1 \tan D_1$. For the above table, $V = 440$ feet/second and $H = 200$ feet, so that $dA_1/dt = (396/\pi)(V/H)\sin A_1 \tan D_1$. For other values of $V$ and $H$, multiply table values by $5V/11H$. 
### TABLE A2 (Continued)

**ANGULAR VELOCITY IN AZIMUTH, \( \frac{dA}{dt} \)**

<table>
<thead>
<tr>
<th>DEGREES/SECOND IN AZIMUTH, ( \frac{dA}{dt} )</th>
<th>Azimuth in Degrees, ( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>60</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
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<td>85</td>
<td>851.5946</td>
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</table>

When velocity \( V \) is not 440 feet/second, and height \( H \) is not 200 feet, multiply table entries by \( 5V/11H \).
### TABLE A3
ANGULAR VELOCITY IN DECLINATION, dD_i/dt

<table>
<thead>
<tr>
<th>Degrees/Second in Declination, dD_i/dt</th>
<th>Azimuth in Degrees, A_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
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<td>40</td>
<td>0.435</td>
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<td>45</td>
<td>0.489</td>
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</table>

Table entries are for \( H = 200 \) feet and \( V = 440 \) feet/second. F or other entries, multiply entries by \((5/11)(V/H)\). Entries calculated from \( dD_i/dt = (150/4)(V/H) \cos A_i \sin^2 D_i \).
APPENDIX 3
EXAMPLE PROBLEMS

Problem 1
An object on the ground is 2,000 feet ahead of an aircraft and is offset from the flight path by 265 feet. The aircraft is flying straight and level at 1,000 feet/second (681.8 miles/hour) at an altitude of 200 feet above ground. Using the equations in Appendix 1, calculate the angle alpha and the object's azimuth and declination. Use the azimuth and declination to calculate alpha again. Then calculate the object's ground range and slant range, its angular velocity in alpha, in azimuth, and in declination. For velocity in alpha, use both Eqn. 15 and Eqn. 16. For velocity in azimuth, use both Eqn. 21 and Eqn. 20. For velocity in declination, use both Eqn. 23 and Eqn. 22. Obtain declination velocity from azimuth velocity by using Eqn. 28. Check angular velocity computations by using velocity values in Eqn. 18, which contains velocities in alpha, azimuth and declination.

In working out the answers, note that what is given is distance ahead \( R_1 = 2,000 \) feet, offset distance \( S = 265 \) feet, aircraft speed \( V = 1,000 \) feet/second, and aircraft height \( H = 200 \) feet.

Solution
The angle alpha \( \alpha_1 \) is obtained from Eqn. 12, from Appendix 1,

\[
\alpha_1 = \arcsin \frac{\sqrt{H^2 + S^2}}{R_1} = \arcsin \frac{\sqrt{200^2 + 265^2}}{R_1} \approx 9.4250 \text{ degrees.}
\]

Azimuth angle \( A_1 \), from triangle B, Fig. 12, \( \tan A_1 = \frac{S}{R_1} \), so that

\[
A_1 = \arctan \left( \frac{S}{R_1} \right) = \arctan \left( \frac{265}{2000} \right) \approx 7.5477 \text{ degrees.}
\]

Declination angle \( D_1 \), from Eqn. 5, \( S = H \sin A_1 \tan D_1 \), so that

\[
D_1 = \arctan \left( \frac{H}{S} \sin A_1 \right) = \arctan \left( \frac{200}{265} \sin 7.5477 \right) \approx 5.6614 \text{ degrees.}
\]

To obtain the angle alpha from azimuth and declination, use Eqn. 14,

\[
\alpha_1 = \arccos \left( \cos A_1 \cos D_1 \right) = \arccos \left( \cos 7.5477 \cos 5.6614 \right) \approx 9.4252 \text{ degrees, as before.}
\]
\[ 4.62806 = (1/\sin 9.42520) \left[ \cos 7.54773 \sin 5.66144)(2.76382) + \\
(\sin 7.54773 \cos 5.66144)(3.73037) \right] = 4.62807. \] This checks out, considering the number of digits (places) used in the calculations.

Summary of worked

- Alpha \( \alpha = \alpha = 9.42520 \) degrees.
- Azimuth \( A = 7.54773 \) degrees.
- Declination \( D = 5.66144 \) degrees.
- Ground range \( R = 2,017.48 \) feet.
- Slant Range \( ri = 2,027.37 \) feet.

Angular velocity in alpha \( \frac{d\alpha}{dt} = 4.62806 \) degrees/second.

Azimuth velocity \( \frac{dA}{dt} = 3.73037 \) degrees/second.

Declination velocity \( \frac{dD}{dt} = 2.76382 \) degrees/second.

Problem 2

An aircraft is flying straight and level at a speed of 500 feet/second at an altitude of 190 feet. The pilot detects a missile launcher at an azimuth of 20 degrees and declination of 15 degrees. What is the launcher's offset distance \( S \), ground range \( R \), slant range \( r \), and distance ahead \( R_i \), and what is the launcher's angular velocity in azimuth, in declination, and in the angle alpha? Find angular rates by using equations from Appendix 1 and also by using tables \( A_1-A_3 \) in Appendix 2.

Solution

By Eqn. 5, offset distance \( S = H \sin A_i / \tan D_i = \\
= (190 \sin 20)/\tan 15 = 242.52 \) feet.

By Eqn. 7, ground range \( R_i = H / \tan D_i = 190/\tan 15 = 709.1 \) feet.

By Eqn. 1, slant range \( ri = H / \sin D_i = 190/\sin 15 = 734.1 \) feet.

By Eqn. 6, Distance ahead \( R_i = H \cos A_i / \tan D_i = \\
= 190 \cos 20/\tan 15 = 666.3 \) feet.

By Eqn. 21, azimuth velocity \( dA_i/dt = 57.296 (V/H) \sin A_i \tan D_i = \\
= 57.296 (500/190) \sin 20 \tan 15 = 13.92 \) degrees/second.
Ground Range $R_i^*$, from Eqn. 9, $R_i^* = S \sin A_i = 265/\sin 7.54773 = 2,017.48$ feet.

Slant Range $r_i$ from Eqn. 1, $r_i = H/\sin D_i = 200/\sin 5.66144 = 2,027.37$ feet.

Angular velocity in alpha, from Eqn. 15,
\[ \frac{d \alpha_i}{dt} = 57.296 \sqrt{H^2 + S^2}/(H^2 + S^2 + R_i^2) = 57.296(1,000) \sqrt{200^2 + 265^2}/(200^2 + 265^2 + 2,000^2) = 4.62806 \text{ degrees/second}. \]

Angular velocity in alpha, from Eqn. 16,
\[ \frac{d \alpha_i}{dt} = 57.296(V/H) \sin D_i \cos D_i \sqrt{\tan^2 D_i + \sin^2 A_i} = 57.296(1,000/200) \sin 5.66144 \cos 5.66144 \sqrt{\tan^2 5.66144 + \sin^2 7.54773} = 4.62806 \text{ degrees/second}, \]

Angular velocity in azimuth $dA_i/dt$, from Eqn. 21,
\[ dA_i/dt = 57.296(V/H) \sin A_i \tan D_i = 57.296(1,000/200) \sin 7.54773 \tan 5.66144 = 3.73037 \text{ degrees/second}. \]

Angular velocity in azimuth, from Eqn. 20, $dA_i/dt = 57.296 V S/(s^2 + R_i^2) = (57.296)(1,000)(265)/(265^2 + 2,000^2) = 3.73037 \text{ degrees/second}.$

Angular velocity in declination $dD_i/dt$, from Eqn. 23,
\[ dD_i/dt = 57.296(V/H) \cos A_i \sin D_i = 57.296(1,000/200) \cos 7.54773 \sin 5.66144 = 2.76382 \text{ degrees/second}. \]

Angular velocity in declination $dD_i/dt$, from Eqn. 22,
\[ dD_i/dt = 57.296(VH)/(H^2 + S^2 + R_i^2) \sqrt{S^2 + R_i^2} = (57.296)(1,000)(200)(2,000)/(200^2 + 265^2 + 2,000^2) \sqrt{265^2 + 2,000^2} = 2.76382 \text{ degrees/second}, \]

To obtain azimuth velocity from declination velocity by using Eqn. 28,
\[ dA_i/dt = (\tan A_i/\cos D_i \sin D_i)(dD_i/dt) = (\tan 7.54773/\cos 5.66144 \sin 5.66144)(2.76382) = 3.73037 \text{ degrees/second}, \]

To check computation of angular velocities by using Eqn. 18,
\[ d \alpha_i/dt = (1/\sin \alpha_i) \left[ (\cos A_i \sin D_i)(dD_i/dt) + (\sin A_i \cos D_i)(dA_i/dt) \right] = \]
By Eqn. 23, declination velocity \( \frac{dD_i}{dt} = 57.296 \frac{(V/H)\cos A_i \sin^2 D_i}{\frac{H}{190}} \)

\[ = 57.296 \left( \frac{500}{190} \right) \cos 20 \sin^2 15 = 9.491 \text{ degrees/second.} \]

By Eqn. 16, Angular velocity in alpha is

\[ \frac{d\alpha_i}{dt} = 57.296 \left( \frac{V}{H} \right) \sin D_i \cos D_i \sqrt{\tan^2 D_i + \sin^2 A_i} = 57.296 \left( \frac{500}{190} \right) \sin 15 \cos 15 \sqrt{\tan^2 15 + \sin^2 20} = 16.38 \text{ deg/sec.} \]

Alternatively, by Eqn. 15, \( \frac{d\alpha_i}{dt} = 57.296 \frac{\sqrt{H^2 + S^2}}{H^2 + S^2 + R_i^2} = 57.296 \left( \frac{500}{190^2 + 242.5^2 + 666.3^2} \right) = \frac{d\alpha_i}{dt} = 16.38 \text{ degrees/second, as above by Eqn. 16.} \]

To find angular velocities by using Tables A1-A3, note that the correction factor for speed and height is \( \frac{5}{11} \left( \frac{500}{190} \right) = 1.1962. \)

From Table A2, \( \frac{dA_i}{dt} = (11.552 \text{ from table}) \times \text{correction factor} = 11.552(1.1962) = 13.82 \text{ degrees/second.} \)

From Table A3, \( \frac{dD_i}{dt} = (7.9345)(1.1962) = 9.491 \text{ degrees/second.} \)

From Table A1, \( \frac{d\alpha_i}{dt} = 13.69(1.1962) = 16.38 \text{ degrees/second.} \)

Note that these angular velocities using the tables are the same as those above calculated from the velocity tables.

Problem 3

For what azimuth \( \alpha_i \) is angular velocity in alpha 36 degrees/second when declination \( D_i \) is 30 degrees for an aircraft flying straight and level at a height \( H \) of 200 feet with a ground speed of 440 feet/second? Check the answer by inserting it back into the equation for angular velocity in alpha, \( \frac{d\alpha_i}{dt} \).

Solution

Eqn. 16 in Appendix 1 for angular velocity in alpha is

\[ \frac{d\alpha_i}{dt} = 57.296(V/H)\sin D_i \cos D_i \sqrt{\tan^2 D_i + \sin^2 A_i}. \]

Dividing both sides of the equation by 57.296(V/H)\sin D_i \cos D_i yields

\[ \frac{H(d\alpha_i}{dt)}{(57.296 \text{ Vsin} D_i \text{ Cos} D_i)} = \sqrt{\tan^2 D_i + \sin^2 A_i}. \]

Squaring both sides of the equation,

\[ \left[ \frac{H(d\alpha_i}{dt)}{57.296 \text{ Vsin} D_i \text{ Cos} D_i} \right]^2 = \tan^2 D_i + \sin^2 A_i. \]

\[ \sin^2 A_i = \frac{H(d\alpha_i}{dt)}{57.296 \text{ Vsin} D_i \text{ Cos} D_i} \]

\[ \sin A_i = \left\{ \left[ \frac{H(d\alpha_i}{dt)}{57.296 \text{ Vsin} D_i \text{ Cos} D_i} \right]^2 \tan^2 D_i \right\}^{1/2} \]

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Taking the inverse of this equation,
\[ A_i = \arcsin \left( \sqrt{\frac{H(d\alpha_i/dt)}{57.296 V \sin D_i \cos D_i}} \right)^2 - \tan^2 D_i \]
\[ A_i = \arcsin \left( \frac{200(36)}{(57.296)(440)(\sin 30 \cos 30)} \right)^2 - \tan^2 30 \]
\[ A_i = \arcsin \frac{318885}{18.596} = 18.596 \text{ degrees.} \]

To check this answer by inserting it into the equation for \( d\alpha_i/dt \),
\[ d\alpha_i/dt = 57.296 \frac{(V/H) \sin D_i \cos D_i \sqrt{\tan^2 D_i + \sin^2 A_i}}{36} = \frac{57.296(440/200)\sin 30 \cos 30 \sqrt{\tan^2 30 + \sin^2 18.596}}{36} = 35.999 \approx 36. \text{ The answer, thus, checks out.} \]

Problem 4

Plot a graph of angular velocity in alpha as a function of declination for an aircraft speed of 350 knots at a height above ground of 200 feet. Use the data of Table A1 corrected for 350 knots. Graph velocity for declination from 0 to 45 degrees and azimuths up to 75 degrees. Use declination for the horizontal axis.

Solution

First, convert 350 knots to feet/second. Now, 350 knots is 350 nautical miles/hour. For a nautical mile of 6070 feet, 350 knots is
\[ (350)(6070) = 2.1277 \times 10^6 \text{ feet/hour} = (2.1266 \times 10^6 \text{ feet/hour})(3600 \text{ seconds/hour}) = 590.722 \text{ feet/second.} \]
The correction factor for the angular velocity table is thus \((5/11)(590.722/200) = 1.343\). Using azimuths of 1, 15, 30, 45, 60, and 75 degrees, calculate angular velocity in alpha for 0 through 45 degrees declination by multiplying table entries by 1.343. For example, for 45 degrees declination and 45 degrees azimuth, the table entry of 77.190 is multiplied by 1.343 to obtain 103.67 degrees/second. The graph obtained by this procedure is shown in Fig. 13.

Since angular velocity for azimuths over 90 degrees mirrors angular velocity for angles less than 90 degrees, each azimuth curve in Fig. 11 has two labels. For example, angular velocity at 180-30 = 150 degrees is the same as for 30 degrees, except for direction.
Fig. 13. Angular velocity in alpha for a height of 200 feet and a speed of 350 knots in straight level flight.
Problem 5

In the field of view of an aircraft pilot there is a line that is the locus of points for which the azimuth and declination velocities are equal. Derive the equation of this line. Using the equation, find the azimuth $A_i$ for a declination of 30 degrees and for a declination of 65 degrees for which azimuth and declination velocities are equal. Using equations 21 and 23 of Appendix 1, verify, for both locations, that the angular velocities are equal to each other.

Solution

Equal angular velocities in azimuth and declination mean that $dA_i/dt = dD_i/dt$. By equations 21 and 23, respectively, of Appendix 1,

$$57.296(V/H)\sin A_i \tan D_i = 57.296(V/H)\cos A_i \sin^2 D_i$$

$$\sin A_i \tan D_i = \cos A_i \sin^2 D_i$$

$$\sin A_i (\sin D_i / \cos D_i) = \cos A_i \sin^2 D_i$$

$$\sin A_i / \cos A_i = \sin D_i \cos D_i$$

$$\tan A_i = \sin D_i \cos D_i$$

This last equation is a solution to the problem. However, a more convenient equation for plotting may be obtained by using the trigonometric identity $\sin D_i \cos D_i = 1/2 \sin 2D_i$. The above equation then becomes $\tan A_i = \sin D_i \cos D_i$, i.e., $\tan A_i = 1/2 \sin 2D_i$.

For the two points specified in the problem, one with $D_i = 30$ and one with $D_i = 60$ degrees, respectively, the corresponding azimuth $A_i$ values are $A_i = \arctan(0.5 \sin 60) = 23.413$, and $A_i = (0.5 \sin 130) = 20.958$. For the first point ($A_i = 23.413$, $D_i = 30$),

$$dA_i/dt = 57.296 \left(440/200\right) \sin 23.412 \tan 30 = 28.92 \text{ degrees/second}.$$  

$$dD_i/dt = 57.286 \left(440/200\right) \cos 23.412 \sin^2 30 = 28.92 \text{ degrees/second}.$$  

For the second point ($A_i = 20.96$, $D_i = 65$),

$$dA_i/dt = 57.296 \left(440/200\right) \sin 20.958 \tan 65 = 96.69 \text{ degrees/second}.$$  

$$dD_i/dt = 57.296 \left(440/200\right) \cos 20.958 \sin^2 65 = 96.69 \text{ degrees/second}.$$  

These calculations show, for each of the two points, that azimuth and declination velocities are equal to each other.
Problem 6

An aircraft is flying straight and level at a height of 220 feet with a ground speed of 220 feet/second. An observer in the aircraft is interested in an object on the ground that is 1,000 feet ahead and is offset from the flight path by 275 feet. Use the equations in Appendix I to calculate the object's azimuth and declination, the angle alpha, slant range and ground range. Calculate azimuth and declination velocities using both the equations with ground distance and offset and the equations containing azimuth and declination. Calculate angular velocity in alpha using all seven of the $\frac{d\alpha}{dt}$ equations in the section on summary of symbol definitions and equations.

Solution

From triangle B of Fig. 12, $\tan A_i = \frac{S}{R_i}$, so that

$$A_i = \arctan \left( \frac{S}{R_i} \right) = \arctan \left( \frac{275}{1000} \right) = A_i = 15.3763$ degrees.

From Eqn. 3, $\sin D_i = \frac{H}{\sqrt{H^2 + S^2 + R_i^2}}$, from which

$$D_i = \arcsin \left( \frac{220}{\sqrt{220^2 + 275^2 + 1000^2}} \right) = 11.9764$ degrees.

From Eqn. (10), $\cos \alpha_i = \frac{R_i}{H \sin D_i} = \frac{(1000/220) \sin(11.9764)}{0.943222}$, so that

$$\alpha_i = \arccos \left( \frac{943222}{194001} \right) = 19.4001$ degrees.

From Eqn. (1), $r_i = \frac{H}{\tan D_i} = \frac{220}{\tan 11.9764} = r_i = 1060.20$ feet.

From Eqn. (5), $S = H \sin A_i / \tan D_i = 220 \sin 15.3763 / \tan 11.9764 = 275.000$, which checks the values of $A_i$ and $D_i$, since $S$ was given as 275 feet.

From triangle A of Fig. 12, $\tan D_i = \frac{H}{R_i}$, from which

$$R_i = \frac{H}{\tan D_i} = 220 / \tan 11.0764 = R_i = 1037.12$ feet.

From Eqn. (21), $\frac{dA_i}{dt} = 57.296 \frac{(V/H) \sin A_i \tan D_i}{S^2 + R_i^2}$

$$\frac{dA_i}{dt} = 57.296 \frac{(440/220) \sin 15.3763 \tan 11.9764}{220 + 275 + 1000} = \frac{dA_i}{dt} = 6.44543$ ft.

From Eqn. (20), $\frac{dA_i}{dt} = 57.296 \frac{V S}{(S^2 + R_i^2)}$

$$\frac{dA_i}{dt} = 57.296 \frac{(440)}{(275^2 + 1000^2)} = \frac{dA_i}{dt} = 6.44538$ degrees/second.

Note that the two equations provide the same azimuth velocities.

From Eqn. (23), $\frac{dD_i}{dt} = 57.296 \frac{(V/H) \cos A_i \sin^2 D_i}{S^2 + R_i^2}$

$$\frac{dD_i}{dt} = 57.296 \frac{(440/220) \cos 15.3763 \sin^2 11.9764}{220 + 275 + 1000} = \frac{dD_i}{dt} = 4.75770$ degrees/second.

From Eqn. (22), $\frac{dD_i}{dt} = 57.296 \frac{H V R_i}{(H^2 + S^2 + R_i^2) \sqrt{(S^2 + R_i^2)}}$

$$\frac{dD_i}{dt} = 57.296 \frac{(220)(440)(1000)}{(220^2 + 275^2 + 1000^2) \sqrt{220^2 + 275^2 + 1000^2}} = \frac{dD_i}{dt} = 4.75766$ degrees/second.
The seven equations for velocity in alpha from the summary section, with inserted values of the variables and constants, are as follows.

1. \( \frac{d}{dt} \alpha_1 = 57.296 \left( \frac{V}{H} \right) \sin D_1 \cos D_1 \sqrt{\tan^2 D_1 + \sin^2 A_1} \)

\( \frac{d}{dt} \alpha_1 = 57.296 \left( \frac{440}{220} \right) \sin 11.9764 \cos 11.9764 \left( \frac{\tan^2 11.9764 + \sin^2 15.3763}{1/2} \right) \)

\( \frac{d}{dt} \alpha_1 = 7.89876 \text{ degrees/second.} \)

2. \( \frac{d}{dt} \alpha_1 = 57.296 \sqrt{(H^2 + S^2)/1/2} \)

\( \frac{d}{dt} \alpha_1 = 57.296 \left( \frac{440}{220} \right) \sqrt{220^2 + 275^2/(220^2 + 275^2 + 1,000^2)} \)

\( \frac{d}{dt} \alpha_1 = 7.89870 \text{ degrees/second.} \)

3. \( \frac{d}{dt} \alpha_1 = 57.296 \left( \frac{V}{H} \right) \sin D_1 \sin A_1 \)

\( \frac{d}{dt} \alpha_1 = 57.296 \left( \frac{440}{220} \right) \sin 11.9764 \sin 19.4001 \)

\( \frac{d}{dt} \alpha_1 = 7.89845 \text{ degrees/second.} \)

4. \( \frac{d}{dt} \alpha_1 = (\cos^2 D_1 / \sin A_1) \sqrt{\tan^2 D_1 + \sin^2 A_1} \left( \frac{dA_1}{dt} \right) \)

\( \frac{d}{dt} \alpha_1 = (\cos 11.9764 / \sin 15.3763) \left( \frac{\tan^2 11.9764 + \sin^2 15.3763}{1/2} \right) \left( \frac{4.75770}{6.44538} \right) \)

\( \frac{d}{dt} \alpha_1 = 7.89870 \text{ degrees/second.} \)

5. \( \frac{d}{dt} \alpha_1 = \left( 1/\sqrt{\tan^2 D_1 + \sin^2 A_1} \right) \left[ (\cos A_1 \tan D_1) \left( \frac{dD_1}{dt} \right) + \sin A_1 \left( \frac{dA_1}{dt} \right) \right] \)

\( \frac{d}{dt} \alpha_1 = \left( 1/\sqrt{\tan^2 11.9764 + \sin^2 15.3763} \right) \left[ (\cos 15.3763 \tan 11.9764)(4.75770) + \sin 15.3763(6.44538) \right] \)

\( \frac{d}{dt} \alpha_1 = 7.89874 \text{ degrees/second.} \)

6. \( \frac{d}{dt} \alpha_1 = \left( 1/ \sin A_1 \right) \left[ (\cos A_1 \sin D_1) \left( \frac{dD_1}{dt} \right) + \sin A_1 \cos D_1 \left( \frac{dA_1}{dt} \right) \right] = \left( 1/ \sin 19.4001 \right) \left[ (\cos 15.3763 \sin 11.9764)(4.75770) + \left( \frac{1}{\sin 11.9764} \right)(6.44538) \right] \)

\( \frac{d}{dt} \alpha_1 = 7.89903 \text{ degrees/second.} \)

7. \( \frac{d}{dt} \alpha_1 = \left( 1/ \sqrt{\tan^2 D_1 + \sin^2 A_1} \right) \left[ (\cos A_1 \tan D_1) \left( \frac{dD_1}{dt} \right) + \sin A_1 \cos D_1 \left( \frac{dA_1}{dt} \right) \right] \)

\( \frac{d}{dt} \alpha_1 = \left( 1/ \sqrt{\tan^2 11.9764 + \sin^2 15.3763} \right) \left[ (\cos 15.3763 \tan 11.9764)(4.75776) + \sin 15.3763(6.44538) \right] \)

\( \frac{d}{dt} \alpha_1 = 7.89870 \text{ degrees/second.} \)

Note that, within rounding errors, all seven equations provide the same answer.
Problem 7

An object on the ground is directly ahead of an aircraft that is flying at 800 feet/second at a height of 300 feet. At what distance ahead and at what declination will the object be when velocity in alpha reaches 35 degrees/second?

Solution

By Eqn. (15), \( \frac{d\alpha}{dt} = \frac{57.296V\sqrt{H^2 + S^2}}{H^2 + S^2 + R_i^2} \). Since \( S = 0 \),

\[
\frac{d\alpha}{dt} = \frac{57.296 VH}{H^2 + R_i^2}
\]

\[
R_i + H^2 = \frac{57.296 VH}{d\alpha/dt}.
\]

Inserting values,

\[
R_i^2 + 300^2 = (57.296)(800)(300)/35 = 392,886.9
\]

\[
R_i = (392,886.9 - 90,000)^{1/2}
\]

\[
R_i = 550.352 \text{ feet.}
\]

By Eqn. (6), \( R_i = H \cos A_i / \tan D_i \). Here, \( A_i = 0 \), so that

\[
R_i = H \cos 0 / \tan D_i = H / \tan D_i,
\]

from which

\[
D_i = \tan^{-1} \left( \frac{H}{R_i} \right) = \tan^{-1} \left( \frac{300}{550.352} \right) = 28.5951 \text{ degrees.}
\]

An alternative to the above solution is instructive. When an object is straight ahead, the angle alpha is declination, i.e., \( D_i = \alpha_i \).

From Fig. 12, \( \tan D_i = \tan \alpha_i = H/R_i = H R_i^{-1} \). Differentiating,

\[
d (\tan^2 D_i / dt) = \sec^2 D_i (dD_i / dt) = (dD_i / dt) / \cos^2 D_i = -(H/R_i^2) (dR_i / dt).
\]

Now, from above, \( R_i = H / \tan D_i \), and, \( dR_i / dt = -V \), relative aircraft velocity,

\[
(dD_i / dt) / \cos^2 D_i = HV / (H^2 / \tan^2 D_i) = (V/H) \tan^2 D_i.
\]

\[
(\tan^2 D_i / \cos^2 D_i = (\sin^2 D_i / \cos^2 D_i) / \cos^2 D_i = \sin^2 D_i = (H/V) (dD_i / dt).
\]

Since \( D_i = \alpha_i \) in this problem, \( \sin \alpha_i = \sqrt{(H/V)} (dD_i / dt) \).

\[
d \alpha_i / dt = 35 \text{ degrees/second} = 35/57.296 \text{ radians/second. Thus, }
\]

\[
\alpha_i = \tan^{-1} \left( \frac{300/800}{35/57.296} \right) = 28.5952 \text{ degrees, as before.}
\]

Since \( \tan D_i = H/R_i \), \( R_i = H / \tan D_i = 300 / \tan 28.5951 = R_i = 550.035 \), as before.

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APPENDIX 4
PROBLEMS IN LOW-ALTITUDE HIGH-SPEED FLIGHT

The present paper looks at the angular velocity of ground points and ground objects since, at low altitudes and high speeds, motion rates pose serious problems for pilots and other crewmembers. It must be kept in mind that at low altitudes at high speed, angular motion is only one of several problems. These include the following:

1. Impact of the aircraft with radio and TV towers, power lines, and even hills, is a serious danger, even in broad daylight in good weather.

2. Almost constant observation of the terrain to avoid obstacles is necessary and this makes detection of airborne missiles and hostile aircraft less likely. Also, keeping track of both hostile and friendly aircraft is more difficult. Still another problem at low altitude is inadequate attention to aircraft instruments.

3. Inability to view a large expanse of terrain except at a grazing angle is serious, particularly in unfamiliar territory, because it causes difficulty in navigation. Pilots easily become lost at low altitudes.

4. Buffeting from air turbulence may be present, causing discomfort, fatigue, and possibly motion sickness, while interfering with vision, aiming and tracking.

5. Objects of interest on the terrain may be masked (concealed) by terrain, trees, and buildings. Masked objects may be unmasked at ranges
too close to take effective action against them, or may be unmasked for so short a time that detection probability is low.

6. High angular rates of the terrain and of objects on it, in conjunction with masking of objects, may result in inadequate search time, may prevent either thorough or systematic search, and cause difficulty in aiming at or tracking those objects that are detected.

7. At low altitude, aircraft are vulnerable to hostile aircraft with look-down shoot-down capabilities. Also, attacking aircraft usually have a height advantage.

8. At low altitude, finding or reaching a suitable clear area for landing in case of pilot injury, vehicle damage, or propulsion failure may be difficult or impossible.

9. Over many types of terrain, low altitude ejection from aircraft is very dangerous.

10. High-speed aircraft, particularly jet-engined aircraft, are not fuel efficient in the dense air at low altitudes. Operating range is restricted and getting back to an airfield may be a problem.

In summary, with low-altitude high-speed flight, the ride is rough and fatiguing, air sickness may occur, it's easy to get lost, ground objects move with high angular rates when close, both hostiles and friendlies are difficult to find and keep track of, objects may be masked until too late, aiming and tracking are difficult, impacting
ground obstacles is an ever-present danger, either safe landing or ejection in use of trouble is difficult or impossible, and low fuel reserve may preclude return to base as glide distance at low altitude is near zero.