The notion of super splines and vertex splines is introduced and studied. Quasi-interpolation formulas for real-time applications are constructed. The method of noncommutative blending of quasi-interpolation and vertex spline interpolation is introduced to yield interpolation schemes which are local, flexible, and of optimal approximation orders. These formulas can be applied to real-time interpolation by means of table-look-up or FIR implementation. Applications to engineering problems such as parallel implementation of the extended Kalman filter and Hankel-norm frequency domain methods are studied. Wavelets are constructed by applying cardinal splines, and hence, they are readily available for real-time interpolation and orthogonal wavelet decompositions and reconstructions.
18. Cont

wavelet orthogonal decompositions and reconstructions
Mathematical Methods and Algorithms for
Real-Time Applications

FINAL REPORT

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September 25, 1990

Sponsored by
SDIO/IST
Managed by
U. S. Army Research Office
Contract Number DAAL03-87-K0025

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I. List of manuscripts submitted or published under SDIO/IST - ARO sponsorship

Books

Papers
7. Shape-preserving interpolation by bivariate \( C^1 \) quadratic splines, (with H.C. Chui and T.X. He) CAT Report # 148.


II. Participating scientific personnel

**Faculty**

Charles K. Chui

**Graduate students and research assistants**

G. Chen (Ph. D. awarded, Aug. 1987)
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T.X. He (Ph. D. expected, May 1991)
M.J. Lai (Ph. D. awarded, Aug. 1989)
X. Li (Ph. D. expected, May 1991)
Q. Liu
J. Wang

**Computer programmers** (Undergraduate students)

Scott Bowers
Susan Fojtasek
III. Brief outline of research findings

The objective of this report is to give a very brief summary of the research findings on the project under Contract Number DAAL 03-87-K-0025 sponsored by SDIO/IST and managed by the U.S. Army Research Office. Technical details are not included since most of them have been reported as semi-annual progress reports. This project covered the period February 1, 1987 - July 30, 1990 which includes a six-month extension at no cost. The list of publications in Section I, page 1 - page 2, will also be used as references. Other references are listed at the end of this summary, We divide our results into five categories as follows:

1. Super splines and vertex splines.

   Our results in this area are found in [4,5,8,9] with applications to interpolation to be discussed in the next section and other applications in Section 4. It is well known that every spline function in one variable can be represented as a $B$-spline series. However, with only a few exceptions, this nice local representation does not generalize to bivariate, and more generally multivariate, splines. One of the reasons is that the $C^r$ piecewise polynomial functions on a triangulation have very complicated local structures, and the one that is crucial to the computational aspects is the extra free parameters on lower-dimensional manifolds that constitute the triangulation. This phenomenon is probably well known to some of the finite element people. In our work [4], we introduced the notion of super splines in order to drop the “useless” parameters. For instance, a $C^1$ quintic bivariate piecewise polynomial is restricted to be $C^2$ at the vertices. In our paper [9], we considered the smallest subspace of super splines in $S_3^{r+2}$ (i.e. $C^r$ piecewise bivariate polynomials with total degree $3r + 2$) and showed that the approximation order is full, that is $3r + 3$ as shown in de Boor - Höllig [S1]. In fact, this super spline subspace has a basis consisting of vertex splines; and hence, a spline series representation is obtained. The notion of vertex splines was introduced by Chui and Lai in [S9]. Later development of super and vertex splines can be found in [5,8,9] and recent work of Alfeld, Höllig, Ibrahim, Piper, Schumaker, and others (cf. the articles in [S10,S12,S13,S28] and references therein).

2. Quasi-interpolation and interpolation formulas.

   The central theme in this area of research is construction of “real-time” interpolation formulas, and real-time implementation of such formulas. In our work [2,3], we developed local quasi-interpolation schemes by introducing the so-called “Neumann-series” approach. Here, quasi-interpolation means optimal-order local linear approximation. In [19], we gave a complete characterization of all quasi-interpolation formulas. The most important application in [19] is construction of quasi-interpolation using arbitrary sample points and various data information. To change quasi-interpolation to actual local interpolation, preserving the optimal order of approximation, we introduced in [14,20] a (noncommutative) “blending” method by using vertex splines or generalized vertex splines. The emphasis is localness and optimality of the order of approximation. Hence, “table-look-up” can be used for real-time interpolation; or equivalently, an FIR (moving-average) scheme can be implemented. The general theory is extended to nonuniform grid and an application to solving nonlinear PDE’s is discussed in [17], and many other applications and examples are included in my tutorial article [15], written for the NATO Graduate Studies on “Computations of Curves and Surfaces”. In our work [6,7], we studied interpolation formulas.
that preserve the shapes of the data. In particular, an almost real-time method is given in [6], and a comprehensive study on shape-criteria is given in [7].

3. General multivariate spline theory.

Our articles in this area include [4,5,7,8,9,12,19,20]. As discussed in the last two sections, the notions of super splines and vertex splines, and the methods of Neumann series approach and noncommutative blending were introduced in this series of papers. To facilitate the construction schemes, the dimensions of the super spline subspaces should be determined. The first results in this area are given in our paper [12], where a dimension criterion is also introduced. The general theory of super spline, vertex splines, quasi-interpolation, and interpolation in higher dimensions is included in my CBMS-NSF monograph [S3]. In the second edition of my encyclopedia article [S2], cardinal spline interpolation is related to wavelets, a topic to be discussed in Section 5.

4. Applications to engineering problems.

Our results in this area are included in [1,10,11,13,16,18,23,24,25,26]. In addition to construction and implementation of real-time approximation and interpolation schemes, we have found almost immediate applications to image reconstruction [1], design of large reflector surfaces [24], and an application to the boundary element method for electromagnetic scattering [25]. On the other hand, real-time methods should be implemented with the Kalman filter. In our work [10], a parallel algorithm is developed to improve the extended Kalman filter significantly and applications to real-time system parameters identification are given. In the second edition of our monograph [S7], which is under preparation, this method is discussed in some details. When frequency-domain methods are considered, the problem of stability must be considered. By using the Hankel-norm approximation, stability is guaranteed. In our work [11,13,16], computational efficiency and accuracy are discussed relative to truncation of the Hankel operator and the estimates are given in terms of s-numbers (or singular values). A related application is the solution of the so-called four-block problem in systems theory. This is studied in [18]. Robust stability are studied in [23] for the one-variable (i.e. SISO) setting, and in [26] for the multi-variable (i.e. MISO) setting.

5. Wavelets.

Perhaps the most exciting development in this project is the introduction of cardinal-spline wavelets in our work [21,22]. This was done toward the end of the extended period of this project which overlaps with the new project (under Contract No. DAAL 03-90-0091). This new project, with the same title, should be considered as continuation of the old one for which this report is prepared. The multiresolution analysis introduced by Meyer [S22] and Mallat [S19, S20, S21] is followed. As is well-known, Daubechies' wavelets [S16, S17] are the only available compactly supported orthonormal wavelets. However, although there are efficient algorithms (cf. [S17]), these wavelets do not have explicit formulations and are certainly not suitable for real-time applications. In addition, since they are not symmetric or anti-symmetric, their non-linear phase property creates distortion. In [21], we introduced the notion of cardinal-spline wavelets. These wavelets are constructed by using cardinal fundamental splines of Schoenberg. Since they are explicitly formulated, symmetric for even-order splines, and antisymmetric for odd-order splines, they are readily
available for interpolation and wavelet decomposition of data in real-time. An application to curve-design is given in [27], and a comprehensive paper on application to real-time signal analysis is under preparation. The general theory of our new wavelets is studied in [S14], partially supported by the new project. This theory includes both the Daubechies wavelets and our cardinal-spline wavelets as special cases. A fairly complete analysis of the minimally supported cardinal spline wavelets (introduced in [22]) is given in [S15]. A survey article [S3] has been written and will be published by Academic Press, and a tutorial article [S4] which is a compilation of the five-hour lecture series I gave in Lancaster, England, last July, will be published by the Oxford University Press.

References

The list of manuscripts in Section 1 is also used as references. In addition, the following supplementary list is relevant to this report.


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1989–Present  Professor of Mathematics, of Electrical Engineering, and of Statistics
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1988–Present  Director, Center for Approximation Theory
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1987–1989  Professor of Mathematics and of Electrical Engineering
Texas A&M University

1984–1986  Head, Division of Applied Mathematics
Texas A&M University

1974–1987  Professor of Mathematics
Texas A&M University

1970–1974  Associate Professor of Mathematics
Texas A&M University

1967–1970  Assistant Professor of Mathematics
State University of New York at Buffalo
Amherst, New York

1967  Post Doctoral Research Associate
University of California at San Diego
La Jolla, California

Visiting Positions: Florence, Italy; Canterbury, New Zealand;
Cambridge, England

IV. JOURNAL EDITORSHIPS:

(1) Editor of Approximation Theory and Its Applications

(2) Editor of Journal of Approximation Theory

(3) Associate Editor of Journal of Mathematical Research and Exposition

(4) Associate Editor of Revista de Matemáticas Aplicadas
V. AWARDS:

Distinguished Achievement Award in Research
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VI. HONORS:

(1) National Science Foundation CBMS Principal Speaker, 1987
(2) Honorary Professor, Ningxia University, China, 1987
(3) Guest Chair Professor, Xiamen University, China, 1987
(4) Erskine Fellow, University of Canterbury, New Zealand, 1987
(5) NATO Advanced Study Institute Principal Speaker, 1989
(6) Principal Speaker, Spline Symposium at Tokyo, Japan, 1990

VII. PROFESSIONAL SOCIETIES:

(1) American Mathematical Society
(2) Mathematical Association of America
(3) Society for Industrial and Applied Mathematics
(4) Institute of Electrical and Electronics Engineers

VIII. RESEARCH AREAS:

(1) Approximation Theory
   (multivariate splines, wavelets)

(2) Applied Mathematics
   (real-time algorithms, systems theory, signal processing,
    boundary element methods, nonlinear optics)

PUBLICATIONS

A. Books:

1. Approximation Theory II, (with G.G. Lorentz and L.L. Schumaker, Editors),

2. Elements of Calculus, (with D. Allen and W. Perry), Brooks-Cole, Monterey,

3. Approximation Theory IV, (with L.L. Schumaker and J.D. Ward, Editors), Aca-


B. Research Papers:


50. On the range of certain locally determined spline projections, (with P.W. Smith and J.D. Ward), in Proc. Conference on Approximation Theory, Bonn, 1976,


147. Shape-preserving interpolation by bivariate $C^1$ quadratic splines, (with H.C. Chui and T.X. He) CAT Report # 148.
149. On bivariate super vertex splines, (with M.J. Lai), Constr. Approx. Vol. 6 (1990), 399–419.
168. Rate of uniform convergence of rational functions corresponding to best approximants of truncated Hankel operators, (with X. Li and J. D. Ward), Math of Control, Signals, and Systems. To appear.
177. Solution of the four-block problem via minimum-norm interpolation, (with X. Li), CAT Report #214.
183. Real-time signal analysis with quasi-interpolatory splines and wavelets (with A. K. Chan), in Curves and Surfaces Ed. by P. J. Laurent, A. Le Méhauté, and L.

185. Approximation by ridge functions and neural networks with one hidden layer, (with X. Li), CAT Report #222, Texas A&M University.


188. Multivariate polynomial natural splines for interpolation of scattered data and other applications, (with L. T. Guan), CAT Report #229, Texas A&M University.

189. Compactly supported box-spline wavelets (with J. Stöckler and J. D. Ward), CAT Report #230, Texas A&M University.