MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

JAMMING PERFORMANCE OF FREQUENCY-HOPPED COMMUNICATION SYSTEMS WITH NONUNIFORM HOPPING DISTRIBUTIONS

P.R. HIRSCHLER-MARCHAND

Group 64

TECHNICAL REPORT 878

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This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Hugh L. Southall

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Chief, ESD Lincoln Laboratory Project Office

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ABSTRACT

This report presents an analysis of the jamming performance of frequency-hopped communication systems in which the user-signal hops according to a nonuniform distribution across the available hopping bandwidth.

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ACKNOWLEDGEMENTS

1. INTRODUCTION

This report analyzes the effect of optimized partial-band Gaussian noise jamming on the performance of communication systems in which the signal frequency-hops according to a nonuniform distribution over the available hopping bandwidth. The analysis is developed for the class of modulations having an exponentially decreasing bit error probability in additive white Gaussian noise. The results can be applied to any modulation since the Chernoff bound provides an exponentially tight upper bound to the bit error probability at high enough signal-to-noise ratios. At low signal-to-noise ratios, the bit error probability can always be approximated by an exponential function, at least over a limited range.

Although ideally it should do so, there are situations in which a signal may not hop according to a uniform distribution. For instance, in a frequency-hopped satellite link, a signal may not be allowed to hop in certain portions of the hopping band and, instead, may be forced to hop more frequently in other portions of the band. This is the case when a wide-band signal, restricted from hopping to the band edges in order to avoid spilling energy outside of the allocated band, more frequently revisit the center of the band. It is also the case when a user signal is restricted from hopping in certains portions of the band because of the presence of other users. Conditions such as these result in a nonuniform distribution of the user signal across the band.

The optimal partial-band jamming strategy and the resulting bit error probability are calculated here for arbitrary nonuniform distributions, and these results are compared to the performance achieved with uniform frequency-hopping. The results confirm intuition in establishing the fact that the more *uniform-like* the distribution, the smaller the jamming loss when compared to uniform hopping. A quantitative measure of the resulting loss is provided for arbitrary *continuous* and *discrete* distributions.

This report is organized as follows.

Section 2 derives the well-known worst-case jamming performance for a uniform distribution.

Section 3 analyses the optimal jamming strategy and worst-case bit error probability performance for arbitrary *nonuniform continuous* distributions. A geometric interpretation is given of the optimal jamming strategy. The performance is compared to that for uniform hopping, at high and low signal-to-noise ratios. A simple example is given to illustrate the analysis.

Section 4 develops the analysis for arbitrary *discrete* distributions, and follows the same outline as Section 3. The equations are derived for staircase-like distributions. Such distributions are important because they arise when several users frequency-hop within the same band [1]. The staircase discontinuities result from the fact that, when the signal revisits certain bands more than once, the probability distribution overlaps itself. The results obtained for *discrete* distributions are similar to those obtained for *nonuniform continuous* distributions. The results from Sections 3 and 4 can be combined to handle any hybrid distribution having continuous portions and staircase-like discontinuities. Finally, a simple example is given to illustrate the *discrete* case results.

Section 5 draws conclusions from the analyses, and provides guidelines for the design of nonuniform frequency-hopping plans.

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2. UNIFORM FREQUENCY DISTRIBUTION

2.1 NOTATIONS AND ASSUMPTIONS

2.1.1 The Signal

Throughout this report. the notation $\{W\}$ with *braces* represents a frequency set. and W *without braces* represents the size or Lebesgue measure $[2]$ of that set.

In a frequency-hopped communication system. the user signal is allowed to hop over all or part of the available hopping bandwidth $\{W\}$. Let $\{W_0\}$ represent the frequency band or set of frequency bands over which the density is defined. For a uniform distribution. the density is a constant equal to $a_0 = 1/W_0$ over the set $\{W_0\}$. and equals 0 elsewhere. $\{W_0\}$ is also called *support* of the distribution throughout this report. $\{W_{n,u}\}$ with subscript u will be used to emphasize that it is the support of a uniform distribution. and similarly subscript n for nonuniform distributions. Figure 2-1 illustrates a uniform hopping distribution.
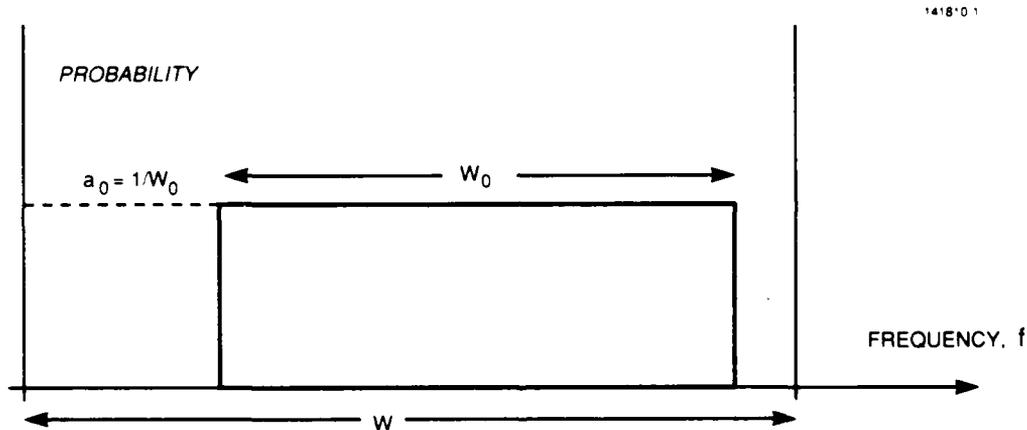
Figure 2-1. Uniform hopping distribution.

2.1.2 The Modulation

The modulation schemes considered are those whose demodulated bit error probability in additive white Gaussian noise (AWGN) is characterized by the equation

$$h(E_b/N_0) = A\, e^{-cE_b/N_0} \tag{2.1}$$

where E_b is the bit energy, N_0 is the single-sided white Gaussian noise power spectral density, and A and c are positive constants which depend upon the modulation. The above equation represents the detection performance for a large class of modulations which includes Binary Frequency

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Shift Keying (BFSK). Differential Phase Shift Keying (DPSK). Differential Quadrature Phase Shift Keying (DQPSK). Differential Offset Quadrature Phase Shift Keying (D-OQPSK), and Differential Minimum Shift Keying (DMSK). to name a few. The analysis is valid for the class of modulations having an exponentially decreasing bit error probability in additive white Gaussian noise. The results can be applied to any modulation since the Chernoff bound provides an exponentially tight upper bound to the bit error probability at sufficiently high signal-to-noise ratios.

2.1.3 The Jammer

The assumed jammer is a single partial-band white Gaussian noise jammer which can spread its total power. J. over the entire hopping band. $\{W\}$. or over just a subset $\{W_j\}$ of $\{W\}$. When the jamming signal covers completely $\{W\}$. the jamming noise power spectral density is equal to $N_j = J/W$. The bit energy-to-jamming noise power spectral density ratio. E_b/N_j. is represented by ρ. For $W_j \leq W$. the effective jamming noise power spectral density is equal to $N_j^{eff} = N_j/(W_j/W)$ over the jammed set of frequencies $\{W_j\}$. and 0 elsewhere.

In this report. the analysis is restricted to the set of jamming strategies described below.

The total jammer power. J. is maintained constant. i.e.. at a given power level. the jammer selects the most effective strategy within the class of strategies it can support. Note that the receiver noise is usually dominated by the jamming noise. and can be neglected.

The jammer is allowed to place its power over one or several frequency bands. When only one frequency band is jammed. the jammer selects the size and the position of that frequency band within the hopping bandwidth. $\{W\}$. When several frequency bands are jammed. the size and position of each one of these bands is selected by the jammer. and the jamming noise power spectral density is constant over the jammed frequency bands. Furthermore. the total jamming power is kept constant and equal to J.

These stategies. referred to as *single-power level partial-band white Gaussian noise jamming*. correspond to a single jammer with one transmitter and a large bank of narrow bandpass filters which can be configured to transmit a signal of constant total power in one or several frequency bands. The number of these frequency bands. the width of each one of them. and their position within $\{W\}$ are chosen by the jammer in order to provide the maximum disruption of the communication performance.

More effective jammers exist. which are not analyzed here. Multi-power level partial-band colored Gaussian noise jamming. in which the jammer spreads its power across several frequency bands. with. possibly. different power levels on each band. Even more general is the jammer which spreads its power across the jammed bandwidth according to a nonuniform spectral density. This jammer would provide more effective jamming strategies. In this report. the terms *optimal jammer* and *optimal jamming strategy* are used with the understanding that the optimality is only for the class of single-power level partial-band white Gaussian noise jamming strategies described above.

Finally, the jammer is assumed to have knowledge of the hopping bandwidth, of the shape and position of the user signal probability distribution across $\{W\}$, and its goal is to maximize the user bit error probability.

2.2 OPTIMAL JAMMING STRATEGY

Using Bayes rule, the probability of error P_u for a uniform distribution can be expressed as

$$P_u = Pr\{e \mid S \text{ is jammed}\}Pr\{S \text{ is jammed}\} \tag{2.2}$$

since there is no transmission error when the signal is not jammed.

The conditional probability that the signal is jammed, given ρ and W_j, is

$$Pr\{S \text{ is jammed} \mid \rho, W_j\} = \begin{cases} W_j/W_0 & \text{for } W_j < W_0 \\ 1 & \text{for } W_j \geq W_0. \end{cases} \tag{2.3}$$

Thus, the bit error probability $P_u(e \mid \rho, W_j)$, given ρ and W_j, is

$$P_u(e \mid \rho, W_j) = \begin{cases} (W_j/W_0)\, h(\rho\, W_j/W) & \text{for } W_j < W_0 \\ h(\rho\, W_j/W) & \text{for } W_j \geq W_0 \end{cases} \tag{2.4}$$

where $\rho = E_b/N_j$ is, by definition, the signal-to-jamming noise ratio.

The optimal jamming strategy is obtained by choosing the value of W_j which maximizes the bit error probability.

For an arbitrary function $h(\cdot)$, a simple calculation yields the optimal jamming strategy W_j^*

$$W_j^* = \begin{cases} W_0 & \text{for } \rho \leq x^* W/W_0 \\ x^* W/\rho & \text{for } \rho > x^* W/W_0 \end{cases} \tag{2.5}$$

where x^* is the solution of the differential equation $h(x) + x\,dh(x)/dx = 0$.

In the case where $h(x) = Ae^{-cx}$, the optimal strategy is given by the expression (2.5) with $x^* = c^{-1}$.

The optimal jamming strategy W_j^* is shown in Figure 2-2, as a function of signal-to-jamming noise ratio ρ. The optimal fractional bandwidth δ_j^* is equal to W_j^*/W. At low signal-to-noise ratios, the jammer has plenty of power to spread over the full hopping bandwidth W_0, and is more effective as a white noise jammer. At high signal-to-noise ratio, the jammer's best strategy is to concentrate its power on just a fraction of the hopping bandwidth, hitting the signal less frequently but more effectively than white noise jamming.

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2.3 PERFORMANCE UNDER OPTIMAL JAMMING

The bit error probability $P_u(e \mid \rho, W_j^*)$ under optimal jamming is obtained by substituting the optimal jamming strategy W_j^* given by (2.5) into Equation (2.4). This gives

$$P_u(e \mid \rho, W_j^*) = \begin{cases} h(\rho W_0/W) & \text{for } \rho \leq x^* W/W_0 \\ (x^* W/W_0)\, h(x^*)\, /\rho & \text{for } \rho > x^* W/W_0. \end{cases} \tag{2.6}$$

For the modulations of interest. $h(x) = Ae^{-cx}$, and Equation (2.4) is given by

$$P_u(e \mid \rho, W_j^*) = \begin{cases} h(\rho W_0/W) & \text{for } \rho \leq c^{-1} W/W_0 \\ (c^{-1} W/W_0)\, h(c^{-1})\, /\rho & \text{for } \rho > c^{-1} W/W_0. \end{cases} \tag{2.7}$$

The bit error probability under optimal jamming is shown on Figure 2-3 as a function of the signal-to-jamming noise ratio ρ. The bit error probability is a decreasing function of the signal-to-jamming noise ratio. The decrease is exponential at signal-to-jamming noise ratios less than W/W_0, and is hyperbolic at signal-to-jamming noise ratios greater than W/W_0.
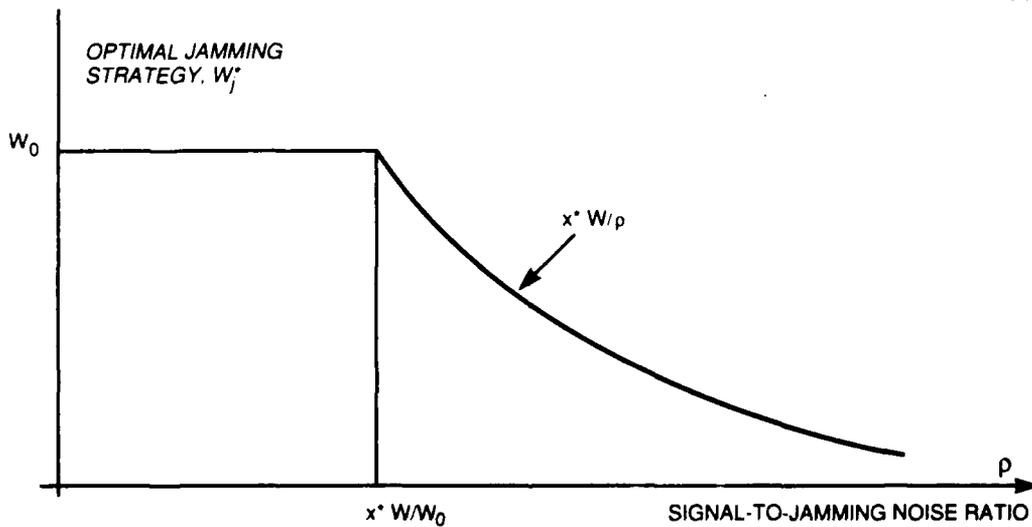

Figure 2-2. *Optimal jamming strategy for uniform hopping.*
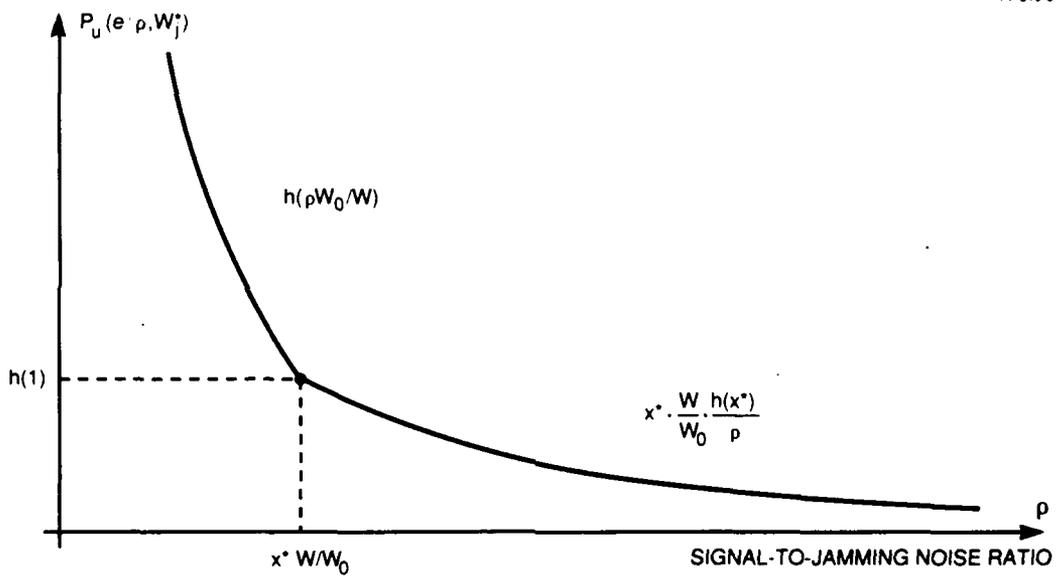
Figure 2-3. Worst-case performance for uniform hopping.

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3. NONUNIFORM CONTINUOUS FREQUENCY DISTRIBUTIONS

3.1 FREQUENCY HOPPING DISTRIBUTION

This section examines the case of arbitrary nonuniform continuous frequency distributions.

The frequency distribution of interest. $\alpha(f)$. is a real-valued function defined over a frequency set or *support* $\{W_0\} = \{f \mid \alpha(f) \geq 0\}$ contained within $\{W\}$. In this section. the function $\alpha(\cdot)$ is assumed to be continuous, with minimum $a_m \geq 0$ and maximum a_M.

The set $\{W_a\}$. defined as

$$\{W_a\} = \{f \mid \alpha(f) \geq a\} \quad for \quad a_m \leq a \leq a_M \tag{3.1}$$

represents the set of frequencies at which the probability density exceeds the value a. In particular, for $a = 0$. $\{W_a\} = \{W_0\}$.

Depending on the shape of the distribution $\alpha(\cdot)$. $\{W_a\}$ may or may not be a *connected* set[1] of frequencies. When $\{W_a\}$ is a connected set. there exists two frequencies f_j and f_j' such that

$$\{W_a\} = \{f \mid f_j' \leq f \leq f_j\}. \tag{3.2}$$

Let us define a real-valued function. $\Gamma(\cdot)$. as follows

$$\Gamma(a) = \mathcal{L}\{W_a\}. \quad for \ all \ a \ such \ that \ a_m \leq a \leq a_M \tag{3.3}$$

where $\mathcal{L}$ is the Lebesgue measure [2]. In other words. $\Gamma(a)$ represents the length or measure of the frequency set $\{W_a\}$ on which the hopping density equals or exceeds a. Figures 3-1 and 3-2 illustrate the definition of the function $\Gamma(\cdot)$ for connected and disconnected sets, respectively. It is assumed, throughout this section, that the function $\Gamma(\cdot)$ and its first derivative $\gamma(\cdot)$ exist on the interval $[a_m, a_M]$.

It results from the above definition that

$$\Gamma(0) = W_0 \tag{3.4}$$

$$\int_{a_m}^{a_M} \Gamma(a) da \;=\; 1. \tag{3.5}$$

Let us define *type-I distributions* as distributions such that

$$\mathcal{L}\{f \mid \alpha(f) = a_M\} = 0 \tag{3.6}$$

---

[1] A set is *connected* if any two of its points can be joined by a line completely contained in the set.

PROBABILITY
DISTRIBUTION
$\alpha(f)$

a_M

$\Gamma(a) = f_2 - f_1$
$\{W_a\} = \{f \mid f_1 \leq f \leq f_2\} = [f_1, f_2]$

a

$\Gamma(a)$

f

f_1 f_2

Figure 3-1.   Function Γ with connected sets.

PROBABILITY
DISTRIBUTION
$\alpha(f)$

$\Gamma(a) = f_2 - f_1 + f_4 - f_3$
$\{W_a\} = [f_1, f_2] \cup [f_3, f_4]$

a_M

a

$\Gamma(a)$

f

f_1 f_2 f_3 f_4
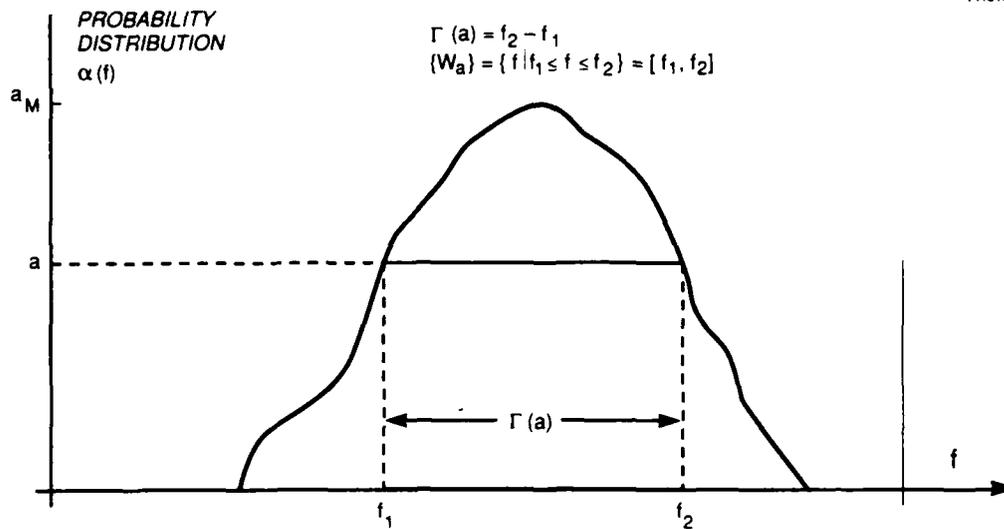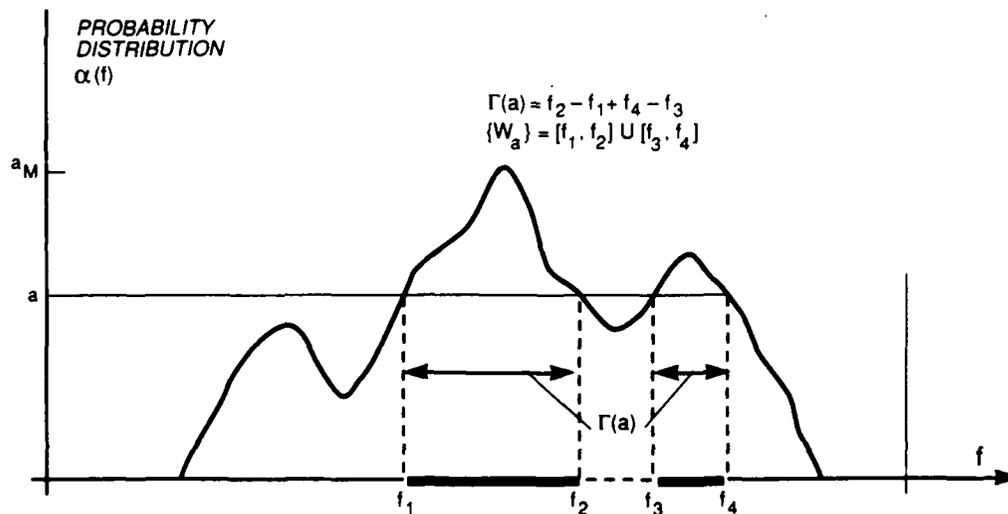
Figure 3-2.   Function Γ with disconnected sets.

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i.e., the peak value a_M is achieved over a frequency set of measure 0, e.g., a single frequency or a countable number of frequencies. Distributions which are not type-I are called *type-II distributions* are illustrated on Figures 3-3 and 3-4. With this definition, $\Gamma(a_M) = 0$ for a type-I distribution, and $\Gamma(a_M) > 0$ for a type-II distribution.
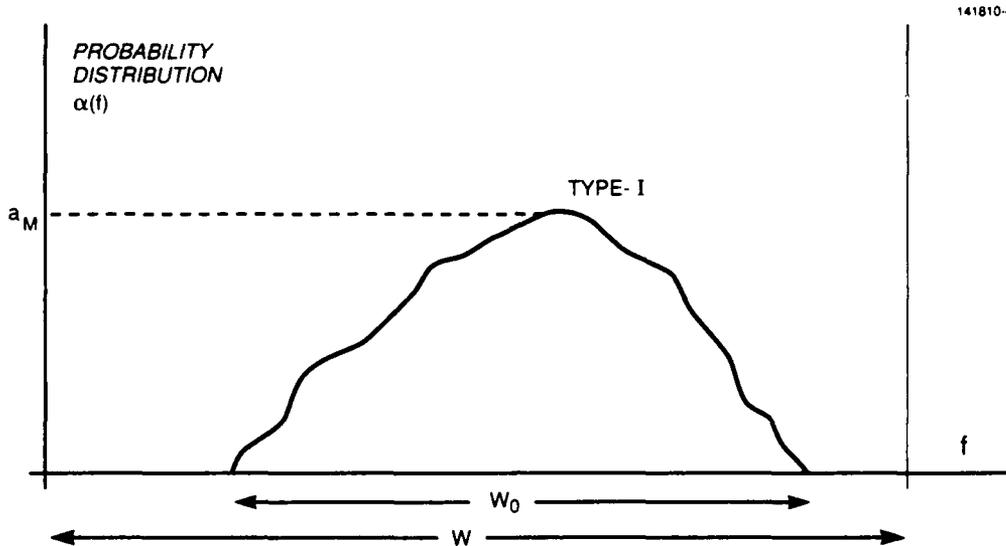
Figure 3-3.   Continuous hopping distributions of type-I.

3.2 OPTIMAL JAMMING STRATEGY

The following notation is used with respect to jammed frequency sets. A jammed frequency set of arbitrary size is represented as $\{W_j\}$, with the subscript j. The frequency set jammed by the optimal jammer is represented as $\{W_j^*\}$, where the $*$ denotes optimality as previously defined.

Consider the class of jammers of equal bandwidth $W_j \leq W_0$ and different jammed sets $\{W_j\} \subseteq \{W_0\}$. The jamming strategy $\{W_{a_j}\}$ such that $\Gamma(a_j) = W_j$ places the jamming power over the set $\{f \mid f \in \{W_{a_j}\}\} = \{f \mid \alpha(f) \geq a_j\}$, i.e., where the user signal is most likely to occur. Among all jammers of bandwidth W_j, this strategy maximizes the probability of jamming the signal. It can be shown, mathematically, that any departure from that choice results in a lower probability of jamming the signal. The probability of jamming the signal, given W_j, is equal to the integral of the frequency distribution over the jammed frequency set. The optimal jamming strategy is one which maximizes the bit error probability. It is shown, in Appendix A, that the optimal jamming strategy W_j^* is

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PROBABILITY
DISTRIBUTION
$\alpha(f)$

TYPE- II

a_M

W_0

W

f
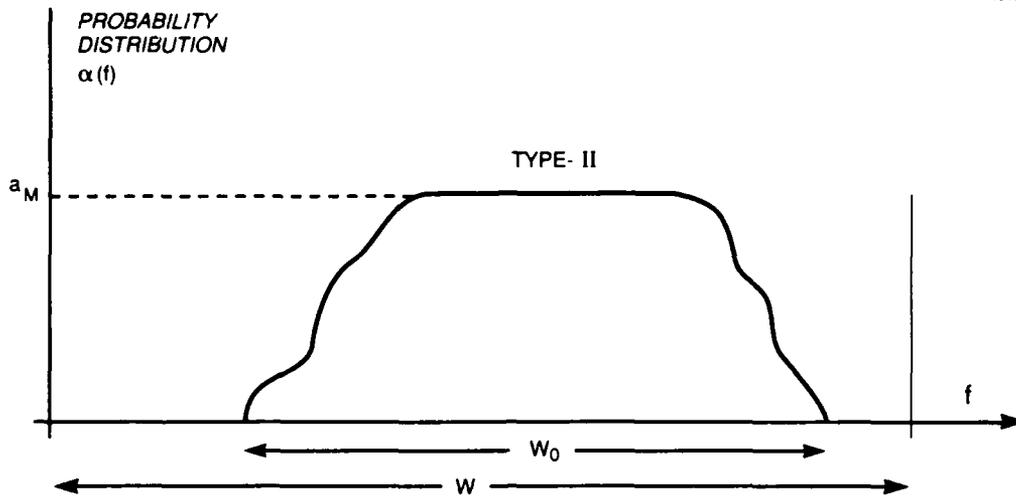
Figure 3-4.   Continuous hopping distributions of type-II.

- For type-I distributions,

$$\{W_j^*\} = \{W_{a_j^*}\} \quad \text{for all values of } \rho \tag{3.7}$$

- For type-II distributions,

$$\{W_j^*\} = \begin{cases} \{W_{a_j^*}\} & \text{for } \rho \le c^{-1}W/\Gamma(a_M) \\ \{c^{-1}W/\rho\} & \text{for } \rho > c^{-1}W/\Gamma(a_M) \end{cases} \tag{3.8}$$

where a_j^* is the implicit solution of the equation

$$\frac{W}{c\rho} = \Gamma(a_j^*) + \frac{1}{a_j^*} \int_{a_j^*}^{a_M} \Gamma(a)da \tag{3.9}$$

and $\{c^{-1}W/\rho\}$ is any subset of $\{W_{a_M}\}$, of size $c^{-1}W/\rho$.

The above equations indicate the size of the optimal jammer bandwidth and the exact portion of the hopping bandwidth where the jammer places its power. The variation of the optimal jammer bandwidth as a function of the signal-to-jamming noise ratio is illustrated on Figure 3-5. The variation of the optimal fractional bandwidth $\delta_j^* = W_j^*/W$ as a function of the signal-to-jamming noise ratio is illustrated on Figure 3-6.   A geometric interpretation of the optimal jamming strategy is given in Section 3.5.

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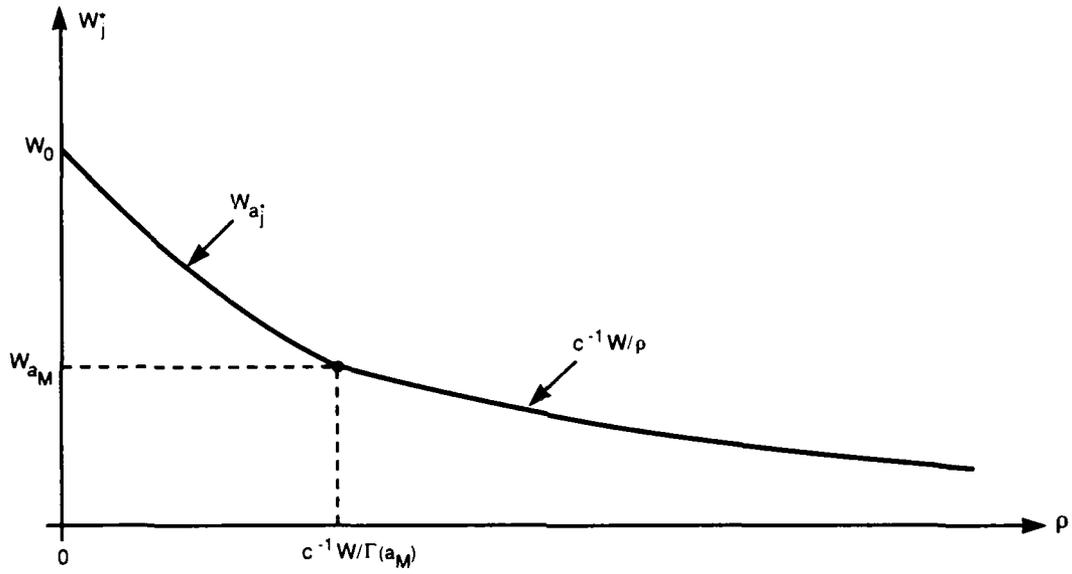
Figure 3-5.   Optimal jamming strategy W_j^* for nonuniform hopping.
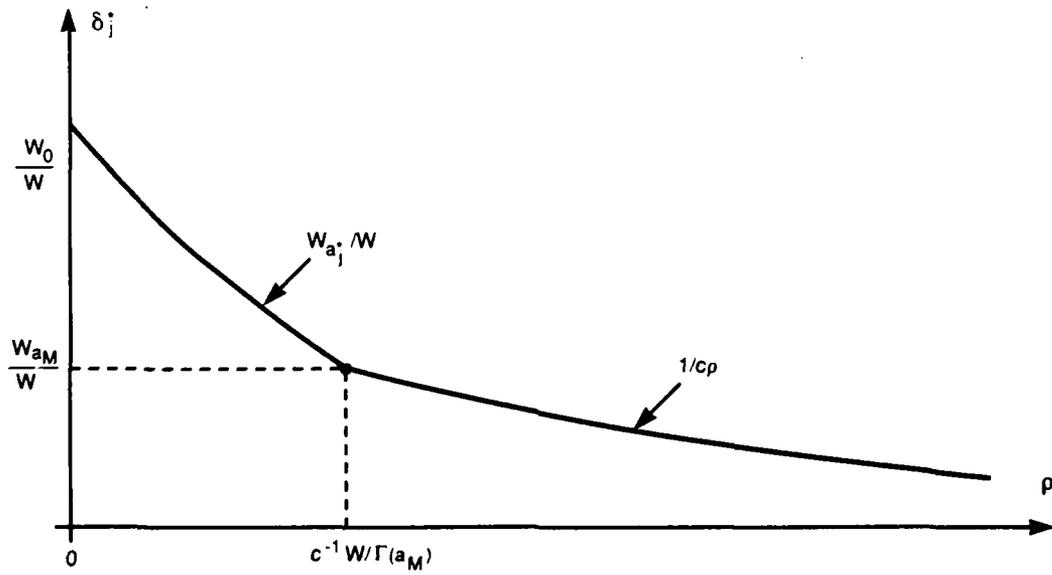
Figure 3-6.   Optimal fractional bandwidth δ_j^* for nonuniform hopping.

3.3 PERFORMANCE UNDER OPTIMAL JAMMING

Equation (A.4) gives the bit error probability for a jammer of arbitrary width W_j. The jamming strategy which maximizes the bit error probability is given by Equations (3.7) and (3.8). Substituting W_j in Equation (A.4) for the optimal jamming strategy W_j^* yields the worst-case performance for type-*I* and type-*II* distributions. This yields the basic expressions for the bit error probability under optimal jamming

$$P_n(e \mid \rho, W_j^*) = [a_j^* W_j^* + \int_{a_j^*}^{a_M} \Gamma(a)da]h(\frac{\rho W_j^*}{W}) \tag{3.10}$$

$$= a_j^* \frac{W}{c\rho} h(\frac{\rho W_j^*}{W}) \tag{3.11}$$

$$= a_j^* \frac{W}{c\rho} h(c^{-1} - \frac{\rho}{a_j^* W} \int_{a_j^*}^{a_M} \Gamma(a)da) \tag{3.12}$$

where a_j^* satisfies Equation (3.9). It is easy to check that the above expressions are equivalent.

In summary, the worst-case bit error probability is

- For type-*I* distributions.

$$P_n(e \mid \rho, W_j^*) = a_j^* \frac{W}{c\rho} h(\frac{\rho W_j^*}{W}) \quad \text{for } \rho > 0 \tag{3.13}$$

where a_j^* satisfies

$$\frac{W}{c\rho} = \Gamma(a_j^*) + \frac{1}{a_j^*} \int_{a_j^*}^{a_M} \Gamma(a)da \tag{3.14}$$

- For type-*II* distributions,

$$P_n(e \mid \rho, W_j^*) = \begin{cases} a_j^* c^{-1} \frac{W}{\rho} h(\frac{\rho W_j^*}{W}) & \text{for } \rho \le c^{-1}W/\Gamma(a_M) \\ \\ a_M c^{-1} \frac{W}{\rho} h(c^{-1}) & \text{for } \rho > c^{-1}W/\Gamma(a_M) \end{cases} \tag{3.15}$$

where a_j^* satisfies Equation (3.9), $\{W_j^*\} = \{W_{a_j^*}\}$, and $W_j^* = \Gamma(a_j^*)$.

The general form of the probability of error as a function of signal-to-jamming noise ratio is shown in Figures 3-7 and 3-8 for type-*I* and type-*II* distributions, respectively.

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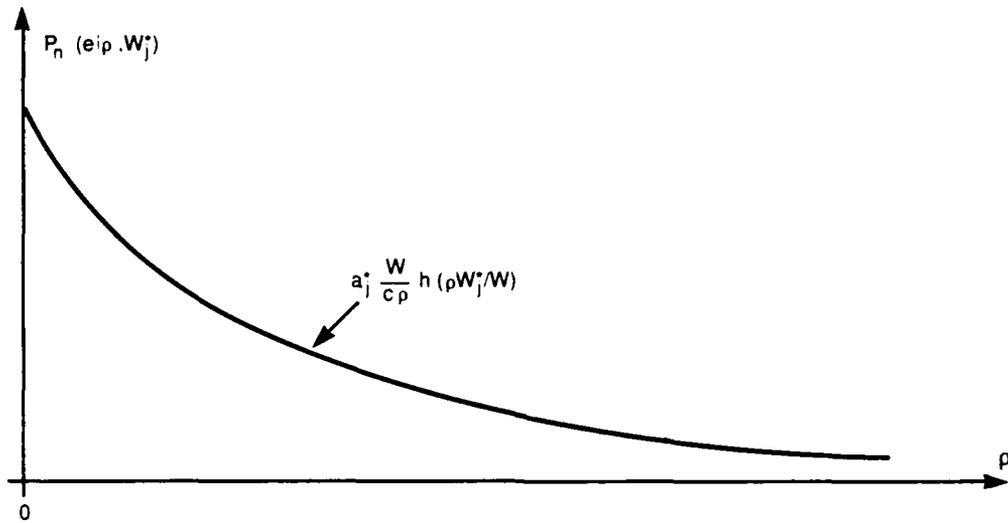
Figure 3-7. *Performance under optimum jamming for type-I distributions.*
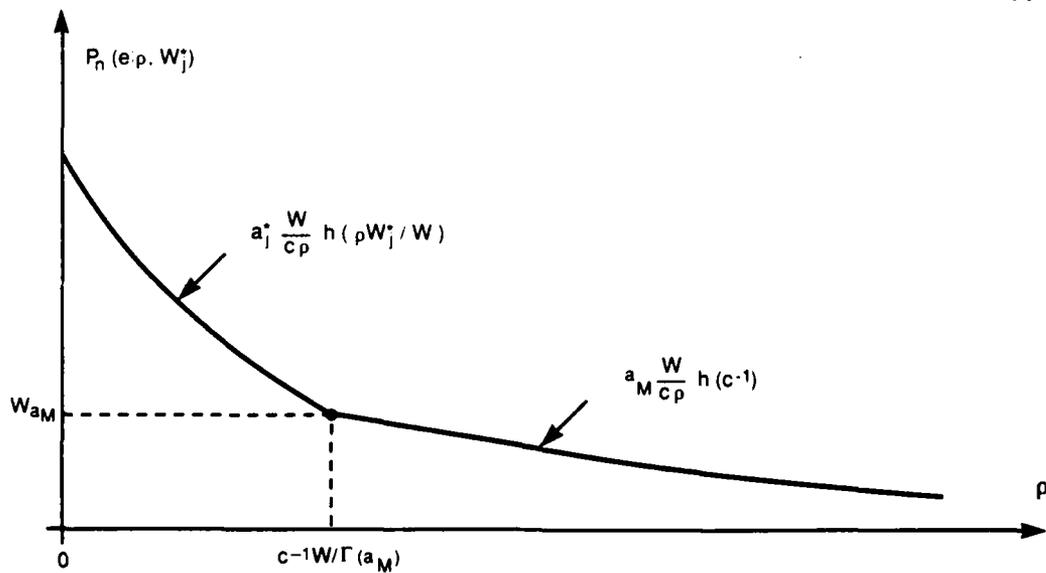
*Figure 3-8.* *Performance under optimum jamming for type-II distributions.*

3.4 PERFORMANCE COMPARISON

3.4.1 Performance at High Signal-to-Jamming Noise Ratio

We now compare the performances, at high signal-to-jamming noise ratios, of a signal hopping according to a nonuniform distribution and facing its optimal jammer $\{W^*_{j,n}\}$, to the performance of the same signal hopping uniformly and facing its optimal jammer $\{W^*_{j,u}\}$. It is shown in Appendix B that the ratio of the nonuniform bit error probability to the uniform bit error probability at the same signal-to-jamming noise ratio ρ is equal to or approaches $a_M/a_{0,u}$, where $a_{0,u} = 1/W_{0,u}$ represents the value of the distribution for uniform frequency-hopping. Specifically,

- For type-I distributions,

$$\frac{P_n(e \mid \rho, W^*_{j,n})}{P_u(e \mid \rho, W^*_{j,u})} \xrightarrow{\rho \to \infty} \frac{a_M}{a_{0,u}} \tag{3.16}$$

- For type-II distributions,

$$\frac{P_n(e \mid \rho, W^*_{j,n})}{P_u(e \mid \rho, W^*_{j,u})} = \frac{a_M}{a_{0,u}} \quad as \ \rho \geq \max(c^{-1}W/W_{0,u}, c^{-1}W/\Gamma(a_M)) \tag{3.17}$$

For both types of distributions, the performance for nonuniform frequency-hopping is worse than that achieved for uniform frequency-hopping at high signal-to-jamming noise ratio. The corresponding increase in bit error probability is by a factor of $a_M/a_{0,u}$. Since the bit error probability increases by a factor of $a_M/a_{0,u}$, the effective signal-to-noise ratio decreases by a corresponding factor. We now prove that, for a same performance level, the loss in signal-to- jamming noise ratio is also equal to $a_M/a_{0,u}$.

The ratio of bit error probabilities is given by Equations (3.16) and (3.17). Let ρ_u and ρ_n be the signal-to-jamming noise ratios at which the uniform and the nonuniform hopping plans achieve the same bit error probability. Since the bit error probabilities are inversely proportional to ρ at large values of ρ, the ratio ρ_n/ρ_u is

$$\frac{\rho_n}{\rho_u} = a_M/a_{0,u}. \tag{3.18}$$

In other words, the signal-to-noise ratio loss is equal to $a_M/a_{0,u}$ for type-I distributions, and asymptotically equal to $a_M/a_{0,u}$ for type-II distributions. It should be pointed out that this last property results from the exponential form of the function $h(.)$ assumed in Equation (2.1).

From the above results, several observations can be made concerning the performance loss due to nonuniform hopping.

At high signal-to-jamming noise ratios, the loss can be very large if the ratio $a_M/a_{0,u}$ is very large. Thus, a frequency distribution with a high narrow peak or a high narrow plateau will generate a large signal-to-noise ratio loss, and this loss will only show up at high operating points, i.e., at

high ρ. This is generally the case if the set $\{\Gamma(a_M)\}$ is small. e.g.. if it is a small percentage of W_0, and if a_M is clearly larger than the rest of the distribution, i.e.. if the plateau or the peak(s) are pronounced. This would be the case of a distribution having a large number of very narrow and tall peaks (resembling a large number of Dirac impulses) with a small total measure, even if these peaks are scattered uniformly over the available band. A scheme which could be used to generate such a large set of Dirac impulses could also be used by the jammer to jam precisely the same set of frequencies.

Similarly. a frequency distribution with a low wide peak, or a low wide plateau, or a large number of low peaks of large total measure [i.e.. $\Gamma(a_M)$ large] will generate a moderate SNR loss. and this loss will show up at lower operating points.

3.4.2   Performance at Low Signal-to-Jamming Noise Ratio

In a similar fashion. it is shown in Appendix C that the ratio of the nonuniform bit error probability to the uniform bit error probability satisfies

$$\frac{P_n(e \mid \rho W_{j,n}^*)}{P_u(e \mid \rho. W_{j,u}^*)} \xrightarrow{\rho \to 0} e^{\left[-\frac{\rho}{W}(W_{o,n}-W_{o,u})\right]} \tag{3.19}$$

where $\{W_{0,n}\}$ and $\{W_{0,u}\}$ represent the support sets of the nonuniform and the uniform frequency distributions. respectively. In the case where both distributions have the same support set, the limit is 1

$$\frac{P_n(e \mid \rho \mid W_{j,n}^*)}{P_u(e \mid \rho. W_{j,u}^*)} \xrightarrow{\rho \to 0} 1 \tag{3.20}$$

i.e.. as the signal-to-jamming noise ratio ρ tends to 0. the performance for uniform and nonuniform frequency-hopping are asymptotically equal.

The above analysis leads to the following observation. In a frequency-hopped communication system. it is preferable to use a uniform frequency-hopping scheme if one wants to minimize the bit error probability in a jammed environment. If a uniform distribution is not possible due to constraints placed on the user signal hopping band. then the distribution should be as uniform as possible, i.e.. the largest peak a_M in the distribution should be kept as close as possible to a_0. A frequency-hopping scheme with a number of small peaks in the probability distribution is preferable to one having even one single larger peak. Such a choice will result in a reduced and small performance loss when compared to the uniform case. The resulting increase in bit error probability is $10 Log_{10}(a_M/a_0)$ dB.

3.5   GEOMETRIC INTERPRETATION

It is possible to give a geometric interpretation to the optimal jamming strategy and to the worst-case bit error probability.

3.5.1 Geometric Interpretation of the Optimal Jamming Strategy

As shown in the previous section, the bandwidth of the optimal jamming strategy W_j^{*} is given by

$$W_j^{*} = \Gamma(a_j^{*}) \tag{3.21}$$

where a_j^{*} satisfies

$$\Gamma(a_j^{*}) = \frac{W}{c\rho} - \frac{1}{a_j^{*}} \int_{a_j^{*}}^{a_M} \Gamma(a)da. \tag{3.22}$$

Thus

$$a_j^{*}\frac{W}{c\rho} = a_j^{*}W_j^{*} + \int_{a_j^{*}}^{a_M} \Gamma(a)da \tag{3.23}$$

$$= \int_{f_j}^{f_j'} \alpha(f)df \tag{3.24}$$

where $\alpha(f_j) = \alpha(f_j') = a_j^{*}$, since the right-hand terms are two equivalent representations of the probability $Pr\{f_j \leq f \leq f_j'\}$.

Let us define

$$\mathcal{A}(\rho) = \int_{f_j}^{f_j'} \alpha(f)df \tag{3.25}$$

as the area under the distribution $\alpha(\cdot)$ between f_j and f_j'.

Similarly, let

$$\mathcal{B}(\rho) = a_j^{*} \, c^{-1}\frac{W}{\rho} \tag{3.26}$$

be the area of the rectangle of width $c^{-1}W/\rho$, and $\mathcal{B}_M(\rho) = a_M c^{-1}W/\rho$. These areas are shown as cross-hatched regions on Figures 3-9 and 3-10. The worst-case jamming strategy is determined by the following geometric conditions:

• For type-I distributions, the optimal jammer places its energy on the set $\{W_j^{*}\} = \{\Gamma(a_j^{*})\}$ where a_j^{*} is such that

$$\mathcal{A}(\rho) = \mathcal{B}(\rho) \quad \text{for all values of } \rho \tag{3.27}$$

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- For type-II distributions.

If $\rho \leq c^{-1}W/\Gamma(a_M)$. the optimal jammer places its energy on the set $\{W_j^*\} = \{\Gamma(a_j^*)\}$ where a_j^* is such that $\mathcal{A}(\rho) = \mathcal{B}(\rho)$:

If $\rho > c^{-1}W/\Gamma(a_M)$. the optimal jammer places its energy on any subset of width $\mathcal{B}_M(\rho)$ contained within $\{\Gamma(a_M)\}$.

With the above interpretation. for a given signal-to-jamming noise ratio. a_j^* is that value which provides equal shaded areas $\mathcal{A}$ and $\mathcal{B}$. The optimal jamming strategy is $\{W_j^*\} = \{W_{a_j^*}\}$, and the bandwidth of the jammed set is equal to $W_j^* = \Gamma(a_j^*)$. as shown on Figures 3-9 and 3-10. The vase filling analogy can be helpful to visualize the condition which determines the optimal jamming strategy as a function of the signal-to-jamming noise ratio.
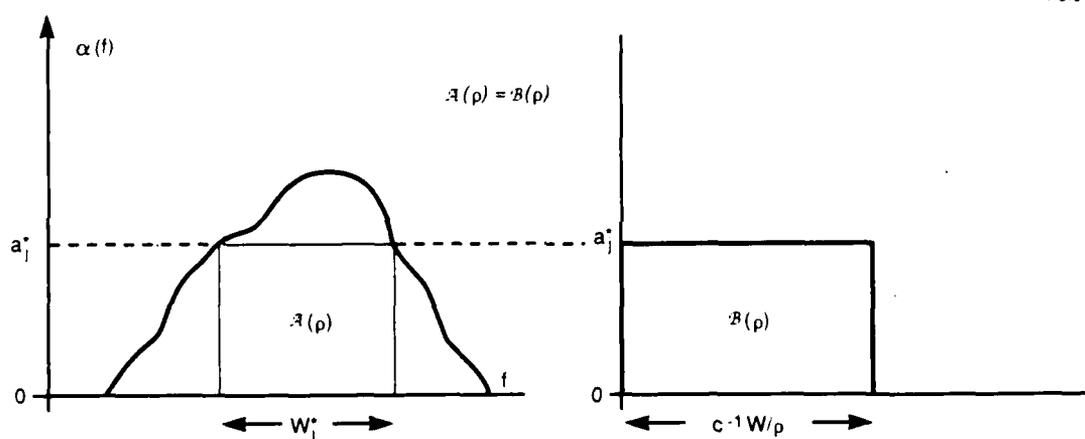
Figure 3-9. *Geometric interpretation of the optimal jamming strategy for type-I distributions.*

3.5.2 Geometric Interpretation of the Worst-Case Performance

- For type-I distributions.

It results from Equation (3.13) that the probability of error under optimal jamming equals

$\rho \le c^{-1} W / \Gamma \, (a_M)$

$\rho > c^{-1} W / \Gamma \, (a_M)$
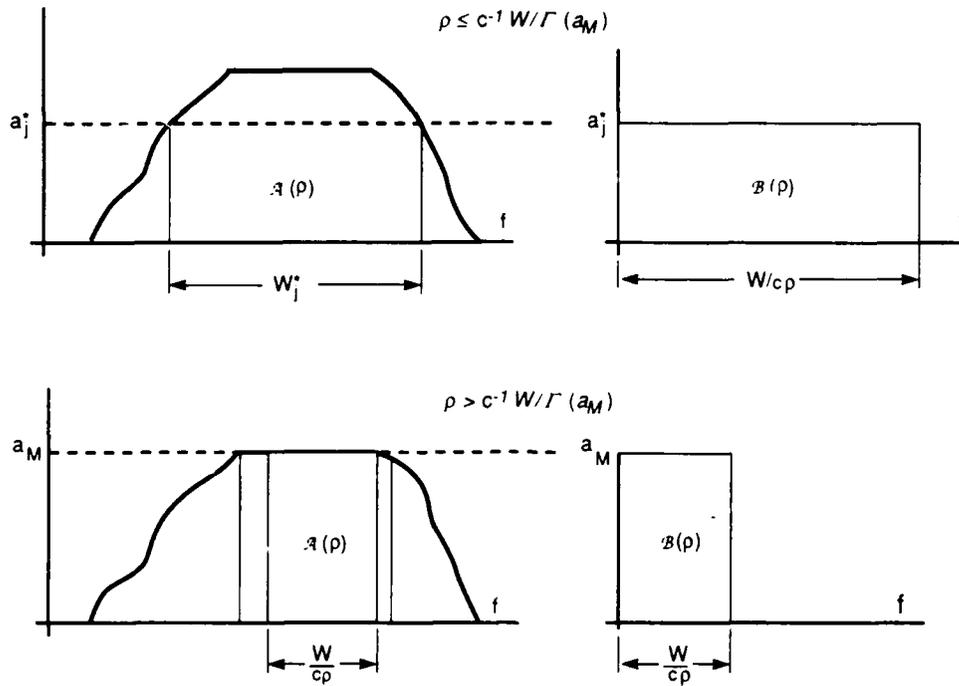
Figure 3-10. Geometric interpretation of the optimal jamming strategy for type-II distributions.

$$P_n(e \mid \rho, W_j^*) = \mathcal{B}(\rho) h \left(c^{-1} \frac{\mathcal{A}_0(\rho)}{\mathcal{B}(\rho)} \right) \tag{3.28}$$

where $\mathcal{B}(\rho)$ was defined above, and $\mathcal{A}_0(\rho) = a_j^* W_j^*$ is the cross-hatched area shown on Figure 3-11. With this geometric interpretation, it is easy to see how the optimal jammer changes as a function of ρ. In particular, as $\rho \to \infty$, $a_j^* \to a_M$, $\mathcal{A}_0(\rho)/\mathcal{B}(\rho) \to 1$, and $P_n(e \mid \rho, W_j^*) \to a_M c^{-1} W h(c^{-1})/\rho$, as expected. Also, as $\rho \to 0$, $a_j^* \to 0$, $\mathcal{B}(\rho) \to 1$, and $P_n(e \mid \rho, W_j^*) \to h(\rho W_0/W)$.

- For type-II distributions.

The bit error probability follows from Equation (3.15)

$$P_n(e \mid \rho, W_j^*) = \begin{cases} \mathcal{B}(\rho) h(c^{-1} \frac{\mathcal{A}_0(\rho)}{\mathcal{B}(\rho)}) & \text{for } \rho \le c^{-1} W / \Gamma(a_M) \, , \\[2ex] \mathcal{B}_M(\rho) h(c^{-1}) & \text{for } \rho > c^{-1} W / \Gamma(a_M). \end{cases} \tag{3.29}$$

20
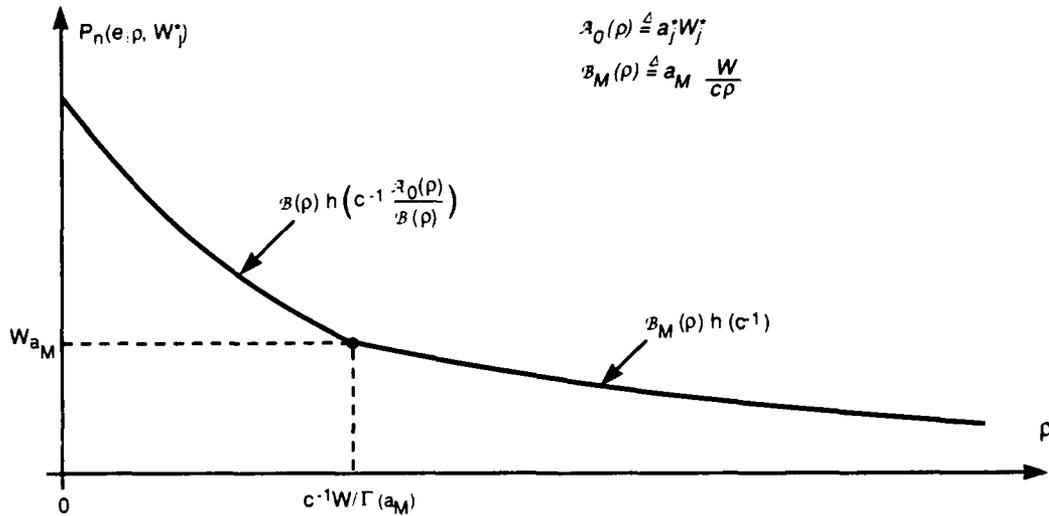
Figure 3-11. *Geometric interpretation of the worst-case performance.*

3.6 EXAMPLE

To illustrate the previous equations. we consider the case of a triangular distribution of type-I, as shown on Figure 3-12. The support set is W_0. and the peak equals $2a_0$. with $a_0 = 1/W_0$. The function Γ is easily seen to be

$$\Gamma(a) = \frac{2a_0 - a}{2a_0} W_0. \tag{3.30}$$

Using Equation (3.23), the optimal jamming strategy is such that

$$a_j^* \frac{W}{c\rho} = a_j^* W_j^* + \frac{W_j^*}{2}(2a_0 - a_j^*) \tag{3.31}$$

or, equivalently,

$$a_j^* = 2a_0 \left[\sqrt{1 + (\frac{W}{c\rho W_0})^2} - (\frac{W}{c\rho W_0}) \right]. \tag{3.32}$$

Consequently, the optimal jamming strategy is

$$W_j^* = W_0 \left[1 + \frac{W}{c\rho W_0} - \sqrt{1 + (\frac{W}{c\rho W_0})^2} \right] \tag{3.33}$$

21
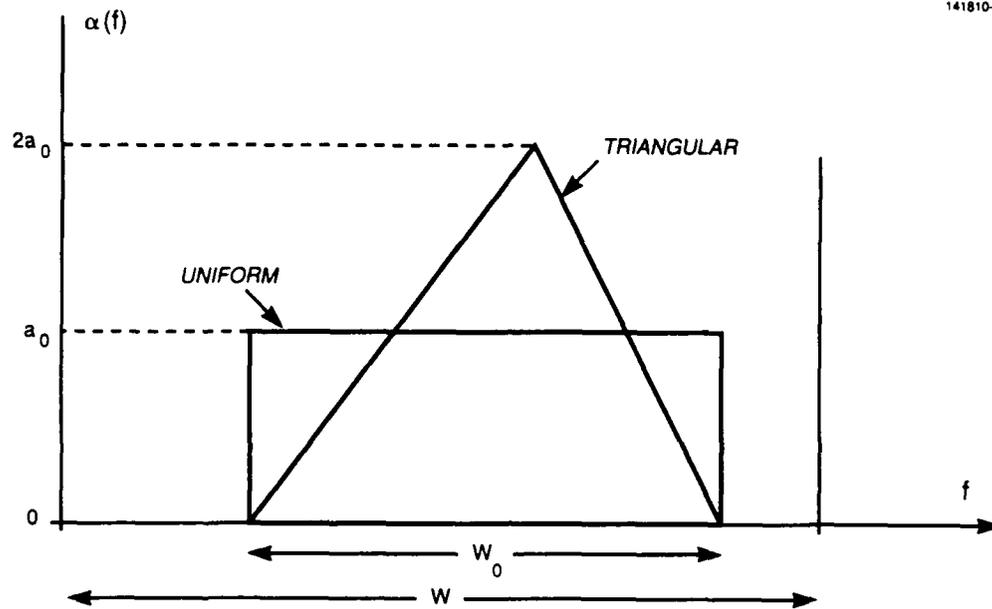
Figure 3-12. Example of triangular distribution.

and from Equation (3.13), the worst-case bit error probability is

$$P_n(e \mid \rho, W_j^*) = \frac{2a_0 W^*}{c\rho} \left[\sqrt{1 + (\frac{W^*}{c\rho W_0})^2} - (\frac{W^*}{c\rho W_0}) \right] A e^{-\frac{c\rho W_0}{W^*} \left[1 - \frac{W^*}{c\rho W_0} - \sqrt{1 + (\frac{W^*}{c\rho W_0})^2} \right]}. \quad (3.34)$$

Figures 3-13, 3-14 and 3-15 represent the variations of the optimal jammer bandwidth, the optimal fractional bandwidth and the worst-case bit error rate, respectively, as a function of ρ. Notice that, at high signal-to-jamming noise ratios, the performance loss is equal to 3 dB when compared to the uniform distribution.

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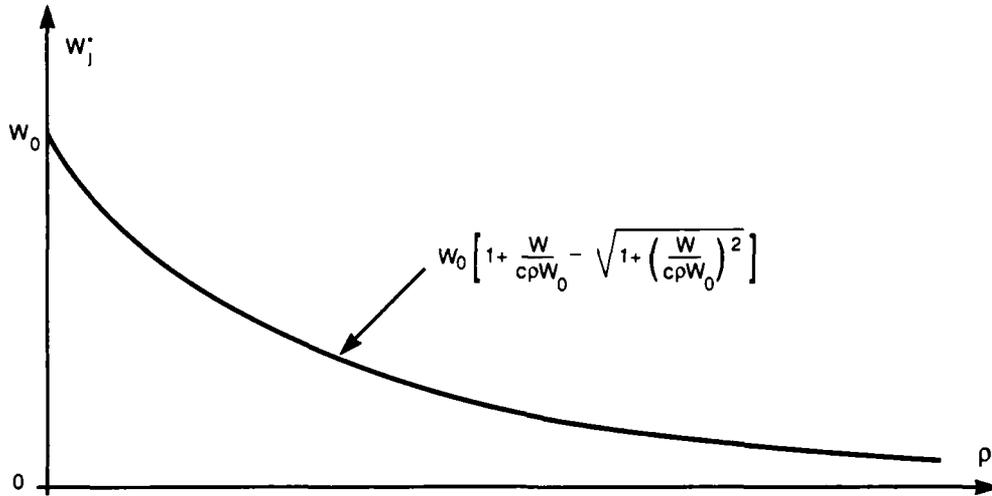
Figure 3-13.   Optimal jammer bandwidth for a triangular distribution.
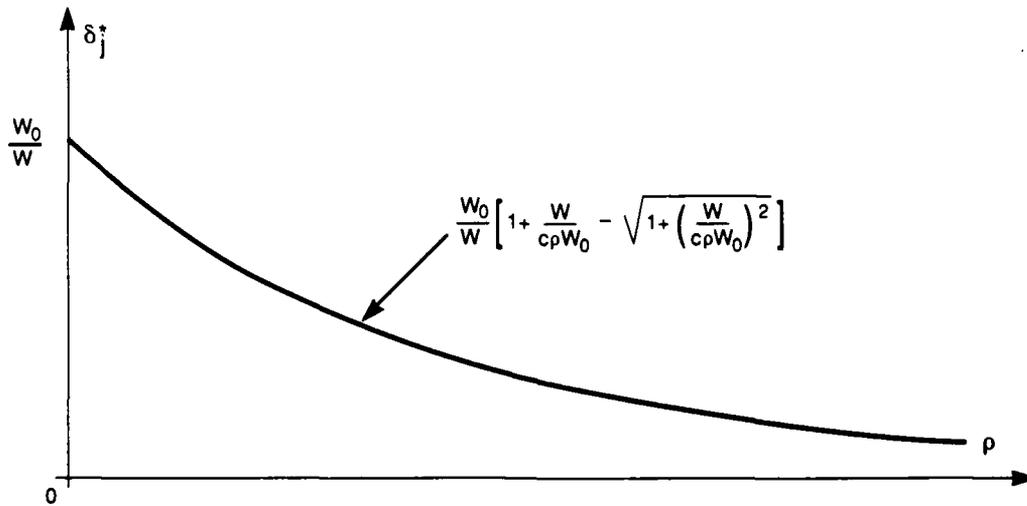
Figure 3-14.   Optimal fractional bandwidth for a triangular distribution.

23
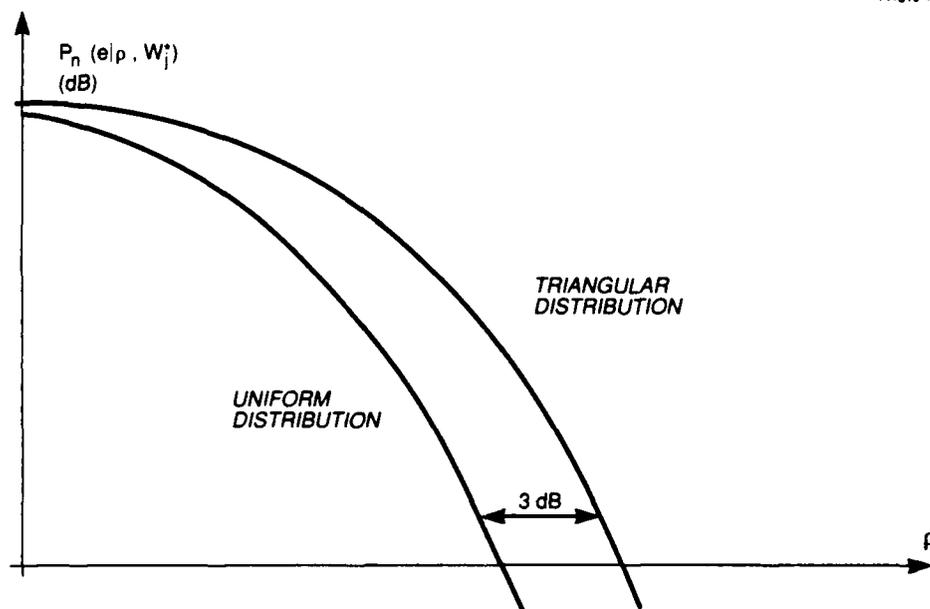
Figure 3-15.   Worst-case bit error rate for a triangular distribution.

4. DISCRETE FREQUENCY DISTRIBUTIONS

4.1 FREQUENCY HOPPING DISTRIBUTION

The frequency distributions considered in this section are discrete distributions which can best be described as *staircase-like*. The analysis for discrete distributions having continuous sections and staircase-like discontinuities can be carried out in a similar fashion using the results derived in this section and in the previous section. The focus in this section is on staircase-like distributions.

Staircase-like distributions take on a finite set of values, a_k, between 0 and a_M. A representative distribution from this class is shown in Figure 4-1. Specifically, the frequency distribution $\alpha(.)$ is
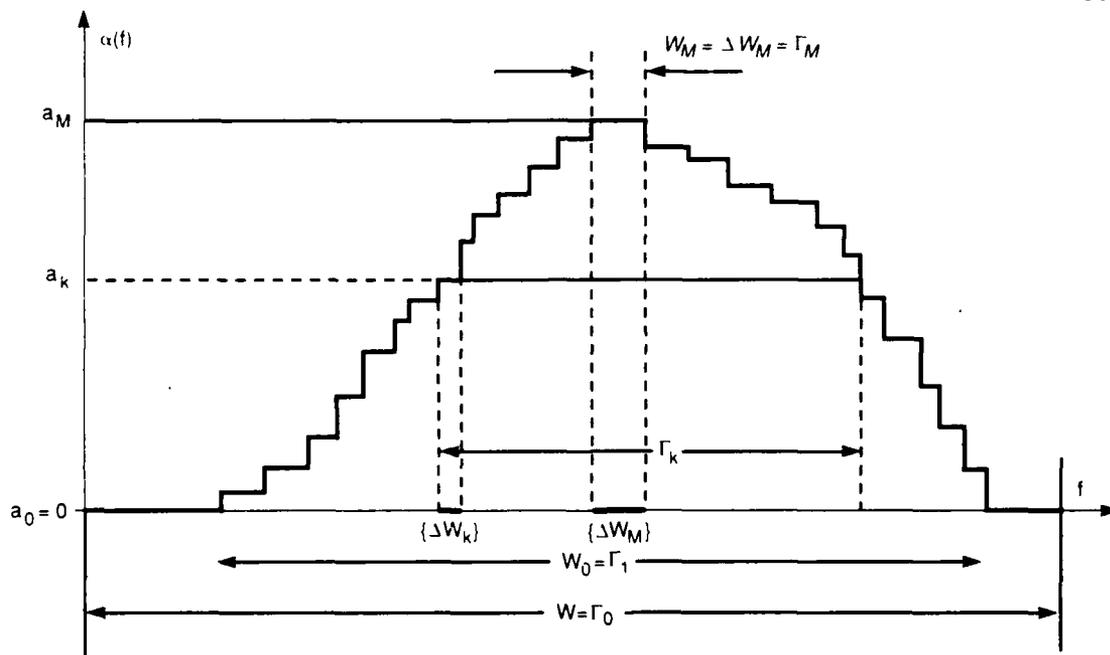
Figure 4-1. Staircase-like discrete distribution.

defined as

$$\begin{cases} \alpha(f) & = & a_k, & for\ f \in \{\Delta W_k\}, & (k = 0, \ldots, M) \\ a_0 & = & 0 \end{cases} \qquad (4.1)$$

where $\{\{\Delta W_k\}; k = 0, \ldots, M + 1\}$ is a collection of finite non-overlapping sets realizing a partition of $\{W_0\}$ such that[1]

---

[1] The symbols $\bigcup$ and $\emptyset$ represent the union of sets and the empty set, respectively.

$$\begin{cases} \bigcup_{k=0}^{M}\{\Delta W_k\} & = & \{W_0\} \\ \{\Delta W_0\} & = & \{\Delta W_{M-1}\} & = & \emptyset. \end{cases} \tag{4.2}$$

The trivial definitions of $\{\Delta W_0\}$ and $\{\Delta W_{M-1}\}$ are introduced because they simplify the formulation of the general solution. As was the case for continuous distributions, the notation with braces $\{\Delta W\}$ represents a set, and ΔW without braces denotes the size or Lebesgue measure [2] of that set.

Since $\alpha(.)$ is a probability density, it also satisfies

$$\sum_{k=1}^{M} a_k \Delta W_k = 1. \tag{4.3}$$

Let us introduce the scalar quantities p_k

$$\begin{cases} p_k = \sum_{i=k}^{M} a_i \Delta W_i, & \text{for } k = 1, \dots, M. \\ p_0 = p_1 = 1, \ p_{M-1} = 0. \end{cases} \tag{4.4}$$

The quantities p_k represent the cumulative probabilities starting from the peak of the distribution.

Finally, the function $\Gamma(.)$ associates the set $\{\Gamma_k\}$ to the scalar a_k as follows

$$\begin{cases} \{\Gamma_k\} = \{f \mid a(f) \geq a_k\} = \bigcup_{i=k}^{M}\{\Delta W_i\}. & \text{for } k = 1, \dots, M. \\ \{\Gamma_0\} = \{W\}, \ \ \{\Gamma_1\} = \{W_0\}, \ \ \{\Gamma_{M-1}\} = \emptyset. \end{cases} \tag{4.5}$$

The function $\Gamma(.)$ is similar to that used for continuous distributions, except that the simpler notation $\{\Gamma_k\}$ is used instead of the previously used notations $\{W_{a_k}\}$ or $\Gamma(a_k)$. As indicated, Γ_k represents the measure of the set $\{\Gamma_k\}$.

The above definition results in the following sequence of inclusions

$$\{W\} = \{\Gamma_0\} \supseteq \{W_0\} = \{\Gamma_1\} \supseteq \dots \supseteq \{\Gamma_M\} = \{\Delta W_M\} \supseteq \{\Gamma_{M-1}\} = \emptyset, \tag{4.6}$$

and their measures satisfy

$$\begin{cases} \Gamma_k = \sum_{k}^{M} \Delta W_i, & \text{for } k = 1, \dots, M. \\ 0 < \Gamma_M \leq \dots \leq \Gamma_k \leq \dots \leq \Gamma_1 = W_0 \end{cases} \tag{4.7}$$

It should be observed that the definition of the sets $\{\Delta W_k\}$ and $\{\Gamma_k\}$ applies to connected as well as non-connected sets.

4.2 OPTIMAL JAMMING STRATEGY

Jammers of the same width. W_j, placing their power in different subbands of $\{W\}$ do not have, in general, the same effectiveness. Maximum effectiveness is achieved when the jammer $\{W_j\}$ positions its energy around the peak of the distribution so as to maximize the probability of jamming the signal. This is possible since the jammer knows the distribution's shape and position, as was previously assumed. For instance, if $W_j \leq \Delta W_M$, then any strategy such that $\{W_j\} \subset \{\Delta W_M\}$ is optimal among strategies of width W_j.

In what follows, the jammer is assumed to always position its energy around the distribution peak so as to maximize the probability of jamming the signal. The remaining optimization left to the jammer is to select its width, W_j, in order to maximize the bit error probability at a given signal-to-jamming noise ratio.

The probability of jamming the signal, conditioned on W_j, is

$$Pr\{S \text{ is jammed} \mid W_j\} = p_{k-1} + a_k(W_j - \Gamma_{k-1})\tag{4.8}$$

for $\{\Gamma_{k-1}\} \subseteq \{W_j\} \subseteq \{\Gamma_k\}$. and $k = 0,\dots,M$.

Consequently, the conditional bit error probability $Q_n(e \mid \rho, W_j)$ is

$$Q_n(e \mid \rho, W_j) = [p_{k-1} + a_k(W_j - \Gamma_{k-1})]\, h(\rho \frac{W_j}{W}).\tag{4.9}$$

It is shown, in Appendix D, that the optimal jamming strategy $\{W_j^*\}$ is given by the expression

$$\{W_j^*\} = \begin{cases} \{W_0\}, & for\ \rho \leq \rho_1 \\ \vdots \\ \vdots \\ \{W_{j,k}\}, & for\ \rho_k \leq \rho \leq \rho_k' & (k = 1,\dots,M-1) \\ \{\Gamma_{k-1}\}, & for\ \rho_k' \leq \rho \leq \rho_{k-1} & (k = 1,\dots,M-1) \\ \vdots \\ \vdots \\ \{W_{j,M}\}, & for\ \rho \geq \rho_M \end{cases}\tag{4.10}$$

where $\{W_{j,k}\}$ is any frequency set such that

$$\{\Gamma_{k-1}\} \subseteq \{W_{j,k}\} \subseteq \{\Gamma_k\}$$

$$W_{j,k} = \frac{W}{c\rho} + \Gamma_{k-1} - \frac{p_{k-1}}{a_k}\tag{4.11}$$

and ρ_k, ρ_k' are defined as

$$\rho_k = c^{-1} \frac{a_k W}{p_{k-1} + a_k(\Gamma_k - \Delta W_{k-1})} \tag{4.12}$$

$$\rho_k' = c^{-1} \frac{a_k W}{p_{k-1} + a_k \Gamma_{k-2}.} \tag{4.13}$$

Figure 4-2 illustrates the variation of W_j^* as a function of the signal-to-jamming noise ratio. Figure 4-3 illustrates the variation of the optimal fractional bandwidth $\delta_j^* = W_j^*/W$ as a function of the signal-to-jamming noise ratio. As ρ increases past ρ_k', W_j^* decreases to Γ_{k+1}, and stays at that value for $\rho_k' \leq \rho \leq \rho_{k-1}$. This behavior then repeats itself, with k replaced with $k + 1$. The curve consists of alternating flat plateaus and hyperbolic sections. At very low signal-to-jamming noise ratio, the jammer has enough power to jam the whole distribution, i.e., $W_j^* = W_0$. As ρ gets larger, the optimal jammer only jams a fraction of W_0, and uses its power more efficiently by jamming the frequency subbands most frequently visited by the user signal. As the signal-to-jamming noise ratio gets very large, the optimal jammer is eventually forced to place all its power over a narrower frequency subband of $\{\Delta W_M\}$ over which the probability of hitting the user signal is largest.
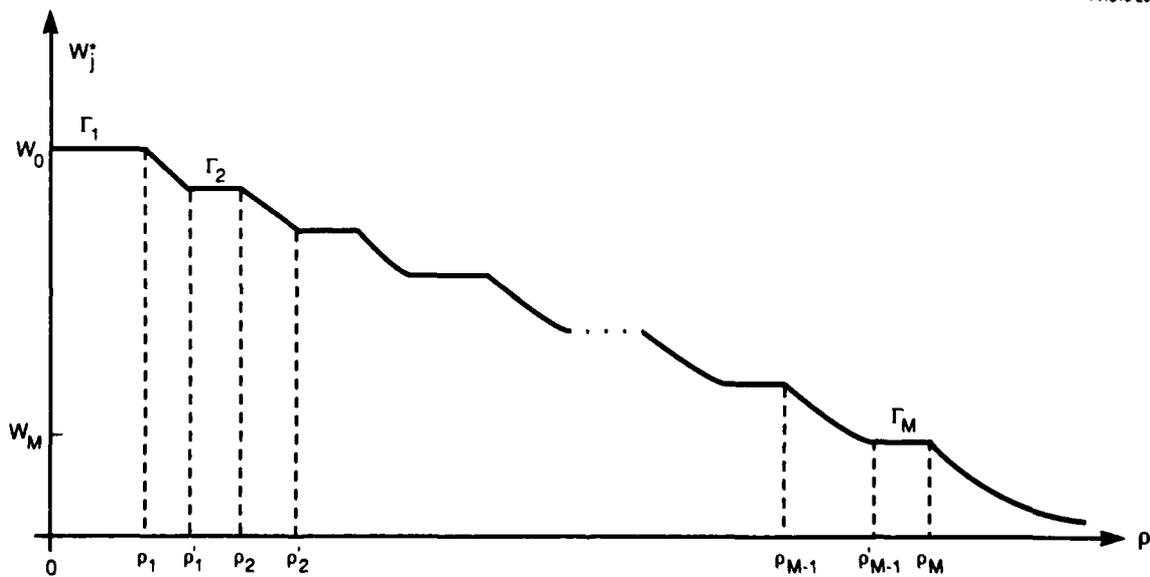
Figure 4-2. Optimal jamming strategy versus signal-to-jamming noise ratio.
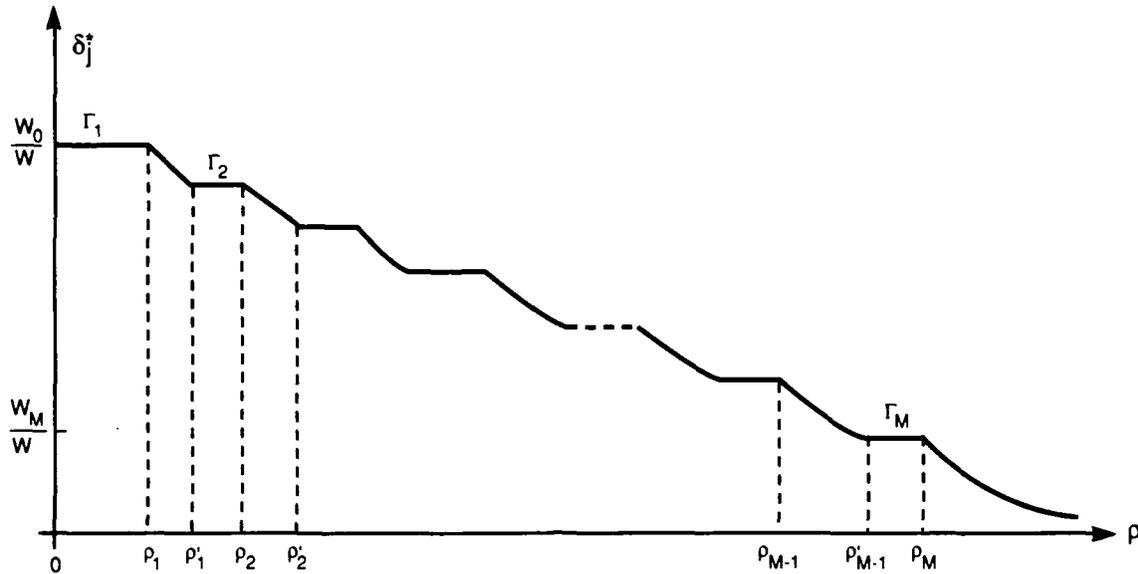
Figure 4-3. Optimal fractional bandwidth versus signal-to-jamming noise ratio.

4.3 PERFORMANCE UNDER OPTIMAL JAMMING

Substituting the optimal jammer given by Equation (4.10) into Equation (4.9) gives the worst-case bit error probability

$$Q_n(e \mid \rho, W_j^*) = \qquad (4.14)$$

$$\begin{cases} h(\rho W_0/W) & \text{for } \rho \leq \rho_1 \\ \vdots & \\ \vdots & \\ a_k c^{-1} \frac{W}{\rho} h(c^{-1} + \frac{\rho}{W}(\Gamma_{k-1} - \frac{p_{k+1}}{a_k})) & \text{for } \rho_k \leq \rho \leq \rho_k' \\ p_{k-1} h(\rho \Gamma_{k-1}/W) & \text{for } \rho_k' \leq \rho \leq \rho_{k-1} \\ \vdots & \\ \vdots & \\ a_M c^{-1} \frac{W}{\rho} h(c^{-1}) & \text{for } \rho \geq \rho_M \end{cases}$$

where p_k, Γ_k, ρ_k, and ρ_k' are defined in equations (4.4), (4.5), (4.12) and (4.13), respectively, and $k = 1, \ldots, M - 1$.

Figure 4-4 represents the variation of $Q_n(e \mid \rho, W_j^*)$ as a function of the signal-to-jamming noise ratio. The performance curve has an exponential shape at low signal-to-noise ratios, and a hyperbolic shape at very large signal-to-noise ratios. For intermediate values of ρ, the shape alternates between those of $(1/\rho)e^{-u\rho}$ and $e^{-v\rho}$, where u and v are positive numbers which depend on k.
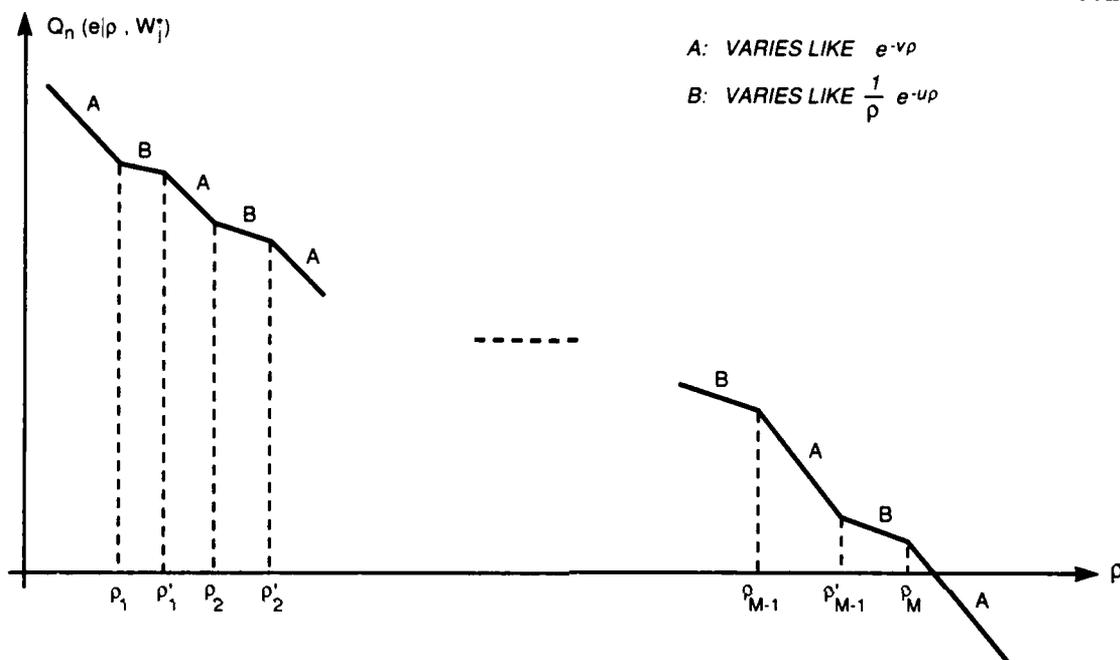
Figure 4-4. Worst-case performance versus signal-to-jamming noise ratio.

4.4 PERFORMANCE COMPARISON

4.4.1 Performance at High Signal-to-Noise Ratio

Let $W_{j,n}^*$ and $W_{j,u}^*$ denote the optimal jammer bandwidth for the discrete and the uniform distributions, respectively. It results from Equation (4.14) that $Q_n(e \mid \rho, W_{j,n}^*)$ equals $a_M c^{-1} W h(c^{-1})/\rho$ for $\rho \geq \rho_M = c^{-1} W/\Delta W_M$.

Comparing this expression to that for $P_u(e \mid \rho, W_{j,u}^*)$ gives

$$\frac{Q_n(e \mid \rho, W_j^*)}{P_u(e \mid \rho, W_j^*)} = \frac{a_M}{a_{0,u}}, \quad for \ \rho \geq max(c^{-1}\frac{W}{\Delta W_M}, c^{-1}\frac{W}{W_{0,u}}) \tag{4.15}$$

and the last condition simplifies to $\rho \geq \rho_M = c^{-1}W/\Delta W_M$.

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Let ρ_u and ρ_n be the signal-to-jamming noise ratios at which the uniform and the nonuniform hopping plans achieve the same bit error probability. Since the bit error probabilities are inversely proportional to ρ at large values of ρ, the ratio ρ_n/ρ_u is

$$\frac{\rho_n}{\rho_u} = a_M/a_{0,u}. \tag{4.16}$$

In other words, the signal-to-noise ratio loss is equal to $a_M/a_{0,u}$ for large values of ρ.

From the above result, several observations can be made concerning the performance loss due to nonuniform hopping. At high signal-to-jamming noise ratios, the loss can be very large if the ratio $a_M/a_{0,u}$ is very large. Thus, a frequency distribution with a high narrow plateau will generate a large signal-to-noise ratio (SNR) loss, and this loss will only show up at high operating points, i.e., at high ρ. This is generally the case if ΔW_M is small, i.e., if it is a small percentage of W_0, and if a_M is clearly larger than the rest of the distribution, i.e., if the plateau is high compared to the rest of the distribution. This is also the case for a distribution having a large number of very narrow and tall plateaus with a small total measure, even if these plateaus are spread across the whole band. It is therefore desirable to use a hopping scheme with a distribution as uniform-like as possible, so as to keep the value of the largest plateau(s) as small as possible. These results are quite similar to those obtained for continuous distributions.

4.4.2 Performance at Low Signal-to-Noise Ratio

From Equation (4.14), the worst-case bit error probability $Q_n(e \mid \rho, W_{j,n}^*)$ equals $h(\rho W_{0,n}/W)$ for $\rho \leq \rho_1 = c^{-1}a_1 W/(p_2 + a_1\Delta W_1)$. Therefore,

$$\frac{Q_n(e \mid \rho, W_{j,n}^*)}{P_u(e \mid \rho, W_{j,u}^*)} = e^{-\left(\frac{c\rho}{W}(W_{0,n}-W_{0,u})\right)}, \quad for \; \rho \leq min(c^{-1}\frac{W}{W_{0,u}}, \rho_1) \tag{4.17}$$

and for distributions with equal support sets,

$$\frac{Q_n(e \mid \rho, W_j^*)}{P_u(e \mid \rho, W_j^*)} = 1, \quad for \; \rho \leq min(c^{-1}\frac{W}{W_{0,u}}, \rho_1). \tag{4.18}$$

Figure 4-5 summarizes the behavior of the bit error probabilities for the uniform and nonuniform hopping plans.

31
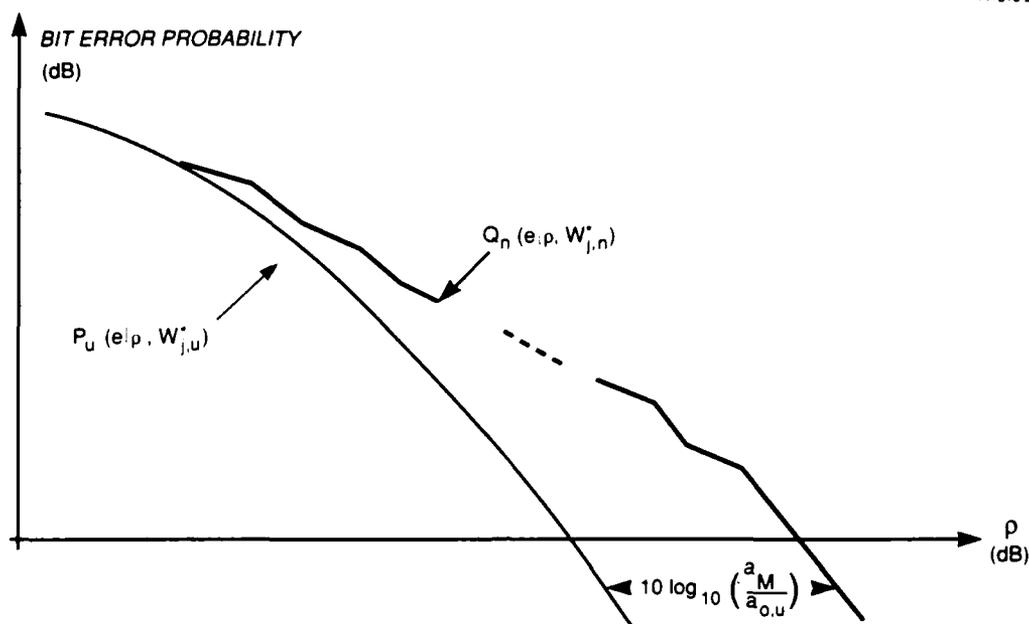
Figure 4-5. Compared performance for uniform and discrete distributions.

4.5 GEOMETRIC INTERPRETATION

4.5.1 Geometric Interpretation of the Optimal Jamming Strategy

The optimal jammer is given by Equation (4.10). For $\rho_k \leq \rho \leq \rho_k'$, $W_j^* = W_{j,k}$, or equivalently,

$$p_{k-1} + a_k(W_j^* - \Gamma_{k-1}) = a_k \frac{W^*}{c\rho}. \tag{4.19}$$

The geometric meaning of this equation is identical to that given for continuous distributions, and is illustrated on Figure 4-6. The left side of Equation (4.19) represents the integral, $A_k(\rho)$, of the distribution over the set $\{W_j\}$, and the right side represents the area, $B_k(\rho)$, of the rectangle of height a_k and width $c^{-1}W^*/\rho$. The same interpretation exists for all values of ρ. With this interpretation, Equation (4.19) is equivalent to

$$A_k(\rho) = B_k(\rho). \tag{4.20}$$

Thus, the jammer's best strategy is such that the two areas $A(\rho)$ and $B(\rho)$ under each curve are equal.

32
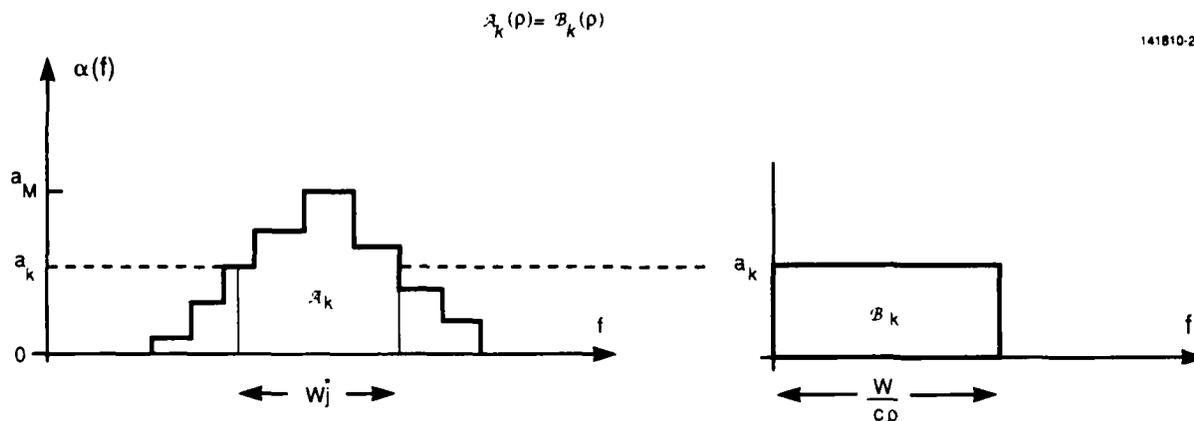
Figure 4-6. Geometric interpretation of the optimal jamming strategy for discrete distributions.

4.5.2 Geometric Interpretation of the Bit Error Probability

Equation (4.14) shows that the worst-case bit error probability equals

$$Q_n(e \mid \rho, W_j^*) = \mathcal{A}_k(\rho) h(c^{-1} \frac{\mathcal{A}_{k,0}(\rho)}{\mathcal{A}_k(\rho)}) \tag{4.21}$$

where $\mathcal{A}_k(\rho)$ was defined above, and $\mathcal{A}_{k,0}(\rho)$ is equal to $a_k W_j^*$. This simple geometric interpretation allows one to visualize the variation of the bit error probability under optimal jamming, as the signal-to-jamming noise ratio changes. This is illustrated on Figure 4-7.

4.6 EXAMPLE

The following example illustrates how to use the equations giving the optimal jammer and worst-case bit error probability in the particular case of a signal hopping according to the frequency distribution shown on Figure 4-8. The distribution is a type-I distribution, defined over a support set $\{W_0\}$. It takes on values a_1 over the set $\{W_1\}$ and a_2 over a set $\{W_2\}$. From Equation (4.4), $p_0 = p_1 = 1$, $p_2 = a_2 W_2$, $p_3 = 0$, and $\{\Gamma_0\} = \{W\}$, $\{\Gamma_1\} = \{W_0\}$, $\{\Gamma_2\} = \{W_2\}$, $\{\Gamma_3\} = \emptyset$. In addition, $\Delta W_1 = W_1$, $\Delta W_2 = W_2$ by definition.

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$$a_k(\rho) = a_{k,0}(\rho) + a_{k,1}(\rho)$$
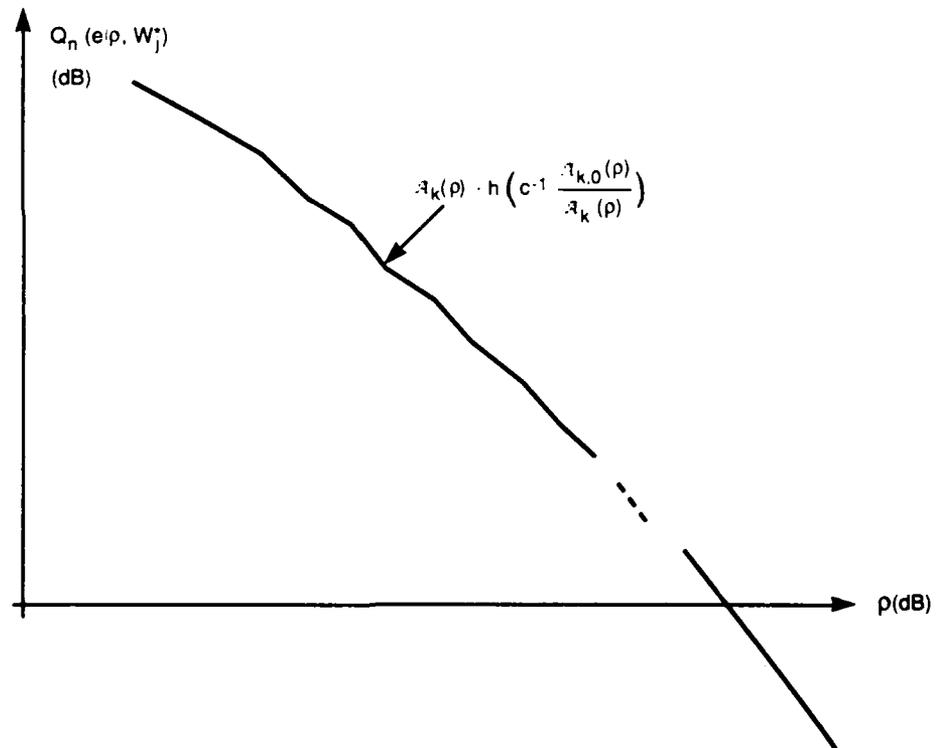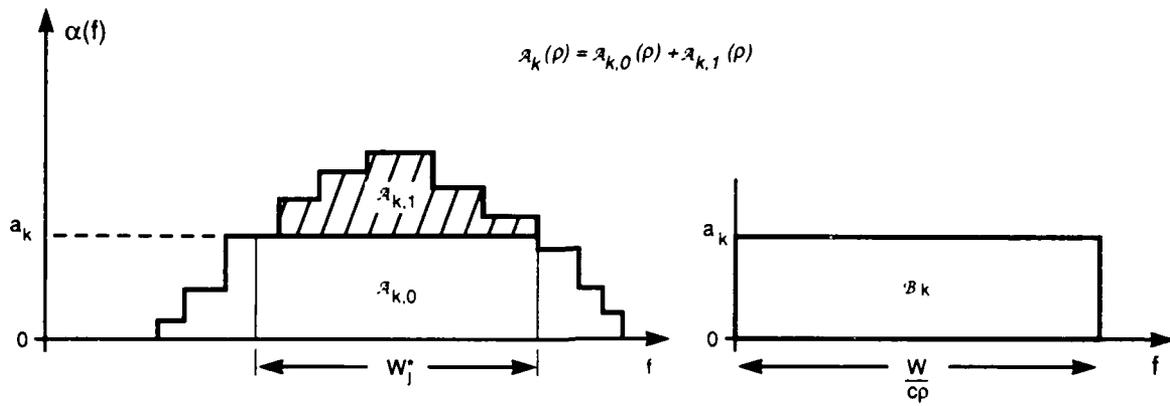
Figure 4-7   *Geometric interpretation of the worst-case performance.*

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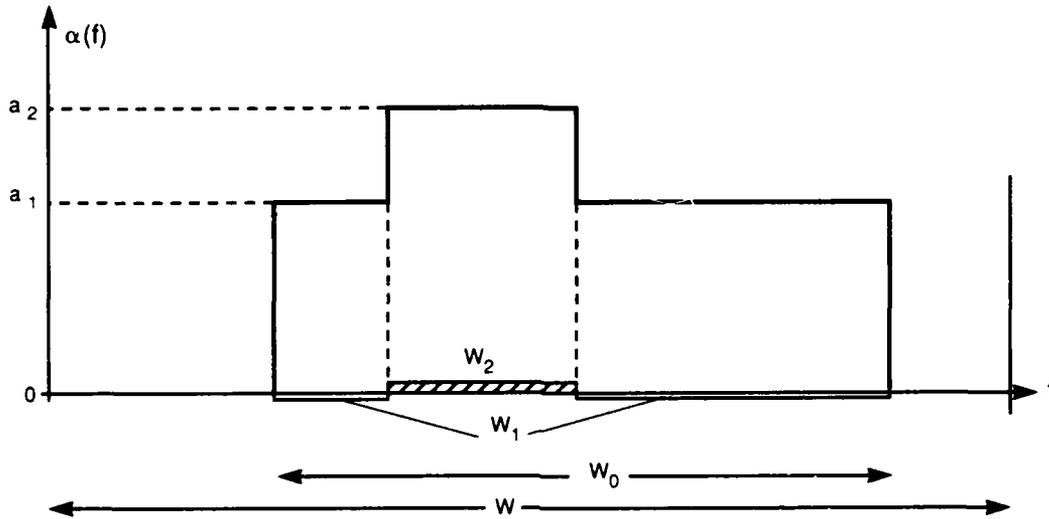
Figure 4-8.   Example of staircase distribution for M=2.

Finally, from equations (4.12) and (4.13). $\rho_1 = a_1 W$. $\rho_1' = c^{-1}a_1 W/a_2 W_2$. $\rho_2 = c^{-1}W/W_2$. Therefore, from Equation (4.10), the optimum jamming strategy is

$$
\{W_j^{**}\} = \begin{cases} \{W_0\}, & for \; \rho \leq c^{-1}a_1 W \\[2ex] \{W_{j,1}\}. & for \; c^{-1}a_1 W \leq \rho \leq c^{-1}a_1 W/a_2 W_2 \\[2ex] \{W_2\}, & for \; c^{-1}a_1 W/a_2 W_2 \leq \rho \leq c^{-1}W/W_2 \\[2ex] \{W_{j,2}\}. & for \; \rho \geq c^{-1}W/W_2 \end{cases}
\tag{4.22}
$$

where $\{W_{j,1}\}$ and $\{W_{j,2}\}$ are two frequency subsets such that

$$
\{W_2\} \subseteq \{W_{j,1}\} \subseteq \{W_0\} \qquad W_{j,1} = c^{-1}\frac{W}{\rho} - W_2\frac{a_2 - a_1}{a_1}
\tag{4.23}
$$

and

$$
\emptyset \subset \{W_{j,2}\} \subseteq \{W_2\} \qquad W_{j,2} = c^{-1}\frac{W}{\rho}
\tag{4.24}
$$

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141810-27
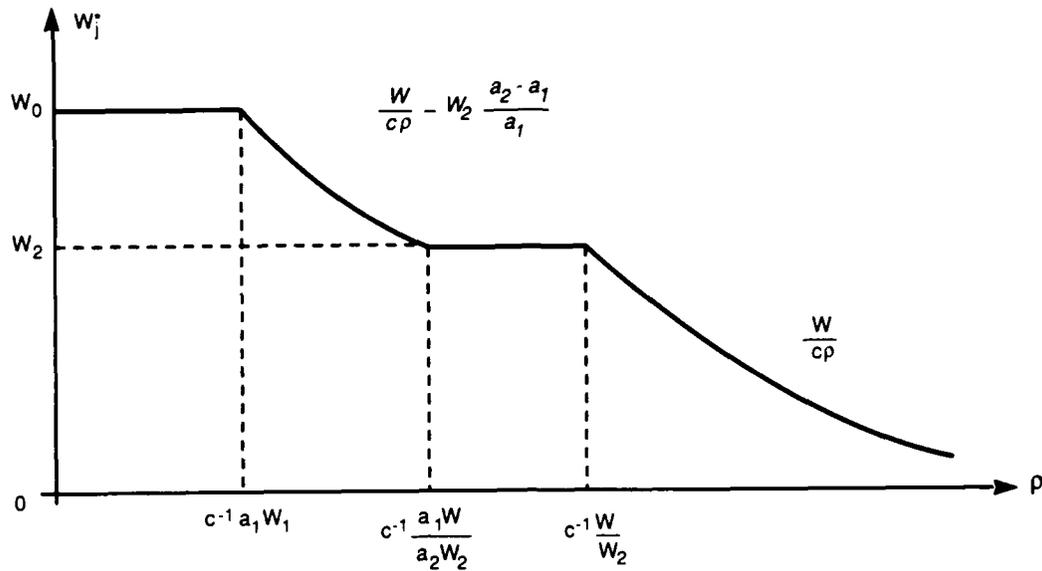
Figure 4-9.   Example of optimal jamming strategy for $M = 2$.

141810-28
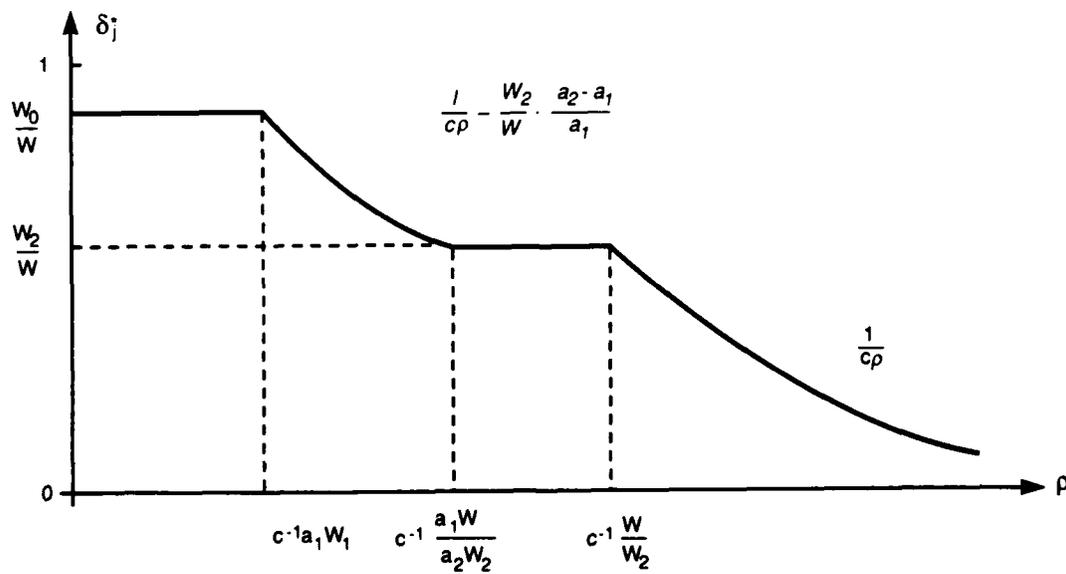
Figure 4-10.   Example of optimal fractional bandwidth for $M = 2$.

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Figure 4-9 illustrates how the measure of the jammed set varies with ρ. Figure 4-10 illustrates the variation of the optimal fractional bandwidth with ρ. The bit error probability under optimized jamming is

$$Q_n(\epsilon \mid \rho, W_j^*) =$$

$$\begin{cases} h(\rho\frac{W_0}{W}) & \text{for } \rho \le c^{-1}a_1 W \\[2ex] a_1\frac{W}{c\rho}h(c^{-1} - \frac{a_2-a_1}{a_1}\frac{W_2}{W}\rho) & \text{for } c^{-1}a_1 W \le \rho \le c^{-1}a_1 W/a_2 W_2 \\[2ex] a_2 W_2 h(\rho\frac{W_2}{W}) & \text{for } c^{-1}a_1 W/a_2 W_2 \le \rho \le c^{-1}W/W_2 \\[2ex] a_2\frac{W}{c\rho}h(c^{-1}) & \text{for } \rho \ge c^{-1}W/W_2. \end{cases} \qquad (4.25)$$

Figure 4-11 represents the probability of error under optimum jamming.
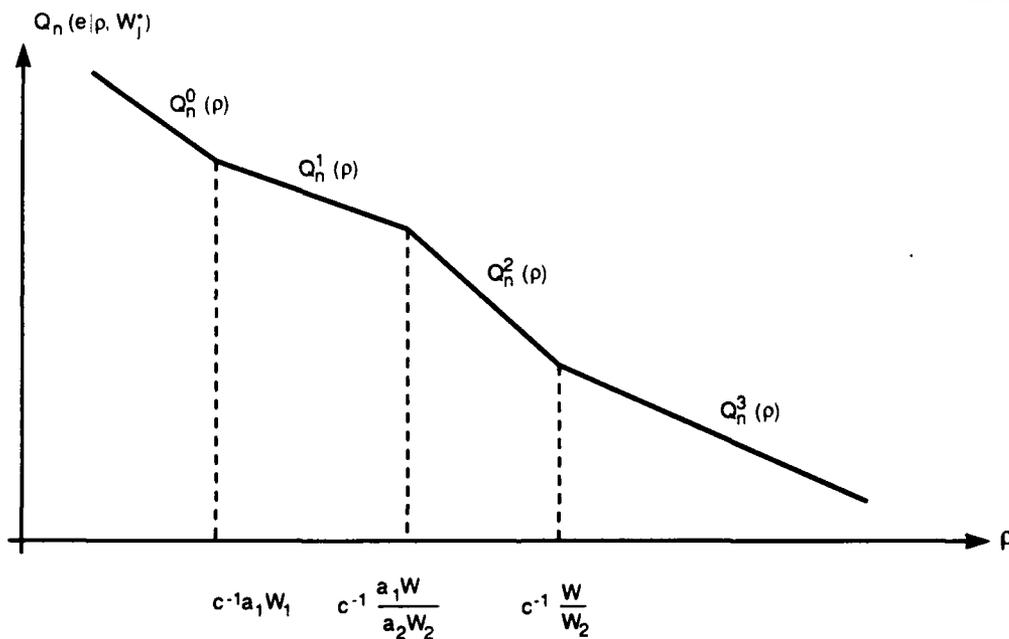
Figure 4-11. Example of worst-case performance for M = 2.

Consider, now, a distribution having one very narrow and large spike somewhere within an otherwise uniform distribution. This is a special case of the discrete distribution just described, where W_2 is very small compared to W_0, and a_2 is very large compared to a_1. It was shown that for large signal-to-jamming noise ratios the jamming loss is equal to $10Log_{10}(a_M/a_0)$ dB. Since the distribution has a very large spike, a_M is much larger than a_0, and the jamming loss is very large. However, the values of ρ for which this expression applies are very large. In fact, ρ must

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exceed $c^{-1}W/W_2$ which, for small values of W_2, is much greater than the value of ρ at which the system operates. Therefore, the maximum jamming loss must be computed at the largest operating signal-to-jamming noise ratio, and could, in a case such as the one described here, be much smaller than $10Log_{10}(a_M/a_0)$. The worst-case two-level distribution for a system operating at a SNR of ρ_1 is one whose overlap size W_2 satisfies $\rho_1 = c^{-1}W/W_2$. When it is the case, the worst-case jamming loss is incurred at the signal-to-jamming noise ratio at which the system operates.

5. CONCLUSION

The jamming performance of frequency-hopped communication systems has been analyzed for arbitrary continuous or discrete hopping distributions. The analysis shows that the worst-case performance loss when compared to uniform hopping is either equal to or bounded above by the expression $10Log_{10}(a_M/a_0)$.

Whenever possible, it is in the interest of the communication designer to select a hopping plan resulting in a frequency distribution which is as uniform as possible, since this leads to the best antijam performance. When this is not possible, two strategies are available to the designer in order to mitigate the additional loss due to the nonuniform distribution.

The first strategy consists of allowing a small number of high narrow spikes or high narrow plateaus in the frequency distribution. As was shown, the maximum jamming loss, given by the ratio a_M/a_0, is very large, but it is only incurred at very large signal-to-jamming noise ratios well above the system's operating point.

The second strategy consists of allowing a number of low wide spikes or low wide plateaus in the frequency distribution. In this case, the maximum jamming loss is incurred at or possibly below the system's operating point but it is quite moderate.

The specific values will depend upon the modulation waveform chosen, the system's operating point, and the user signal bandwidth.

Finally, the use of Chernoff bounding techniques [3] makes it possible to extend these results to arbitrary modulations.

REFERENCES

1. P.R. Hirschler-Marchand, *"Performance of Nonuniform Frequency-Hopping Plans in a Jamming Environment,"* Private Communication.

2. P.R. Halmos, *Measure Theory*, New York: Van Nostrand (1950).

3. R.G. Gallager, *Information Theory and Reliable Communication.* New York: John Wiley and Sons, Inc. (1968).

APPENDIX A
CALCULATION OF THE OPTIMAL JAMMER FOR NONUNIFORM CONTINUOUS FREQUENCY DISTRIBUTIONS

A.1 CONDITIONAL BIT ERROR PROBABILITY

Since the jammer knows the shape of the distribution $\alpha(\cdot)$ and takes advantage of that information, the probability that the signal is jammed, conditioned on W_j, is

$$Pr\{S \ is \ jammed \mid W_j\} = \begin{cases} \int_{\{W_{a_j}\}} \alpha(f)df & \text{for } W_j < W_0 \\ 1 & \text{for } W_j \geq W_0 \end{cases} \tag{A.1}$$

with $W_j = \Gamma(a_j)$, or equivalently

$$Pr\{S \ is \ jammed \mid W_j\} = \begin{cases} a_j W_j + \int_{a_j}^{a_M} \Gamma(a)da & \text{for } W_j < W_0 \\ 1 & \text{for } W_j \geq W_0 \end{cases} \tag{A.2}$$

with $W_j = \Gamma(a_j)$, and $a_m \leq a_j \leq a_M$.

If the sets $\{W_a\}$ associated with the distribution $\alpha(.)$ are connected sets, the integral over a set is equivalent to an integral between two integration limits

$$Pr\{S \ is \ jammed \mid W_j\} = \begin{cases} \int_{f_j}^{f'_j} \alpha(f)df & \text{for } W_j < W_0 \\ 1 & \text{for } W_j \geq W_0 \end{cases} \tag{A.3}$$

with $\alpha(f_j) = \alpha(f'_j) = a_j$, and $W_j = \Gamma(a_j)$. For the sake of clarity and without loss of generality, the notation valid for connected sets will be used.

Using Bayes' rule, the conditional bit error probability $P_n(e \mid \rho, W_j)$, given a jammer bandwidth W_j, is equal to

$$P_n(e \mid \rho, W_j) = \begin{cases} [a_j W_j + \int_{a_j}^{a_M} \Gamma(a)da]h(\rho W_j/W) & \text{for } W_j < W_0 \\ h(\rho W_j/W) & \text{for } W_j \geq W_0 \end{cases} \tag{A.4}$$

with $W_j = \Gamma(a_j)$, and $a_m \leq a_j \leq a_M$.

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A.2 OPTIMAL JAMMING STRATEGY

The optimal jamming strategy $\{W_j^*\}$ maximizes the bit error probability $P_n(e \mid \rho, W_j)$ with respect to W_j. Let us examine the expression for the derivative of $P_n(e \mid \rho, W_j)$ with respect to W_j. For the class of modulations for which $h(x) = Ae^{-cx}$, this derivative equals

$$
\begin{aligned}
\frac{\partial P_n(e|\rho,W_j)}{\partial W_j} &= h(\rho W_j/W)\left[a_j - c\frac{\rho}{W}\left(a_j\Gamma(a_j) + \int_{a_j}^{a_M}\Gamma(a)da\right)\right] \\
&= h(\rho W_j/W)a_j\frac{\rho}{W}\left[\frac{W}{\rho} - c\Gamma(a_j) - \frac{c}{a_j}\int_{a_j}^{a_M}\Gamma(a)da\right]
\end{aligned}
\tag{A.5}
$$

for $W_j < W_0$. The sign of $\partial P_n(e \mid \rho, W_j)/\partial W_j$ is given by the sign of the implicit expression $T(a_j)$

$$
T(a_j) = \frac{W}{\rho} - c\Gamma(a_j) - \frac{c}{a_j}\int_{a_j}^{a_M}\Gamma(a)da.
\tag{A.6}
$$

Since $T(0)$ is $-\infty$, and $T(a_M) = W/\rho - c\Gamma(a_M)$, the derivative only changes sign if $T(a_M)$ is positive.

Consequently,

- For $W/\rho - c\Gamma(a_M) > 0$, $T(a_j)$ equals 0 for some value a_j^* between a_m and a_M. The value of a_j^* at which the derivative equals 0 is the implicit solution of the equation

$$
\frac{W}{c\rho} = \Gamma(a_j^*) + \frac{1}{a_j^*}\int_{a_j^*}^{a_M}\Gamma(a)da.
\tag{A.7}
$$

- For $W/\rho - c\Gamma(a_M) < 0$, $T(a_j)$ is negative over the interval (a_n, a_M), and the maximum of P_n is achieved for $a_j^* = a_M$.

It is now possible to express the optimal jamming strategy:

- For type-I distributions,

$$
\{W_j^*\} = \{W_{a_j^*}\} \quad \text{for all values of } \rho
\tag{A.8}
$$

where a_j^* is the implicit solution of Equation (A.7).

- For type-II distributions,

$$
\{W_j^*\} = \begin{cases} \{W_{a_j^*}\} & \text{for } \rho \leq c^{-1}W/\Gamma(a_M) \\ \{W/c\rho\} & \text{for } \rho > c^{-1}W/\Gamma(a_M) \end{cases}
\tag{A.9}
$$

where a_j^* satisfies Equation (A.7), and $\{c^{-1}W/\rho\}$ is any subset of $\{W_{a_M}\}$, of size $c^{-1}W/\rho$. The optimal fraction of the bandwidth to be jammed is given by the expression $\delta_j^* = W_j^*/W$.

APPENDIX B
PERFORMANCE AT HIGH SIGNAL-TO-JAMMING NOISE RATIO FOR A NONUNIFORM CONTINUOUS DISTRIBUTION

Since $W_j^* > 0$, it results from Equation (3.9) that

$$0 \leq \frac{1}{a_j^*} \int_{a_j^*}^{a_M} \Gamma(a) da \leq \frac{W}{c\rho} \tag{B.1}$$

or

$$\frac{1}{a_j^*} \int_{a_j^*}^{a_M} \Gamma(a) da \to 0, \quad as \ \rho \to \infty. \tag{B.2}$$

Thus, $a_j^* \to a_M$ as $\rho \to \infty$.

For type-I distributions, $\Gamma(a_M) = 0$. Then, from equations (3.13), (3.9), and (B.2), it results that $a_j^* \to a_M$, and $\Gamma(a_j^*) \to c^{-1}W/\rho$.

Consequently,

$$P_n(e \mid \rho, W_j^*) \to a_M c^{-1} \frac{W}{\rho} h(c^{-1}), \quad as \ \rho \to \infty \tag{B.3}$$

and the ratio of the nonuniform bit error probability to the uniform one is asymptotically equal to

$$\frac{P_n(e \mid \rho \mid W_j^*)}{P_u(e \mid \rho, W_j^*)} \xrightarrow{\rho \to \infty} \frac{a_M}{a_{0,u}} \tag{B.4}$$

where $a_{0,u} = 1/W_{0,u}$ is the value of the uniform distribution with support $W_{0,u}$.

For type-II distributions, $\Gamma(a_M) > 0$, and for $\rho > c^{-1}W/\Gamma(a_M)$,

$$P_n(e \mid \rho, W_j^*) = a_M c^{-1} \frac{W}{\rho} h(c^{-1}). \tag{B.5}$$

For the uniform distribution and $\rho \geq c^{-1}W/W_0$, we have from Equation (2.7)

$$P_u(e \mid \rho, W_j^*) = a_{0,u} \frac{W}{c\rho} h(c^{-1}). \tag{B.6}$$

Therefore, for $\rho \geq \max(c^{-1}W/W_{0,u}, c^{-1}W/\Gamma(a_M))$, the ratio of the nonuniform bit error probability to the uniform one is equal to

$$\frac{P_n(e \mid \rho, W_j^*)}{P_u(e \mid \rho, W_j^*)} = \frac{a_M}{a_{0,u}}. \tag{B.7}$$

This completes the proof.

APPENDIX C
PERFORMANCE AT LOW SIGNAL-TO-JAMMING NOISE RATIO FOR NONUNIFORM CONTINUOUS DISTRIBUTIONS

It follows from Equation (3.9) and integration by parts that

$$a_j^* = \frac{c\rho}{W} \int_{a_M}^{a_j^*} a\gamma(a)da \tag{C.1}$$

and therefore, $a_j^* \to 0$, and $W_j^* = \Gamma(a_j^*) \to W_0$, as $\rho \to 0$.

Equation (3.9) further implies

$$\frac{a_j^* W}{c\rho} = a_j^* W + \int_{a_j^*}^{a_M} \Gamma(a)da \to \int_0^{a_M} \Gamma(a)da = 1. \tag{C.2}$$

Thus, from Equation (3.10)

$$P_n(e \mid \rho, W_j^*) \to h(\rho\frac{W_0}{W}), \quad as\ \rho \to 0. \tag{C.3}$$

Comparing the asymptotic performances for uniform and nonuniform hopping leads to the result

$$\frac{P_n(e \mid \rho \mid W_j^*)}{P_u(e \mid \rho, W_j^*)} \xrightarrow{\rho \to 0} 1\ . \tag{C.4}$$

If the uniform and nonuniform distributions have different support sets $\{W_{0,n}\}$ and $\{W_{0,u}\}$ respectively, the limit of the above ratio is

$$\frac{P_n(e \mid \rho \mid W_j^*)}{P_u(e \mid \rho, W_j^*)} \longrightarrow e^{\left[-\frac{\rho}{W}(W_{o,n} - W_{o,u})\right]}, \quad as\ \rho \to 0. \tag{C.5}$$

Q.E.D.

APPENDIX D
CALCULATION OF THE OPTIMAL JAMMING STRATEGY FOR DISCRETE DISTRIBUTIONS

D.1 DEFINITION OF THE FUNCTIONS $\{Q_n^k(W_j)\}$

Consider the real functions $\{Q_n^k(W_j); k = 0, 1, \cdots, M)$, defined for $0 \leq W_j \leq W_0$. These functions are of interest here because they coincide with the probability distribution $Q_n(e \mid \rho, W_j)$ over a subset of $(0, W)$. Therefore their behavior on each subset determines that of the probability distribution. They are as follows.

- Q_n^0, defined as

$$Q_n^0(W_j) = h(\rho\frac{W_j}{W}), \tag{D.1}$$

equals $Q_n(e \mid \rho, W_j)$ for $\{W_0\} = \{\Gamma_1\} \subseteq \{W_j\} \subseteq \{\Gamma_0\} = \{W\}$.

- $Q_n^k(W_j)$, defined for $k = 1, \ldots, M - 1$, as

$$Q_n^k(W_j) = [p_{k-1} + a_k(W_j - \Gamma_{k-1})] \, h(\rho\frac{W_j}{W}), \tag{D.2}$$

equals $Q_n(e \mid \rho, W_j)$ for $\{\Gamma_{k-1}\} \subseteq \{W_j\} \subseteq \{\Gamma_k\}$.

- $Q_n^M(W_j)$, defined as

$$Q_n^M(W_j) = a_M W_j h(\rho\frac{W_j}{W}), \tag{D.3}$$

equals $Q_n(e \mid \rho, W_j)$ for $\emptyset = \{\Gamma_{M-1}\} \subseteq \{W_j\} \subseteq \{\Gamma_M\} = \{\Delta W_M\}$.

As can be seen from Equation (4.9), $Q_n(e|\rho, W_j)$ consists of sections of the functions $\{Q_n^k(.); k = 0, \ldots, M\}$ pieced together, and coincides with Q_n^k only for $\{\Gamma_{k-1}\} \subseteq \{W_j\} \subseteq \{\Gamma_k\}$.

D.2 OPTIMUM OF $Q_n^k(W_j)$

Calculation of the jamming strategy which maximizes the probability of error $Q_n(e \mid \rho, W_j)$ requires the study of the maxima of the functions $Q_n^k(W_j)$.

For the class of modulations considered, the partial derivative of $Q_n^k(W_j)$ with respect to W_j is

$$\frac{\partial Q_n^k(W_j)}{\partial W_j} = [a_k - \frac{c\rho}{W}(p_{k+1} + a_k(W_j - \Gamma_{k+1}))]h(\rho\frac{W_j}{W}) \tag{D.4}$$

$$for \quad \{\Gamma_{k+1}\} \subseteq \{W_j\} \subseteq \{\Gamma_k\}. \tag{D.5}$$

From this expression, it follows that

$$\frac{\partial Q_n^k(W_j)}{\partial W_j} \begin{cases} \geq & 0, \quad for \ \rho < a_k c^{-1} W \left[p_{k-1} + a_k (W_j - \Gamma_{k-1}) \right]^{-1} \\[2ex] = & 0. \quad for \ \rho = a_k c^{-1} W \left[p_{k-1} + a_k (W_j - \Gamma_{k-1}) \right]^{-1} . \\[2ex] \leq & 0, \quad for \ \rho > a_k c^{-1} W \left[p_{k-1} + a_k (W_j - \Gamma_{k+1}) \right]^{-1} \end{cases} \tag{D.6}$$

The maximum of $Q_n^k(W_j)$ over $(0, W)$ is achieved for $W_j = W_{j,k}$, where

$$\{\Gamma_{k-1}\} \subseteq \{W_{j,k}\} \subseteq \{\Gamma_k\}, \quad and$$

$$W_{j,k} = \frac{W}{c\rho} + \Gamma_{k-1} - \frac{p_{k-1}}{a_k} \qquad . \tag{D.7}$$

D.3 OPTIMUM OF $Q_n(e \mid \rho, W_j)$

Since $Q_n(e \mid \rho, W_j)$ coincides with $Q_n^k(W_j)$ for $(\{\Gamma_{k-1}\} \subseteq \{W_j\} \subseteq \{\Gamma_k\})$, the optimal jammer over that interval is given by

$$\{W_j^*\} = \begin{cases} \{\Gamma_{k-1}\}, & if \ \frac{\partial Q_n^k}{\partial W_j}(\Gamma_{k-1}) \leq 0, \ and \ \frac{\partial Q_n^k}{\partial W_j}(\Gamma_k) \leq 0 \\[2ex] \{W_{j,k}\}, & if \ \frac{\partial Q_n^k}{\partial W_j}(\Gamma_{k-1}) \geq 0. \ and \ \frac{\partial Q_n^k}{\partial W_j}(\Gamma_k) \leq 0 \\[2ex] \{\Gamma_k\}, & if \ \frac{\partial Q_n^k}{\partial W_j}(\Gamma_{k-1}) \geq 0, \ and \ \frac{\partial Q_n^k}{\partial W_j}(\Gamma_k) \geq 0 \end{cases} \tag{D.8}$$

or equivalently, for $k = 1, \ldots, M$

$$\{W_j^*\} = \begin{cases} \{\Gamma_{k-1}\}, & if \ \rho \geq \rho_k' \ and \ \rho \geq \rho_k \\[2ex] \{W_{j,k}\}, & if \ \rho_k \leq \rho \leq \rho_k' \\[2ex] \{\Gamma_k\}, & if \ \rho \leq \rho_k' \ and \ \rho \leq \rho_k \end{cases} \tag{D.9}$$

where

$$\rho_k = c^{-1} \frac{a_k W}{p_{k-1} + a_k(\Gamma_k - \Delta W_{k+1})} \tag{D.10}$$

and

$$\rho_k' = c^{-1} \frac{a_k W}{p_{k+1} + a_k \Gamma_{k+2}}. \tag{D.11}$$

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It is a simple matter to show that $\rho_k < \rho_k' < \rho_{k-1}$ for $k = 1, \cdots, M$. It results that the optimal jamming strategy, $\{W_j^*\}$, which is obtained by piecing together the above solutions valid over each sub-interval, is given by the expression

$$\{W_j^*\} = \begin{cases} \{W_0\}, & for \; \rho \leq \rho_1 \\ \vdots \\ \vdots \\ \{W_{j,k}\}, & for \; \rho_k \leq \rho \leq \rho_k' \quad (k = 1, \ldots, M-1) \\ \{\Gamma_{k-1}\}, & for \; \rho_k' \leq \rho \leq \rho_{k-1} \quad (k = 1, \ldots, M-1) \\ \vdots \\ \vdots \\ \{W_{j,M}\}, & for \; \rho \geq \rho_M \end{cases} \tag{D.12}$$

where $W_{j,k}$ is as shown in Equation (D.7). and ρ_k and ρ_k' are defined in equations (D.10) and (D.11).

In summary. the optimal jamming strategy $\{W_j^*\}$ consists of $M+1$ expressions, each one being valid over a specific range of signal-to-jamming noise ratios.

This completes the calculation of the optimal jamming strategy for arbitrary discrete distributions.

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