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The Role of the Topography in Gravity Gradiometer Reductions and in the Solution of the Geodetic Boundary Value Problem Using Analytical Downward Continuation

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September 1989

Final Report
1 April 1986-30 September 1989

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AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
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Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved
OMB No 0704-0188

1a REPORT SECURITY CLASSIFICATION
Unclassified

1b RESTRICTIVE MARKINGS

2a SECURITY CLASSIFICATION AUTHORITY

2b DECLASSIFICATION/DOWNGRADING SCHEDULE

3 DISTRIBUTION/AVAILABILITY OF REPORT
Approved for public release; distribution unlimited

4 PERFORMING ORGANIZATION REPORT NUMBER(S)
OSU/DGSS Report No. (RF 765306/718188)

5 MONITORING ORGANIZATION REPORT NUMBER(S)
GL-TR-90-0001

6a. NAME OF PERFORMING ORGANIZATION
Department of Geodetic Science and Surveying

6b OFFICE SYMBOL (If applicable)

7a NAME OF MONITORING ORGANIZATION
Geophysics Laboratory

6c. ADDRESS (City, State, and ZIP Code)
The Ohio State University
Columbus, Ohio 43210

7b ADDRESS (City, State, and ZIP Code)
Hanscom AFB
Massachusetts 01731-5000

8a NAME OF FUNDING/SPONSORING ORGANIZATION
Air Force Geophysics Laboratory

8b OFFICE SYMBOL (If applicable)
(AFGL)

9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
F-19628-86-K-0016

8c. ADDRESS (City, State, and ZIP Code)
Hanscom Air Force Base, Massachusetts 01731-5000

10 SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO	PROJECT NO.	TASK NO	WORK UNIT ACCESSION NO
62101F	7600	03	AO

11 TITLE (Include Security Classification)
The Role of the Topography in Gravity Gradiometer Reductions and in the Solution of the Geodetic Boundary Value Problem Using Analytical Downward Continuation

12 PERSONAL AUTHOR(S)
Yan Ming Wang

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	FROM 4/1/86 TO 9/30/89	1989, September	54

16 SUPPLEMENTARY NOTATION

17 COSATI CODES

FIELD	GROUP	SUB-GROUP

18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Gravity, Gradiometer, Downward Continuation

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

The topography of the earth plays an important role in solving the geodetic boundary values problem. In this report the effect of topography on gravity gradient data is considered and the effect of topography on the solution of the geodetic boundary value problem by using analytical downward continuation is also investigated.

The validity of solving Molodensky's problem by using the analytical downward continuation is inspected. Even though it has been shown that the analytical downward continuation solution is equivalent to Molodensky's solution which is considered theoretically perfect, a very small topographic effect exists. This effect is trivial and can be neglected in the numerical computations.

It is also shown that a spherical harmonic expansion cannot exactly represent the disturbing potential outside the Brillouin sphere and nearby the earth at the same time. If the points are near by the earth (between the Brillouin sphere and the earth's surface), there is a topographic effect to the geopotential represented by a spherical harmonic

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS

21 ABSTRACT SECURITY CLASSIFICATION
Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL
Christopher Jekeli

22b TELEPHONE (Include Area Code)

22c. OFFICE SYMBOL
GL/LWG

DD Form 1473, JUN 86 Previous editions are obsolete

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

expansion whose coefficients are determined by using the gravity anomalies analytically downward continued onto the ellipsoid. This effect is the same as to the solving of the Molodensky's problem by using the analytical downward continuation.

The convergence problem of the analytical downward continuation is also investigated under planar approximation. It is shown that the downward continuation is convergent almost everywhere, except at the infinite point of the circular frequency $\omega = \infty$. This is important for geopotential modeling. Provided the downward continuation is convergent, then the geopotential can be expanded into a spherical harmonic series up to very high degree and order without any theoretical difficulty.

Foreword

This report was prepared by Wang Yan Ming, Research Associate, Department of Geodetic Science and Surveying at The Ohio State University. This research was sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, Geophysics Laboratory, under Contract Number AFOSR F19628-86-K-0016, The Ohio State University Research Foundation Project 718188, project supervisor, Richard H. Rapp. The contract covering this research is administered by the Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Dr. Christopher Jekeli, Scientific Program Officer.

Computer resources were provided by contract funds and by the funds supplied by the Instruction and Research Computer Center, through the Department of Geodetic Science and Surveying.

Acknowledgements

I am very grateful to Professor Richard H. Rapp for his support and help with this research. I wish to thank Ms. Lisa Schneck for her excellent typing.

Table of Contents

1. Introduction

The analytical downward continuation (ADC) is widely used in the solving of the geodetic boundary value problem. One example is the use of the ADC for the solving of Molodensky's problem (Moritz, 1980, Section 45). Recently, the airborne and satellite gradiometry have been getting substantial improvement and they will supply a huge gravity gradient data for the determination of the Earth's gravity field. To process such data the ADC method has to be used and some reference surface, such as the sea level, the ellipsoid, may be chosen. The topographic effect is becoming an important problem in the use of the gradient data. This study is addressed on the problem of the topographic effect on the using of the ADC method.

In Chapter 2 the classic topographic reductions - the Bouguer reduction and the Helmert second condensation will be introduced into the airborne gradiometry. The reductions are studied in the space domain and the frequency domain. The reduction in space domain is tedious but they give a clear view of the physical meaning. In the frequency domain, this problem becomes much easier to solve.

In Chapter 3 the topographic effect for the use of the analytical downward continuation method is studied. The remove-restore of the mass above the ellipsoid is used. The goal of the study is to find the topographic corrections, in another word, to find the difference between the disturbing potential on the sea level and disturbing potential which is downward continued to the sea level from the outside of the earth.

In Chapter 4 we study the topographic effect on the solving of the Molodensky's problem. It was proved that the ADC solution is equivalent to the Molodensky's solution (ibid, p. 388). In this chapter we show that the topographic effect still exists in the solution of Molodensky's problem by using the ADC method, even if this effect is very small. The topographic effect on the determination of the coefficients of the spherical harmonics of the geopotential by using the ADC method is also considered.

2. Topographic Correction on the Gravity Gradient Data Measured by an Airborne Gradiometer

2.1 Introduction

In this chapter we consider the topographic correction on the gravity gradient data measured by an airborne gradiometer. We assume that the airborne flights are at a constant altitude above the sea level. In comparison with the satellite gradiometry the airborne flights are much lower and can sense much higher frequencies of the gravity field. It is well known that the gravity gradient depends strongly on the roughness of the topography. Because the airborne gradiometer flies close to the topography, we will try to eliminate the topography effect directly from the measured gravity gradient data.

The topographic correction for airborne gravity and gradient data has been discussed extensively in the literature; e.g., Chinnery (1961), Parker (1972), Dorman and Lewis (1974), Hammer (1976). Recently an algorithm was developed for eliminating the "topographic noise" (Tziavos et al., 1988). The fast Fourier transformation was used and this algorithm is very efficient to compute the topographic effect on the gravity and gradient data.

Now we want to make the statement "eliminating the effect of the topography" more clear.

Tziavos et al., (1988) developed an algorithm to compute the gradient of the attraction of the topography. The topographic effect was considered as "topographic noise" (ibid) and subtracted from the measured gradient data.

Actually the topographic effect is a part of the gravitational potential of the earth and cannot be simply eliminated by subtracting its contribution from the measured gravity and gradient data. Before we consider the effect of the earth's topography, we must ask ourselves, what is the goal and what will be determined by using the gravity gradient data or the reduced data. This is very important for the start of the topography reduction procedure.

If the contribution of the topography is subtracted from the gravity and gradient data, it means the topography mass is removed, then the disturbing potential and its derivatives are harmonic above the sea level and can be analytically downward continued to the sea level without any theoretical difficulty. Obviously, the reduced gravity anomaly is the Bouguer anomaly and the topography reduction is the Bouguer reduction.

In this chapter, we assume that the geoid and the deflections of the vertical will be determined in a mountain area by using the aerial gravity gradient data. The topographic effect will be eliminated from such data. For this purpose an instinctive action is to expand the Bouguer reduction and the Helmert's second condensation into the airborne gradiometry.

2.2 Topographic Correction Studied in Space Domain

2.2.1 Bouguer Reduction

The Bouguer reduction in airborne gradiometry is to eliminate the "topographic noise" by subtracting the contribution of the topography to the gravity and gradient data (e.g., Tziavos et al., 1988). The airborne gradiometry is taken in a local area. For such area a plane approximation for the computation is suitable (Jekeli et al., 1985).

The geometry of the computation of the topographic effect in the airborne gravity gradiometry is illustrated in Fig. 1.

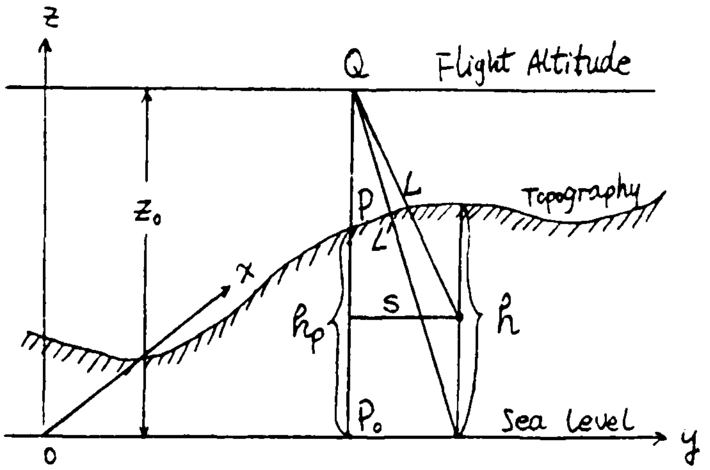

Fig. 1. Geometry of the Computation of the Topographic Correction

We define the local coordinate system whose origin is located at sea level and the x axis points to the north and y axis points to the east.

The potential of the mass above the sea level at the point Q is given by:

$$V_Q = G \iint_\tau \int_0^h \frac{\rho}{L}\, dz\, dx\, dy \tag{2-1}$$

with

$$L^2 = s^2 + (z_0 - z)^2 \tag{2-2}$$

where G is the gravitational constant and the density of the topography ρ (x, y, z) is a function of the location of the current point, τ is a 2D plane which coincides with the local sea level and $s^2 = (x-x_Q)^2 + (y-y_Q)^2$. At first we assume that the density ρ is a constant. Notice that

$$s^2 + z_0^2 \geq 2z_0 z - z^2 \tag{2-3}$$

then eq. (2-1) can be expanded by

$$V_Q = G\rho \iint_\tau \frac{h}{L'} dxdy + \frac{1}{2} G\rho \iint_\tau \int_0^h \frac{2z_0 z - z^2}{L'^3} dzdxdy + \frac{3}{8} G\rho \iint_\tau \int_0^h \frac{\left(2z_0 z - z^2\right)^2}{L'^5} dzdxdy \tag{2-4}$$

where $L'^2 = s^2 + z_0^2$ is the distance between the point Q and the current point on the sea level τ.

The equation (2-4) gives the potential of the topography at the point Q. The terms which contain the inverse of the power of L' higher than 5 are omitted. In the next discussion we separate the topography into a Bouguer plate with thickness h_p and a terrain undulation around the point P.

The Bouguer plate generates a potential at the point Q:

$$V_Q^B = G\rho \iint_\tau \int_0^{h_p} \frac{dz}{L} dxdy$$

$$= \lim_{s\to\infty} 2\pi G\rho h_p s - 2\pi G\rho\left(z_0 h_p - \frac{1}{2} h_p^2\right) \tag{2-5}$$

where V_Q^B denotes the potential of the Bouguer plate. The first term in (2-5) becomes infinite when $s\to\infty$. This infinity is due to the plane approximation. In spherical approximation this term is a constant (cf. Moritz, 1968). From (2-5) we can see that the potential of the Bouguer plate is a function of the flight altitude, z_0. Obviously, the potential of the Bouguer plate is not a regular potential because it does not vanish at infinity. The terrain undulation has the contribution to the geopotential at the point Q:

$$V_Q^T = G\rho \iint_\tau \int_{h_p}^{h} \frac{dz}{L}\, dxdy$$

$$= G\rho \iint_\tau \frac{h-h_p}{L'}\, dxdy + \frac{1}{2} G\rho \iint_\tau \int_{h_p}^{h} \frac{2z_0 z - z^2}{L'^3}\, dxdy \tag{2-6}$$

where V_Q^T denotes the potential of the terrain undulation around the point Q.

Now we consider the first and second derivatives of the potential of the topography.

The vertical derivative of the potential of the topography is given by

$$V_z = \frac{\partial V_Q}{\partial z_0} = G\rho \iint_\tau \int_0^h \frac{\partial}{\partial z_0} \frac{1}{L}\, dzdxdy$$

$$= V_z' + V_z'' \tag{2-7}$$

with

$$V_z' = G\rho \iint_\tau \int_0^{h_p} \frac{\partial}{\partial z_0} \frac{1}{L}\, dzdxdy \tag{2-8}$$

$$V_z'' = G\rho \iint_\tau \int_{h_p}^{h} \frac{\partial}{\partial z_0} \frac{1}{L}\, dzdxdy \tag{2-9}$$

Obviously, V_z' is the contribution of the Bouguer plate and V_z'' is the contribution of the terrain undulation. Notice that

$$\iint_\tau \frac{1}{\sqrt{s^2+(z_0-\alpha)^2}}\, dxdy = \int_0^{2\pi} d\theta \int_0^{\infty} \frac{sds}{\sqrt{s^2+(z_0-\alpha)^2}}$$

$$= 2\pi \lim_{s\to\infty} s - 2\pi\,(z_0 - \alpha) \tag{2-10}$$

where $\alpha > 0$ is a parameter, then we obtain

$$V_z' = G\rho \iint_\tau \left(-\frac{1}{L}\right)\Bigg|_0^{h_p} dxdy$$

$$= G\rho \iint_\tau \left[\frac{1}{\sqrt{s^2+z_0^2}} - \frac{1}{\sqrt{s^2+(z_0-h_p)^2}} \right] dxdy$$

$$= -2\pi G\rho h_p \tag{2-11}$$

This is nothing else than the attraction of the Bouguer plate, where the computation point is not only on the Bouguer plate but also at the flight altitude. Equation (2-11) means that the attraction of a Bouguer plate is a constant on and above the Bouguer plate and it does not change with the flight altitude z_0.

The effect of the terrain undulation is given by

$$V_z'' = G\rho \iint_\tau \left(-\frac{1}{L} \right) \Bigg|_{h_p}^{h} dxdy$$

$$= -G\rho z_0 \iint_\tau \frac{h-h_p}{L'^3} dxdy - \frac{1}{2} G\rho \iint_\tau \frac{h^2-h_p^2}{L'^3} dxdy \tag{2-12}$$

The attraction of the mass above the sea level at the point Q is then

$$V_z = -2\pi G\rho h_p - \frac{1}{2} G\rho \iint_\tau \frac{2z_0(h-h_p) + (h^2-h_p^2)}{L'^3} dxdy \tag{2-13}$$

In a mountainous area the first term in (2-13) is the principal value and much bigger than the last term in eq. (2-13).

In the same way we can get the horizontal derivatives of the potential of the topography. The horizontal derivatives of the potential of the topography are

$$V_x = \frac{\partial V_Q}{\partial x_p} = G\rho \iint_\tau \int_0^h \frac{\partial}{\partial x_p} \frac{1}{L} dzdxdy \tag{2-14}$$

$$V_y = \frac{\partial V_Q}{\partial y_p} = G\rho \iint_\tau \int_0^h \frac{\partial}{\partial y_p} \frac{1}{L} dzdxdy \tag{2-15}$$

We define the horizontal derivatives of the potential of the Bouguer plate as:

$$\iint_\tau \int_0^{h_p} \frac{\partial}{\partial x_p} \frac{1}{L} dzdxdy = \iint_\tau \int_0^{h_p} \frac{x-x_p}{L^3} dzdxdy \tag{2-16a}$$

$$\iint_\tau \int_0^{h_p} \frac{\partial}{\partial y_p} \frac{1}{L} dzdxdy = \iint_\tau \int_0^{h_p} \frac{y-y_p}{L^3} dzdxdy \tag{2-16b}$$

It is easy to prove that the Bouguer plate has no effect on the horizontal derivatives:

Introduce the polar coordinate system:

$$X = x - x_p = s \cos \theta$$

$$Y = y - y_p = s \sin \theta \tag{2-17}$$

so that have

$$\iint_\tau \int_0^{h_p} \frac{x-x_p}{L^3} dzdxdy = \int_0^{h_p} dz \int_0^{2\pi} \cos \theta \, d\theta \int_0^{\infty} \frac{s^2 \, ds}{\left[s^2+(z_0-z)^2\right]^{3/2}}$$

$$= 0 \tag{2-18}$$

In the same way we have

$$\iint_\tau \int_0^{h_p} \frac{y-y_p}{L^3} dzdxdy = 0 \tag{2-19}$$

Therefore the horizontal derivatives of the potential of the topography are given by

$$V_x = G\rho \iint_\tau \frac{h-h_p}{L^3} (x-x_p) dxdy + \frac{3}{2} G\rho \iint_\tau \int_{h_p}^{h} \frac{2z_0 z-z^2}{L^5} (x-x_p) dzdxdy \tag{2-20}$$

$$V_y = G\rho \iint_\tau \frac{h-h_p}{L^3} (y-y_p) dxdy + \frac{3}{2} G\rho \iint_\tau \int_{h_p}^{h} \frac{2z_0 z-z^2}{L^5} (y-y_p) dzdxdy \tag{2-21}$$

Furthermore, we consider the second derivatives of the potential of the topography at the point Q. It is easy to show that a Bouguer plate has no effect on the gravity gradient. As shown above, the horizontal derivatives of the potential of al Bouguer plate are zero. Therefore, the derivatives such as V^B_{xx}, V^B_{xy}, V^B_{xz}, V^B_{yz}, V^B_{yy}, are equal to zero. The superscript B denots the effect of the Bouguer plate.

It is also not difficult to prove the derivatives V^B_{zz} is equal to zero:

$$V^B_{zz} = \iint_\tau \int_0^{h_p} \frac{\partial^2}{\partial z_0^2} \frac{1}{L} dzdxdy$$

$$= \frac{\partial}{\partial z_0} \iint_\tau \left. \frac{-1}{L} \right|_0^{h_p} dxdy$$

$$= \frac{\partial}{\partial z_0} \left(- 2\pi G\rho h_p\right) = 0 \tag{2-22}$$

Based on the above discussion we can see that the Bouguer plate has no effect on the gradient data. Therefore, it is not possible to recover the effect of the Bouguer plate on gravity and the potential by processing the aerial gravity gradient data only.

By using eqs. (2-18) and (2-22) we have the second derivatives of the potential of the topography

$$V_{xx} = \frac{\partial^2 V_Q}{\partial x_p^2} = - G\rho \iint_\tau \int_{h_p}^{h} \frac{dz}{L^3} dxdy + 3\, G\rho \iint_\tau \int_{h_p}^{h} \frac{(x-x_p)^2}{L^5} dzdxdy \tag{2-23}$$

$$V_{xy} = \frac{\partial^2 V_Q}{\partial x_p \partial y_p} = 3\, G\rho \iint_\tau \int_{h_p}^{h} \frac{(x-x_p)(y-y_p)}{L^5} dzdxdy \tag{2-24}$$

$$V_{yy} = \frac{\partial^2 V_Q}{\partial y_p^2} = - G\rho \iint_\tau \int_{h_p}^{h} \frac{dz}{L^3} dxdy + 3\, G\rho \iint_\tau \int_{h_p}^{h} \frac{(y-y_p)^2}{L^5} dzdxdy \tag{2-25}$$

$$V_{xz} = \frac{\partial^2 V_Q}{\partial x_p \partial z_0} = - 3\, G\rho \iint_\tau \int_{h_p}^{h} \frac{(x-x_p)(z_0-z)}{L^5} dzdxdy \tag{2-26}$$

$$V_{yz} = \frac{\partial^2 V_Q}{\partial y_p \partial z_0} = - 3\, G\rho \iint_\tau \int_{h_p}^{h} \frac{(y-y_p)(z_0-z)}{L^5} dzdxdy \tag{2-27}$$

$$V_{zz} = \frac{\partial^2 V_Q}{\partial z_p^2} = - G\rho \iint_\tau \int_{h_p}^{h} \frac{dz}{L^3} dxdy + 3\, G\rho \iint_\tau \int_{h_p}^{h} \frac{(z_0-z)^2}{L^5} dzdxdy \tag{2-28}$$

Between the six components V_{xx}, V_{xy}, ..., there are five independent components. A strict condition is

$$V_{xx} + V_{yy} + V_{zz} = 0 \tag{2-29}$$

The derivation of the formulas are routine and tedious, but it helps us to understand better the physical meaning of the reduction and helps us to figure out the role of the Bouguer plate and the terrain undulation in the airborne gravity gradiometry.

Next we consider a more general case where the density of the topography is not a constant. We separate the density function into a constant part and a variable part:

$$\rho(x, y, z) = \rho_0 + \delta\rho(x, y, z) \tag{2-30}$$

where ρ_0 is a constant and $\delta\rho$ is the variable part of the density of the topography which is sometimes called the density anomaly (Forsberg, 1984). We assume that this density function is known. For the constant density ρ_0 all formulas derived above remain valid. Therefore we need only to consider the potential caused by the variable part of the density.

The potential caused by the density anomaly $\delta\rho$ is given by

$$\Delta V_Q = G \iint_\tau \int_0^h \frac{\delta\rho}{L}\, dz\,dx\,dy \tag{2-31}$$

$$\Delta V_Q = G \iint_\tau \int_0^h \frac{\delta\rho}{L'} \left(1 + \frac{1}{2}\,\frac{2z_0 z - z^2}{L'^2}\right) dz\,dx\,dy \tag{2-32}$$

Define the new functions

$$\delta\rho_M = \int_0^h \delta\rho(x, y, z)\, dz \tag{2-33}$$

$$\delta\rho_N = \int_0^h \left(2z_0 z - z^2\right) \delta\rho(x, y, z)\, dz \tag{2-34}$$

Then (2-32) can be written as

$$\Delta V_Q = G\, \delta\rho_M * \frac{1}{L'} + \frac{1}{2} G\, \delta\rho_N * \frac{1}{L'^3} \tag{2-35}$$

where "*" denotes the convolution of two functions. Therefore the numerical evaluation of (2-35) can be done by using the efficient fast Fourier transformation. The first derivatives of the potential ΔV_Q are given by:

$$\Delta V_x = \frac{\partial \Delta V_Q}{\partial x_p} = G \iint_\tau \frac{x - x_p}{L'^3}\, \delta\rho_M(x, y)\, dx\,dy + \frac{3}{2} G \iint_\tau \frac{x - x_p}{L'^5}\, \delta\rho_N(x, y)\, dx\,dy \tag{2-36}$$

$$\Delta V_y = \frac{\partial \Delta V_Q}{\partial y_p} = G \iint_\tau \frac{y-y_p}{L'^3} \delta\rho_M(x,y)\,dxdy + \frac{3}{2} G \iint_\tau \frac{y-y_p}{L'^5} \delta\rho_N(x,y)\,dxdy \tag{2-37}$$

$$\Delta V_z = \frac{\partial \Delta V_Q}{\partial z_p} = -G z_0 \iint_\tau \frac{\delta\rho_M}{L'^3}\,dxdy - \frac{3}{2} G z_0 \iint_\tau \frac{\delta\rho_N}{L'^5}\,dxdy \tag{2-38}$$

If the density is not a constant, then the second derivatives of the Bouguer plate is no longer equal to zero. Therefore we have:

$$\Delta V_{xx} = \frac{\partial^2 \Delta V_Q}{\partial x_p^2} = G \iint_\tau \delta\rho_M \left[-\frac{1}{L'^3} + 3\frac{(x-x_p)^2}{L'^5} \right] dxdy + \frac{1}{2} G \iint_\tau \delta\rho_N \left[-\frac{3}{L'^3} + 15\frac{(x-x_p)^2}{L'^7} \right] dxdy \tag{2-39}$$

$$\Delta V_{xy} = \frac{\partial^2 \Delta V_Q}{\partial x_p \partial y_p} = 3G \iint_\tau \frac{(x-x_p)(y-y_p)}{L'^5} \delta\rho_M\,dxdy + \frac{15}{2} G \iint_\tau \frac{(x-x_p)(y-y_p)}{L'^7} \delta\rho_N\,dxdy \tag{2-40}$$

$$\Delta V_{yy} = \frac{\partial^2 \Delta V_Q}{\partial y_p^2} = G \iint_\tau \delta\rho_M \left[-\frac{1}{L'^3} + 3\frac{(y-y_p)^2}{L'^5} \right] dxdy + \frac{1}{2} G \iint_\tau \delta\rho_N \left[-\frac{3}{L'^3} + 15\frac{(y-y_p)^2}{L'^7} \right] dxdy \tag{2-41}$$

$$\Delta V_{xz} = \frac{\partial^2 \Delta V_Q}{\partial x_p \partial y_p} = -3G z_0 \iint_\tau \frac{x-x_p}{L'^5} \delta\rho_M\,dxdy - \frac{15}{2} z_0 \iint_\tau \frac{x-x_p}{L'^7} \delta\rho_N\,dxdy \tag{2-42}$$

$$\Delta V_{yz} = \frac{\partial^2 \Delta V_Q}{\partial y_p \partial z_p} = -3G z_0 \iint_\tau \frac{y-y_p}{L'^5} \delta\rho_M \, dxdy -$$

$$-\frac{15}{2} z_0 \iint_\tau \frac{y-y_p}{L'^7} \delta\rho_N \, dxdy \tag{2-43}$$

$$\Delta V_{zz} = \frac{\partial^2 \Delta V_Q}{\partial z_0^2} = G \iint_\tau \left[-\frac{1}{L'^3} + 3\frac{z_0^2}{L'^5} \right] \delta\rho_M \, dxdy +$$

$$+\frac{1}{2} G \iint_\tau \left[-\frac{3}{L'^3} + 15\frac{z_0^2}{L'^7} \right] \delta\rho_N \, dxdy \tag{2-44}$$

The inverse of the higher powers of the distance L' appearing in the formulas means that the derivatives, such as ΔV_{xx}, ΔV_{yy}, ..., depend strongly on the value of the innermost zone. The effect of the remote zone is very small and vanishes very fast. In the numerical computation the integration region can be taken small, but the data should be dense enough around the computation point, in order to ensure the computation accuracy.

The higher power of the inverse of the distance L' are omitted in the above formulas. These higher power terms contain very high frequencies of the potential. In the numerical computation the omitted terms may not cause serious error in the results.

The formulas derived above are in convolution form and the efficient numerical computation method - fast Fourier transformation can be used.

2.2.2 Helmert's Second Condensation

It is a well-known fact that the indirect effect of the Bouguer reduction on the geoid is very large (Heiskanen and Moritz, 1967, p. 142) and it is not suitable for the determination of the geoid.

In this paragraph we introduce the Helmert's second condensation into airborne gradiometry reduction. The Helmert's second condensation is to press the topographic mass onto the geoid, or, in a good approximation, onto the sea level.

We assume the density is constant, so that the density of the condensed layer is ρh.

The potential of the condensed layer at the point Q is given by (cf. Figure 1):

$$W_Q = G \iint_\tau \frac{\rho h}{L'} \, dxdy \tag{2-45}$$

First we consider the condensation of a Bouguer plate with thickness h_p into a layer.

The density of the layer condensed from a Bouguer plate is $\rho \cdot h_p$ and its potential at the point Q is

$$W_Q^B = G\rho \iint_\tau \frac{h_p}{L'}\, dxdy$$

$$= 2\pi G\rho\, h_p \lim_{s\to\infty} s - 2\pi z_0 h_p \tag{2-46}$$

Combining (2-5) and (2-46) we obtain the variation (change) of the potential of the Bouguer plate due to Helmert's condensation:

$$\delta V_Q^B = - V_Q^B + W_Q^B$$

$$= - \pi\, G\rho\, h_p^2 \tag{2-47}$$

The variation of the potential δV_Q^B is independent of the flight altitude and is a constant above and on the Bouguer plate. Eq. (2-47) means that if a Bouguer plate is pressed into a layer, the potential will change at point Q a constant $-\pi G\rho h_p^2$. This term is small. If we take $\rho = 2.67$ g/cm^3, $h_p = 1000$ meters, the term $-\pi G\rho h_p^2$ causes a geoid undulation change of 5 cm.

Roughly speaking, the condensed Bouguer layer generates almost the same potential as the Bouguer plate does. The difference $\pi G\rho h_p^2$ is a small quantity and it can be neglected for the first approximation. But if the geoid is needed to be precisely determined, this term is also important and cannot be omitted.

Equation (2-45) can be written as

$$W_Q = G\rho \iint_\tau \frac{h-h_p}{L'}\, dxdy + G\rho \iint_\tau \frac{h_p}{L'}\, dxdy$$

$$= W_Q^B + G\rho \iint_\tau \frac{h-h_p}{L'}\, dxdy \tag{2-48}$$

Combining eqs. (2-5), (2-6) and (2-46), (2-48) we get the change (at point Q) of the potential of the topography after Helmert's condensation:

$$\delta V_Q = - V_Q + W_Q$$

$$= - \pi G\rho\, h_p^2 - \frac{1}{2} G\rho \iint_\tau \frac{z_0\left(h^2-h_p^2\right) - \frac{1}{3}\left(h^3-h_p^3\right)}{L'^3}\, dxdy \tag{2-49}$$

The last term in (2-49) is one order smaller than the first term and can be neglected except in some extreme cases.

Now let us consider the variation of the derivatives of the potential due to the Helmert's second condensation.

The vertical derivative of the potential of the condensed topography at point Q is

$$W_z = \frac{\partial W_Q}{\partial z_0} = -G\rho z_0 \iint_\tau \frac{h}{L'^3} dxdy \tag{2-50}$$

We separate (2-50) into

$$W_z = W_z^B + W_z^T \tag{2-51}$$

with

$$W_z^B = -G\rho z_0 \iint_\tau \frac{h_p}{L'^3} dxdy \tag{2-52}$$

$$W_z^T = -G\rho z_0 \iint_\tau \frac{h - h_p}{L'^3} dxdy \tag{2-53}$$

Obviously (2-52) is the attraction of the condensed Bouguer plate and (2-53) is the attraction of the condensed terrain variations with respect to the Bouguer plate.

Notice that

$$\iint_\tau \frac{1}{L'^3} dxdy = \int_0^{2\pi} d\theta \int_0^\infty \frac{sds}{\left(s^2+z_0^2\right)^{3/2}}$$

$$= \frac{2\pi}{z_0} \tag{2-54}$$

we have

$$W_z^B = -2\pi G\rho h_p \tag{2-55}$$

Comparing (2-55) with (2-11), we find that the condensed Bouguer plate layer generates the same attraction as the Bouguer plate does.

Therefore the attraction of the condensed topography is given by combining (2-51), (2-53) and (2-55):

$$W_z = -2\pi G\rho h_p - G\rho z_0 \iint_\tau \frac{h-h_p}{L'^3} dxdy\,, \tag{2-56}$$

and the difference in attraction between the actual potential and tne potential of the condensed topography is

$$\delta V_z = - V_z + W_z = \frac{1}{2} G \rho \iint_\tau \frac{h^2 - h_p^2}{L'^3} dxdy \tag{2-57}$$

It is interesting to compare (2-57) with the result of Vaniček and Kleusberg (1987, eq. (14)). They are almost the same. The difference is just due to the location of the computation point. If the computation point Q on the flight altitude is down to the point P on the topography, then eq. (2-57) is reduced to the Vaniček-Kleusberg result. If the computation point Q is down to the condensed layer, δV_z is reduced to the classic terrain correction (cf. Wang and Rapp, 1989).

Because we have:

$$\iint_\tau \frac{x - x_p}{L'^3} dxdy = \int_0^{2\pi} \cos\theta \, d\theta \int_0^\infty \frac{s^2 ds}{\left(s^2 + z_0^2\right)^{3/2}} = 0, \tag{2-58}$$

$$\iint_\tau \frac{y - y_p}{L'^3} dxdy = \int_0^{2\pi} \sin\theta \, d\theta \int_0^\infty \frac{s^2 ds}{\left(s^2 + z_0^2\right)^{3/2}} = 0, \tag{2-59}$$

then we have the horizontal derivatives of the potential of the condensed topography can be written as:

$$W_x = \frac{\partial W_Q}{\partial x_p} = G\rho \iint_\tau \frac{h - h_p}{L'^3} \left(x - x_p\right) dxdy \tag{2-60}$$

$$W_y = \frac{\partial W_Q}{\partial y_p} = G\rho \iint_\tau \frac{h - h_p}{L'^3} \left(y - y_p\right) dxdy \tag{2-61}$$

Combining eqs. (2-20) and (2-21) with (2-60) and (2-61) we get the change (actual topography minus condensed) of the horizontal derivatives:

$$\delta V_x = - V_x + W_x = -\frac{3}{2} G\rho \iint_\tau \frac{z_0\left(h^2 - h_p^2\right) - \frac{1}{3}\left(h^3 - h_p^3\right)}{L'^5} \left(x - x_p\right) dxdy \tag{2-62}$$

$$\delta V_y = - V_y + W_y = -\frac{3}{2} G\rho \iint_\tau \frac{z_0\left(h^2 - h_p^2\right) - \frac{1}{3}\left(h^3 - h_p^3\right)}{L'^5} \left(y - y_p\right) dxdy \tag{2-63}$$

The variations are one order smaller than the horizontal derivatives of the potential of the topography. It is easy to show that the condensed Bouguer plate has no effect on the gradient data.

From eqs. (2-58) and (2-59) we know that the second derivatives of the potential of a condensed Bouguer plate, such as W^B_{xx}, W^B_{xy}, ..., defined by

$$W^B_{xx} = G\rho \iint_\tau h_p \frac{\partial^2}{\partial x_p^2} \frac{1}{L'} dxdy \tag{2-64a}$$

$$V^B_{xy} = G\rho \iint_\tau h_p \frac{\partial^2}{\partial x_p \partial y_p} \frac{1}{L'} dxdy \tag{2-64b}$$

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are equal to zero.

For the component W^B_{zz} we have

$$\begin{aligned} W^B_{zz} &= G\rho \iint_\tau h_p \frac{\partial^2}{\partial z_0^2} \frac{1}{L'} dxdy \\ &= - G\rho\, h_p \frac{\partial}{\partial z_0} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{z_0}{L'^3} dxdy \\ &= - 2\pi G\rho\, h_p \frac{\partial}{\partial z_0} (1) = 0 \end{aligned} \tag{2-65}$$

Here we need to point out that the elevation h_p is considered as a constant in all integrals. Obviously, h_p is a function of the point P, therefore the order of the elevation h_p with the partial differential, such as it is in (2-64a) and (2-64b), cannot be changed. By way of exception the vertical differential $\partial/\partial z_0$ and $\partial^2/\partial z_0^2$ are changable with the h_p because the elevation h_p (x_p, y_p) is not a function of the flight altitude z_0. The second derivatives of the potential of the condensed topography are given by

$$W_{xx} = \frac{\partial^2}{\partial x_p^2} W_Q = - G\rho \iint_\tau \frac{h-h_p}{L'^3} dxdy + 3G\rho \iint_\tau \frac{h-h_p}{L'^5} (x-x_p)^2 dxdy \tag{2-66}$$

$$W_{xy} = \frac{\partial^2}{\partial x_p \partial y_p} W_Q = 3\, G\rho \iint_\tau \frac{h-h_p}{L'^5} (x-x_p)(y-y_p) dxdy \tag{2-67}$$

$$W_{yy} = \frac{\partial^2}{\partial y_p^2} W_Q = -G\rho \iint_\tau \frac{h-h_p}{L'^3}\,dxdy + 3G\rho \iint_\tau \frac{h-h_p}{L'^5}(x-x_p)^2\,dxdy \tag{2-68}$$

$$W_{xz} = \frac{\partial^2}{\partial x_p \partial z_0} W_Q = -3\,G\rho\, z_0 \iint_\tau \frac{h-h_p}{L'^5}(x-x_p)\,dxdy \tag{2-69}$$

$$W_{yz} = \frac{\partial^2}{\partial x_p \partial z_0} W_Q = -3\,G\rho\, z_0 \iint_\tau \frac{h-h_p}{L'^5}(y-y_p)\,dxdy \tag{2-70}$$

$$W_{zz} = \frac{\partial^2}{\partial z_0^2} W_Q = -G\rho \iint_\tau \frac{h-h_p}{L'^3}\,dxdy + 3G\rho \iint_\tau \frac{h-h_p}{L'^5} z_0^2\,dxdy \tag{2-71}$$

By combining the above formulas (2-66) - (2-71) with the second derivatives of the potential of the topography, eqs. (2-23) - (2-28), we obtain the change of the second derivatives of the potential due to the topography condensation:

$$\delta V_{xx} = -V_{xx} + W_{xx} = \frac{3}{2} G\rho \iint_\tau \frac{H}{L'^5}\,dxdy - \frac{15}{2} G\rho \iint_\tau \frac{(x-x_p)^2}{L'^7} H\,dxdy \tag{2-72}$$

$$\delta V_{xy} = -V_{xy} + W_{xy} = -\frac{15}{2} G\rho \iint_\tau \frac{H}{L'^7}(x-x_p)(y-y_p)\,dxdy \tag{2-73}$$

$$\delta V_{yy} = -V_{yy} + W_{yy} = \frac{3}{2} G\rho \iint_\tau \frac{H}{L'^5}\,dxdy - \frac{15}{2} G\rho \iint_\tau \frac{(y-y_p)^2}{L'^7} H\,dxdy \tag{2-74}$$

$$\delta V_{xz} = -V_{xz} + W_{xz} = -\frac{3}{2} G\rho \iint_\tau \frac{h^2-h_p^2}{L'^5}(x-x_p)\,dxdy +$$

$$+\frac{15}{2} G\rho \iint_\tau \frac{x-x_p}{L'^7}\left[z_0^2\left(h^2-h_p^2\right) - z_0\left(h^3-h_p^3\right) + \frac{1}{4}\left(h^4-h_p^4\right)\right] dxdy \tag{2-75}$$

$$\delta V_{yz} = -V_{yz} + W_{yz} = -\frac{3}{2} G\rho \iint_\tau \frac{h^2-h_p^2}{L'^5}(x-x_p)\,dxdy +$$

$$+\frac{15}{2}G\rho\iint_{\tau}\frac{x-x_p}{L'^7}\left[z_0^2\left(h^2-h_p^2\right)-z_0\left(h^3-h_p^3\right)+\frac{1}{4}\left(h^4-h_p^4\right)\right]dxdy \tag{2-76}$$

$$\delta V_{zz}=-V_{zz}+W_{zz}=-\frac{3}{2}G\rho\iint_{\tau}\frac{1}{L'^5}\left[-3z_0\left(h^2-h_p^2\right)+\right.$$

$$\left.+\left(h^3-h_p^3\right)\right]dxdy-\frac{15}{8}G\rho\iint_{\tau}\frac{1}{L'^7}\left[4\,z_0\left(h^2-h_p^2\right)-\right.$$

$$\left.-8z_0^2\left(h^3-h_p^3\right)+5z_0\left(h^4-h_p^4\right)-\left(h^5-h_p^5\right)\right]dxdy \tag{2-77}$$

with

$$H=\int_{h_p}^{h}\left(2z_0\,z-z^2\right)dz=\left(h^2-h_p^2\right)-\frac{1}{3}\left(h^3-h_p^3\right) \tag{2-78}$$

The above formulas are complicated. But they can be written in the convolution form and the fast computation method - fast Fourier transformation can be applied. Now we consider the case in which the density of the topography is not a constant.

Following eq. (2-30) we separate the density of the topography into a constant part and a variaiable part. We consider only the effect of the variable density. Using Helmert's second condensation we condense the topography whose density varies into the sea level with the surface density $\delta\rho_M$ defined by eq. (2-33). Then the potential of this layer at point Q is

$$\Delta W_Q=G\iint_{\tau}\frac{\delta\rho_M}{L'}dxdy \tag{2-79}$$

Comparing (2-79) with (2-32) we find the change of the potential due to the condensation is

$$\Delta\delta V_Q=-\Delta V_Q+\Delta W_Q$$

$$=G\frac{1}{2}\iint_{\tau}\frac{\delta\rho_N}{L'^3}dxdy \tag{2-80}$$

where $\delta\rho_M$, $\delta\rho_N$ where defined by (2-33) and (2-34). Here we need to point out that the attraction of a Bouguer plate with a variable density is no longer a constant above the Bouguer plate and the second derivatives of the Bouguer plate are no longer zero.

If the density anomaly $\delta\rho$ is a function of the variable x and y, only, then eqs. (2-33) and (2-34) become:

$$\delta\rho_M = h \cdot \delta\rho\,(x,y) \tag{2-81}$$

$$\delta\rho_N = \left(h^2 z_0 - \frac{1}{3} h^2\right) \delta\rho\,(x,y) \tag{2-82}$$

Just as in the above derivations we give the corrections to the first and second derivatives of the potential which were generated by the density anomaly $\delta\rho$:

$$\delta\Delta V_x = \frac{\partial}{\partial x_p} \Delta V_Q = -\frac{3}{2} G \iint_\tau \frac{x-x_p}{L'^5} \delta\rho_N \, dxdy \tag{2-83}$$

$$\delta\Delta V_y = \frac{\partial}{\partial y_p} \Delta V_Q = -\frac{3}{2} G \iint_\tau \frac{y-y_p}{L'^5} \delta\rho_N \, dxdy \tag{2-84}$$

$$\delta\Delta V_z = \frac{\partial}{\partial z_p} \Delta V_Q = G \iint_\tau \int_0^h z\,\delta\rho\,(x,y,z)\,dz \frac{1}{L'^3} \, dxdy -$$

$$-\frac{3}{2} G \iint_\tau \frac{1}{L'^5} \int_0^h \left(2z_0 z - z^2\right)(z_0 - z)\,\delta\rho \, dz\, dxdy \tag{2-85}$$

Similar to eqs. (2-72) - (2-77) we have

$$\delta\Delta V_{xx} = \frac{3}{2} G \iint_\tau \frac{\delta\rho_N}{L'^5} dxdy - \frac{15}{2} G \iint_\tau \frac{(x-x_p)^2}{L'^7} \delta\rho_N \, dxdy \tag{2-86}$$

$$\delta\Delta V_{xy} = -\frac{15}{2} G \iint_\tau \frac{\delta\rho_N}{L'^7} (x-x_p)(y-y_p) \, dxdy \tag{2-87}$$

$$\delta\Delta V_{yy} = \frac{3}{2} G \iint_\tau \frac{\delta\rho_N}{L'^5} dxdy - \frac{15}{2} G \iint_\tau \frac{(y-y_p)^2}{L'^7} \delta\rho_N \, dxdy \tag{2-88}$$

$$\delta\Delta V_{xz} = -3G \iint_\tau \frac{x-x_p}{L'^5} \int_0^h z\,\delta\rho\, dz\, dxdy +$$

$$+\frac{15}{2} G \iint_\tau \frac{x-x_p}{L'^7} \int_0^h \left(2z_0 z - z^2\right)(z_0 - z)\,\delta\rho\, dz\, dxdy \tag{2-89}$$

$$\delta\Delta V_{yz} = -3G \iint_\tau \frac{y-y_P}{L'^5} \int_0^h z\,\delta\rho\, dz\; dxdy +$$

$$+\frac{15}{2} G \iint_\tau \frac{x-x_P}{L'^7} \int_0^h \left(2z_0 z - z^2\right)\left(z_0 - z\right) \delta\rho\, dz\; dxdy \tag{2-90}$$

$$\delta\Delta V_{zz} = \frac{9}{2} G \iint_\tau \frac{1}{L'^5} \int_0^h \left(-2z_0 z + z^2\right) \delta\rho\, dz\; dxdy +$$

$$+\frac{15}{2} G \iint_\tau \frac{z_0}{L'^7} \int_0^h \left(2z_0 z - z^2\right)\left(z_0 - z\right) \delta\rho\, dz\; dxdy \tag{2-91}$$

If the density anomaly is known everywhere, then we can compute the change of the potential and its derivatives at the flight altitutde by using the formulas as derived above. For the airborne gradiometry case, the second derivatives of the potential of the topography can be computed by using the Bouguer reduction or the Helmert's second condensation and are added to the measured gradient data to obtain terrain corrected gradients. By using the gradient data, the gravity disturbance (anomalies) can be recovered. The disturbing potential on the ground then can be determined by using the recovered gravity data. The bias problem - which occurs in the airborne and satellite gradiometry, can be solved by combining other data.

2.2.3 Indirect Effect

Now we consider the indirect effect on the potential of the topography reduction to the geoid.

Basically, the topography reduction in this report is the same as the topography reduction in any textbook. Therefore, the indirect effect is the same as it is described in (Heiskanen and Moritz, 1967, p. 142, Wichiencharoen, 1982).

In the Bouguer reduction the mass above the sea level is removed and the indirect effect is very large.

The indirect effect of Helmert's condensation is given by (Vaniček and Kleusberg, 1987):

$$\delta V_I = -\pi G\rho h_p^2 - \frac{1}{6} G\rho \iint_\tau \frac{h^2 - h_p^2}{s^3}\, dxdy \tag{2-92}$$

where $s = \sqrt{(x - x_p)^2 + (y - y_p)^2}$,

the distance between computation point P and current point on the sea level.

Compare (2-92) with (2-49), we find that the change of the potential due to the Helmert's condensation has the same magnitude order as the indirect effect. If the geoid is determined precisely, the indirect effect must be carefully considered.

2.3 Topographic Correction Studied in the Frequency Domain

2.3.1 Bouguer Reduction

The derivation of the formulas for the Bouguer reduction and Helmert's second condensation for airborne gradiometry in the space domain is time consuming. If the reductions are studied in the frequency domain, the problem becomes easier.

We define the 2-D Fourier transformation and its inverse as follows:

$$G(u,v) = F\{f(x,y)\} = \iint_{-\infty}^{\infty} f(x,y)\, e^{2\pi j(xu+yv)}\, dxdy \tag{2-93}$$

$$f(x,y) = F^{-1}\{G(u,v)\} = \iint_{-\infty}^{\infty} G(u,v)\, e^{-2\pi j(xu+yv)}\, dudv \tag{2-94}$$

where F and F^{-1} denote the Fourier transformation and its inverse and u, v are frequency variables. This definition is a little different from its definition in most text books (e.g., Papoulis, 1968, Bracewell, 1965), but it is more convenient for the numerical computation if the subroutines are written for the discrete Fourier transformation which is defined similarly as (2-93) and (2-94).

Applying 2-D Fourier transformation to eq. (2-1) we obtain the Fourier transform of the potential V_Q (cf. Parker, 1972):

$$F\{V_Q\} = G e^{-2\pi\omega z_0} \frac{1}{2\pi\omega^2} \sum_{n=1}^{\infty} \frac{(2\pi\omega)^n}{n!} F\{\rho\, h^n\} \tag{2-95}$$

here we assume $\rho = \rho\,(x, y)$, is a function of the variables x, y, and the circular frequency $\omega=(u^2+v^2)^{1/2}$; z_0 is the flight altitude of the airborne above the sea level. If the density ρ is assumed to be a constant it can be taken out from the Fourier transformation, and eq. (2-95) is reduced to Parker's solution.

As it was shown (Parker, 1972) that the series convergent with a convergence rate better than $(h_{max}/z_0)^n$, where h_{max} is the maximum of the elevation of the topography. The physical meaning of the series is very clear. Writing (2-95) in the form

$$F\{V_Q\} = G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{-2\pi\omega(z_0-h)} - e^{-2\pi\omega z_0}\right)\right\}$$

$$= G \frac{1}{2\pi\omega^2} \sum_{n=1}^{\infty} \frac{(2\pi\omega)^n}{n!} F\left\{\rho\,(h-z_0)^n - \rho\,(-z_0)^n\right\} \tag{2-96}$$

and taking the first two terms, we get

$$F\{V_Q|_0\} = G\left[\frac{1}{\omega}F\{h\rho\} + \pi F\left\{\left(h^2 - 2z_0h\right)\rho\right\}\right] \tag{2-97}$$

Assuming that the density of the topography is constant, then eq. (2-97) is nothing other than the Fourier transformation of the potential of a Bouguer plate with thickness h plus a correction due to the variation of the heights. We show this in the following.

Taking the inverse Fourier transformation of eq. (2-97) it gives

$$\begin{aligned} F^{-1}\{F\{V_Q|_0\}\} &= G\iint_\tau \frac{\rho h}{s}\,dxdy - \pi G\rho\left(2z_0 h - h^2\right) \\ &= G\rho\iint_\tau \frac{h_P}{s}\,dxdy - \pi G\rho\left(2z_0 h - h^2\right) + G\rho\iint_\tau \frac{h-h_P}{s}\,dxdy \\ &= V_Q^B + G\rho\iint_\tau \frac{h-h_P}{s}\,dxdy \end{aligned} \tag{2-98}$$

where s is the distance between the computation point and current point on the sea level. Here we have used the relationship:

$$F\left\{\iint_\tau \frac{f}{s}\,dxdy\right\} = \frac{1}{\omega}F\{f\}$$

and the density ρ is assumed to be a constant.

Therefore the terms n>2 in eq. (2-96) represent the terrain effect. The terrain effect is smaller than the effect of the Bouguer plate, but it is rough when the topography is rugged.

Above computations are referenced to the sea level. In order to speed the convergence of the series (2-95), reference surfaces other than sea level can be chosen. One of the choices is the mean elevation level, $z = h_m$, so that eq. (2-95) becomes

$$F\{V_Q\} = G\,e^{2\pi\omega(z_P - h_m)}\frac{1}{2\pi\omega^2}\sum_{n=1}^{\infty}\frac{(2\pi\omega)^n}{n!}F\left\{(h-h_m)^n\rho - (-h_m)^n\rho\right\} \tag{2-99}$$

Because we have

$$\begin{aligned} &F^{-1}\left\{\omega^n F\left\{(-h_m)^n\right\}\right\} \\ &= F^{-1}\left\{\omega^n (-h_m)^n \delta(u,v)\right\} \end{aligned}$$

$$= \iint_\tau \omega^n (-h_m)^n \delta(u, v) e^{-2\pi j(xu + yv)} dudv$$

$$= 0, \qquad n = 1, 2, \ldots \tag{2-100}$$

where we have used the property of the δ - function

$$\iint_\tau f(u, v)\, \delta(u, v)\, dudv = f(0, 0), \tag{2-101}$$

then we can build up an artificial formula to replace eq. (2-98) if the density ρ is also a constant:

$$F\{V_Q\}_c = G\rho\, e^{-2\pi\omega(z_\sigma h_m)} \left\{ \frac{1}{\omega} F\{h\} + \pi F\{h^2 - 2h\, h_m\} + \right.$$

$$\left. + \frac{1}{2\pi\omega^2} \sum_{n=3}^{\infty} \frac{(2\pi\omega)^n}{n!} F\{(h - h_m)^n\} \right\} \tag{2-102}$$

The function $F\{V_Q\}_c$ is not exactly the Fourier transformation of the potential of the topography with a constant density. Based on (2-100), the Fourier transformation of (2-102) gives the same potential V_Q as the equation (2-99) does. If the density ρ is not a constant, then eq. (2-95) has to be used. The advantage of eq. (2-102) is that the series in equation converges faster than the series in eq. (2-94).

The parameter h_m in (2-102) is defined as the mean elevation in the area. Based on Parker (1972) the parameter $h_m = 1/2\ (h_{max} - h_{min})$, where h_{max} and h_{min} is the maximum and minimum elevation in the area, can be the best choice for the convergence of the series in (2-102). But, if the series converges very fast, the choice of h_m as a mean elevation or the average value of the difference between the maximum and minimum elevation should not play an important role.

In the similar way as Parker did (ibid) one can show that the convergence rate of the series in (2-102) is better than the series

$$\sum \left(\frac{H}{z_\sigma h_m} \right)^n ,$$

where $H = \max\ ((h_{max} - h_m), - (h_{min} - h_m))$.

The vertical derivative of V_Q is easy to get by spectral analysis. As in many text books (e.g., Bracewell, 1965) the relationship between the differential operators and the spectra are given in the following:

$$\frac{\partial}{\partial x} \quad \Rightarrow j2\pi u$$

$$\frac{\partial}{\partial y} \Rightarrow j2\pi v$$

$$\frac{\partial}{\partial z} \Rightarrow -2\pi\omega \tag{2-103}$$

$$\frac{\partial^n}{\partial x^k \partial y^s \partial z^p} \Rightarrow (-1)^p j^{k+s} (2\pi)^n u^k v^s \omega^p$$

where $n = k + s + p$. $k, s, p = 1, 2, 3, \ldots$, and j is unit imaginary number.

The Fourier transformation of the attraction of the topography is given by

$$F\{V_z\} = -2\pi\omega F\{V_Q\} \tag{2-104}$$

Insert eq. (2-96) into eq. (2-104) and taking the first term:

$$F\{V_z\}_0 = -2\pi G F\{\rho h\} \tag{2-105}$$

The inverse Fourier transformation of (2-105) is

$$V_z^0 = F^{-1}\{F\{V_z\}_0\} = -2\pi G\rho h \tag{2-106}$$

It is nothing else but the attraction of a Bouguer plate with thickness h.

By using eq. (2-103) we get the Fourier transformation of the first derivatives of the potential at the flight altitude:

$$F\{V_x\} = j2\pi u F\{V_Q\} \tag{2-107}$$

$$F\{V_y\} = j2\pi v F\{V_Q\} \tag{2-108}$$

where $F\{V_Q\}$ is defined by eq. (2-99) or (2-102).

In the same way we obtain the Fourier transformation of the second derivatives of the potential at the flight altitude:

$$F\{V_{xx}\} = -(2\pi u)^2 F\{V_Q\} \tag{2-109}$$

$$F\{V_{xy}\} = -(2\pi)^2 uv F\{V_Q\} \tag{2-110}$$

$$F\{V_{yy}\} = -(2\pi v)^2 F\{V_Q\} \tag{2-111}$$

$$F\{V_{xz}\} = -j(2\pi)^2 u\omega F\{V_Q\} \tag{2-112}$$

$$F\{V_{yz}\} = -j(2\pi)^2 v\omega F\{V_Q\} \tag{2-113}$$

$$F\{V_{zz}\} = (2\pi\omega)^2 F\{V_Q\} \tag{2-114}$$

Obviously we have

$$F\{V_{xx}\} + F\{V_{yy}\} + F\{V_{zz}\} = 0 \tag{2-115}$$

so that there are five independent second derivatives of the potential.

2.3.2 Helmert's Second Condensation

It is also easy to apply the Fourier transformation to the Helmert's second condensation.

Applying the Fourier transformation to eq. (2-45), we get

$$F\{W_Q\} = G\frac{e^{-2\pi\omega z_0}}{\omega} F\{\rho h\} \tag{2-116}$$

This is the Fourier transform of the potential generated by the condensed topography layer on the sea level. Obviously, the Fourier transform of the change of the potential due to the Helmert's second condensation is (cf. eqs. (2-49) and (2-95)):

$$\begin{aligned} F\{\delta V\} &= F\{-V_Q + W_Q\} \\ &= -G\frac{1}{2\pi\omega^2} e^{-2\pi\omega z_0} \sum_{n=2}^{\infty} \frac{(2\pi\omega)^n}{n!} F\{\rho h^n\} \end{aligned} \tag{2-117}$$

If we expand the exponent in (2-117) in series, then we have

$$F\{\delta V\} = -G\frac{1}{2\pi\omega^2} \sum_{m=0}^{\infty} \sum_{n=2}^{\infty} \frac{(2\pi\omega)^n(-2\pi\omega z_0)^m}{m!\,n!} F\{\rho h^n\} \tag{2-118}$$

The first term of the series in (2-118) is (m=0, n=2)

$$-G\pi F\{\rho h^2\}$$

and its inverse Fourier transformation is

$$-G \pi \rho h^2$$

which is nothing else than the first term in eq. (2-49). Therefore we can expect that the terms (n>3) in eq. (2-117) may be ignored in most computations. This agrees with our earlier study in space domain.

If the topography is condensed on the sea level, then the first derivatives of the geopotential changes at the flight altitude by

$$\begin{aligned} \delta V_x &= F^{-1}\left\{ j2\pi u \, F\left\{\delta V\right\}\right\} \\ \delta V_y &= F^{-1}\left\{ j2\pi v \, F\left\{\delta V\right\}\right\} \\ \delta V_z &= F^{-1}\left\{ -2\pi\omega \, F\left\{\delta V\right\}\right\} \end{aligned} \tag{2-119}$$

where δV_x, δV_y, δV_z are the change of the first derivatives of the geopotential, δV is given by eq. (2-117). The effect of the Helmert's second condensation on the gravity gradient data is also easy to get by

$$\begin{aligned} \delta V_{xx} &= F^{-1}\left\{ -(2\pi u)^2 F\left\{\delta V\right\}\right\} \\ \delta V_{xy} &= F^{-1}\left\{ -(2\pi)^2 uv \; F\left\{\delta V\right\}\right\} \\ \delta V_{yy} &= F^{-1}\left\{ -(2\pi u)^2 \; F\left\{\delta V\right\}\right\} \\ \delta V_{xz} &= F^{-1}\left\{ -j(2\pi)^2 u\omega \, F\left\{\delta V\right\}\right\} \\ \delta V_{yz} &= F^{-1}\left\{ -j(2\pi)^2 v\omega \, F\left\{\delta V\right\}\right\} \\ \delta V_{zz} &= F^{-1}\left\{ (z\pi\omega)^2 \; F\left\{\delta V\right\}\right\} \end{aligned} \tag{2-120}$$

In comparison with eqs. (2-66) - (2-71), eq. (2-120) is simple and easy to use for the computation.

3. Effect of Topography on Determination of the Geoid by Using Analytical Downward Continuation

3.1 Introduction

In the preceeding chapter we studied the effect of the topography on the airborne gradiometry; some corrections are added to the aerial gradient data to eliminate the topographic effect. Now we consider the topography effect in satellite gradiometry. This situation is different from airborne gradiometry because a satellite flies much higher and the high frequencies of the gravity field, which mostly come from the terrain variation of the earth, become small or negligable at the satellite flight altitude. In this chapter we want to find out a topography correction which is not related to the flight altitude of an airborne or a satellite.

In processing the satellite gravity gradient data one can ignore the mass above the reference surface (e.g., the ellipsoid) at first, the gravity disturbance is recovered on the reference surface by using the analytical downward continuation. If such recovered gravity data are used to determine the coefficients of the spherical harmonics of the earth's gravitational potential, no topographic correction is needed. This is the same as the analytical downward continuation of the free-air anomaly from the earth's surface to the ellipsoid for the determination of the coefficients of the spherical harmonics expansion. If these recovered data are used for determination of the geoid, the topographic effect has to be considered. The question is what is the difference between the analytically downward continued potential and the true one on the ellipsoid. In other words: What is the difference between the geoid determined by analytically downward continuation and the true one?

The following work is addressed on this aspect and we focus our attention on the topographic correction to the geoid which is determined by using the analytical downward continuation which is the method used in the processing of the aerial gravity gradient data.

3.2. Remove-Restore Technique - Its Physical Meaning and Mathematical Formulation

In the following we will find the topographic correction to the geopotential which is determined by using the analytical downward continuation procedure. In order to do this, we remove the mass above the reference surface first, then add it back. This technique is called remove-restore technique. This technique is illustrated in the following.

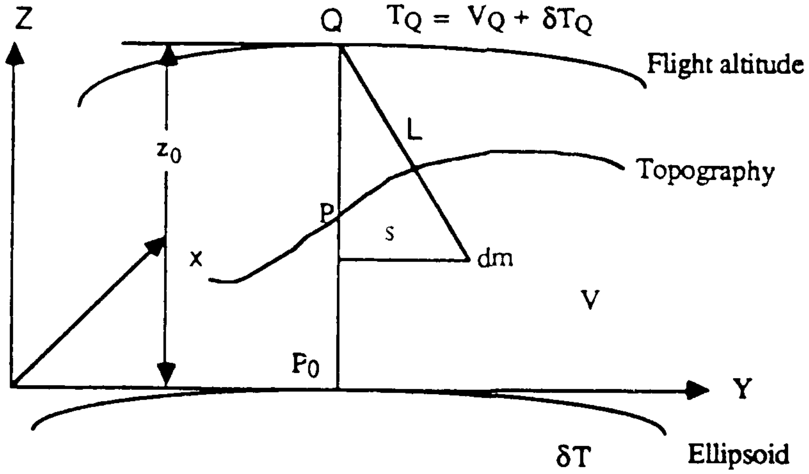

Figure 2. Geometry of computation of the disturbing potential

The disturbing potential T at the flight altitude Q consists of the gravitational potential (V) of the mass above the ellipsoid and the potential (δ_T) caused by the density anomaly under the ellipsoid.

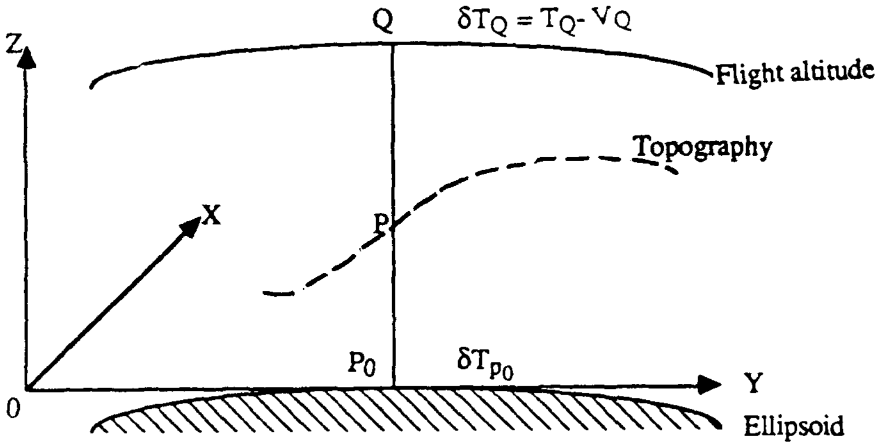

Figure 3. Removal of the mass above the ellipsoid

The mass above the ellipsoid is removed. The residual disturbing potential $\delta T_Q = T_Q - V_Q$ is harmonic above the ellipsoid and can be analytically downward continued to the ellipsoid without any theoretical problem. δT_{p0} is the downward continued residual potential on the ellipsoid.

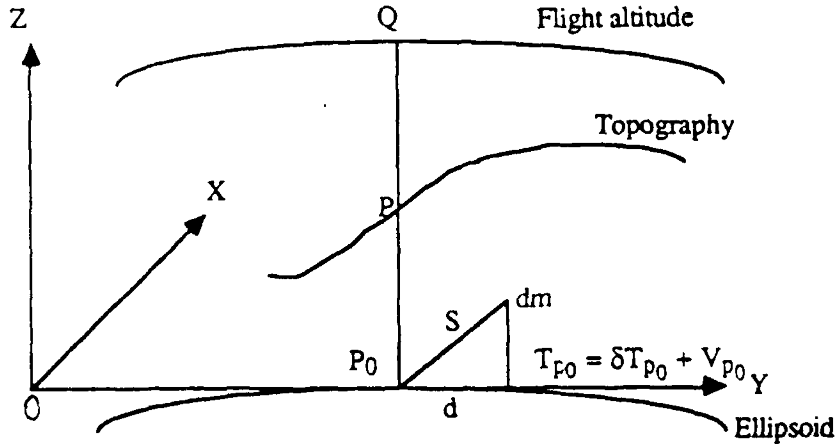

Figure 4. Restoration of the mass above the ellipsoid.

The mass above the ellipsoid is added back. The disturbing potential at point P_0 located on the ellipsoid is $T_{p0} = \delta T_{p0} + V_{p0}$, where V_{p0} is the gravitational potential of the mass above the ellipsoid at point P_0.

Now we assume that the disturbing potential T_Q is analytically downward continued to the ellipsoid: $T^*_{P_0} = \delta T_{P_0} + V^*_{P_0}$, where $T^*_{P_0}$ and $V^*_{P_0}$ are the downward continued potentials of the T_Q, V_Q at the flight altitude, respectively.

Compare $T^*_{P_0}$ with the disturbing potential T_{P_0} which is the original disturbing potential on the ellipsoid, we get

$$\delta V = T_{P_0} - T^*_{P_0} = \delta T_{P_0} + V_{P_0} - \delta T_{P_0} - V^*_{P_0}$$

$$= - V^*_{P_0} + V_{P_0} \tag{3-1}$$

Obviously, δV defined by (3-1) is the difference between the true gravitational potential of the topographic mass at point P_0 and an artificial potential $V^*_{P_0}$ which is the gravitational potential of the topographic mass at point Q (outside the Earth's surface) analytically downward continued to point P_0.

From the above short description of the remove-restore technique we can see, δV is independent of the flight altitude; therefore it is suitable for computation of the topographic effect on the determination of the geoid by using analytical downward continuation. For instance, the disturbing potential T_Q can be computed by processing the satellite or the airborne gravity gradient data analytically downward continued onto the ellipsoid. So the disturbing potential T_{P_0} can be obtained by adding the δV,and at the same time the geoid is also determined from Brun's equation.

We still call δV the topographic correction because it comes from the earth's topography. If the earth is exactly an ellipsoid which coincides with the reference ellipsoid, then δV is equal to zero.

Before we compute the topographic correction δV, we first consider the correction of a spherical Bouguer plate.

3.3 Effect of A Spherical Bouguer Plate on the Analytical Downward Continuation

We approximate the ellipsoid by a sphere with the radius R, the mean radius of the earth. A spherical Bouguer plate has the boundaries $r = R$ and $r = R + h_P$, h_P is the thickness of the Bouguer plate.

The geopotential of the Bouguer plate, just like a point mass or a homogeneous sphere, is given by

$$V_Q = \frac{GM}{R+z_0} \tag{3-2}$$

where V_Q is the geopotential of the Bouguer plate at the flight altitude and z_0 is the flight altitude above the ellipsoid; G is the gravitational constant and M is the mass of the Bouguer plate which is given by

$$M = \frac{4}{3}\pi\rho\left[(R + h_P)^3 - R^3\right]$$

$$= 4\pi\rho\left(R^2 h_p + R h_p^2 + \frac{1}{3} h_p^3\right) \tag{3-3}$$

where we have assumed that the density of the Bouguer plate is a constant.

If we analytically downward continue the potential V_Q from the flight altitude to the ellipsoid, we obtain

$$V^{*}_{P_0} = \frac{GM}{R} = 4\pi G\rho\left(Rh_p + h_p^2 + \frac{1}{3}\frac{h_p^3}{R}\right) \tag{3-4}$$

where $V^{*}_{P_0}$ is the downward continued potential on the ellipsoid at P_0.

At the same time, the gravitational potential of the Bouguer plate at point P_0 is given by

$$V_{P_0} = 2\pi G\rho\left[(R + h_p)^2 - R^2\right]$$

$$= 4\pi G\rho R\, h_p + 2\pi G\rho h_p^2 \tag{3-5}$$

The mean radius of the earth, R, is much larger than the thickness of the Bouguer plate. For example, if we take the maximum thickness of the Bouguer plate as $h_{max} = 10$ km, then the ratio of the thickness of the Bouguer plate and the mean radius of the earth is

$$\frac{h_p}{R} < \frac{10}{6.4 \times 10^3} = 1.6 \times 10^{-3} \tag{3-6}$$

Therefore the third term ($2\pi G\rho h_p^2$) in (3-5) is one thousand times smaller than the second one. Comparing (3-4) with (3-5), we find the potential difference

$$\delta V = - V^{*}_{P_0} + V_{P_0} = - 2\pi G\rho h_p^2 \tag{3-7}$$

The term on the order of h_p/R in δV is neglected.

Therefore the analytically downward continued geopotential of a Bouguer plate from outside onto the ellipsoid differs from the original geopotential by $- 2\pi G\rho\, h_p^2$. This result has been known from many previous publications (e.g., Wichiencharoen, 1982, p. 16, Table 1). Here we emphasize the fact that the potential difference $-2hG\rho h_p^2$ is due to the use of the analytical downward continuation.

3.4 Topographic Effect on the Determination of to the Disturbing Potential on the Ellipsoid by Using Analytical Downward Continuation

Now we consider the topographic effect on the disturbing potential which is analytically downward continued from outside earth's surface onto the ellipsoid. The geometry of the computation of the potential of the topography is drawn in Figure 5.

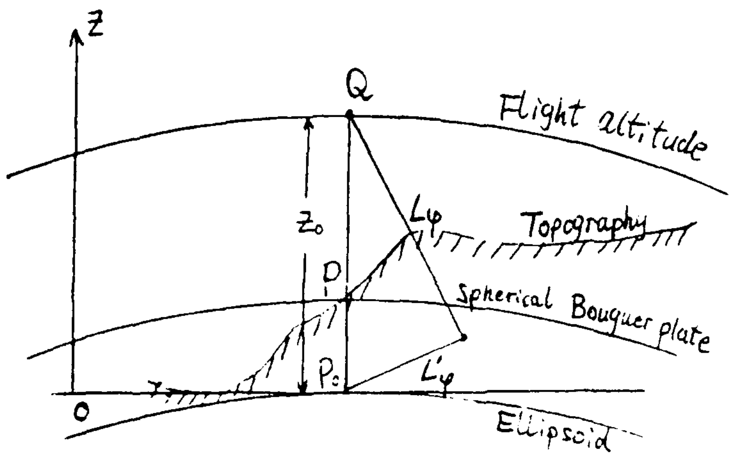

Figure 5. Geometry of the Computation of the Potential of the Topography

The gravitational potential of the mass above the ellipsoid at the flight altitude is given by (cf. Moritz, 1968):

$$V_Q = G\rho R^2 \iint_\sigma \int_0^h \frac{dz}{L_\varphi} d\sigma \tag{3-8}$$

where the density ρ is assumed to be a constant, L_φ is the distance between the point Q and the current point inside the topography; σ is the unit sphere and R is the mean earth radius.

The potential V_Q can be separated into

$$V_Q = V_1 + V_2 \tag{3-9}$$

with

$$V_1 = G\rho R^2 \iint_\sigma \int_0^{h_p} \frac{dz}{L_\varphi} d\sigma \tag{3-10}$$

$$V_2 = G\rho R^2 \iint_\sigma \int_{h_p}^h \frac{dz}{L_\varphi} d\sigma \tag{3-11}$$

where V_1 is the potential of the spherical Bouguer plate and V_2 is the effect of the terrain around point P. Without loss of accuracy (3-11) can be approximated by a plane approximation

$$V_2 = G\rho \iint_\tau \int_{h_p}^h \frac{dz}{L} dxdy \tag{3-12}$$

where τ is the tangential plane of the ellipsoid at point P_0, and

$$L^2 = s^2 + (z_0 - z)^2 \tag{3-13}$$

where L and s were previously defined (p. 3). Now we analytically downward continue V_Q to the ellipsoid to get the potential difference δV. Based on last section the potential difference of a Bouguer plate is $-2\pi G\rho h_p^2$. In the following we consider only the terrain effect V_2.

We assume that the potential V_2 is analytically downward continued to the ellipsoid by the Taylor's series:

$$V_2^{*} = V_2 - z_0 \frac{\partial V_2}{\partial z_0} + \frac{1}{2} z_0^2 \frac{\partial^2 V_2}{\partial z_0^2} - \frac{1}{6} z_0^3 \frac{\partial^3 V_2}{\partial z_0^3} \tag{3-14}$$

For this series we have

$$\frac{\partial}{\partial z_0} \frac{1}{L} = - \frac{z_0 - z}{L^3} \tag{3-15}$$

$$\frac{\partial^2}{\partial z_0^2} \frac{1}{L} = - \frac{1}{L^3} + 3 \frac{(z_0 - z)^2}{L^5} \tag{3-16}$$

$$\frac{\partial^3}{\partial z_0^3} \frac{1}{L} = 9 \frac{z_0 - z}{L^5} + 0\left(\frac{1}{L^7}\right) \tag{3-17}$$

where $0\,(L^{-7})$ denotes the term contains L^{-7}. Putting (3-15), (3-16), (3-17) and (3-12) into (3-14) we get

$$V_2^{*} = G\rho \iint_\tau \int_{h_p}^{h} \frac{dz}{L^{*}}\, dxdy \tag{3-18}$$

with

$$\frac{1}{L^{*}} = \frac{1}{L} + \frac{z_0^2 - 2z_0 z}{2L^3} + \frac{3}{2} \frac{- z_0^3 z + z_0^2 z^2}{L^5} \tag{3-19}$$

Because we have always

$$s^2 + z_0^2 > 2z_0 z - z^2, \text{ or } \frac{2z_0 z - z^2}{s^2 + z_0^2} < 1 \tag{3-20}$$

one can write

$$\frac{1}{L} = \frac{1}{L'}\left[1 + \frac{1}{2}\frac{2z_0 z - z^2}{L'^2} + \frac{3}{8}\frac{\left(2z_0 z - z^2\right)^2}{L'^4} + \ldots\right] \tag{3-21}$$

$$\frac{1}{L^3} = \frac{1}{L'^3}\left[1 + \frac{3}{2}\frac{2z_0 z - z^2}{L'^2} + \ldots\right] \tag{3-22}$$

$$\frac{1}{L^5} = \frac{1}{L'^5}\left[1 + \ldots\right] \tag{3-23}$$

with

$$L'^2 = s^2 + z_0^2 \tag{3-24}$$

Insert (3-21), (3-22) and (3-23) into (3-19), we obtain

$$\frac{1}{L^\bullet} = \frac{1}{L'}\left[1 + \frac{z_0^2 - z^2}{2L'^2} + \frac{3}{8}\frac{-2z_0^2 z + z^4}{L'^4}\right] \tag{3-25}$$

and the potential $V_2^\bullet$ is given by

$$V_2^\bullet = G\rho \iint_\tau \int_{h_p}^{h} \frac{1}{L'} dzdxdy + \frac{1}{2} G\rho \iint_\tau \int_{h_p}^{h} \frac{z_0^2 - z^2}{L'^3} dzdxdy +$$

$$+ \frac{3}{8} G\rho \iint_\tau \int_{h_p}^{h} \frac{-2z_0^2 z^2 + z^4}{L'^5} dzdxdy \tag{3-26}$$

Here we need to point out that the quantity $V_2^\bullet$ has no physical meaning. If we say it has a meaning, it means that it is a fictitious potential created by analytically downward continuing the potential of the terrain from outside the earth onto the ellipsoid.

The potential of the topography at point P_0 is given by

$$V_{p_0} = G\rho R^2 \iint_\sigma \int_0^h \frac{dz}{L'_\varphi}\, d\sigma \tag{3-27}$$

where L'_φ is the distance between the point P_0 and the current point inside the topography.

The potential V_{p_0} can be split into

$$V_{p_0} = V'_{p_0} + V''_{p_0} \tag{3-28}$$

with

$$V'_{p_0} = G\rho R^2 \iint_\sigma \int_0^{h_p} \frac{dz}{L'_\varphi}\, d\sigma \tag{3-29}$$

$$V''_{p_0} = G\rho R^2 \iint_\sigma \int_{h_p}^{h} \frac{dz}{L'_\varphi}\, d\sigma \tag{3-30}$$

The potential V'_{p_0} is the potential of the spherical Bouguer plate, but the computation point p_0 is under the Bouguer plate on the ellipsoid. Therefore (3-29) is the same as eq. (3-5). For the potential V''_{p_0} we take the plane approximation:

$$V''_{p_0} = G\rho \iint_\tau \int_{h_p}^{h} \frac{dz}{S}\, dxdy \tag{3-31}$$

with

$$S^2 = s^2 + z^2 \tag{3-32}$$

We expand (3-31) into a series:

$$V''_{p_0} = G\rho \iint_\tau \int_{h_p}^{h} \frac{1}{L'}\, dzdxdy + \frac{1}{2} G\rho \iint_\tau \int_{h_p}^{h} \frac{z_0^2 - z^2}{L'^3}\, dzdxdy +$$

$$+ \frac{3}{8} G\rho \iint_\tau \int_{h_p}^{h} \frac{\left(z_0^2 - z^2\right)^2}{L'^5}\, dzdxdy \tag{3-33}$$

putting (3-33) and (3-26) together, we get

$$\delta V_2 = - V_2^{\bullet} + V''_{p_0}$$

$$= \frac{3}{8} G\rho\, z_0^4 \iint_\tau \frac{h-h_p}{L'^5} dxdy \tag{3-34}$$

Completely we have the potential difference δV is a sum of the δV_2 and a term $-2\pi G\rho h_p^2$ which is from the downward continuation of the potential through a Bouguer plate:

$$\delta V = -2\pi G\rho h_p^2 + \frac{3}{8} G\rho z_0^4 \iint_\tau \frac{h-h_p}{L'^5} dxdy \tag{3-35}$$

The potential difference δV represented by (3-35) is a function of the flight altitude z_0. This is in contradiction with our earlier statement that the potential difference δV is independent of flight altitude z_0. This contradiction comes from the approximation used in the derivation of δV. In frequence domain the problem can be studied without approximation, and we can get the δV independent of flight altitude as expected (cf. next section).

Equation (3-35) gives the potential difference in space domain approximately. For this equation we have two more things to say:

1. The expression of δV is an approximation, the integrals which contain terms higher than S^{-5} were omitted. The flight altitude is assumed above the topography. The above derivations are also valid, if the computation point is chosen on the topography of the Earth. In this case, eq. (3-35) becomes:

$$\delta V = -2\pi G\rho h_p^2 + \frac{3}{8} G\rho h_p^4 \iint_\tau \frac{h-h_p}{\left(s^2+h_p^2\right)^{5/2}} dxdy \tag{3-36}$$

2. The last term in (3-36) is a very small quantity. A numerical test (Wang, 1989) showed that the last term in (3-36) contributed to the geoid at the millimeter level in a rough mountain area so that it can be neglected in most cases;

The change of the geoid caused by δV is then given by the Brun's formula:

$$\delta N = \frac{\delta V}{\gamma_0} \tag{3-37}$$

where γ_0 is the normal gravity on the ellipsoid.

The first term in (3-36) is primary and its contribution to the geoid can reach 1-2 meters in high mountains, therefore the topographic correction is significant and cannot be neglected.

Now we consider the effect of the topography to the deflections of the vertical.

It is well known that a Bouguer plate with constant density has no contribution to the deflection of the vertical. Here an implicit assumption is made that the elevation h_p is a constant.

Therefore h_p can be viewed as a constant in eq. (3-36) too, when the derivatives are applied. The corrections to the deflections of the vertical due to the δV are given then

$$\delta\xi = -\frac{1}{\gamma_0}\frac{\partial \delta V}{\partial x_p}$$

$$= \frac{15}{8}\frac{G\rho}{\gamma_0} h_p^4 \int_{-\infty}^{\infty}\int \frac{h - h_p}{\left(s^2+h_p^2\right)^{7/2}} (x - x_p)\, dx\, dy \tag{3-38}$$

$$\delta\eta = -\frac{1}{\gamma_0}\frac{\partial \delta V}{\partial y_p}$$

$$= \frac{15}{8}\frac{G\rho}{\gamma_0} h_p^4 \int_{\infty}^{\infty}\int \frac{h - h_p}{\left(s^2+h_p^2\right)^{7/2}} (y - y_p)\, dx\, dy \tag{3-39}$$

3.5 Topographic Correction in the Frequency Domain

Now we study the topographic correction studied in frequency domain.

The geometry of the computation of the topographic correction is drawn in Figure 6.

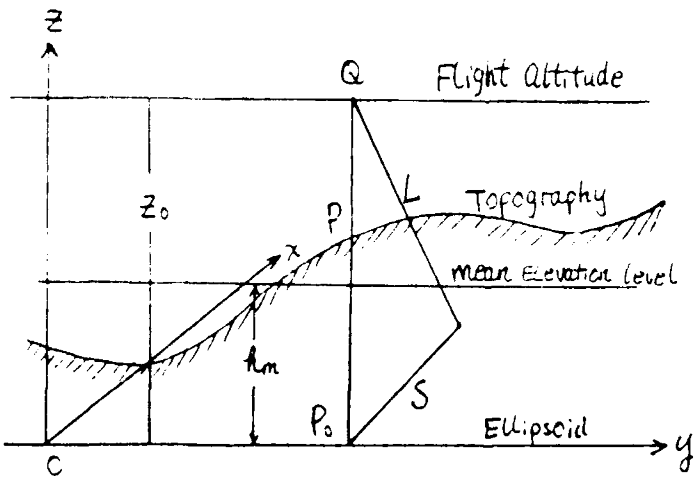

Figure 6. Geometry of the computation of Topographic Correction

In the plane approximation the gravitational potential of the topography at points Q and P_0 is given by:

$$V_Q = G \iint_\tau \rho(x, y) \int_0^h \frac{dz}{L}\, dxdy \tag{3-40}$$

$$V_{P_0} = G \iint_\tau \rho(x, y) \int_0^h \frac{dz}{S}\, dxdy \tag{3-41}$$

Here we have assumed the density of the topography is a function of the the horizontal variables x, y, and the distances between the current point and the computation points Q and P_0 are

$$L^2 = s^2 + (z_0 - z)^2$$

$$S^2 = s^2 + z^2$$

They were previously defined.

Applying 2-D Fourier transformation to (3-40), we get (cf. Schwarz et al., 1989):

$$F\{V_Q\} = G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{-2\pi\omega(z_0-h)} - e^{-2\pi\omega z_0}\right)\right\}$$

$$= G\, e^{-2\pi\omega z_0} \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{2\pi\omega h} - 1\right)\right\} \tag{3-42}$$

Analytical downward continuing the potential V_Q onto the ellipsoid, then its Fourier transformation is given by:

$$F\{V_Q^*\} = G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{2\pi\omega h} - 1\right)\right\} \tag{3-43}$$

In the same manner we obtain the Fourier transformation of V_{P_0}:

$$F\{V_{P_0}\} = -G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{-2\pi\omega h} - 1\right)\right\} \tag{3-44}$$

Combine eqs. (3-43) and (3-44) we get the Fourier transform of the potential difference δV:

$$F\{\delta V\} = F\left\{-V_Q^* + V_{P_0}\right\}$$

$$= - G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{2\pi\omega h} + e^{-2\pi\omega h} - 2\right)\right\} \tag{3-45}$$

Eq. (3-45) is independent of the altitude z_0 as expected. Expanding (3-45) into a series, then we get

$$F\left\{\delta V\right\} = - G \frac{1}{2\pi\omega^2} \sum_{n=1}^{\infty} \frac{(2\pi\omega)^n}{n!} F\left\{\rho h^n\right\}\left[1 + (-1)^n\right]$$

$$= - G \frac{1}{\pi\omega^2} \sum_{n=1}^{\infty} \frac{(2\pi\omega)^{2n}}{(2n)!} F\left\{\rho h^{2n}\right\}$$

$$= - 2 G F\left\{\rho h^2\right\} - G \frac{1}{\pi\omega^2} \sum_{n=2}^{\infty} \frac{(2\pi\omega)^{2n}}{(2n)!} F\left\{\rho h^{2n}\right\} \tag{3-46}$$

The inverse Fourier transformation of the first term of eq. (3-46)

$$- 2\pi G\, F\left\{\rho h^2\right\} \tag{3-47}$$

is equal to $-2\pi G\rho h^2$ which is nothing other than the correction of a Bouguer plate (cf. eq. (3-7)).

Clearly the magnitude of the series in (3-46) depends very strongly on the roughness of the topography: if the topography is smooth, the series has little contribution to the change of the potential δV, if h = constant, the series in (3-46) has no contribution to δV.

The last sum in (3-46) supplies mostly very high frequencies in δV, therefore it is not important to δV which is assumed that the very high frequencies have few contribution. This agrees with our earlier study in space domain.

If we compute (3-46) and its inverse Fourier transformation, the discrete Fourier transformation has to be used. In this case the influence of aliasing and leakage becomes a very serious problem, because both phenomenan have the most effect on high frequencies. If such wrong high frequencies are amplified by ω^n, then results will not be reliable. Therefore the low pass filters is needed for such computation.

The first derivatives of δV is given by

$$\begin{aligned} \delta V_x &= j2\pi\, F^{-1}\left\{uF\left\{\delta V\right\}\right\} \\ \delta V_y &= j2\pi\, F^{-1}\left\{vF\left\{\delta V\right\}\right\} \\ \delta V_z &= -2\pi\, F^{-1}\left\{\omega F\left\{\delta V\right\}\right\} \end{aligned} \tag{3-47}$$

where $F\{\delta V\}$ is given by eq. (3-46).

In the above discussion we have not considered whether the analytical downward continuation is convergent. We believe the above derivation is justified based on the assumption that the series in eqs. (3-46) is at least numerical convergent. We cannot guarantee that the series in eq. (3-45) is convergent. The terms ($n\geq 2$) in the series may be very rough, but they could be very small after some smoothing. Ignoring the terms ($n\geq 2$) in eq. (3-45) may not cause serious error in the computation.

In the next section we will study the convergence problem of the analytical downward continuation under planar approximation.

3.6 Convergence Study of the Analytical Downward Continuation Under Planar Approximation

Even though the convergence problem of analytical downward continuing a harmonic function into the mass does not have too much meaning on the practical point of view, it is still interesting to know the mathematical property of the analytical downward continuation under planar approximation.

Equation (3-46) is directly related to the convergence or divergence of the analytical downward continuation of the geopotential inside the earth. If (3-46) is convergent, then the analytical downward continuation is also convergent. Obviously the convergence problem depends very strongly on the roughness of the topography. If $h=0$, then $F\{\delta V\}=0$; if h = constant, then $F^{-1}\{F\{\delta V\}\} = -2\pi G\rho h^2$, the topographic correction of a Bouguer plate with thickness h. In both cases we do not have any divergence problem.

From eq. (3-45) we have

$$\left|F\{\delta V\}\right| = \frac{G}{\pi\omega^2}\sum_{n=1}^{\infty}\frac{(2\pi\omega)^{2n}}{(n+1)(n+2)\dots 2n}\left|F\left\{\frac{\rho h^{2n}}{n!}\right\}\right| \tag{3-48}$$

we assume that the Fourier transformation of $\rho h^{2n}/n!$ exists, it means, the following condition holds:

$$\iint_\tau \frac{\rho h^{2n}}{n!}\,dxdy = M_n < \infty \tag{3-49}$$

From (3-49) one can get

$$M_{n+1} = \iint_\tau \frac{\rho h^{2n+2}}{(n+1)!}\,dxdy \leq \frac{h_{max}^2}{n+1}\iint_\tau \frac{\rho h^{2n}}{n!}\,dxdy$$

$$= \frac{h_{max}^2}{n+1}M_n \tag{3-50}$$

This means that the following inequality holds:

$$\frac{M_{n+1}}{M_n} \le \frac{h_{max}^2}{n+1} \tag{3-51}$$

For the Fourier transformation of the quantity $\rho h^{2n}/n!$ we have

$$\left| F\left\{ \frac{\rho h^{2n}}{n!} \right\} \right| = \left| \iint_\tau e^{2\pi j(xu+yv)} \frac{\rho h^{2n}}{n!} dxdy \right|$$

$$\le \iint_\tau \left| \frac{\rho h^{2n}}{n!} \right| dxdy = M_n \tag{3-52}$$

therefore for eq. (3-48) we have

$$\left| F\{\delta V\} \right| \le \frac{G}{\pi\omega^2} \sum_{n=1}^{\infty} \frac{(2\pi\omega)^{2n}}{(n+1)(n+2)\dots 2n} \cdot M_n \tag{3-53}$$

The convergence radius of the series in (3-53) is easy to get by using D'Alembert's criterion

$$d = \lim_{n\to\infty} \left[\frac{(n+1)(n+2)\dots 2n}{(n+1)(n+2)\dots(2n+2)} \cdot \frac{(2\pi\omega)^{2n+2}}{(2\pi\omega)^{2n}} \frac{M_{n+1}}{M_n} \right]$$

$$\le \lim_{n\to\infty} \frac{(2\pi\omega)^2}{2n+2} \cdot \frac{h_{max}^2}{n+1} \tag{3-54}$$

Therefore the Fourier transformation of the potential difference potential δV is convergent in the frequency domain $0 \le \omega < \infty$. Note that the infinitive point $\omega = \infty$ is not included in the convergence domain. In the same way we can show that the inverse Fourier transformation of eq. (3-46) is convergent, provided the infinite point $\omega = \infty$ is not included in $F\{\delta V\}$.

We assume the maximum frequency of the Fourier transformation of δV is ω_0, where ω_0 can be an arbitrarily large number, but not equal to infinity. From eq. (3-46) we have

$$|\delta V| = \left| F^{-1} F\{\delta V\} \right|$$

$$= \frac{G}{\pi} \left| \iint_\tau e^{-j2\pi(x'u+y'v)} \frac{1}{\omega^2} \sum_{n=1}^{\infty} \frac{(2\pi\omega)^{2n}}{(2n)!} \cdot \right.$$

$$\cdot \iint_\tau e^{j2\pi(xu+yv)} \rho h^{2n} \, dxdy \, dudv \Bigg|$$

$$\leq 2\pi G \sum_{n=1}^{\infty} \left| \int_0^{2\pi} d\alpha \int_0^{\omega_0} e^{j[(x-x')u+(y-y')v]} \frac{(2\pi\omega)^{2n-1}}{(n+1)(n+2)\ldots 2n} d\omega \, . \iint_\tau \frac{\rho h^{2n}}{n!} dxdy \right|$$

$$\leq 2\pi G \sum_{n=1}^{\infty} \int_0^{2\pi} d\alpha \int_0^{\omega_0} \frac{(2\pi\omega)^{2n-1}}{(n+1)(n+2)\ldots 2n} d\omega \, . \, M_n$$

$$= 2\pi G \sum_{n=1}^{\infty} \frac{(2\pi\omega_0)^{2n-1}}{(n+1)(n+2)\ldots 2n \, . \, 2n} M_n \tag{3-55}$$

Here we have used the polar coorcinates system in the frequence domain. The relationship between the frequences variable u, v and the polar coordinates system is defined as:

$$\begin{cases} u = \omega \cos \alpha \\ v = \omega \sin \alpha \end{cases} \tag{3-55a}$$

The convergence radius of the series in (3-61) is given by

$$d = \lim_{n\to\infty} \frac{(n+1)(n+2)\ldots 2n \cdot 2n}{(n+1)(n+2)\ldots(2n+1)(2n+2)} \frac{(2\pi\omega_0)^{2n+2}}{(2\pi\omega)^{2n}} \frac{M_{n+1}}{M_n}$$

$$= \lim_{n\to\infty} \frac{(2\pi\omega_0)^2}{(2n+2)^2} \frac{h_{max}}{n+1} \tag{3-56}$$

From eq. (3-56) we know that the series in (3-55) is convergent for all ω_0: $0 \leq \omega_0 < \infty$.

From the above derivation we can say that the Fourier transformation of the δV and the potential difference δV are convergent everywhere, except at the point $\omega = \infty$. That means the analytical downward continuation (ADC) of a harmonic function into the earth is also convergent, except at the point $\omega = \infty$. This result is slightly a surprise, because the ADC may be assumed (considered) as divergent (Sjöberg, 1977). We have shown that the analytical downward continuation of the external potential into the earth is almost convergent everywhere! Here "almost everywhere" means the infinite point $\omega = \infty$ is not included.

Indeed, the convergence of analytical downward continuing the geopotential outside the earth's surface into the mass of the earth is a delicate problem. From eqs. (3-43) or (3-46) one can see that the convergence problem depends on the roughness of the topography. Based on the above derivation we know that the series in (3-46) is covnergent for all ω: $0 \leq w < \infty$. At the infinite point $\omega=\infty$, the convergence of (3-46) is not defined. Obviously, the series is convergent at the infinite point $\omega=\infty$, only if

$$\lim_{\omega \to \infty} \left| (2\pi\omega)^{2n-2} F\left\{ \rho h^{2n} \right\} \right| < \infty \tag{3-57}$$

exists.

Note that the Laplace operator in spectral domain is corresponding to

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Rightarrow -(2\pi\omega)^2 \tag{3-58}$$

the eq. (3-57) is equivalent to

$$\lim_{\omega \to \infty} F\left\{ (-1)^n \Delta^{n-1} \left(\rho h^{2n} \right) \right\} < \infty \tag{3-59}$$

where $\Delta^{n-1} = \Delta\Delta\Delta ...\Delta$ (n - 1 times).

Eq. (3-59) indicates that the function ρh^{2n} must be an analytical function (an analytical function has infinite derivatives). This is not the case for the real earth, because there are steep cliffs and overhanging rocks on the earth's surface. In this meaning the ADC is devergent.

But one has to keep in mind that the real world can only be approximated by some mathematical models. If the elevation h is approximated by an analytical surface, or, more practical, be approximated by point or mean blocks values, whose Fourier transformation has finite frequencies, then (3-46) is convergent. Let us now imagine in the future: during the developement of the technology the topography of the earth is measured more and more in detail, we get more and more elevation data which includes high and very high frequencies of the topography. But the number of the data is always limited and its Fourier transformation has always finite frequencies. In this case, eq. (3-46) is always convergent. Therefore we will say that the ADC is convergent for all practical use.

The above conclusion is also important for geopotential modeling. If the ADC is convergent, we can expand the spherical harmonics of the geopotential up to high and very high degree and order without any theoretical problem. In practice the geopotential can be modeled with a finite degree and order which correspond to a certain frequency $\omega = \omega_0$, therefore the ADC can be considered as convergent for modeling the spherical harmonic of the geopotential.

4. Effect of the Topography on Solving Molodensky's Problem and the Determination of the Coefficients of the Spherical Harmonics of the Geopotential

4.1 Introduction

In solving Molodensky's problem and the determination of the coefficients of the spherical harmonics of the gravitational potential of the earth the analytical downward continuation (ADC) is widely used. Moritz (1980, section 45) used ADC to continue the gravity anomaly to the point level and the disturbing potential is determined by using the Stokes' integral. Rapp et al., (1986) also used ADC to continue the gravity anomaly to the ellipsoid for the determination of the coefficients of the spherical harmonics. It is always assumed that ADC is correct for the determination of the disturbing potential on the earth's surface. The disturbing potential on the earth's surface and outside the earth can also be represented by the spherical harmonic expansion determined by using ADC of the gravity anomaly to the ellipsoid. In this chapter we will look at this problem in more detail. We wish to answer the questions: Can the spherical harmonic represent the disturbing potential on and outside the earth exactly? This problem is different from the divergence or convergence of the spherical harmonics on the earth's surface. Based on the derivation in chapter 3 we assume that an spherical harmonic expansion of the geopotential, such as used by Rapp (1984, eq. (1)), be convergent on the earth's surface and even so on the ellipsoid. This assumption may be argued. But we think it is reasonable for the practical use. The convergence or divergence of the series is meaningless in practice because a small "grain of sand" can make a series divergent (Moritz, 1980, p. 64), and the potential of a grain of sand is so small that it can never be included in a gravity model related to the earth's gravity field. Now the question is: does the expansion (ibid) represent the gravity field of the earth on and outside the earth's surface well? This problem is important for today's gravitational modeling. Up to now we have various geopotential models (e.g., GEMT1, OSU89, ...) generated by different groups and by using different data sets. If the spherical harmonics defined by (Rapp, 1984, eq. (1)) represents the geopotential on and outside the earth's surface exactly, then the geopotential modeling can be expanded into high and very high degree and order. An explicit example: if the spherical harmonics is expanded in degree and order 10000, then the geopotential is modeled with resolution 2 km. This model will represent a global uniform detailed gravity field. But if the spherical harmonics (ibid) do not represent the geopotential on and outside the earth's surface well, then such expansion is meaningless, because the high degree and order of the spherical harmonics may be totally wrong.

In practice the high degree and order of the spherical harmonics are effected mostly by different errors and are not as reliable as the lower degree and order. But if the expansion (ibid) gives the geopotential on and outside the earth's surface as well as we wish, one can expect that the more and more gravity data will be gathered with more and more better accuracy and data coverage, during the development and improvement of the high technology and space technology, and an expansion of spherical harmonics to high degree and order will approximate the geopotential on and outside the earth's surface to a great extent.

In the following we will study the effect of the topography on solving the Molodensky's problem and determination of the coefficients of the spherical harmonics of the earth's gravitational potential. More exactly, to study the error of the analytical downward continuation for solving Molodensky's problem and the determination of the coefficients of the spherical harmonics. A similar study on "error of the analytical downward continuations" was given by Sjöberg (1977, 1980). The error was studied on a sphere earth and the spherical harmonics were used for the study. Here we will study this problem for a flat earth. The planar approximation should not cause serious problem for our study. Note that we use the term "effect of the topography" instead the "error of the downward continuation", in the following.

4.2 Analytical Downward Continuation for Solving Molodensky's Problem

ADC was used for solving the Molodensky's problem (Moritz, 1980 section 45). This method corresponds to the analytical continuation of the external potential T into the earth's interior (ibid, p. 378) on the point level.

Now let us look at this problem in more detail. This method is illustrated in Figure 6.

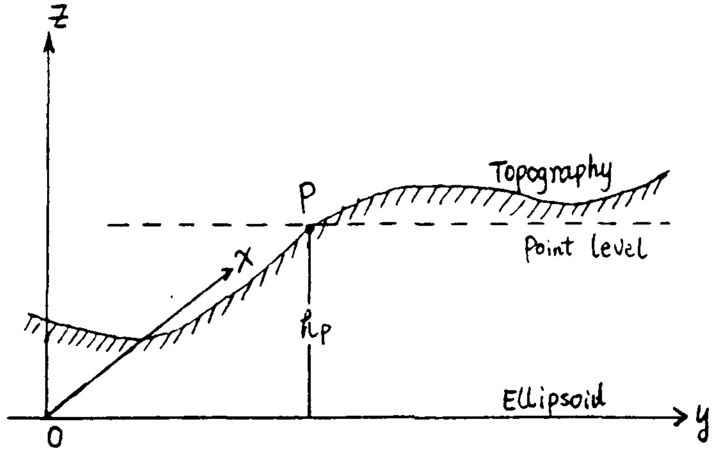

Figure 7. Geometry of Solving Molodensky's Problem by Using Analytical Downward Continuation

Gravity data are given on the earth's surface and then analytically downward to the point level. The disturbing potential at the point P is then determined by using Stokes' integral. This method is equivalent to the analytical downward continuation of the external potential down to the point level so that the effect of the topography in this procedure can be obtained from the derivation in Chapter 3.

Based on the description in section 3.2, the difference between the disturbing potential determined by using the ADC and the original one is given by

$$\delta V_p = - V_p^* + V_p \tag{4-1}$$

where V_p^*, V_p are the potential determined by using ADC and the potential of the mass above the point level respectively.

Comparing Figure 7 with Figure 6, we get the V_p^*, V_p similar to eqs. (3-43) and (3-44):

$$F\left\{V_p^*\right\} = G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{2\pi\omega\Delta h} - 1\right)\right\} \tag{4-2}$$

$$F\{V_p\} = -G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{-2\pi\omega\Delta h} - 1\right)\right\} \tag{4-3}$$

with

$$\Delta h = \begin{cases} h-h_p & h>h_p \\ 0 & h\leq h_p \end{cases} \tag{4-4}$$

Combining (4-2) and (4-3) we obtain

$$F\{\delta V_p\} = -G \frac{1}{2\pi\omega^2} F\left\{\rho\left(e^{2\pi\omega\Delta h} + e^{-2\pi\omega\Delta h} - 2\right)\right\}$$

$$= -G \frac{1}{\pi\omega^2} \sum_{n=1}^{\infty} \frac{(2\pi\omega)^{2n}}{(2n)!} F\left\{\rho\,(\Delta h)^{2n}\right\} \tag{4-5}$$

(4-5) shows the difference between the potential obtained by using ADC and the true one. Obviously, (4-5) is the effect of the topography which is neglected in the solving of the Molodensky's problem by using the ADC.

Theoretically, (4-5) plays a role of the convergence of the ADC of a harmonic function onto the earth. It is clear that the convergence of the series in (4-5) depends on the roughness of the topography. As is shown in section 3.5, one can show that the topographic correction δV_p and its Fourier transformation are convergent almost everywhere except at the infinite point of the circle frequency $\omega = \infty$. Obviously, the δV_p could be rough, but if some kinds of smoothing are taken, (4-5) could be very small and be neglected. This is also the reason why we assume that the ADC is always correct for the practical using. Approximately we take the first two terms in (4-5):

$$F\{\delta V_p\} \approx -2\pi G\, F\left\{\rho\,(\Delta h)^2\right\} - \frac{2}{3} G\pi^3 \omega^2 F\left\{\rho\,(\Delta h)^4\right\} \tag{4-6}$$

For point P on the topographic surface, the first term has no contribution to δV_p. We have then

$$\delta V_p \approx -\frac{2}{3} G\pi^3 F^{-1}\left\{\omega^2 F\left\{\rho\,(\Delta h)^4\right\}\right\} \tag{4-7}$$

This term should be very small after the smoothing and can be omitted.

Therefore we can say: The using of the ADC to solve the Molodensky's problem is reasonable and correct. The topography effect δV_p in this procedure represents very high frequency in the disturbing potential and in practice it can be neglected.

4.3 Analytical Downward Continuation for Determination of the Spherical Harmonics of the Earth's Gravitational Potential

If we analytically downward continue the gravity anomaly to the ellipsoid and this reduced gravity anomaly is used for the determination of a spherical harmonic function, we always assume that this spherical harmonic function represents the disturbing potential on and outside the earth very well. In the following we will show that this spherical harmonic function gives the disturbing potential above and on the Brillouin sphere (Moritz, 1980, p. 431) exactly if this series is convergent. There is an effect of the topography in the vicinity of the earth's surface (the space between the topography surface and the Brillouin sphere) to the geopotential represented by this spherical harmonic expansion. This means that a single spherical harmonic expression for the representation of the disturbing potential outside the earth's surface is not enough. The attempt to include the topographic effect in a series for the representation of the disturbing potential in the whole space can be found in (Petrovskaya, 1976, 1977). This method seems not so promising for the practical use.

Here we assume we have a spherical harmonic expansion determined by using ADC and we intend to find the topographic effect near by the earth's surface.

In the first place we will show that if the analytical downward continuation of the spherical harmonics to the ellipsoid exists (convergent), and if the point P is on the Brillouin sphere, then the downward continued spherical harmonics represent the disturbing potential at point P exactly.

We denote the radius of the Brillouin sphere by R_B, and the semimajor axis of the ellipsoid by a. On the Brillouin sphere the spherical harmonic expansion of the disturbing potential is convergent (no mass above the Brillouin sphere):

$$T_B = \frac{kM}{R_B} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda\right) \bar{P}_{nm}(\sin \varphi) \tag{4-8}$$

where T_B is the disturbing potential on the Brillouin sphere, kM is the gravitational constant times the earth's mass; $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are fully normalized potential coefficients. If we analytically downward continue the T_B to a sphere with radius a, then we have

$$T(r, \varphi, \lambda) = \frac{kM}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda\right) . \left(\frac{R_B}{a}\right)^{n+1} \left(\frac{a}{r}\right)^{n+1} P_{nm}(\sin \varphi)$$

$$= \frac{kM}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\bar{C}'_{nm} \cos m\lambda + \bar{S}'_{nm} \sin m\lambda\right) . \left(\frac{a}{r}\right)^{n+1} P_{nm}(\sin \varphi) \tag{4-9}$$

where

$$\left\{ \begin{matrix} \bar{C}'_{nm} \\ \bar{S}'_{nm} \end{matrix} \right\} = \left\{ \begin{matrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{matrix} \right\} \left(\frac{R_B}{a}\right)^{n+1} \tag{4-10}$$

It is difficult to say whether the series in (4-9) is divergent or covergent. If the series in (4-9) is convergent, on the Brillouin sphere we have $r=R_B$, and (4-9) returns to (4-8).

For the points above the Brillouin sphere we have the same conclusion.

Therefore, the spherical harmonic expansion determined by using the ADC method represents the disturbing potential on and above the Brillouin sphere (the bounding sphere) exactly, provided the downward continued potential is convergent. Even if the series in (4-9) is divergent, based on the Runge's theorem, we can find a spherical harmonic expansion similar to (4-9) to approximate the disturbing potential as well as we wish (cf. Moritz, 1980, p. 67).

Now we are going to find out the difference between the true gravitational potential at point P and the gravitational potential represented by a spherical harmonic expansion like eq. (4-9). If point P is on the earth's surface, the point level separates the mass of the earth into two parts - one is under the point level and one is above the point level. Based on the above discussion the disturbing potential generated by the mass under the point level can be represented exactly at point P by an expansion similar to (4-9) provided the downward continued potential is convergent. The potential of the mass above the point level cannot be represented exactly by an expansion like (4-9).

In order to simplify this problem, we take the planar approximation. Obviously, the potential of the mass above the point level continued downward to the ellipsoid is given by (cf. eq. (4-2)):

$$V_e^{\bullet} = F^{-1}\left\{ G e^{2\pi\omega h_p} \frac{1}{2\pi\omega^2} F\left\{ \rho \left(e^{2\pi\omega\Delta h} - 1 \right) \right\} \right\} \tag{4-11}$$

where Δh is defined by eq. (4-4); $V_e^{\bullet}$ denotes the potential on the ellipsoid.

Upward continue the $V_e^{\bullet}$ to the point level, we obtain

$$V_p^{\bullet} = F^{-1}\left\{ G \frac{1}{2\pi\omega^2} F\left\{ \rho \left(e^{2\pi\omega\Delta h} - 1 \right) \right\} \right\} \tag{4-12}$$

It is the same as eq. (4-2). At the same time the potential of the mass above the point level is given by eq. (4-3). The difference between the $V_p^{\bullet}$ and V_p is given by eq. (4-5). Therefore, the disturbing potential on the point level (which is partly inside the earth) represented by a spherical harmonics like eq. (4-9) differs the true disturbing potential by a quantity δV which is given by eq. (4-5). Therefore the representation of the disturbing potential by an expansion of spherical harmonics, which is determined by using the gravity anomalies downward continued from outside Earth's surface or from Earth's surface to the ellipsoid, is just as good as solving the Molodensky's problem by using the ADC procedure.

The effect of the topography on the computation of the geoid from a spherical harmonic expansion, such as eq. (4-9), is referenced to Wang (1989).

5. Conclusion

The earth's surface is a complicated surface. It is not suitable for the computations of the physical geodesy. Therefore the different reference surfaces such as the sphere, the ellipsoid or the point level have been used for solving the geodetic boundary value problem. It is clear, if such reference surfaces are chosen, the analytical downward continuation has to be used and the effect of the topography has to be considered. Up to date the airborne and satellite gradiometry have been reaching great improvement and will supply a huge gravity gradient data for the geodetic boundary value problem. To process such data the reference surface has to be chosen and the ADC method has to be used. Therefore the effect of the topography must be taken into consideration.

In Chapter 2 the topographic correction to the airborne gravity gradient data was considered. In this chapter the Helmert's second condensation was introduced into the airborne gradiometry. This is much more complex than it is in the classical meaning because the gravity gradient has six components, the computations of the topographic correction to it needs much more computation effort. It was found that the Bouguer plate has no contribution to the gravity gradient. The topographic correction is mostly due to the terrain of the earth.

In Chapter 3 we introduced the remove-restore technique. Our focus was still on how the geoid inside the earth can be determined by using the method of the analytical downward continuation. A simple way to do this is to obtain the disturbing potential at the flight altitude by processing the aerial gradient data, and then downward continue it to the sea level. Obviously, such potential is not the original one. Some correction has to be considered. By using the remove-restore technique we found that the topographic correction consists mostly by a term $-2\pi G\rho h_p^2$, and a terrain effect which can be very small after the smoothing. The first term is important for the determination of the geoid in a mountain area and the terrain effect (eq. (3-34)) is very rough and becomes small when some kind of smoothing are applied. Because the remove-restore technique does not change the location of the mass of the topography, the indirect effect is equal to zero. In a planar approximation the change of the potential δV and its Fourier transformation was shown convergent almost everywhere, except at the infinite point of the circle frequency $\omega=\infty$. This means that the ADC is also convergent almost everywhere. This is important for developing the geopotential into harmonic series up to high or very high degree and order.

In Chapter 4 the topographic effect on the solution of Molodensky's problem by using the ADC was considered. Theoretically, the topographic effect still exists, even if the analytical downward continuation solution is equivalent to the Molodensky's solution which is considered theoretically perfect. Because the topographic effect (eq. (4-7)) can be very small after smoothing, this effect can be neglected in the practical computations.

The same topographic effect exists also for the geopotential represented by a spherical harmonic expansion. One should not be surprised about this because the ADC is used in the solving of Molodensky's problem and the determination of the spherical harmonics. The discussion in Chapter 4 shows that a simple spherical harmonic function, like eq. (4-9), cannot represent exactly the disturbing potential nearby the earth and on the earth's surface. By using the ADC method we can obtain a spherical harmonic function, it represents exactly the disturbing potential on and above the boundary sphere of the earth, provided such a spherical harmonics exists (convergent); it represents practically well the disturbing potential nearby the earth and on the earth's surface, even though the topographic effect (correction) exists.

In practice the spherical harmonic expansion is taken to a finite degree and order and some kind of smoothing are always involved in the practical computation. Therefore the topographic effect can be very small, even though it looks very rough and unstable. Here we must not be confused by the word "topographic effect". The topographic effect is defined as the difference of the true geopotential on the point level and the potential determined by using analytical downward continuation. Because the difference is caused by the topography of the earth, we call this difference topographic effect (correction). This effect is totally different from the classic topography-reduction or the topographic correction to the spherical harmonic expansion of the geopotential which has been done by Sjöberg (1988).

Theoretically, the topographic effect is important, because it gives the idea of how good a spherical harmonics represents the disturbing potential on and outside the earth. The equivalent problem is how good can the disturbing potential on the earth's surface be determined by using the ADC method.

There is only one gravity field but it can only be approximated by using the numerically computed different gravity models. Based on the results in Chapter 4 we should not worry about the expanding of the spherical harmonics into a high degree and order, if the gravity data are dense enough and accurate enough for the needs. In such a case the spherical harmonic expansion of high degree and order will represent the disturbing potential better than one in lower degree and order. If the accuracy of the spherical harmonic representation is so high that the topographic effect has to be taken into account, the topographic effect can be computed with a proper smoothing.

Note that the analytically downward continuation of a spherical harmonics into the earth is assumed. Theoretically it is difficult to prove this series convergent or divergent in a spherical case even though it was shown in Chapter 3 that the ADC is convergent almost everywhere in a plane approximation case. As Moritz (1980) pointed out, such a problem is meaningless in practice. We can still believe that a spherical harmonic expansion, like eq. (4-9), can represent the gravity field as well as we wish. The derivation in Chapter 3 helped us to heighten our confidence.

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