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A LYAPUNOV BOUND FOR SOLUTIONS OF POISSON'S EQUATION

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A LYAPUNOV BOUND FOR SOLUTIONS OF POISSON'S EQUATION

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ABSTRACT:

Suppose that X is a positive recurrent Harris chain with invariant measure π. We develop a Lyapunov function criterion that permits one to bound the solution g to Poisson's equation for X. This bound is then applied to obtain sufficient conditions that guarantee that the solution be an element of $L^p(\pi)$. When $p = 2$, the square integrability of g implies the validity of a functional central limit theorem for the Markov chain. We illustrate the technique with applications to the waiting time sequence of the single-server queue and autoregressive sequences.

KEYWORDS:

Markov chain, Poisson's equation, Lyapunov function, functional central limit theorem.

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1. INTRODUCTION

Let $X = (X_n : n \geq 0)$ be a (temporally homogeneous) Markov chain living on a measurable space $(S, \mathcal{S})$. Given a real-valued $\mathcal{S}$-measurable function f, **Poisson's equation** involves solving

$$g = Pg + f \tag{1.1}$$

for the unknown function g, where P is the transition kernel of X. The solution g to Poisson's equation is fundamental to the analysis of the additive function

$$S_n = \sum_{k=0}^{n-1} f(X_k).$$

Set $S_0 = 0$. The principal observation that underlies the analysis of $(S_n : n \geq 0)$ is that the behavior of such an additive functional is closely related to that of a certain martingale. Specifically, let

$$M_n = g(X_n) + S_n \tag{1.2}$$

for $n \geq 0$. Set $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Proceeding formally, we note that since g solves (1.1),

$$\begin{aligned} E_x\left[M_{n+1} | \mathcal{F}_n\right] &= (Pg)(X_n) + S_{n+1} \\ &= g(X_n) + S_n = M_n \quad P_x \ a.s. \end{aligned}$$

Hence, under suitable integrability conditions on f and g, it follows that $M = (M_n : n \geq 0)$ is a P_x-martingale with respect to the filtration $(\mathcal{F}_n : n \geq 0)$. In fact, the martingale M is just the discrete-time analog of the well-known continuous-time martingale $M(t) = g(X(t)) + \int_0^t f(X(s))\, ds$, where $f = -Ag$ and A is the (infinitesimal) generator of the Markov process X.

Returning to (1.2), it is evident that the asymptotic behavior of $(S_n : n \geq 0)$ will typically mimic that of the martingale M. In particular, laws of large numbers, central limit theorems, and laws of the iterated logarithm can often be derived for $(S_n : n \geq 0)$ by applying appropriate martingale theorems. For example, MAIGRET (1978) and KURTZ (1981) use this approach to obtain functional central limit theorems (FCLT's) for recurrent discrete- time Markov chains, whereas BHATTACHARAYA (1982) uses similar techniques to derive FCLT's and laws of the iterated logarithm (LIL) for recurrent continuous-time Markov processes. However, in order to successfully apply this technique to a given chain, one must basically show that the martingale difference sequence $D_n = g(X_n) - (Pg)(X_{n-1})$ is appropriately square integrable.

In this paper, we develop, for positive recurrent Harris chains, a Lyapunov function criterion that permits one to verify that the solution $g \epsilon L^2(\pi)$, where π is the (unique) invariant measure of X. This guarantees that the martingale differences are well-behaved, so that the above mentioned FCLT's apply. Thus, the main goal of the paper is to derive an easily applied Lyapunov function technique that permits one to verify the hypotheses necessary in order to assert that $(S_n : n \geq 0)$ satisfies a FCLT.

2. THE MAIN RESULT

Throughout the remainder of this paper, we adopt the notation and terminology of NUMMELIN (1984) and REVUZ (1984). The function h that appears in the following proposition is called a (stochastic) **Lyapunov function.**

PROPOSITION 1. Let K be a non-negative σ-finite kernel on $(S, \mathcal{S})$. Suppose that there exists $\epsilon > 0$ and a finite-valued non-negative function h such that

$$(Kh)(x) \le h(x) - \epsilon |f(x)| \tag{2.1}$$

for $x \in S$. Then, the equation $g = Kg + f$ has a solution $g^* = \sum_{n=0}^{\infty} K^n f$ that satisfies $|g^*(x)| \le h(x)/\epsilon$ for $x \in S$.

Proof. Since $\epsilon|f(x)| + (Kh)(x) \le h(x)$, it follows that

$$\epsilon K^n|f| + K^{n+1}h \le K^n h. \tag{2.2}$$

Since $K^0 h = h$ is finite-valued, induction shows that $K^n h$ is finite-valued for all $n \ge 0$. Hence, we may re-write (2.2) as

$$\epsilon K^n|f| \le K^n h - K^{n+1}h. \tag{2.3}$$

Summing both sides of (2.3) over n, we find that $\epsilon \sum_{n=0}^{m} K^n|f| \le h - K^{m+1}h$. Since $h \ge 0$, it is evident that $\epsilon \sum_{n=0}^{m} K^n|f| \le h$, from which it follows that $g^* = \sum_{n=0}^{\infty} K^n f$ converges, satisfies $g^* \le h/\epsilon$, and is a solution of $g = Kg + f$.

We note that Proposition 1 does not typically apply to recurrent Markov chains. In particular, suppose that there exists a positive invariant measure π such that $\pi h < \infty$. By applying π to both sides of (2.1), we find that (2.1) can only be satisfied globally if f vanishes π a.e.

However, (2.1) can provide useful information for substochastic and transient kernels.

EXAMPLE 1. Suppose that $X_n = X_0 + \epsilon_1 + \ldots + \epsilon_n$, where the ϵ_i's are i.i.d. real-valued r.v.'s with $E\exp(-\lambda\epsilon_i) < 1$ for some $\lambda > 0$. Suppose $f(x) = O(\exp(-\lambda x))$ as $x \to \infty$. Then, if $h(x) = \exp(-\lambda x)$, we find that

$$\begin{aligned}(Ph)(x) &= e^{-\lambda x} E e^{-\lambda \epsilon_1} \\ &\le e^{-\lambda x} - \epsilon|f(x)| = h(x) - \epsilon|f(x)|,\end{aligned}$$

provided that $\epsilon = (1 - E\exp(-\lambda\epsilon_1))(\beta+1)^{-1}$, where $\beta = \sup\{\exp(\lambda x)|f(x)| : x \in \mathsf{R}\}$. Hence, for f satisfying the above growth condition, Proposition 1 applies.

We henceforth assume that P is the transition kernel of a Harris recurrent Markov chain with non-trivial invariant measure π. Then, there exists an integer $n \geq 1$, a non-negative $\mathcal{S}$-measurable function s, and a probability measure ν such that

$$P^m \geq s \otimes \nu \tag{2.4}$$

with $\pi s > 0$ (see Section 2.3 of NUMMELIN (1984)).

Suppose, for the moment, that m = 1 in the minorization condition (2.4) and put

$$G_{s,\nu} = \sum_{n=0}^{\infty} [P - s \otimes \nu]^n.$$

According to Theorem 3.1 ii) of NUMMELIN (1985), it follows that if

$$(G_{s,\nu}|f|)(x) < \infty$$

for all $x \in S$ and $\nu G_{s,\nu} f = 0$, then $f^* = G_{s,\nu} f$ is a solution of Poisson's equation

$$g(x) = (Pg)(x) + f(x)$$

for all $x \in S$. We note that the condition $\nu G_{s,\nu} f = 0$ is equivalent to requiring that $\pi f = 0$ (see p. 73 of NUMMELIN (1984)).

Our goal is now to obtain a Lyapunov bound on g. Proposition 2 is an immediate consequence of Proposition 1.

PROPOSITION 2. Suppose that there exists $\epsilon > 0$ and a finite-valued non-negative function h such that

$$((P - s \otimes \nu)h)(x) \leq h(x) - \epsilon|f(x)| \tag{2.5}$$

for $x \in S$. If $\pi f = 0$, then $g^* = G_{s,\nu} f$ satisfies $g = Pg + f$ and the bound $|g^*(x)| \leq h(x)/\epsilon$.

Before proceeding, we note that the solution g^* is often unique, in a certain sense.

PROPOSITION 3. Suppose that $g^* \epsilon L^1(\pi)$ is a solution of $g = Pg + f$. Then, if $g_* \epsilon L^1(\pi)$ is another solution, there exists a constant c such that $g_* = g^* + c\pi$ a.e.

PROOF. Note that $h = Ph$, where $h = g^* - g_* \in L^1(\pi)$. Then, it follows that $h = Ph = P^2 h = \ldots = P^n h$ so $h(x) = n^{-1} \sum_{k=1}^{n} (P^k h)(x)$. But $n^{-1} \sum_{k=1}^{n} (P^k h)(x) \to \pi h$ as $n \to \infty$ for π a.e. x (Proposition 3.5 and Theorem 7 of NUMMELIN (1978)). Hence, $h(x) = \pi h$ π a.e. To prove that the solution $g^* = G_{s,\nu} f$ is an element of $L^p(\pi)$, we can use the following result; the proof is an immediate consequence of Theorem 1 of TWEEDIE (1983).

PROPOSITION 4. Let X be a positive recurrent Harris chain and let $g^* = G_{s,\nu} f$. Suppose that there exists $h \geq 0$ satisfying (2.5), and a non-negative function k, $\eta > 0$, and a subset $B \in \mathcal{S}$ such that

$$(Pk)(x) \leq k(x) - \eta h^p(x)$$

for $x \in B^c$. Then, if π is normalized to be a probability,

$$\int_S |h(x)|^p \pi(dx) \leq \|I_B h\|^p + \eta^{-1} \|I_B P k\|,$$

where $\|r\| \triangleq \sup\{|r(x)| : x \in S\}$.

Unfortunately, Proposition 2 is difficult to apply in practice. There are two problem. Firstly, the function f that is of interest is rarely given explicitly as a centered functional for which $\pi f = 0$. Also, condition (2.5) can be difficult to verify because of the global character of the inequality. Theorem 1 remedies these problems.

THEOREM 1. Let X be a Markov chain for which there exists a set $A \in \mathcal{S}$ and $\lambda > 0$ such that

$$P(x, \cdot) \geq \lambda \nu(\cdot) \tag{2.6}$$

for all $x \in A$. Suppose that there exists $\epsilon, \eta > 0$ and non-negative finite-valued $\mathcal{S}$-measurable functions h and k such that

$$(Ph)(x) \leq h(x) - \epsilon(|f(x)| + 1) \tag{2.7}$$

$$(Pk)(x) \leq k(x) - \eta h(x)^p \tag{2.8}$$

for $x \in A^c$ and $p > 0$. In addition, assume that

$$\|I_A f\|, \ \|I_A P h\|, \ \|I_A h\|, \ \|I_A P k\|$$

are all finite. Then:

i) X is a positive recurrent Harris chain;

ii) If π is the (unique) invariant probability measure of X, then $\pi|f| < \infty$ and $G_{s,\nu}(|f|+1) < \infty$;

iii) If $\hat{f} = f - \pi f$, the function $g^* = G_{s,\nu}\hat{f}$ satisfies Poisson's equation $g^*(x) = (Pg^*)(x) + \hat{f}(x)$ for all $x \in S$:

iv) There exist constants α and β, independent of x, such that $|g^*(x)| \leq \alpha h(x) + \beta$ for all $x \in S$;

v) $g^* \epsilon L^p(\pi)$.

PROOF. Let K be a non-negative σ-finite kernel. For $A \in \mathcal{S}$, set

$$K_A = \sum_{n=0}^{\infty} (I_{A^c}K)^n I_A$$

$$H^A = \sum_{n=0}^{\infty} (I_{A^c}K)^n I_{A^c}$$

$$G = \sum_{n=0}^{\infty} K^n$$

$$F = \sum_{n=0}^{\infty} (K_A K)^n [H^A + K_A].$$

We start by showing that $G = F$. Note that $G = I + R + I_{A^c}KG$, where $R = I_AKG$. Iterating this equality n times, we find that

$$G = \sum_{j=0}^{n} (I_{A^c}K)^j (I+R) + (I_{A^c}K)^{n+1}G.$$

Hence, $G \geq \sum_{j=0}^{n} (I_{A^c}K)^j (I+R)$. Letting $n \to \infty$, we find that

$$\begin{aligned} G &\geq \sum_{n=0}^{\infty} (I_{A^c}K)^n (I + I_AKG) \\ &= \sum_{n=0}^{\infty} (I_{A^c}K)^n (I_{A^c} + I_A + I_AKG) \\ &= H^A + K_A + K_AKG. \end{aligned}$$

Iterating this last inequality n times, we get

$$\begin{aligned} G &\geq \sum_{j=0}^{n} (K_AK)^j (H^A + K_A) + (K_AK)^{n+1}G \\ &\geq \sum_{j=0}^{n} (K_AK)^j (H^A + K_A). \end{aligned}$$

Letting $n \to \infty$, we obtain the inequality $G \geq F$. On the other hand,

$$\begin{aligned} F &= H^A + K_A + K_A K F \\ &= \sum_{n=0}^{\infty} (I_{A^c} K)^n [I_{A^c} + I_A + I_A K F] \\ &= I + I_A K F + \sum_{n=0}^{\infty} (I_{A^c} K)^n [I_{A^c} + I_A + I_A K F] \\ &= I + I_A K F + I_{A^c} K F = I + K F. \end{aligned}$$

By iterating the equality n times, we conclude that $F = \sum_{j=0}^{n} K^j + K^{n+1} F$. Thus, $F \geq \sum_{j=0}^{n} K^j$. By letting $n \to \infty$, we conclude that $F \geq \sum_{n=0}^{\infty} K^n$, proving that $G = F$.

We now let $K = P - s \otimes \nu$. Note that

$$I_{A^c} P h \geq I_{A^c} P I_{A^c} h \geq I_{A^c} K I_{A^c} h.$$

Combining this inequality with (2.7), we conclude that

$$\widetilde{K} \tilde{h} \leq \tilde{h} - \epsilon \tilde{f},$$

where $\widetilde{K} = I_{A^c} K$, $\tilde{h} = I_{A^c} h$, and $\tilde{f} = I_{A^c}(|f| + 1)$. Proposition 1 then implies that

$$\sum_{n=0}^{\infty} \widetilde{K}^n \tilde{f} = H^A(|f| + 1) \leq \tilde{h}/\epsilon = I_{A^c} h/\epsilon. \tag{2.9}$$

For a non-negative kernel W, let $\|W\| = \sup\{W(x, S) : x \in S\} < \infty$. Since $K_A(|f| + 1) = K_A I_A(|f| + 1)$, it is evident that $\|K_A(|f| + 1)\| \leq \|P_A I_A(|f| + 1)\| \leq \|I_A f\| + 1$. We now combine this bound with (2.9) and the fact that $G_{s,\nu} = F$ to obtain

$$\begin{aligned} G_{s,\nu}(|f| + 1) &= \sum_{n=0}^{\infty} (K_A K)^n (H^A + K_A)(|f| + 1) \\ &\leq \sum_{n=0}^{\infty} (K_A K)^n (h/\epsilon + \|I_A f\| + 1) \\ &= h/\epsilon + \sum_{n=0}^{\infty} (K_A K)^n [K_A K h/\epsilon + \|I_A f\| + 1] \\ &\leq h/\epsilon + \sum_{n=0}^{\infty} (K_A/K)^n [\|I_A K h\|/\epsilon + \|I_A f\| + 1] \end{aligned}$$

But $\|K_A K\| \leq \|K_A\| \; \|K\| \leq \|P_A\|(1-\lambda) = (1-\lambda)$. This provides us with the bound

$$G_{s,\nu}(|f|+1) \leq h/\lambda\epsilon + \lambda^{-1}[\|I_A Ph\|/\epsilon + \|I_A f\| + 1]. \tag{2.10}$$

We also observe that condition (2.7) guarantees that $I_{A^c}Ph \leq I_{A^c}h - \epsilon I_{A^c}1$. The test function criterion developed by TWEEDIE (1976), in conjunction with (2.6), then proves that X is a positive recurrent Harris chain. (See ATHREYA and NEY (1978) for a discussion of the definition of a Harris chain that is suitable for our present purposes.) As a consequence, Theorem 1 of TWEEDIE (1943) implies that

$$|\pi f| \leq \|I_A f\| + \epsilon^{-1}\|I_A Ph\|.$$

Thus, by applying (2.10), we get

$$\begin{aligned}
|G_{s,\nu}\hat{f}| &\leq G_{s,\nu}(|f| + |\pi f| \cdot 1) \\
&\leq G_{s,\nu}|f| + |\pi f| G_{s,\nu} \cdot 1 \\
&\leq (1 + |\pi f|) G_{s,\nu}(|f| + 1) \\
&\leq \alpha h + \beta
\end{aligned}$$

for suitable constants α and β proving iv). The bound on the L^p-norm of the solution $g^* = G_{s,\nu}\hat{f}$ to Poisson's equation then follows immediately, upon application of Theorem 1 of TWEEDIE (1983) to the function h.

Although Theorem 1 covers a great many applications, there are certain settings in which the minorization condition (2.4) does not hold with $m = 1$. The remainder of this section is devoted to discussing the modifications necessary in order to deal with $m > 1$.

Set

$$G_{m,s,\nu} = \sum_{n=0}^{\infty} [P^m - s \otimes \nu]^n \sum_{j=0}^{m-1} P^j. \tag{2.11}$$

According to Theorem 3.5 to NUMMELIN (1985), the function $g^* = G_{s,m,\nu}f$ satisfies Poisson's equation $g(x) = (Pg)(x) + f(x)$ for π a.e. x whenever $\pi f = 0$. (Actually, the statement of the result is that g^* solves Poisson's equation on a closed domain of S. However, Proposition 2.5 of NUMMELIN (1984) asserts that the complement of any closed domain has π-measure zero.)

To carry out the program followed in the case $m = 1$ for the case $m > 1$ is now straightforward. We need to bound $|G_{m,s,\nu}f|$ by a Lyapunov function of some kind. Note that

$$G_{m,s,\nu}f = \sum_{n=0}^{\infty} K^n \bar{f},$$

where $K = P^m - s \otimes \nu$ and $\bar{f} = \sum_{j=0}^{m-1} P^j f$. A similar analysis to that used to derive Theorem 1 then yields the next result.

THEOREM 2. Let X be a Markov chain for which there exists a set $A \in \mathcal{S}$ and $\lambda > 0$ such that

$$P^m(x, \cdot) \geq \lambda \nu(\cdot) \tag{2.12}$$

for all $x \in A$. Suppose that there exists ϵ, $\eta > 0$ and non-negative finite-valued $\mathcal{S}$-measurable functions h and k such that

$$(P^m h)(x) \leq h(x) - \epsilon \left(\sum_{k=0}^{m-1} (P^k |f|)(x) + 1 \right)$$
$$(Pk)(x) \leq k(x) - \eta h(x)^p$$

for $x \in A^c$ and $p > 0$. In addition, assume that $\|I_A P^k |f|\| (0 \leq k \leq m-1)$, $\|I_A P^m h\|$, $\|I_A h\|$, $\|I_A Pk\|$ are all finite. Then:

i) X is a positive recurrent Harris chain;
ii) If π is the (unique) invariant probability measure of X, then $\pi|f| < \infty$ and $G_{m,s,\nu}(|f|+1) < \infty$ for all $x \in S$;
iii) If $\hat{f} = f - \pi f$, the function $g^* = G_{m,s,\nu}\hat{f}$ satisfies Poisson's equation $g^*(x) = (Pg^*)(x) + \hat{f}(x)$ at π a.e. x;
iv) There exists constants α, β, independent of x, such that $|g^*(x)| \leq \alpha h(x) + \beta$ for all $x \in S$;
v) $g^* \in L^p(\pi)$.

We note that there always exists a set A satisfying (2.12) if X is a Harris chain. Furthermore, in many applications settings, the set A will be compact and f, h, and k continuous. If the transition function P is Feller (i.e, Pr is bounded and continuous whenever r is), then $P^i f$, $P^j h$, and $P^\ell k$ will usually be continuous (this is automatic if the functions are bounded), and the finiteness of $\|I_A P^i f\|$, $\|I_A P^j h\|$, and $\|I_A P^\ell k\|$ is then immediate.

We conclude this section with an application of our results to the question of when a FCLT holds for an additive functional of a Markov chain X.

THEOREM 3. Suppose that either Theorem 1 or Theorem 2 is in force with $p = 2$. Then, there exist constants r and σ such that $Z_n \Rightarrow \sigma B$ P_μ-weakly as $n \to \infty$ in $D[0, \infty)$ (for μ is an arbitrary probability measure on $(S, \mathcal{S})$), where

$$Z_n(t) = n^{1/2}\left(\frac{1}{n}\sum_{k=0}^{\lfloor nt\rfloor} f(X_k) - rt\right).$$

Furthermore, the constants r and σ^2 can be defined as $r = \pi f$ and $\sigma^2 = \pi(g^*)^2 - \pi(Pg^*)^2$, when g^* is defined as in Theorems 1 and 2.

This basically follows from MAIGRET (1978). Our theorem actually requires a slightly strengthened version of her result, because Theorem 3 asserts that the weak convergence holds for any initial distribution μ. To obtain our result from MAIGRET (1978), let $A_{n,\ell}(B) = \{n^{-1/2}\left(\sum_{k=\ell}^{\lfloor n\cdot\rfloor} f(X_k) - r\cdot\right) \in B\}$, for B a (measurable) subset of $D[0,\infty)$. If X is an aperiodic Harris chain, then μP^ℓ converges to π in total variation norm. As a consequence,

$$\sup_{\substack{n\geq 1\\ B}} |P_\mu(A_{n,\ell}(B)) - P_\pi(A_{n,\ell}(B))| \to 0$$

as $\ell \to \infty$. Since MAIGRET (1978) does establish weak convergence under P_π, it is evident that $P_\pi(A_{n,\ell}(\cdot)) \Rightarrow P\{\sigma B \epsilon \cdot\}$ as $n \to \infty$. Noting that

$$n^{-1/2}\left(\sum_{k=0}^{\ell-1} f(X_k) - \ell r\right) \Rightarrow 0$$

P_μ-weakly in $D[0,\infty)$, we may therefore conclude that $Z_n \Rightarrow \sigma B$ P_μ-weakly. Thus, Theorem 3 holds under any initial distribution μ. (A slight variation in this argument handles the case when X is periodic.)

3. EXAMPLES

In this section, we illustrate our Lyapunov function techniques with a couple of examples.

EXAMPLE 2. Let $X_{n+1} = [X_n + \eta_{n+1}]^+$, where $(\eta_n : n \geq 1)$ is i.i.d. For simplicity of exposition, we require that η_n have a distribution that satisfies $P\{\eta_n \leq x\} > 0$ for all x. The chain $X = (X_n : n \geq 0)$ can be viewed as the waiting-time sequence of a single-server queuing system (see ASMUSSEN (1987)). It is well known that if $E|\eta_1| < \infty$, then it is necessary that $E\eta_1 < 0$ in order that X be a positive recurrent Harris chain. Furthermore, it is easy to see that condition (2.6) holds for any compact set $A \subseteq [0, \infty)$ (we can then take ν to be a point mass at zero).

Suppose that $f(x) = x$. Then, if $E\eta_1 < 0$ and $E|\eta_1|^5 < \infty$, we can choose $h(x) = x^2$ and $k(x) = x^5$. Note that

$$(Ph)(x) = h(x) + 2f(x)E\eta_1 + 0(1)$$

$$(Pk)(x) = k(x) + 5h(x)^2 E\eta_1 + 0(x^3)$$

as $x \to \infty$. By choosing ϵ, η, and K appropriately, it is evident that (2.7) and (2.8) can be made to hold for $x \geq K$. By choosing $A = [0, K]$, we find that Theorems 1 and 3 then apply. Hence, $S_n = X_1 + \ldots + X_n$ satisfies a FCLT if $E\eta_1 < 0$, $E|\eta_1|5 < \infty$, and $P\{\eta_1 \leq x\} > 0$ for all x. (By arguing more carefully, one can easily drop this last hypothesis.)

It turns out that the fifth moment hypothesis tat we have imposed is close to the right condition for guaranteeing that S_n satisfy a FCLT. DALEY (1968) shows that for the $GI/G/1$ queue, it is necessary and sufficient that $E\max(0, \eta^4) < \infty$ in order that the series

$$\sum_{n=0}^{\infty} \operatorname{cov}_\pi(X_0, X_n)$$

be absolutely summable. This absolute summability condition is implicit in many FCLT's for stationary processes (see ETHIER and KURTZ (1986), p. 351). This suggests that the "correct" condition to require for validity of the FCLT in this setting is $E\eta_1^4 < \infty$. It is reasonable to question why our proof requires an additional moment on η_1.

The basic reason is that the martingale CLT basically applies to the additive functional $(S_n : n \geq 0)$ whenever $E_\pi D_k^2 < \infty$, where $D_k = g(X_k) - (Pg)(X_{k-1})$. Suppose that $g \in L^2(\pi)$. Then, $g(X_k) - (Pg)(X_{k-1})$ is orthogonal to $(Pg)(X_{k-1})$, so that

$$E_\pi D_k^2 = E_\pi g^2(X_0) - E_\pi (Pg)(X_0)^2.$$

The approach that we have followed in this paper to show that $E_\pi D_k^2 < \infty$ is to establish sufficient conditions, using Lyapunov functions, for the finiteness of $E_\pi g^2(X_0)$. This is the standard technique used in the literature to verify square integrability of D_k. However, in this example, we believe that if $E\eta_1^4 < \infty$ with $E|\eta_1|^5 = \infty$, then $g\epsilon L^2(\pi)$ but $E_\pi D_k^2 < \infty$. Thus, our analysis does not obtain the appropriate moment condition because it is not a fine enough tool to be able to pick up "cancellation" that occurs between $g(X_n)$ and $(Pg)(X_{n-1})$. However, it seems likely that in most examples, the cancellation effect is probably insignificant. Thus, one can expect that in general, the Lyapunov function method described in this paper should be capable of obtaining moment conditions that are close to optimal. This is illustrated in our next example.

EXAMPLE 3. Let $X_{n+1} = \rho X_n + \eta_{n+1}$, where $(\eta_n : n \geq 1)$ is i.i.d. To simplify our discussion, we shall require that η_n has a continuous Lebesque density that is strictly positive everywhere. The chain $X = (X_n : n \geq 0)$ given in this example is, of course, just a first-order autoregressive sequence. For stability of X, we shall demand that $|\rho| < 1$ and $E|\eta_1| < \infty$. As in Example 2, it is easily checked that condition (2.6) holds whenever A is a compact subset of $(-\infty, \infty)$ (we can take ν equal to Lebesque measure on $[0,1]$).

Suppose that $f(x) = x$. We claim that if $E\eta_1^2 < \infty$, then $S_n = X_1 + \ldots + X_n$ satisfies a FCLT. To see this, we put $h(x) = |x|$ and $k(x) = x^2$. Then,

$$\begin{aligned}(Ph((x) &\leq |\rho \cdot |x| + E|\eta_1| \\ &= h(x) - |f(x)|(1 - |\rho|) + O(1)\end{aligned}$$

and

$$\begin{aligned}(Pk)(x) &= \rho^2 x^2 + 2\rho x E\eta_1 + E\eta_1^2 \\ &= k(x) + (\rho^2 - 1)h^2(x) + O(x)\end{aligned}$$

as $x \to \infty$. Again, an appropriate choice of ϵ, η, and K prove that conditions (2.6)-(2.8) hold for $A = [-K, K]$. Thus, the condition $E\eta_1^2 < \infty$ suffices to guarantee that $(S_n : n \geq 0)$ satisfies a FCLT. We note that the Lyapunov function method that we have applied produces the minimal moment condition required for this FCLT.

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A LYAPUNOV BOUND FOR SOLUTIONS OF POISSON'S EQUATION

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ABSTRACT:

Suppose that X is a positive recurrent Harris chain with invariant measure π. We develop a Lyapunov function criterion that permits one to bound the solution g to Poisson's equation for X. This bound is then applied to obtain sufficient conditions that guarantee that the solution be an element of $L^p(\pi)$. When $p = 2$, the square integrability of g implies the validity of a functional central limit theorem for the Markov chain. We illustrate the technique with applications to the waiting time sequence of the single-server queue and autoregressive sequences.