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ERROR CANCELLATION IN HPBL DERIVATION OF ELASTIC HOLE-PRESSURE ERROR[1]

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ABSTRACT

Analytical studies of the hole-pressure error for non-Newtonian creeping flows over a transverse slot are pursued with particular interest in the theory of Higashitani, Pritchard, Baird and Lodge (HPBL). To correctly apply the HPBL theory a modified hole-pressure relation (MHPR) is employed. Some important mathematical properties of the MHPR are presented. By studying the MHPR in streamline coordinate formulation we find a fortuitous *error cancellation phenomenon* in the derivation of HPBL formula. For second-order fluid and Tanner's "viscometric model" (under certain assumptions) the error cancellation is proved to be exact. It is this error cancellation phenomenon that provides a complete theoretical explanation for the paradox between an apparently flawed derivation and the fortunate success of the HPBL prediction.

AMS (MOS) Subject Classifications: 76A05, 76A10, 76D07, 73B05.
Key Words: Hole Pressure Error, Error Cancellation,
Non Newtonian Fluid, Second Order fluid,
HPBL Theory, Modified Hole Pressure Relation.

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1. Introduction

In this paper we consider shear flow of some liquids between parallel walls which are at a separation of h. One wall contains a hole whose cross section is a transverse slot (of narrow dimension w parallel to the main flow direction). There is no net flow through the hole. A transducer at point a on the bottom of the hole records a normal thrust $P_a = -\hat{T}_{22}|_a$, exerted by the fluid. Another transducer mounted flush with the wall face opposite the hole records a normal thrust $P_b = -\hat{T}_{22}|_b$ at a location b on the hole centerline $\mathbb{C}$ shown schematically in Figure 1, is a well-established experimental procedure for determining the fluid thrust component normal to a solid wall in the undisturbed shear flow. For slow flows of Newtonian fluids, it is well known that the two readings, P_a and P_b, are the same, i.e. no error is introduced by this procedure. But for non-Newtonian fluids, by contrast, the difference between P_a and P_b can be very substantial as shown both experimetally and numerically (see, among others, references [1] to [12]). This difference is called hole-pressure error in the literature and usually denoted as P_H, namely

$$P_H = P_b - P_a = -(\hat{T}_{22}|_b - \hat{T}_{22}|_a) \ . \tag{1.1}$$

The physical explanation of the existence of hole-pressure error P_H can be found in references [13,14]. Roughly speaking, the stretching of the N_1- spring, as shown in Figure 1, indicates how an extra tension T along a streamline lifts the fluid out of the hole resulting in a low reading of P_b. An analytical calculation of the hole-pressure error, P_H, was first made by Tanner and Pipkin [2] for the particular case of slow, two-dimensional flows of second-order fluids past a transverse slot. They showed that P_H is equal in magnitude to a quarter of the first normal stress difference, N_1, of the fluid.

The hole-pressure problem has been a research topic of considerable interest in recent years because of the possibility of obtaining elasticity measurements for viscoelastic liquids. Of particular interest has been the theory of Higashitani, Pritchard, Baird and Lodge (HPBL) [15,5,6] which leads to expressions relating P_H to the shear stress $\hat{T}_{12}$, the wall shear stress σ_w and the first normal stress difference N_1. Based on the phenomenon of hole-pressure error, Lodge and de Vargas [5] constructed a slit die rheometer using the hole-pressure difference P_H to measure the first normal stress difference N_1 for molten polymers. This application has made it more important to understand the theory behind the hole-pressure phenomenon, and the hole-pressure problem itself has been proved to be a highly controversial issue.

The big conflict, that has been puzzling people in this field for many years, is that although very good agreement between the theory of HPBL and the published experimental & numerical results has been obtained for several polymer solutions, polymer melts and some non-Newtonian constitutive models, it has been known that at least two of the key assumptions in the theory are violated [16]. Some attempts have been made in the literature trying to correct the flaws in HPBL theory and to explain the paradox, but not much progress has been achieved so far.

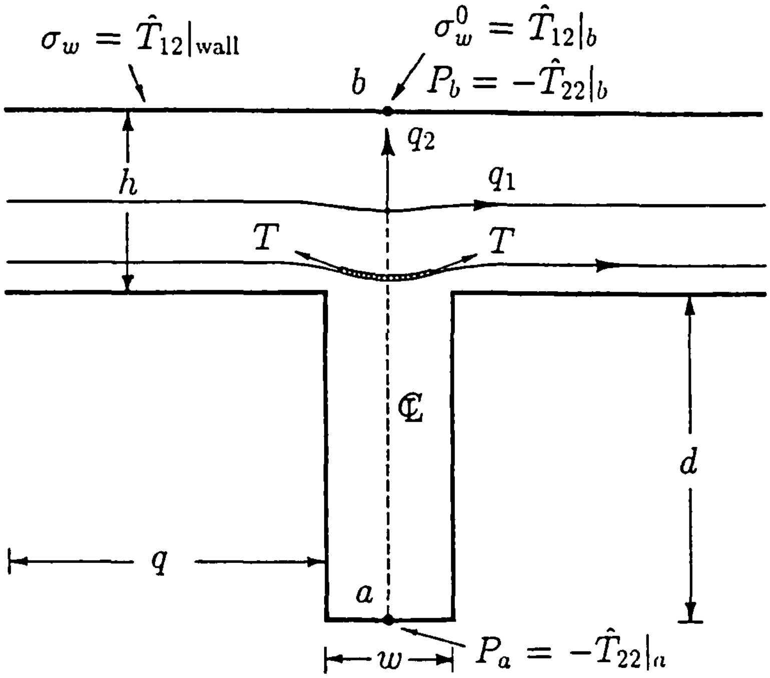

Figure 1. Schematic illustration of hole-pressure problem and definition of streamline coordinate system for planar flows past a transverse slot.

In order to correctly apply the HPBL theory, a *modified hole-pressure relation* (MHPR) in path integral form is used in our work. This MHPR contains an extra term, i.e. the contribution of the stress gradient $\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1}$. This term is non-zero in general, according to our numerical observation on the excess pressure rise phenomenon [17], however it was assumed to be zero and was neglected in the original derivation of HPBL theory. To investigate the two key flaws of the HPBL theory, we found that it is sufficient for the present to consider the second-order fluid, since both flaws exist there. By studying the MHPR based on the streamline coordinate formulation, we found a very fortuitous and interesting phenomenon behind the success of HPBL prediction : *the exact error cancellation for the second-order fluids*, i.e. the error caused by one key flaw can be exactly canceled out by the error introduced by another key flaw. By using the streamline coordinate formulation, we are also able to prove analytically the equivalence of the MHPR with the re-interpreted HPBL prediction. That is, although the two relations have different forms and contain different terms, the MHPR and the re-interpreted HPBL theory predict the same exact value of $N_{1w}^0/4$ for the second-order fluid. Furthermore, the conclusion of error-cancellation has also been generalized to include Tanner's "viscometric model." These theoretical results enable us to eventually solve the puzzle about the success of HPBL prediction and to explain the controversy in the derivation of the HPBL formula.

The outline of this paper is as follows. In section 2 we review the HPBL theory, some recent work in the literature and the modified hole-pressure relation; in section 3 we develop the streamline coordinate formulation of the MHPR for a second-order fluid; in section 4 we prove the exact error cancellation in MHPR for second-order fluid and for Tanner's viscometric model; finally, in section 5 we discuss our conclusions.

2. The HPBL Theory and Modified Hole-Pressure Relation

2.1 HPBL Theory of Creeping Flow

By considering the equation of motion in streamline coordinate form and making some assumptions about the flow field, Higashitani and Pritchard [15] deduced a proposed relation between hole-pressure difference, P_H, and the first normal-stress difference, N_1. The final relation is attributable to Higashitani, Pritchard, Baird, and Lodge [5,6], and it has two forms for creeping flow:

academic form

$$P_H = \int_0^{\sigma_w} \frac{N_1}{2\tau}\, d\tau \ , \tag{2.1}$$

practical form

$$N_{1w} = 2P_H \frac{d\ln P_H}{d\ln \sigma_w} \quad \text{or} \quad N_{1w} = 2\sigma_w \frac{dP_H}{d\sigma_w} \tag{2.2}$$

where σ_w and N_{1w} are the values of $\hat{T}_{12}$ and N_1 at the channel walls as shown in Figure 1. P_H is the hole-pressure difference defined by equation (1.1). The practical form (2.2) is obtained by differentiating (2.1) with respect to σ_w. It is important to note that in the original HPBL formula N_1 & τ correspond to the steady unidirectional shear flows only, and $N_1 = \nu_1\dot{\gamma}^2$ & $\tau = \eta\dot{\gamma}$ are the strict viscometric functions. However, in this paper we shall define N_1 as the generalized viscometric part of $\hat{T}_{11} - \hat{T}_{22}$ for more general flows which may not necessarily be a unidirectional shear (viscometric) flow.

For convenience of further discussion and reference, the assumptions made either explicitly or implicitly by Higashitani, Pritchard, Baird and Lodge in their derivation of equations (2.1) & (2.2) [15,5,6] are summarized as following:

(A1) Incompressible, isothermal, creeping Couette flow (plate-driven) is primarily concerned. (Re is very small)

(A2) The flow patterns are symmetric about the hole centerline $\mathbb{C}$. Particularly, the streamlines are symmetric about $\mathbb{C}$, which gives the symmetry conditions

$$\frac{\partial h_i}{\partial q_1} = 0. \quad (i = 1,2) \quad \forall\,(q_1, q_2)\ \in \mathbb{C} \tag{2.3}$$

(A3) An orthogonal curvilinear coordinate system based on the streamlines of the flow field can be established in the local area around the hole centerline.

(A4) The stresses are symmetrically distributed about the hole centerline $\mathbb{C}$, consequently

$$\frac{\partial \hat{T}_{12}}{\partial q_1} = 0 \qquad \forall\,(q_1, q_2)\, \in \mathbb{C} \tag{2.4}$$

$$\frac{\partial \hat{T}_{11}}{\partial q_1} = 0 \qquad \forall\,(q_1, q_2)\, \in \mathbb{C} \tag{2.5}$$

(A5) The motion at the hole centerline is that of a shear (viscometric) flow. Therefore, the quantities N_1 and $\hat{T}_{12}$ in equation (2.1) are the same as the viscometric functions.

(A6) The path integral

$$\int_{q_2^a}^{q_2^b} \frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} \frac{\partial \hat{T}_{12}}{\partial q_2}\, dq_2 \tag{2.6}$$

exists and the change of variable from streamline coordinate q_2 to shear stress σ should be valid.

It is known from [17] that the streamwise gradients, $\frac{1}{h_1}\frac{\partial p}{\partial q_1}$ and $\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1}$, are non-zero on $\mathbb{C}$ even for the Stokes solution of Couette base flow over a slot. Flow visualization [6] and analytical studies [17] have shown that $\hat{T}_{11} - \hat{T}_{22}$ is not viscometric on $\mathbb{C}$ due to the existence of slot. These two important facts suggest that the derivation of HPBL prediction (2.1), especially when it is desired to apply HPBL theory to Poiseuille base flow, is flawed mainly in two ways:

(i) The assumption of $\hat{T}_{11,1} = 0$ on $\mathbb{C}$ is incorrect, i.e. *the streamwise stress gradient* is in general non-zero. (We will see later on in section 4 that the term $\hat{T}_{11,1}$ plays an important role in the error cancellation.)

(ii) The assumption (A5) is not true, i.e. $\hat{T}_{11} - \hat{T}_{22}$ is in general not viscometric. Therefore, the change of variable required in getting the rheological space integral (2.1) from the path integral (2.6) is invalid. The change of variable requires that $\hat{T}_{11} - \hat{T}_{22}$ can be written as a unique function of $\tau = \hat{T}_{12}(q_2)$. This is not possible in general as will be seen in equation (3.17). It is also easy to see that the correspondence between the values of $\hat{T}_{11} - \hat{T}_{22}$ and the values of $\hat{T}_{12}$ is not *one to one*.

These two flaws have been pointed out by Malkus & Yao in reference [16]. From assumption (A1) we know that only creeping Couette flow is primarily concerned when Higashitani and Pritchard derived the relation (2.1) in [15]. Since that time, the possibility of applying relation (2.1) & (2.2) to Poiseuille flow, and to flows with inertial effects, etc. has been explored both experimentally and numerically in references [5-12,18]. There is now considerable experimental and computational evidence to suggest that (2.1) is a reasonably good approximation to the observations.

Srinivasan, in a recent paper [19], attempted to extend the HPBL theory by including the inertial effects and the effect of asymmetry in flow field, and the integral was theoretically evaluated along a generalized path of zero slope of the streamlines (this path may not coincide with the hole centerline in the most general situation). Unfortunately, although

Srinivasan recognized that the assumption of (2.5) would be a significant source of error, he generally accepted the assumption $\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} = 0$ in his formulation.

2.2 Tanner's Recent Work - an Alternative Approach

To avoid the paradoxes and problems with the original HPBL analysis, Tanner used an alternative approach in reference [20] to formulate the hole-pressure difference. In his work, Tanner made the following fundamental assumptions:

(T1) Assume the fluid is desrcibed by the "*viscometric*" constitutive equation [21]

$$\mathbf{T} = -p\mathbf{I} + \eta\mathbf{A} + (\nu_1 + \nu_2)\mathbf{A}^2 - \frac{1}{2}\nu_1\mathbf{B} \tag{2.7}$$

where $\eta(\dot\gamma)$, $\nu_1(\dot\gamma)$, $\nu_2(\dot\gamma)$ are functions of $\dot\gamma$.

(T2) Assume that the normal stress coefficient ν_1 is proportional to the viscosity $\eta(\dot\gamma)$, viz.

$$\nu_1 = \alpha\eta \tag{2.8}$$

where α is a constant.

(T3) The following identity

$$\nabla\cdot[\nu_1(\mathbf{B} - \mathbf{A}^2)] + \nabla\cdot\left(\mathbf{A}\frac{D\nu_1}{Dt}\right) = \nabla\left(\alpha\frac{Dp^o}{Dt} + \int_0^{\dot\gamma}\nu_1\dot\gamma\,d\dot\gamma\right) \tag{2.9}$$

is used with the postulation that $D\nu_1/Dt$ is small and negligible.

With the above three assumptions, a closed-form solution of the modified pressure can be obtained, viz.

$$p = p^o - \frac{\alpha}{2}\frac{Dp^o}{Dt} + \frac{1}{2}\left(\nu_1\dot\gamma^2 - \int_0^{\dot\gamma}\nu_1\dot\gamma\,d\dot\gamma\right) + \nu_2\dot\gamma^2 \tag{2.10}$$

where p^o is the pressure of the generalized Newtonian flow, i.e. the flow that has the same symmetry as Stokes flow but its viscosity, $\eta(\dot\gamma)$ can be more general. By calculating the hole-pressure difference directly *via.* (2.10), Tanner found the formula

$$P_H = \frac{1}{2}N_{1w} - \frac{1}{2}\int_0^{\dot\gamma}\nu_1\dot\gamma\,d\dot\gamma\ . \tag{2.11}$$

When the flow is in the second-order regime, equation (2.11) gives the well known result [2]

$$P_H = N_{1w}/4\ . \tag{2.12}$$

Based on this result, Tanner concluded that the HPBL equation is applicable to the hole-pressure problem, even though when flow is not strictly a viscometric flow assumed in (T1). Furthermore, Tanner predicted that the Maxwell and Oldroyd-B models, which have constant viscosity and normal stress coefficients in truly viscometric flows, are expected to show the second-order fluid result (2.12).

Tanner's work is important in verifying and supporting the conclusion about the validity of the HPBL theory from an alternative approach. But his work did not directly resolve the paradoxes and the problems in the derivation of HPBL theory.

2.3 Modified Hole-Pressure Relation (MHPR)

If one goes through the details in deriving equation (2.1) by following the way described in [15], one would find that the equation in streamline coordinates which is integrated along the hole centerline to give the thrust difference, P_H, is not

$$\frac{\partial \hat{T}_{12}}{\partial q_2} + \frac{2\hat{T}_{12}}{\hat{T}_{11} - \hat{T}_{22}} \frac{\partial \hat{T}_{22}}{\partial q_2} = 0 \tag{2.13}$$

but

$$\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} + \frac{2}{h_2}\frac{\hat{T}_{12}}{\hat{T}_{11} - \hat{T}_{22}} \frac{\partial \hat{T}_{22}}{\partial q_2} = 0, \tag{2.14}$$

where $\hat{T}_{ij}$ are the physical components of stress in streamline coordinates and h_i are the metric coefficients. The first term in (2.14) is the contribution of the streamwise stress gradient. It is assumed to be zero in the HPBL theory by assumption (A4). If we leave the stress gradient term in (2.14) and continue the derivation, we will get a modified hole-pressure relation [16]

$$P_H = \int_{q_2^a}^{q_2^b} \left(\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} \right) \frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} h_2 \, dq_2 \ , \tag{2.15}$$

which looks quit different from the HPBL predition (2.1). (2.15) is in path integral form and contains an extra term, while (2.1) is an integral in the rheological stress space. Since it has been known that the HPBL prediction (2.1) is quite successful for some non-Newtonian fluids, and it also seems to work even under less restrictive assumptions [5,6,17,18,24,25], the questions which arise are: ***What is the relation between MHPR and HPBL prediction? How does the MHPR*** (2.15) ***work? Does*** (2.15) ***predict the same result as the HPBL theory?*** We shall try to answer these questions in the following sections.

2.4 Some Mathematical Properties of MHPR

We have skipped over some important details in getting the path integral (2.15) from (2.14). We need to look into the derivation more deeply and try to define the MHPR rigorously.

After considering the symmetric conditions (2.3) & (2.4), the equations of motion are simplified as follows

$$\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} + \frac{2\hat{T}_{12}}{h_1 h_2}\frac{\partial h_1}{\partial q_2} = 0, \tag{2.16}$$

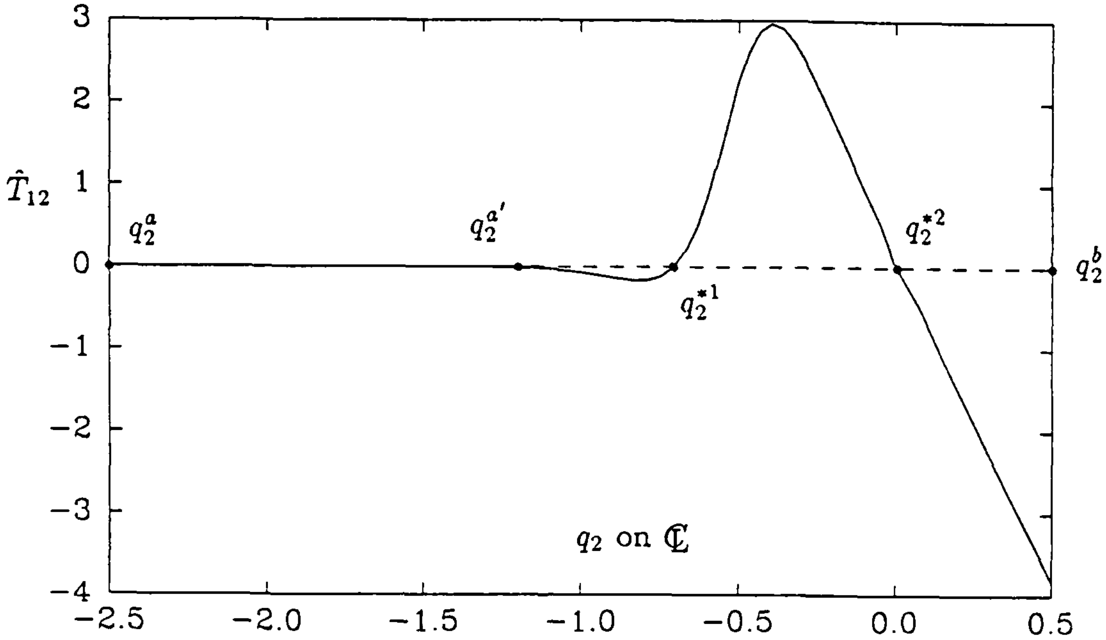

Figure 2. An example of the possible change of shear stress $\hat{T}_{12}$ on the hole centerline $\mathfrak{C}$ for a Poiseuille base flow of second-order fluid over a slot (De ≈ 0.2). Note $q_2 = 0$ is the center of channel.

$$\frac{1}{h_1}\frac{\partial h_1}{\partial q_2} = \frac{1}{\hat{T}_{11} - \hat{T}_{22}}\frac{\partial \hat{T}_{22}}{\partial q_2}\ . \tag{2.17}$$

To obtain (2.15), we eliminated $\partial h_1/\partial q_2$ from (2.16) & (2.17), multiplied both sides of (2.14) by $h_2(\hat{T}_{11} - \hat{T}_{22})/2\hat{T}_{12}$, and then integrated it on $\mathfrak{C}$. By doing this, we have introduced singularities to the path integral of MHPR, because $\hat{T}_{12}$ can be zero at some places on $\mathfrak{C}$. Figure 2 shows an example of the possible change of $\hat{T}_{12}$ on $\mathfrak{C}$ for a Poiseuille flow of second-order fluid. We can classify the places where $\hat{T}_{12} = 0$ into two different cases: a closed interval such as the part $[q_2^a,\ q_2^{a'}]$ shown in Figure 2; or some isolated points, like the points q_2^{*i}, $(i = 1, 2)$ in Figure 2. These two cases need to be considered separately and the results are given in the following two propositions.

Proposition 2.1 For incompressible, steady, creeping, 2-D flows over a slot (Re = 0), if the shear stress $\hat{T}_{12} \equiv 0$ on a closed interval $[q_2^a, q_2^{a'}]$ of $\mathfrak{C}$, then the hole-pressure difference between q_2^a and $q_2^{a'}$ is zero, i.e.

$$\begin{aligned} P_H|_{q_2^a - q_2^{a'}} &= -(\hat{T}_{22}|_{q_2^{a'}} - \hat{T}_{22}|_{q_2^a}) \\ &= \int_{q_2^a}^{q_2^{a'}} \left(\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2}\right) \frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} h_2\, dq_2 = 0\ . \end{aligned} \tag{2.18}$$

Proof: Since $\hat{T}_{12} \equiv 0$ on $[q_2^a, q_2^{a'}]$, we have $\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} = 0$ on $[q_2^a, q_2^{a'}]$ according to (2.16). This immediately leads to the result in equation (2.18). $\diamondsuit$

Proposition 2.2 Let q_2^* be an isolated point on $\mathfrak{C}$ where $\hat{T}_{12} = 0$. Assume $\hat{T}_{11}$ and $\hat{T}_{12}$ are C_2 at point q_2^*. Then the integrand of (2.15) has only a removable singularity at q_2^* and its limit at q_2^* exists.

Proof: $\hat{T}_{12} = 0$ at q_2^* implies that

$$\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} = 0 \qquad \text{at} \;\; q_2^*$$

from equation (2.16). Do the Taylor series expansion in the vicinity of point q_2^*, we obtain

$$\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} = \frac{\partial}{\partial q_2}\left(\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2}\right)_{q_2^*} \Delta q_2 + O(|\Delta q_2|^2)$$

$$\hat{T}_{12} = \frac{\partial \hat{T}_{12}}{\partial q_2}\,|_{q_2^*}\, \Delta q_2 + O(|\Delta q_2|^2)$$

where $\Delta q_2 = q_2 - q_2^*$. Substitution of the above two equations into the integrand of (2.15) gives

$$\begin{aligned} &\lim_{q_2 \to q_2^*}\left(\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2}\right)\frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} \\ &= \frac{\partial}{\partial q_2}\left(\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2}\right)_{q_2^*}\left(2\frac{\partial \hat{T}_{12}}{\partial q_2}\right)_{q_2^*}^{-1}(\hat{T}_{11} - \hat{T}_{22})_{q_2^*} \le M \end{aligned} \tag{2.19}$$

where constant M is a finite real number. $\diamondsuit$

Remark: Since the singularity of the integrand of (2.15) is removable at the isolated pole q_2^*, the integrand will become a continuous function at q_2^* if we define its value to be its limit (2.19). In this way there will be no real singularity involved in the MHPR and the path integral (2.15) can be well defined. Based on Propositions 2.1 and 2.2, one can directly prove the following proposition regarding the existence of MHPR path integral.

Proposition 2.3 Assume $\hat{T}_{11}$ and $\hat{T}_{12}$ be C_2 continuous. Assume $\hat{T}_{12} = 0$ on $[q_2^a, q_2^{a'}]$ and at some isolated points q_2^{*i} $(i = 1, 2, \ldots)$. If we interprete the path integral of MHPR (2.15) as

$$P_H = \int_{q_2^{a'}}^{q_2^b}\left(\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2}\right)\frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} h_2\, dq_2 \tag{2.15a}$$

where the values of integrand at the removable poles $q_2^{a'}$, q_2^{*i}, $(i = 1, 2, \ldots)$, should be replaced by its limits (2.19), then the path integral (2.15) exists and is finite.

Remark: In practice, it may not be necessary to assume as much smoothness as we have to obtain the same results. For example, the C_2 continuous condition for $\hat{T}_{12}$ & $\hat{T}_{11}$ assumed here may be replaced by some weaker assumption if the distribution theory is used.

Finally, as pointed out by Higashitani & Pritchard in [15], the path integral in (2.15) may pass through some points, such as the center of a vortex, at which the streamline coordinate system is not well defined. The path integral should then be carried out in a piecewise manner.

3. Formulation of the MHPR for a Second-Order Fluid

In this section we shall confine our attention to the slow flows of a second-order fluid and try to formulate analytically the MHPR in terms of the Stokes solution. First it is worth noting that the MHPR path integral in equation (2.15) is based on the streamline coordinate system; the stresses involved are the physical stress components in streamline coordinates and $\partial/\partial q_1$ is the streamwise derivative. An important fact we will use in our development is that for the assumed symmetric velocity field the streamline coordinate systems on $\mathfrak{C}$ are orthogonal curvilinear systems (by assumptions (A2), (A3) in section 2 and by choosing the q_2 axis properly), and these systems coincide with the local rectangular Cartesian coordinate systems. As defined in reference [22], the physical components of a vector or second-order tensor at a point P relative to a system of orthogonal curvilinear coordinates are simply the Cartesian components in a local set of Cartesian axes tangent to the coordinate curves through P. Any tensor equation involving only tensor values at one point and no derivatives with respect to the coordinates has the same form in terms of the physical components in orthogonal curvilinear coordinates as it has in terms of rectangular Cartesians. However the constitutive equation we are going to consider here does involve derivatives with respect to the streamline coordinates. Accordingly, we need to work on differentiations with respect to orthogonal curvilinear coordinates.

3.1 Formulation of Stress

By the Tanner–Giesekus theorem, the Stokes solution for the velocity, $\hat{\mathbf{u}}$, with a modified pressure, p, satisfies the second-order fluid equations [14,15,23]. The resulting stress field is

$$\begin{aligned}\hat{\mathbf{T}} = &-[\, p^o - T(\hat{u}_1 p^o_{,1} + \hat{u}_2 p^o_{,2})]\,\mathbf{I} + \eta\mathbf{A} \\ &- \eta T(\mathbf{B} - \mathbf{A}^2 - \frac{1}{2}\dot{\gamma}^2\,\mathbf{I}) + \eta T^*(\mathbf{A}^2 - \dot{\gamma}^2\mathbf{I}) \end{aligned} \tag{3.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the first two Rivlin–Ericksen tensors; η, T and T^* are the constants of the second-order fluid expansion, $\hat{\mathbf{u}}$ and p^o are the velocity and pressure fields of Stokes flow, and $\dot{\gamma}^2 = tr\mathbf{A}^2/2$.

Define the streamline coordinate system as shown in Figure 1. Particularly, we can choose

$$q_1 = q_1(x_1, x_2), \qquad q_2 = x_2 \tag{3.2}$$

$$h_1 = h_1(q_1, q_2), \qquad h_2 = 1\ . \tag{3.3}$$

From the symmetry assumption of the flow field, it is not difficult to prove the following symmetry conditions:

(a) The q_1-component of velocity, $\hat{u}_1(q_1, q_2)$, is an even function of q_1, consequently

$$\frac{1}{h_1}\frac{\partial \hat{u}_1}{\partial q_1} = 0 \qquad \text{on } \mathfrak{C} \tag{3.4}$$

(b) There is no flow across the streamline, i.e.

$$\hat{u}_2 \equiv 0 \qquad \forall\,(q_1, q_2) \in \Omega \tag{3.5}$$

(c) $p^o(q_1, q_2)$ is an odd function of q_1, viz.

$$p^o = 0 \qquad \text{on } \mathfrak{C} \tag{3.6}$$

(d) $\hat{T}_{12}(q_1, q_2)$ is an even function of q_1

$$\frac{1}{h_1}\frac{\partial \hat{T}_{12}}{\partial q_1} = 0 \qquad \text{on } \mathfrak{C} \tag{3.7}$$

(e) The derivatives of scale factors are

$$\frac{\partial h_1}{\partial q_1} = 0, \qquad \frac{\partial h_1}{\partial q_2} = \text{unknown} \neq 0 \tag{3.8a}$$

$$\frac{\partial h_2}{\partial q_1} = 0, \qquad \frac{\partial h_2}{\partial q_2} = 0 \tag{3.8b}$$

By applying equations (3.4), (3.5) and (3.8), we have the strain-rate tensor on $\mathfrak{C}$

$$\mathbf{A} = 2\mathbf{D} = \begin{bmatrix} 0 & \dot{\gamma} \\ \dot{\gamma} & 0 \end{bmatrix} \tag{3.9}$$

where $\dot{\gamma}$ is the shear strain rate and defined by

$$\dot{\gamma} = \frac{1}{h_2}\frac{\partial \hat{u}_1}{\partial q_2} - \frac{\hat{u}_1}{h_1 h_2}\frac{\partial h_1}{\partial q_2} . \tag{3.10}$$

Hence we have

$$\mathbf{A}^2 = \dot{\gamma}^2 \mathbf{I} . \tag{3.11}$$

The material derivative of $\mathbf{A}$ is reduced to

$$\left[\frac{D\mathbf{A}}{Dt}\right] = 2\hat{u}_1 \begin{bmatrix} \frac{1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial q_2}\dot{\gamma} & 0 \\ 0 & -\left(\frac{1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} + \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial q_2}\dot{\gamma}\right) \end{bmatrix} . \tag{3.12}$$

In getting (3.12), the following relation is used

$$\frac{\hat{u}_1}{h_1^2 h_2}\frac{\partial^2 h_2}{\partial q_1^2} = -\frac{1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} \qquad \text{on } \mathfrak{C} \tag{3.13}$$

which can be verified by differentiating the continuity equation. The convective term is

$$
\begin{aligned}
\mathbf{A}\nabla\hat{\mathbf{u}} &= \begin{bmatrix} 0 & \dot\gamma \\ \dot\gamma & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{h_1}\frac{\partial \hat{u}_1}{\partial q_2} \\ -\frac{\hat{u}_1}{h_1 h_2}\frac{\partial h_1}{\partial q_2} & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\hat{u}_1}{h_1 h_2}\frac{\partial h_1}{\partial q_2}\dot\gamma & 0 \\ 0 & \frac{1}{h_1}\frac{\partial \hat{u}_1}{\partial q_2}\dot\gamma \end{bmatrix}
\end{aligned}
\tag{3.14}
$$

Using (3.12)-(3.15) we obtain the second-order Rivlin-Ericksen tensor on $\mathfrak{C}$

$$
\mathbf{B} = 2\begin{bmatrix} \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} & 0 \\ 0 & -\frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} + \dot\gamma^2 \end{bmatrix} . \tag{3.15}
$$

Inserting (3.9)-(3.11), (3.15) into (3.1) yields the physical components of Cauchy stress tensor on $\mathfrak{C}$

$$
\begin{aligned}
\hat{\mathbf{T}} =& -\left(p^o - T\frac{\hat{u}_1}{h_1}\frac{\partial p^o}{\partial q_1}\right)\mathbf{I} + \eta\begin{bmatrix} 0 & \dot\gamma \\ \dot\gamma & 0 \end{bmatrix} \\
& - 2\eta T\begin{bmatrix} \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} & 0 \\ 0 & -\frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} + \dot\gamma^2 \end{bmatrix} + \frac{3}{2}\eta T\begin{bmatrix} \dot\gamma^2 & 0 \\ 0 & \dot\gamma^2 \end{bmatrix}
\end{aligned}
\tag{3.16}
$$

From equation (3.16) we can obtain some important results:

I. On $\mathfrak{C}$, the physical stress components

$$
\left.
\begin{aligned}
\hat{T}_{12} &= \eta\left(\frac{1}{h_2}\frac{\partial \hat{u}_1}{\partial q_2} - \frac{\hat{u}_1}{h_1 h_2}\frac{\partial h_1}{\partial q_2}\right) = \eta\dot\gamma \\
\hat{T}_{11} - \hat{T}_{22} &= 2\eta T\left(\dot\gamma^2 - 2\frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}\right) = N_1 - 4\eta T\frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}
\end{aligned}
\right\}
\tag{3.17}
$$

II. The hole-pressure error

$$
P_H = -\hat{T}_{22}|_{q_2^b} = \frac{1}{2}\eta T(\dot\gamma_w^0)^2 = N_{1w}^0/4 \tag{3.18}
$$

where $\dot\gamma_w^0$ and N_{1w}^0 are the values of disturbed $\dot\gamma$ and N_1 at wall & $\mathfrak{C}$ opposte to the slot. It is necessary to make the following comments before we proceed further.

(1) The numerical results show that the streamwise derivative, $\partial^2\hat{u}_1/\partial q_1^2$, is non-zero on $\mathfrak{C}$. Therefore, the last equation in (3.17) indicates that the flow is not strictly viscometric, owing to the disturbance induced by the slot. Consequently the assumption (A5) in the original HPBL theory is inappropriate.

(2) For second-order fluid under the symmetry conditions, the shear stress on $\mathfrak{C}$ has the same form as that in viscometric flow, the flow is locally shear flow and $\dot\gamma$ is the usual shear strain rate on $\mathfrak{C}$. However the shear flow on $\mathfrak{C}$ may not be unidirectional.

(3) The notation N_1 in the last equation of (3.17) has different meaning from convention and here $N_1 \neq \hat{T}_{11} - \hat{T}_{22}$ unless the flow is strictly viscometric. In this paper we define the first part in the last equation of (3.17)

$$N_1 = \nu_1 \dot{\gamma}^2 \tag{3.19}$$

as ***the generalized viscometric part*** of $\hat{T}_{11} - \hat{T}_{22}$. We call it the "***viscometric part***" because it has the same form as the conventional viscometric function in the unidirectional shear flow. Note that even when flow is symmetric, $\hat{T}_{11} - \hat{T}_{22}$ still may not be the same as N_1 in the viscometric base flow because in general $\partial^2 \hat{u}_1 / \partial q_1^2 \neq 0$ on $\mathfrak{C}$.

(4) Result II applies when the depth of pressure-hole, d, is deep enough. This will be the case for $d = \infty$ and approximately so for $d \gg w$.

(5) It is known that in general $N_{1w}^0 \neq N_{1w}$ for a given flow field. Strictly speaking, for second-order fluid the correct hole-pressure error should be $N_{1w}^0/4$ as given by (3.18) and the HP result of (2.12) is only an approximation. However the results (2.12) and (3.18) give only a slight difference in the predicted values of P_H for a given flow field (see item (8), below).

(6) In order to make the HPBL prediction be consistent with the directly-calculated values of P_H, we define ***the re-interpreted HPBL formula*** as

$$P_H = \int_0^{\sigma_w^0} \frac{N_1}{2\tau}\, d\tau \tag{2.1'}$$

where σ_w^0 is the disturbed wall shear stress at point q_2^b shown in Figure 1. In (2.1′) the original HPBL formula (2.1) is re-interpreted in two places: first σ_w is changed to σ_w^0; second N_1 represents the generalized viscometric part of $\hat{T}_{11} - \hat{T}_{22}$. Note that the original HPBL prediction (2.1) is actually an approximation for the re-interpreted formula (2.1′). (2.1′) is precise in theory and (2.1) is convenient for application.

(7) The slight difference between (2.1′) and (2.1) does not affect the application of HPBL theory. (2.1) and (2.1′) lead to the same differential-form N_1-P_H relation. As a matter of fact, differentiating bothsides of (2.1′) with respect to σ_w^0 yields

$$N_{1w}^0 = 2\sigma_w^0 \frac{dP_H(\sigma_w^0)}{d\sigma_w^0} \tag{2.2'}$$

here we use $P_H(\sigma_w^0)$ to emphasize that P_H is a function of σ_w^0. It is easy to see, by comparing (2.2′) with (2.2), that once we obtain the differential relation between N_1 & P_H, either in the form of (2.2) or (2.2′), and apply the relation in the rheological space, the values of σ_w & σ_w^0 in (2.2) & (2.2′) become unimportant, they only correspond to two neighboring points on the $\hat{T}_{12}$-axis in the rheological stress space and hence they can be simply replaced by a dummy variable, say, σ or τ. Therefore

(2.2) and (2.2′) actually represent the same N_1-P_H relation in the rheological stress space and we can simply rewrite them as

$$N_1 = 2\sigma \frac{dP_H}{d\sigma} \ . \tag{2.2''}$$

Obviously, (2.2″) is the more general differential-form expression for the N_1-P_H relation in the rheological stress space, and (2.2) & (2.2′) are two special cases for (2.2″) when σ takes the values of σ_w & σ_w^0, respectively.

(8) (2.2″) is only useful for theoretical analysis. In practice the values of σ_w^0 (or $\dot{\gamma}_w^0$) are not available experimentally and some approximation has to be made. Usually the following combination of (2.2) and (2.2′) is employed, i.e.

$$N_{1w} \doteq 2\sigma_w \frac{dP_H(\sigma_w^0)}{d\sigma_w} \tag{2.2'''}$$

where σ_w is the undisturbed wall shear stress, while $P_H(\sigma_w^0)$ is the measured hole-pressure depending on σ_w^0. (2.2‴) is a good approximation for (2.2″) or (2.2′) if $|\sigma_w - \sigma_w^0|$ is small. According to Malkus & Webster [17] $|\sigma_w - \sigma_w^0|$ can be minimized by proper die design with $h/w \geq 1$. The full instrument simulation results presented in [17] also show that the difference by using either σ_w or σ_w^0 in predicting N_1 is rather small.

3.2 Formulation of Streamwise Stress Gradient $\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1}$

The general expression of $\hat{T}_{11}$ is given by equation (3.1). We need to differentiate $\hat{T}_{11}$ with respect to streamline coordinate q_1. Reader is referred to reference [17] for the details of formulation. Here we just simply present the final expression of the streamwise gradient

$$\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} = -\frac{1}{h_1}\frac{\partial p^o}{\partial q_1} + 2\eta \frac{1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} \ . \qquad \text{on } \mathfrak{C} \tag{3.20}$$

4. Error Cancellation – A Fortuitous Phenomenon

In order to make a distinction between the modified hole-pressure prediction defined by (2.15) and the original HPBL prediction, we shall use $\bar{P}_H$ to denote the MHPR and reserve P_H for the original HPBL prediction given in (2.1). With new notations, (2.15) can be re-written as

$$\bar{P}_H = P_H^1 + P_H^2 \tag{4.1}$$

where

$$P_H^1 = \int_{q_2^a}^{q_2^b} \frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} h_2 dq_2 \ , \tag{4.2}$$

$$P_H^2 = \int_{q_2^a}^{q_2^b} \frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} \frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1} h_2 dq_2 \ . \tag{4.3}$$

It is easy to see that P_H^1 is the path integral corresponding to the original HPBL prediction P_H (but they are not equal, we will see it later) and P_H^2 is the streamwise stress gradient term neglected by the HPBL theory.

In reference [12], Baird *et al.* reported their observation on some partial-cancellation of the HPBL integral (2.1) itself. They suggested that the success of HPBL prediction was attributed to the cancellation of certain contributions to the integral in (2.1) until reaching the (channel) centerline at which point one obtain just the contributions from viscometric flow (p.642 [12]). We believe that their picture was incomplete. By studying the MHPR based on the streamline coordinate formulation and the two key flaws of the HPBL theory, we found a more important error cancellation phenomenon in the derivation of the HPBL prediction, i.e. the error caused by neglecting the stress gradient term, $\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1}$, can be canceled out by the error introduced through the viscometric flow assumption (A5). By using the streamline coordinate formulation, we can prove that this error cancellation is *exact* for the second-order fluid. Furthermore, the conclusion of error cancellation is also generalized to include the viscoelastic liquids described by the viscometric constitutive equation (2.7).

4.1 Basic Results for Second-Order Fluid

In order to show the exact error cancellation in MHPR for the second-order fluid, we need some primary results to proceed.

Proposition 4.1 For incompressible, steady, creeping flows over a transverse slot, the hole-pressure difference

$$P_H = -(\hat{T}_{22}|_{q_2^b} - \hat{T}_{22}|_{q_2^a}) = \int_0^{\sigma_w^0} \frac{N_1}{2\sigma}\, d\sigma = N_{1w}^0/4$$

is exact for the second-order fluid. Here N_1 is interpreted as the generalized viscometric part of $\hat{T}_{11} - \hat{T}_{22}$, i.e. $N_1 = \nu_1\dot{\gamma}^2$.

The proof of Proposition 4.1 has already been given in the formulation in section 3.

Proposition 4.2 Let p^o be the pressure of Stokes solution. The following relation

$$\frac{1}{h_1}\frac{\partial p^o}{\partial q_1} = \frac{2\eta}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} + \frac{2\hat{T}_{12}}{h_1 h_2}\frac{\partial h_1}{\partial q_2} + \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2} \tag{4.4}$$

holds on the hole centerline $\mathfrak{C}$.

Proof : Since p^o is the pressure of Stokes solution, the equation of motion gives

$$\nabla p^o = \eta \nabla \cdot \hat{\mathbf{A}} \tag{a}$$

where $\hat{\mathbf{A}} = 2\hat{\mathbf{D}}$, and the components of $\nabla \cdot \hat{\mathbf{A}}$ can be obtained by replacing $\hat{\mathbf{T}}$ with $\hat{\mathbf{A}}$ in equation of motion. The first equation of (a) becomes

$$\frac{1}{h_1}\frac{\partial p^o}{\partial q_1} = \frac{\eta}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_2\hat{A}_{11}) + \frac{\partial}{\partial q_2}(h_2\hat{A}_{21}) + \hat{A}_{12}\frac{\partial h_1}{\partial q_2} - \hat{A}_{22}\frac{\partial h_2}{\partial q_1}\right] \tag{b}$$

By using the symmetry conditions (2.8), (b) can be easily reduced to (4.4). $\diamond$

Proposition 4.3 For second-order fluid, we have

$$P_H^1 = N_{1w}^0/4 - P_H^3 \tag{4.5}$$

where

$$P_H^3 = 2\eta T \int_{q_2^a}^{q_2^b} \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}\frac{1}{\hat{T}_{12}}\frac{\partial \hat{T}_{12}}{\partial q_2}\, dq_2 \ . \tag{4.6}$$

Proof: Substituting (3.17) into (4.2) yields

$$\begin{aligned} P_H^1 &= \int_{q_2^a}^{q_2^b} \left(N_1 - 4\eta T \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}\right)\frac{1}{2\hat{T}_{12}}\frac{\partial \hat{T}_{12}}{\partial q_2}\, dq_2 \\ &= \int_{q_2^a}^{q_2^b} 2\eta T \dot{\gamma}^2 \frac{\eta}{2\eta\dot{\gamma}}\frac{\partial \dot{\gamma}}{\partial q_2} dq_2 - \int_{q_2^a}^{q_2^b} 2\eta T \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}\frac{1}{\hat{T}_{12}}\frac{\partial \hat{T}_{12}}{\partial q_2} dq_2 \\ &= \frac{1}{2}\eta T \left(\dot{\gamma}_w^0\right)^2 - P_H^3 = N_{1w}^0/4 - P_H^3 \ \diamond \end{aligned}$$

Remark : Note that P_H^3 can be considered as the error term introduced by the viscometric flow assumption (A5). According to our numerical results, P_H^3 is in general non-zero. Therefore Proposition 4.3 indicates that in general $P_H^1 \neq N_{1w}^0/4$ and the error included in P_H^1 term comes from the viscometric flow assumption.

Proposition 4.4 For the second-order fluid, we have

$$P_H^2 = N_{1w}^0/4 - P_H^1 \ . \tag{4.7}$$

Proof: By using (3.20) and (4.4) in (4.3)

$$\begin{aligned} P_H^2 &= \int_{q_2^a}^{q_2^b} \left(-\frac{1}{h_1}\frac{\partial p^o}{\partial q_1} + \frac{2\eta}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}\right)\frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} h_2 dq_2 \\ &= \int_{q_2^a}^{q_2^b} \left(-\frac{2\hat{T}_{12}}{h_1 h_2}\frac{\partial h_1}{\partial q_2} - \frac{1}{h_2}\frac{\partial \hat{T}_{12}}{\partial q_2}\right)\frac{\hat{T}_{11} - \hat{T}_{22}}{2\hat{T}_{12}} h_2 dq_2 \\ &= -\int_{q_2^a}^{q_2^b} \frac{1}{h_1}\frac{\partial h_1}{\partial q_2}(\hat{T}_{11} - \hat{T}_{22}) dq_2 - P_H^1 \\ &= -\int_{q_2^a}^{q_2^b} \frac{\partial \hat{T}_{22}}{\partial q_2}\, dq_2 - P_H^1 = N_{1w}^0/4 - P_H^1 \end{aligned}$$

where equation (2.17) and Proposition 4.1 have been used. $\diamond$

Remark : Keep in mind that P_H^2 is actually the error term caused by neglecting the streamwise stress-gradient, $\frac{1}{h_1}\frac{\partial \hat{T}_{11}}{\partial q_1}$. Proposition 4.4 tells us that for second-order fluid the

MHPR predicts the same result as the HPBL theory, namely, $\bar{P}_H = P_H^1 + P_H^2 = N_{1w}^0/4 = P_H$.

Based on Propositions 4.3 & 4.4 we can easily prove the following proposition regarding the ***exact error cancellation in the MHPR*** for the second-order fluids.

Proposition 4.5 For incompressible, steady, creeping flows of the second-order fluids over a transverse slot, we have the following relation

$$P_H^2 = P_H^3 \,. \tag{4.8}$$

Proof: By simply combining Propositions 4.3 & 4.4. $\diamondsuit$

Remark : The result presented in Proposition 4.5 is interesting and important. It concludes that for second-order fluid the error caused by one key flaw in the original HPBL derivation is exactly equal to, hence ***is exactly cancelled out by***, the error introduced through another key flaw. In order to make this point more explicitly we rewrite (4.1) in the following form:

$$\bar{P}_H = P_H^1 + P_H^2 = N_{1w}^0/4 \underbrace{- P_H^3 + P_H^2}_{=0} = P_H \,.$$

This fortuitous error cancellation phenomenon has been unfortunately hidden behind the success of the HPBL prediction for a long time, and now it eventually becomes clear to us. Based on Proposition 4.5 the following conclusion can also be drawn, namely:

For incompressible, steady, creeping flows of the second-order fluid, the MHPR in path integral form, (2.15) is equivalent to the re-interpreted HPBL prediction in stress integral form, (2.1′), due to the exact error cancellation.

4.2 Generalized Results for the Viscometric Model

Having obtained the basic results for the second-order fluid, we now turn to the fluids described by the viscometric constitutive equation (2.7), and try to generalize the error cancellation theorem to include more fluid models. Similar to the case of second-order fluid, we begin by presenting some generalized results for the viscometric model.

Proposition 4.6 For the viscometric model described by (2.7), the physical stress components on the hole centerline $\mathfrak{C}$ are

$$\left.\begin{aligned} \hat{T}_{12} &= \eta(\dot{\gamma})\dot{\gamma} \\ \hat{T}_{11} &= -p + (\nu_1 + \nu_2)\dot{\gamma}^2 - \nu_1 \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} \\ \hat{T}_{22} &= -p + \nu_2\dot{\gamma}^2 + \nu_1 \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2} \\ \hat{T}_{11} - \hat{T}_{22} &= \nu_1\left(\dot{\gamma}^2 - 2\frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}\right) \end{aligned}\right\} \tag{4.9}$$

where $\hat{u}_1$, etc. is now the generalized Newtonian solution associated with $\eta(\dot{\gamma})$.

Proof : Simply inserting (3.9), (3.11) and (3.15) into (2.7) directly gives the above results. ◇

Remark : (i) It should be borne in mind that η, ν_1, ν_2 are generally functions of $\dot{\gamma}$. When η = constant, $\nu_1 = 2\eta T$ and $\nu_2 = -\eta T/2$, (4.9) reduces to the second-order fluid result (3.16) & (3.17). (ii) Although equation (2.7) is called "***viscometric***" model, the stresses still may not be viscometric due to the disturbance of the slot, as one can see from (4.9). When flow is not viscometric, ν_1 is only the viscometric part of the first normal stress coefficient.

Proposition 4.7 For the viscometric model under Tanner's assumptions (T1) to (T3), the hole-pressure difference can be expressed as

$$P_H = \frac{1}{2}N_{1w}^0 - \frac{1}{2}\int_0^{\dot{\gamma}_w^0} \nu_1 \dot{\gamma} d\dot{\gamma} \ . \tag{4.10}$$

Proof : From (4.9) and (2.10) we have

$$\begin{aligned} \hat{T}_{22}|_{q_2^a} &= -p^o(0, q_2^a) \\ \hat{T}_{22}|_{q_2^b} &= -p^o(0, q_2^b) - \frac{\nu_1}{2}\left(\dot{\gamma}_w^0\right)^2 + \frac{1}{2}\int_0^{\dot{\gamma}_w^0} \nu_1 \dot{\gamma}\, d\dot{\gamma} \end{aligned} \tag{c}$$

Noting $p^o(0, q_2) \equiv 0$ on $\mathfrak{C}$, (4.10) can be immediately obtained. ◇

Remark : (4.10) was first given by Tanner in [20]. As a matter of fact, (4.10) has another equivalent form.

Proposition 4.8 For the viscometric model under Tanner's assumptions (T1) to (T3), the Tanner's result (4.10) can also be written as the following equivalent form:

$$P_H = \frac{1}{4}N_{1w}^0 + \frac{\alpha}{4}\int_0^{\dot{\gamma}_w^0} \dot{\gamma}^2 d\eta(\dot{\gamma}) \ . \tag{4.11}$$

Proof : Integrating (4.10) by parts yields

$$P_H = \frac{1}{2}N_{1w}^0 - \frac{1}{4}\nu_1\left(\dot{\gamma}_w^0\right)^2 + \frac{1}{4}\int_0^{\dot{\gamma}_w^0} \dot{\gamma}^2 d\nu_1$$

where $\nu_1 = \alpha\eta$. On the other hand, integrating (4.11) by parts gives (4.10). ◇

Next, we prove that results (4.10) and (4.11) are equivalent to the re-interpreted HPBL prediction (2.1′).

Proposition 4.9 For the viscometric model under Tanner's assumptions (T1) to (T3), the Tanner's hole-pressure difference results (4.10) and (4.11) are equivalent to the re-interpreted HPBL prediction (2.1′), when N_1 in (2.1′) is interpreted as only the viscometric part of $\hat{T}_{11} - \hat{T}_{22}$.

Proof : Substituting $N_1 = \alpha\eta\dot\gamma^2$, $\sigma = \eta\dot\gamma$ into (2.1′)

$$\int_0^{\sigma_w^0} \frac{N_1}{2\sigma} d\sigma = \int_0^{\sigma_w^0} \frac{\alpha\eta\dot\gamma^2}{2\eta\dot\gamma} d(\eta\dot\gamma) = \frac{\alpha}{2}\int_0^{\dot\gamma_w^0} \dot\gamma d(\eta\dot\gamma)$$
$$= \frac{\alpha}{2}\eta\left(\dot\gamma_w^0\right)^2 - \frac{\alpha}{2}\int_0^{\dot\gamma_w^0} \eta\dot\gamma d\dot\gamma$$

which gives (4.10). On the other hand, by using the following relations

$$\frac{1}{4}N_{1w}^0 = \frac{1}{4}\int_0^{\dot\gamma_w^0} d(\alpha\eta\dot\gamma^2) = \frac{\alpha}{4}\int_0^{\dot\gamma_w^0} (\dot\gamma^2 d\eta + 2\eta\dot\gamma d\dot\gamma) \tag{d}$$

$$\int_0^{\sigma_w^0} \frac{N_1}{2\sigma} d\sigma = \frac{\alpha}{2}\int_0^{\dot\gamma_w^0} (\dot\gamma^2 d\eta + \eta\dot\gamma d\dot\gamma) \tag{e}$$

in (4.11) we can obtain (2.1′). ◇

Now it is the time to consider the MHPR for the viscometric model. To proceed, it is essential to have the expression for stress gradient.

Proposition 4.10 For the viscometric model under Tanner's assumptions (T1) to (T3), the streamwise stress gradients on hole centerline $\mathfrak{C}$ are

$$\begin{aligned} \frac{1}{h_1}\frac{\partial p}{\partial q_1} &= \frac{1}{h_1}\frac{\partial p^o}{\partial q_1} \\ \frac{1}{h_1}\frac{\partial \hat T_{11}}{\partial q_1} &= -\frac{1}{h_1}\frac{\partial p^o}{\partial q_1} + 2\eta\frac{1}{h_1^2}\frac{\partial^2 \hat u_1}{\partial q_1^2} \end{aligned} \tag{4.12}$$

where p is the modified pressure given by (2.10) and p^o is the pressure for the generalized Newtonian flow.

Proof : Differentiating (2.10) with respect to q_1 yields

$$\frac{1}{h_1}\frac{\partial p}{\partial q_1} = \frac{1}{h_1}\frac{\partial p^o}{\partial q_1} - \frac{\alpha}{2h_1}\frac{\partial}{\partial q_1}\left(\frac{\hat u_1}{h_1}\frac{\partial p^o}{\partial q_1}\right) + \frac{1}{h_1}\frac{\partial}{\partial q_1}\left(\nu_1\dot\gamma^2/2 + \nu_2\dot\gamma^2 - \frac{1}{2}\int_0^{\dot\gamma} \nu_1\dot\gamma d\dot\gamma\right)$$

Keeping in mind that $\hat u_1$, h_1, $\dot\gamma$, $\partial p^o/\partial q_1$ are even functions of q_1 on $\mathfrak{C}$, consequently the last two terms in above equation are all zero. Then the derivation of $\frac{1}{h_1}\frac{\partial \hat T_{11}}{\partial q_1}$ is similar as in the second-order fluid. ◇

Remarks : Comparison of (4.12) with (3.20) tells us that the streamwise stress gradient of the viscomertic model is the same as that of the second-order fluid. Therefore, the proof of the error cancellation theorem for the viscometric model will be very similar to that of second-order fluid. Since we have assumed that the velocity field for the viscometric model (2.7) are the same as the generalized Newtonian flow field, Proposition 4.2 is still useful, while Propositions 4.3 & 4.4 need to be slightly modified as follows.

Proposition 4.11 For the viscometric model under Tanner's assumptions (T1) to (T3), we have the following results for the path integral of MHPR

$$P_H^1 = N_{1w}^0/2 - \frac{1}{2}\int_0^{\dot{\gamma}_w^0} \nu_1 \dot{\gamma} d\dot{\gamma} - P_H^3 \tag{4.13a}$$

$$P_H^2 = N_{1w}^0/2 - \frac{1}{2}\int_0^{\dot{\gamma}_w^0} \nu_1 \dot{\gamma} d\dot{\gamma} - P_H^1 \tag{4.13b}$$

$$P_H^2 = P_H^3 \tag{4.13c}$$

where

$$P_H^3 = \int_{q_2^a}^{q_2^b} \frac{\hat{u}_1}{h_1^2}\frac{\partial^2 \hat{u}_1}{\partial q_1^2}\frac{\nu_1}{\hat{T}_{12}}\frac{\partial \hat{T}_{12}}{\partial q_2} dq_2 \, . \tag{4.6a}$$

The proof of Proposition 4.11 is straightforward and has been omitted here. On the basis of Proposition 4.11, one can easily prove the following result regarding the equivalent expressions for P_H.

Proposition 4.12 For incompressible, steady, creeping flows of the viscometric model (2.7) with Tanner's assumptions (T1)-(T3), there are four equivalent expressions that predict the hole-pressure error, P_H. They are the MHPR (2.15), the Tanner's results (4.10) & (4.11), and the re-interpreted HPBL formula (2.1′).

Remarks : From section 2 we know that the MHPR is derived from the equations of motion, without any assumptions on the fluid model. Therefore (2.15) is model independent and hence the more general expression for the hole-pressure error. For the viscometric model, all the other three relations can be derived from the MHPR under the Tanner's assumptions (T1)-(T3). It is also easy to see that when $\eta\,,\nu_1$ =constant, all the four expressions give the same result $P_H = N_{1w}^0/4$.

5. Conclusions

The correct application of the HPBL theory in the hole-pressure problem involves the *modified hole-pressure relation* (MHPR) in path integral form. Some important mathematical properties of the MHPR, such as the singularities and the existence of the path integral, were studied.

To investigate the two key flaws of the HPBL theory, it is sufficient to consider the second-order fluid. By studying the MHPR in terms of the streamline coordinate formulation, we found the fortuitous phenomenon of error cancellation in the derivation of HPBL formula. For second-order fluid and Tanner's viscometric model under certain assumptions, we have proved analytically that the error cancellation is exact and the MHPR is equivalent to the re-interpreted HPBL prediction. We also proved that there are four equivalent expressions that predict the same hole-pressure error P_H for Tanner's viscometric model. The theoretical results of this paper provided a complete explanation for the fortunate success of the HPBL prediction and the controversy in the derivation of the HPBL formula.

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REFERENCES

[1] Kaye, A., Lodge, A.S. and Vale, D.G., Determination of normal stress differences in steady shear flow, II. Flow birefringence, viscosity, and normal stress data for a polyisobutene liquid, *Rheol. Acta*, 7 (1968) 368-379.

[2] Tanner, R.I. and Pipkin, A.C., Intrinsic errors in pressure-hole measurements, *Trans. Society of Rheology*, 13:4 (1969), 471-484.

[3] Broadbent, J.M. and Lodge, A.S., Determination of normal stress differences in steady shear flow, III. A wide-gap concetric cylinder apparatus, *Rheol. Acta*, 10 (1971) 557-573.

[4] Pritchard, W.G., *Phil. Trans. Roy. Soc.* London A., 270 (1971) 507.

[5] Lodge, A.S. and Vargas, L.de, Positive hole-pressures and negative exit pressure generated by molten low-density polyethylene flowing through a slit die, *Rheologica Acta* 22 (1983), 151-170.

[6] Pike, R. D. and Baird, D. G., Evaluation of the Higashitani and Pritchard analysis of the hole pressure using flow birefringence, *J. Non-Newtonian Fluid Mech.* 16 (1984), 211.

[7] Jackson, N.R. and Finlayson B.A., Calculation of hole pressure: I. Newtonian fluids, *J. Non-Newtonian fluid Mech.*, 10 (1982) 55-69.

[8] Jackson, N.R. and Finlayson B.A., Calculation of hole pressure: II. Viscoelastic fluids, *J. Non-Newtonian fluid Mech.*, 10 (1982) 71-84.

[9] Richards, G. D. and Townsend, P., A finite element computer model of the hole-pressure problem, *Rheol. Acta* 20 (1981), 261–269.

[10] Webster, M.F., The hole-pressure problem, *Rheologica Acta*, 23 (1984), 582-590.

[11] Lodge, A.S., Newtonian liquid hole pressures at small Reynolds numbers, *J. of Rheology*, 27(5), (1983) 497-501.

[12] Baird, D.G., Malkus, D.S. and Reddy, J.N., Comparison of flow birefringence results with numerical simulation of the hole pressure, Proceedings IX Intl. Congress on Rheology, Mexico, (1984) 641-646.

[13] Pipkin, A.C. and Tanner, R.I., A survey of theory and experiment in viscometric flows of viscoelastic liquids, **Mechanics Today**. Vol. 1, edited by S. Nemat-Nasser, Pergamon, 1972, 262-321.

[14] Bird, R.B., Armstrong, R.C. and Hassager, O., **Dynamics of Polymeric Liquids**, Vol. 1 **Fluid Mechanics**, *2nd edn.*, John Wiley & Sons, (1987).

[15] Higashitani, K. and Pritchard, W.G., A kinematic calculation of intrinsic errors in pressure measurements made with holes, *Trans. Soc. Rheology* 16 (1972), 687–696.

[16] Malkus, D.S. and Yao, M., On hole-pressure in plane Poiseuille flow over transverse slots, Mathematics Research Center Technical Summary Report # 2943, Mathematics Research Center, Univ. Wisconsin-Madison, (1986).

[17] Yao, Minwu, A numerical and analytical study of normal stesses and pressure differences in non-Newtonian creeping flows, *Ph.D. Thesis*, University of Wisconsin-Madison, Wisconsin, U.S.A. (1989).

[18] Malkus, D.S. and Webster, M.F., On the accuracy of finite element and finite difference predictions of non-Newtonian slot pressures for a Maxwell fluid, *J. Non-Newtonian Fluid Mech.*, 25 (1987), 93-127.

[19] Srinivasan, R., The hole-pressure problem: on the Higashitani-Pritchard theory for transverse and axial slots, *Rheologica Acta*, 26 (1987), 107-118.

[20] Tanner, R.I., Pressure-hole errors – an alternative approach, *J. Non-Newtonian Fluid Mech.*, 28 (1988) 309-318.

[21] Tanner, R.I., **Engineering Rheology**, Clarendon Press · Oxford, 1985.

[22] Malvern, L.E., **Introduction to the Mechanics of a Continuous Medium**, Prentice-Hall, (1969).

[23] Lodge, A.S., **Elastic Liquids**, Academic Press, New York, (1964).

[24] Sugeng, F., Phan-Thien, N. and Tanner, R.I., A boundary-element investigation of the pressure-hole effect, *J. Rheology*, 32 (3) (1988), 215-233.

[25] Lodge, A.S., Chapter 11 in **Rheological Measurement**, edited by Collyer Clegg, Elsevier (1988).