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ON THE EXISTENCE OF LOCAL TIMES:

A GEOMETRIC STUDY

by

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On the Existence of Local Times: A Geometric Study[1]

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Abstract

We present a general study relating the geometry of the graph of a real function to the existence of local times for the function. The general results obtained are applied to Gaussian processes, and we show that with probability 1 the sample functions of a non-differentiable stationary Gaussian process with local times will be Jarnik functions. This extends earlier works of Lifschitz and Pitt, which gave examples of Gaussian processes without local times. An example is given of a Jarnik function without local times thus answering negatively a question raised by Geman and Horowitz.

Section 1: Introduction

Let $f\colon [0,1] \to \mathbb{R}^1$ be a Lebesgue function and define the two measures

(1.1) $\mu(A) \equiv \mu_f(A) \equiv |\{t\colon f(t) \in A\}|$, and

(1.2) $\nu(B) \equiv \nu_f(B) \equiv |\{t\colon (t,f(t)) \in B\}|$.

Here $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^1$ and $A \in B(\mathbb{R}^1)$ is a Borel set of $\mathbb{R}^1$ while $B \in \mathcal{B}(\mathbb{R}^2)$ is a Borel set of $\mathbb{R}^2$. The measure μ is called the occupation measure of f and ν will be called the image of Lebesgue measure on the graph of f. A fundamental question is, when is μ absolutely continuous with respect to Lebesgue measure? If $d\mu \ll dx$, then the occupation density $d\mu(x)/dx$ is called the local time at x. Following [5], we describe this by saying f has local times.

The survey article [5] gave a full account of what was known concerning the existence of local times in 1980, and raised several open questions. This paper addresses two areas of investigation that were raised there. What is the exact role that the Jarnik condition $J_1(t)$:

$$\operatorname*{ap\,lim}_{h\to 0} \frac{|f(t+h)-f(t)|}{|h|} = +\infty$$

plays in the existence of local times? Also, if f(t) has local times, what can be said about the existence of local times for perturbed functions $f_\alpha(t) = f(t)+\alpha t$?

Our approach to both of these questions is through geometric measure theory and consideration of the measure μ_f as a projection of the measure ν_f. The general setup here is: for $z = (t,x) \in \mathbb{R}^2$, the function $P_\theta(z) \equiv x \operatorname{Cos} \theta - t \operatorname{Sin} \theta$ is viewed as the orthogonal projection onto the line L_θ through 0 and $(-\operatorname{Sin} \theta, \operatorname{Cos} \theta)$. The two measures μ_f and ν_f are related through the projection P_0 by the equation

$$\mu_f(A) = \nu_f(P_0^{-1}(A)).$$

Although it is often the case that little is known about the specific measure $\nu_f \circ P_0^{-1}$, there is much classical geometric information about the family of measures $\{\nu_f \circ P_\theta^{-1}\}$.

Two particularly useful results which are both exposed in Falconer [4], chapter 6, are

Besicovitch (1939). If $B \subseteq \mathbb{R}^2$ is an irregular 1-set, then for a.a. θ, $|P_\theta(B)| = 0$ holds.

Kaufmann (1968). If $\nu(dz)$ is a finite Borel measure on $\mathbb{R}^2$ with

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nu(dz)\nu(d\xi)}{|z-\xi|} < \infty,$$

then for a.a. θ the measure $\nu \circ P_\theta^{-1}$ is absolutely continuous.

These results can become an effective analytic tool for discussing the occupation measures μ_α of the functions

$f_\alpha(t) = f(t)+\alpha t$. The first necessary step here is a simple change of variables argument. The second key ingredient is the following geometric lemma proved in Section 2.

Proposition 2.2. Let $D = \{t: J_1(t) \text{ does not hold}\}$. That is, $t \in D$ iff

$$\operatorname{ap}\liminf_{h\to 0} \frac{|f(t+h)-f(t)|}{|h|} < \infty.$$

Let

$$G(f:D) = \{(t,f(t)): t \in D\}$$

denote the graph of f above D. The set G(f:D) has σ-finite linear Hausdorff measure $\mathcal{H}^1$ and the restrictions $\nu_f|G(f:D)$ and $\mathcal{H}^1|G(f:D)$ to G(f:D) of $\nu_f(dz)$ and $\mathcal{H}^1(dz)$ are mutually absolutely continuous. Moreover, if $J \equiv [0,1]-D$ is the class of Jarnik points, then each subset $E \subset G(f:J)$ with $\nu_f(E) > 0$ has non-σ-finite Λ_1 measure.

The following minor modification of Theorem 5.2 is proved in Section 5.

Corollary 5.2′. **Suppose that f is a non-Jarnik function in the strong sense that $|J| = 0$ where $J = \{t: J_1(t) \text{ is satisfied}\}$. Suppose also that the approximate derivative**

$$f'_{ap}(x) = \operatorname{ap}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists at most on a set of Lebesgue measure 0. Then, for a.a. α, the occupation measures μ_α of $f_\alpha(t) = f(t)+\alpha t$ are purely continuous singular.

An idea that goes back to R. Klein (1976) allows us to lift theorems about perturbations to almost sure results about Gaussian processes. Thus, for example, in Section 9 we prove

<u>Theorem 9.1</u>. Let $\{X(t): t \in \mathbb{R}^1\}$ be a real continuous stationary Gaussian process. Set

$$\Delta_1(h) = E|X(t+h)-X(t)|^2,$$

and

$$\Delta_2(h) = E|X(t+h)+X(t-h)-2X(t)|^2.$$

If

(1.3) $$\sup_{h>0} \frac{\Delta_1(h)}{h^2} = +\infty,$$

and

(1.4) $$\sup_{h>0} \frac{\Delta_2(h)}{h^2} < \infty,$$

then with probability 1 the occupation measure of $\{X(t)\}$ is a purely continuous singular measure.

<u>Remark</u>. It is shown in Section 8 that condition (1.4) has a spectral equivalent. Namely, if $EX_tX_0 = \int_{-\infty}^{\infty} e^{it\cdot\lambda}\Delta(d\lambda)$, then (1.4) is equivalent to

$$\sup_{T>0} T^2\Delta\{\lambda: |\lambda| > T\} < \infty.$$

It is also known that (1.3) is equivalent to $\int\lambda^2\Delta(d\lambda) = +\infty$. Thus, examples of measures Δ which correspond to $\{X(t)\}$ satisfying (1.3) and (1.4) are easy to give, e.g.,

$\Delta(d\lambda) = \sum_{n\neq 0} \frac{1}{|n|^3} \delta_n(d\lambda)$. Other examples satisfying more stringent conditions are in [9] and [10].

We now outline this paper.

In Section 2, we derive Proposition 2.2 and related results concerned with decomposing the graph G(f) of f into a countable union of 1-sets and a piece which is essentially larger than 1-dimensional.

Section 3 looks closer at this decomposition, breaking the 1-set parts of G(f) into a regular piece which is the graph above the set $\{t: f'_{ap}(t) \text{ exists}\}$ and an irregular piece. Applying the results of Besicovitch and Kaufman to these pieces in Section 4, we derive several theorems on the absolute continuity or singularity of the measures $\nu_f \circ P_\theta^{-1}$ which hold for a.a. θ.

Section 5 translates the work of Section 4 into the language of the perturbations f_α and their occupation measures μ_α. In addition to Corollary 5.2′ and related results we prove

Theorem 5.3. Suppose that

$$\int_0^1 \frac{dt}{\sqrt{(t-s)^2+(f(t)-f(s))^2}} < \infty$$

holds for a.a. $s \in [0,1]$. Then, for a.a. α, the function $f(t)+\alpha t$ has local times.

Formally, this result is related to, but distinct from, the L^2 Fourier theory of local times that was explored by Berman starting in [3].

Section 6 contains examples showing that the exceptional sets of α in Theorems 5.2 and 5.3 may not be non-empty. It also answers in the negative a question raised by Geman and Horowitz in [5]: Does every Jarnik function have local times?

In Section 7, we turn to probability proper and we show how an idea of Klein may be used to translate perturbation results into a.s. theorems about Gaussian processes and other related processes.

Section 8 presents a class of stochastic processes which are stochastic analogues of the Zygmund space Λ^* of quasi-smooth functions. Several spectral characterizations of these processes are also developed.

Section 9 combines the earlier results, especially those of Sections 4 and 8 to prove our basic result, Theorem 9.1. The methods also lead in Section 10 to a further example which settles in the negative another question from [5]. We exhibit a discontinuous stationary Gaussian process with no local times.

Section 2: On the Geometry of Graphs

We let f(t) be Borel measurable, and we denote the set of approximate discontinuities of f with

$$N = \bigcup_{\varepsilon>0} \{t: \liminf_{h\downarrow 0} |\{s: |s-t| < h, |f(s)-f(t)| < \varepsilon\}|/2h < 1.$$

It is elementary to show that N is a Borel set, and by the theorem of Denjoy [12], p. 132, $|N| = 0$. Thus, $G(f:N)$ is a Borel subset of $G(f)$ with $\nu_f(G(f:N)) = 0$.

Introduce the quantity

$$|D|f(t) = \operatorname{ap}\liminf_{s\to t} \frac{|f(t)-f(s)|}{|t-s|},$$

and the set of Jarnik points not in N,

$$J = \{t:\ |D|f(t) = +\infty\}-N.$$

The set of points of linear condensation for f is

$$D = \{t:\ |D|f(t) < \infty\}-N = [0,1]-(J\cup N).$$

We also introduce the upper linear density of ν_f at $z = (t,x)$

$$\overline{D}_1\nu_f(z) = \limsup_{r\downarrow 0} \frac{\nu_f(B(z,r))}{2r}.$$

Here $B(z,r)$ is the closed disk of radius r and center z. $\overline{D}_1\nu_f$ and $|D|f$ are related by

<u>Lemma 2.1</u>. For $x = f(t)$ and $z = (t,f(t))$,

$$\overline{D}_1\nu_f(z) > 0 \quad \text{iff } |D|f(t) < \infty.$$

<u>Proof</u>. For $c > 0$ and $k = \sqrt{1+c^2}$,

$$\{(s,f(s)):\ |s-t| < r \text{ and } |f(s)-f(t)| < c|s-t|\} \subseteq B(z,kr).$$

Thus $|D|f(t) < \infty$ implies that for some $c < \infty$,

$$\limsup_{r\downarrow 0} \frac{|\{s:\ |s-t| < r \text{ and } |f(s)-f(t)| < c|t-s|\}|}{2r} > 0,$$

so $\limsup\limits_{h\downarrow 0} \dfrac{\nu_f(B(z,kr))}{2r} > 0$ and $\overline{D}_1\nu_f(z) > 0$.

Conversely, we observe that for $0 < \lambda < 1$ and $r > 0$,

$$\{s: (s,f(s)) \in B(z,r)\} - \{s: |s-t| < \frac{\lambda}{2} r\}$$

$$\subseteq \{s: |s-t| < r,\ |f(s)-f(t)| < \frac{2}{\lambda}|s-t|\}.$$

Thus, $\overline{D}_1\nu_f(z) > \lambda$ implies

$$\limsup_{r\downarrow 0} \frac{1}{2r} |\{s: |s-t| < r \text{ and } |f(s)-f(t)| < \frac{2}{\lambda} |s-t|\}| \geq \frac{\lambda}{2} > 0,$$

and

$$|D|f(t) < \frac{2}{\lambda} < \infty.$$

In Lemma 2 of [11], Rogers and Taylor show that the set $\{z: \overline{D}_1\nu_f(z) > 0\}$ is a Borel subset of $\mathbb{R}^2$. Hence,

$$G(f:D) = [G(f) \cap \{z: \overline{D}_1\nu_f(z) > 0\}] - G(f:N)$$

is also a Borel subset of $\mathbb{R}^2$.

We now turn to the

__Proof of Proposition 2.2__. We again invoke the Lemma 2 of [11] where it is established that there is a finite constant k such that for each $\lambda > 0$ and each finite Borel measure m on $\mathbb{R}^2$, the set $\mathcal{C}_\lambda \equiv \{z: \overline{D}_1 m(z) > \lambda\}$ is a $\mathcal{G}_{\delta\sigma}$ set and satisfies

$$\mathcal{H}^1(\mathcal{C}_\lambda) \leq \frac{k}{\lambda} m(\mathbb{R}^2).$$

It follows that each open set $O \subseteq \mathbb{R}^2$ satisfies

$$\mathcal{H}^1(O \cap \mathcal{C}_\lambda) \le \frac{k}{\lambda} m(O),$$

from which it follows that the restriction of $\mathcal{H}^1$ to C_λ is absolutely continuous with respect to the measure m.

Setting $m = \nu_f$ and taking the union of the sets $\mathcal{C}_{(1/n)}$ shows that $G(f:D) \subseteq \bigcup_n \{\mathcal{C}_{(1/n)}: n > 0\}$ is a Borel set of σ-finite $\mathcal{H}^1$ measure and that $\mathcal{H}^1 | G(f:D) << \nu_f$. On the other hand, the obvious inequality $\nu_f(B(z,r)) \le 2r$ shows that ν_f satisfies $\nu_f \le \text{const.}\ \mathcal{H}^1$ for some constant. Thus $\mathcal{H}^1 | G(f:D)$ and $\nu_f | G(f:D)$ are equivalent as claimed.

Finally, suppose $E \subset G(f:J)$ satisfies $\nu_f(E) > 0$. Lemma 3 of [11] shows that $\mathcal{H}^1(E) = +\infty$, and hence E cannot be of σ-finite $\mathcal{H}^1$-measure.

Section 3: Regular and Irregular Parts of G(f:D)

We apply the classical theory of 1-sets to obtain a decomposition of G(f:D) into a regular part, an irregular part, and a negligible part. A brief self-contained treatment of the general theory is in Falconer's book [4], Chapters 2, 3, and 6.

If $z = (t,x) \in \mathbb{R}^2$ and $A \subseteq \mathbb{R}^2$, the upper and lower linear densities of A at z are defined as

$$\overline{D}_1(A,z) \equiv \limsup_{r \downarrow 0} \mathcal{H}^1(A \cap B(z,r))/2r,$$

and

$$\underline{D}_1(A,z) \equiv \liminf_{r \downarrow 0} \mathcal{H}^1(A \cap B(z,r))/2r,$$

respectively. If A is a 1-set, it is known that $\overline{D}_1(A,z) \leq 1$ for $\mathcal{H}^1$-a.a. z in A. If $z \in A$ and $\underline{D}_1(A,z) = \overline{D}_1(A,z) = 1$, then z is called a regular point of A. Otherwise, z is called an irregular point of A. The 1-set A is called regular (resp. irregular) if $\mathcal{H}^1$-a.a. z in A are regular (resp. irregular). If $A_r \equiv \{z \in A: z \text{ is regular}\}$ then A_r is a regular 1-set if $\mathcal{H}^1(A_r) > 0$ while $A_i = A-A_r$ is an irregular 1-set if $\mathcal{H}^1(A_i) > 0$. See [4], Ch. 2.

<u>Proposition 3.1</u>. Define two subsets of D by

$$D_r = \{t\colon f'_{ap}(t) \text{ exists and is finite}\};$$

$$D_i = \{t\colon f'_{ap}(t) \text{ does not exist, finite or infinite}\}.$$

(a) Then D_r and D_i are Borel sets with,

$$D = D_r \cup D_i. \tag{3.1}$$

(b) If $|D_r| > 0$, then $G(f{:}D_r)$ is a countable union of regular 1-sets.

(c) If $|D_i| > 0$, then $G(f{:}D_i)$ is a countable union of irregular 1-sets.

<u>Proof</u>. The argument showing that D_i is a Borel set is routine, while $f'_{ap}(t) = \pm\infty$ implies $|D|f(t) = +\infty$. Thus $\{t \in D\colon f'_{ap}(t) = \pm\infty\} = \emptyset$, and (3.1) follows.

The theorem of Denjoy on the bottom of p. 237 in [12] shows that the restriction of f to D_r is of generalized bounded variation, and in particular there exists a sequence $\{f_n\}$ of functions of bounded variation on [0,1] with

$$G(f:D_r) \subseteq \bigcup_n G(f_n:[0,1]).$$

But $G(f_n:[0,1])$ is a regular 1-set, so the Borel subset,

$$A_n \equiv G(f:D_r) \cap G(f_n:[0,1])$$

of $G(f_n:[0,1])$ is a regular 1-set or

$$|\{t \in D_r: f_n(t) = f(t)\}| = 0,$$

see [4], p. 26.

The proof of (b) is completed by invoking the elementary fact that a measurable subset of a regular set is regular, [4], p. 26.

To prove (c), we observe that $G(f:D_i) \subseteq \bigcup_n G(f:D_i) \cap \mathcal{C}_{1/n}$, where $\mathcal{C}_\alpha$ is defined as in paragraph one of the proof of Proposition 2.2. If $|D_i| > 0$, then for some $\alpha > 0$,

$$0 < \mathcal{H}^1(G(f:D_i) \cap \mathcal{C}_\alpha) < \mathcal{H}^1(\mathcal{C}_\alpha) < \infty,$$

and it suffices to show that

$$E_\alpha \equiv G(f:D_i) \cap \mathcal{C}_\alpha$$

is an irregular 1-set. If E_α were not irregular, then by Theorem 3.25 of [4], there would exist a rectifiable curve

$\mathcal{Z}$: $\{z(s) = (t(s), x(s)): 0 \leq s \leq L\}$, parametrized by arc length with $\mathcal{H}^1(\mathcal{Z} \cap E_\alpha) > 0$.

We introduce the sets

$$T_1 = \{s: z'(s) \text{ does not exist}\},$$

$$T_2 = \{s: z'(s) \text{ exists and } t'(s) \neq 0\},$$

$$T_3 = \{s: z'(s) \text{ exists and } t'(s) = 0\},$$

and

$$\mathcal{Z}_i = \{z(s): s \in T_i\}, \quad i = 1,2,3.$$

The $\mathcal{Z}_i$ are disjoint, Borel measurable, and satisfy $\mathcal{Z} = \bigcup_{i=1}^{3} \mathcal{Z}_i$. We must show that $\mathcal{H}^1(\mathcal{Z}_i \cap E_\alpha) = 0$ for each i.

Since $|z(s)-z(\sigma)| \leq |\sigma-s|$, z is differentiable a.e. and $\mathcal{H}^1(\mathcal{Z}_1 \cap E_\alpha) \leq |T_1| = 0$.

To treat the sets $\mathcal{Z}_2 \cap E_\alpha$, we use the property that for all rational a and b with $a < b$ if $S_2 = T_2 \cap \{s: z(s) \in E_\alpha\}$ and if $|(a,b) \cap S_2| > 0$, then $T_{a,b} \equiv \{t = t(s) \text{ for some } s \in (a,b) \cap S_2\}$ has positive Lebesgue measure, see [14], Theorem 1, and almost all $s \in (a,b) \cap S_2$ are such that $t = t(s)$ is a point of density of $T_{a,b}$. For such an s and $t = t(s)$, we have

$$f'_{ap}(t) = \lim_{\substack{\tau \to t \\ \tau \in T_{a,b}}} \{f(\tau)-f(t)\}/(\tau-t) = \frac{x'(s)}{t'(s)} .$$

By definition of D_i, $f'_{ap}(t)$ exists for no $t \in D_i$. Hence, $|S_2| = 0$, and we have $\mathcal{H}^1(\mathcal{Z}_2 \cap E_\alpha) \leq |S_2| = 0$.

For $\mathcal{Z}_3 \cap E_\alpha$, we will show that

(3.2) $\overline{D}_1(E_\alpha, z) > 1$ for $\mathcal{H}^1$-a.a. $z \in \mathcal{Z}_3 \cap E_\alpha$.

Since $\overline{D}_1(E_\alpha; z) \leq 1$ for $\mathcal{H}^1$-a.a. z, this implies $\mathcal{H}^1(\mathcal{Z}_3 \cap E_\alpha) = 0$.

For this end, we note that for $\mathcal{H}^1$-a.a. $z = (t,x) = (t(s),x(s)) \in \mathcal{Z}_3 \cap E_\alpha$,

$$1 = \lim_{r\downarrow 0} \mathcal{H}^1(\mathcal{Z}_3 \cap E_\alpha \cap B(z,r))/2r.$$

Fixing $M > 0$ and setting

$$C(z,M) = \{(\tau,y): |y-x| < M|\tau-t|\},$$

we see from $t'(s) = 0$ and $x'(s) \neq 0$ that

$$\{\sigma: |\sigma-s| < \varepsilon \text{ and } z(s) \in C(z,M)\}$$

is empty provided only that ε is sufficiently small.

From this, it follows that

$$1 = \lim_{r\downarrow 0} \mathcal{H}^1(\mathcal{Z}_3 \cap E_\alpha \cap B(z,r))-C(z,M))/2r,$$

for each $M < \infty$. But $z \in G(f:D)$, so $|D|f(x) < \infty$, and thus, for some $M < \infty$,

$$0 < \limsup_{r\downarrow 0} m(B(z,r) \cap C(z,M))/2r$$

$$\leq \limsup_{r\downarrow 0} \mathcal{H}^1(E_\alpha \cap C(z,M) \cap B(z,r))/2r.$$

Thus,

$$1 < \limsup_{r\downarrow 0} \mathcal{H}^1(E_\alpha \cap B(z,r))/2r$$

$$= \overline{D}_1(E_\alpha;P),$$

which completes the proof. ∎

__Section 4: Projection Properties of__ ν_f.

We break the measure ν_f into three parts:

$$\nu_R(A) \equiv \nu_f(A \cap G(f:D_r)),$$

$$\nu_I(A) \equiv \nu_f(A \cap G(f:D_i)),$$

$$\nu_J(A) \equiv \nu_f(A \cap G(f:J)).$$

Observe that $\nu_f = \nu_R+\nu_I+\nu_J$. For $\theta \in [0,2\pi)$, the projection operator P_θ mapping $\mathbb{R}^2$ prependicularly onto the line L_θ spanned by 0 and $(-\text{Sin}\ \theta,\ \text{Cos}\ \theta)$ is,

$$P_\theta(t,x) = x\ \text{Cos}\ \theta - t\ \text{Sin}\ \theta.$$

The corresponding projections of the measures ν_R, ν_I, and ν_J are

$$\mu_{R,\theta}(A) \equiv \nu_R(P_\theta^{-1}(A)),$$

$$\mu_{I,\theta}(A) \equiv \nu_I(P_\theta^{-1}(A)),$$

$$\mu_{J,\theta}(A) \equiv \nu_J(P_\theta^{-1}(A)).$$

Note the occupation measure μ of f has the decomposition

$$\mu = \mu_{R,0} + \mu_{I,0} + \mu_{J,0}.$$

__Theorem 4.1__. Except for a countable set Θ_R of exceptional values of θ the measures $\mu_{R,\theta}$ are absolutely continuous. Moreover,

(4.1) $$\Theta_R = \{\theta:\ |\{x:\ f'_{ap}(x) = \text{Tan}\ \theta\}| > 0\}.$$

<u>Theorem 4.2</u>. Except for a Lebesgue null set Θ_I of exceptional values of θ, the measures $\mu_{I,\theta}$ are singular, and for all θ, the measure $\mu_{I,\theta}$ has no discrete part.

<u>Theorem 4.3</u>. For $M \leq \infty$, define

$$F_M = \{z: \int_0^1 \frac{\nu_J(B,z,r))}{r^2} dr < M\}.$$

If $\mu_J(F_\infty) > 0$, then for a.a. θ, the meausre $\mu_{J,\theta}$ has a nonzero absolutely continuous part. If $\nu_J(G(f:J)-F_\infty) = 0$, then for a.a. θ, $\mu_{J,\theta}$ is absolutely continuous.

<u>Proof of Theorem 4.1</u>. As described in the proof of Proposition 3.1 there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of functions of bounded variation with

$$G(f:D_r) \subseteq \bigcup_n G(f_n:[0,1]).$$

Thus, ν_f is absolutely continuous w.r.t. the measure $m = \sum_{n=1}^{\infty} 2^{-n} \nu_{f_n}$. From this, it suffices to consider the special case when f is a bounded variation and to show in this case that

$$\mu_{R,\theta}(A) = |\{t: -t \operatorname{Sin} \theta + f(t) \operatorname{Cos} \theta \in A\}|$$

is absolutely continuous unless

$$|\{t: -\operatorname{Sin} \theta + \operatorname{Cos} \theta \cdot f'(t) = 0\}| > 0.$$

Since this is satisfied for at most a countable set of θ, the result will follow. But, by Theorem 1 of [14], $\mu_{R,\theta}$ is absolutely continuous iff

$$|\{t: \frac{d}{dt}[-t \operatorname{Sin} \theta + f(t) \operatorname{Cos} \theta] = 0\}| = 0.$$

This is automatic if $\operatorname{Cos} \theta = 0$ while for $\operatorname{Cos} \theta \neq 0$, this is the same as

$$|\{t: f'(t) = \operatorname{Tan} \theta\}| = 0. \tag{4.2}$$

The result follows in this case, and the extension to the general case is elementary.

<u>Proof of Theorem 4.2</u>. The set $G(f:D_i)$ is contained in a countable union $\bigcup_n A_n$ of irregular 1-sets. Hence, by Besicovitch's fundamental theorem [4], p. 89, $|P_\theta A_n| = 0$ for a.a. θ. Letting $E_n = \{\theta: |P_\theta A_n| > 0\}$ and $\Theta_i = \bigcup_n E_n$, we see that whenever $\theta \notin \Theta_i$ the measure $\mu_{I,\theta}$ is carried by the set $\bigcup_n P_\theta A_n$ which has Lebesgue measure 0. Thus, $\mu_{I,\theta}$ is singular for a.a. θ.

If $\mu_{I,\theta}$ were to have a nonzero atom at y_0, then

$$|\{t: -t \operatorname{Sin} \theta + f(t) \operatorname{Cos} \theta = y_0\}| > 0.$$

Letting $L = \{(t,x): -t \operatorname{Sin} \theta + x \operatorname{Cos} \theta = y_0\}$, it would follow that $L \cap G(f:D_i)$ is a regular 1-set, see [4], page 33. This is impossible, since either $|D_i| = 0$ or $G(f:D_i)$ is an irregular 1-set.

<u>Proof of Theorem 4.3</u>. The proof is easily reduced to the special case in

<u>Lemma 4.4</u>. Suppose that for some $M < \infty$, $\nu_J(F_M) = \nu_J(\mathbb{R}^2)$. Then for a.a. θ, $\mu_{J,\theta}$ is absolutely continuous with

$$\int_{-\infty}^{\infty} \left|\frac{d\mu_{J,\theta}(y)}{dy}\right|^2 dy < \infty.$$

<u>Proof</u>. By definition, $\mu_J(\mathbb{R}_2) \leq 1$, and by hypothesis,

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{d\nu_J(p)d\nu_J(q)}{|p-q|} &\leq \sup_{p\in F_M} \int_{\mathbb{R}^2} \frac{d\nu_J(q)}{|p-q|} \\
&= \sup_{p\in F_M} \int_0^{\infty} \frac{d\nu_J\{B(p,r)\}}{r} \\
&= \sup_{p\in F_M} \int_0^{\infty} \frac{\nu_J(B(p,r))}{r^2}\, dr \\
&\leq M + \int_1^{\infty} \frac{dr}{r^2} \\
&= M+1 < \infty.
\end{aligned}$$

Now Kaufman's arguments [7] or [4], Sec. 6.3, show that

$$\int_0^{\infty} \left\{\int_{-\infty}^{\infty} \left|\frac{d\mu_{J,\theta}(\lambda)}{d\lambda}\right|^2 d\lambda\right\} d\theta < \infty,$$

and the proof is complete.

Section 5: Perturbations

If we choose $\theta = \mathrm{Tan}^{-1}(-\alpha)$, then the occupation measure μ_α of $f_\alpha(t) \equiv -\alpha t + f(t)$ is related to the projection $\nu_f \circ P_\theta^{-1}$ of ν_f by

$$\nu_f(P_\theta^{-1}A) = \mu_\alpha((\mathrm{Sec}\ \theta)\cdot A).$$

From this identity and Theorems 4.1, 4.2, and 4.3, we can deduce at once

Theorem 5.1. Suppose $|D_r| = 1$. Then, except for at most a countable set of values of α, the function f_α has local times.

Theorem 5.2. Suppose $|D_i| = 1$. Then, except for a Lebesgue null set of values of α, the function f_α has a continuous singular occupation measure.

Theorem 5.3. Suppose that

$$\int_0^1 \frac{ds}{\sqrt{(t-s)^2+(f(t)-f(s))^2}} < \infty \tag{5.1}$$

holds for a.a. $t \in [0,1]$. Then for a.a. α, f_α has local times.

Remark. Corollary 5.2′ follows directly from Theorem 5.2 if we observe that $|D_i| = 1$ is equivalent to $|J| = 0$ and $|\{t: f'_{ap}(t)$ exists$\}| = 0$.

The particular perturbation of f by αt is not essential in Theorems 5.1, 5.3, and 5.3. In fact, if $\varphi(t)\colon [0,1] \to \mathbb{R}^1$ is a continuously differentiable increasing function with $\varphi'(t) > 0$ for all t, it is elementary to use the change of variables $u = \varphi(t)$ and show that

$$\alpha\varphi(t)+f(t)$$

will have local times iff the function

$$\alpha u+f\circ\varphi^{-1}(u)$$

has local times. From this observation, routine arguments lead to the following theorems. We will call $\varphi(t)$ a "regular perturbation" if $\varphi(t)$ is continuously differentiable, not necessarily monotonic, but with $\varphi'(x) \neq 0$ a.e.

Theorem 5.4. Suppose φ is a regular perturbation and that $|D_r| = 1$. Then, except for a countable set of λ, the function

$$\lambda\varphi(t)+f(t)$$

has local times.

Theorem 5.5. If φ is a regular perturbation and $|D_i| = 1$, then the function

$$\lambda\varphi(t)+f(t)$$

has a continuous singular occupation measure for a.a. λ.

Theorem 5.6. If φ is a regular perturbation, and if

$$\int_0^1 \frac{ds}{\sqrt{(s-t)^2+(f(s)-f(t))^2}} < \infty$$

holds for a.a. $t \in [0,1]$, then $\lambda\varphi(t)+f(s)$ has local times for a.a. λ.

We can now easily prove Corollary 5.2′ which was stated in the introduction. In fact, the assumption $|J| = 0$ implies $\nu_J = 0$, while the assumption $f'_{ap}(t)$ exists almost nowhere implies $|D_1| = 1$. Invoking Theorem 5.2 gives the result.

__Remarks__. In the next section, we will give examples which show that the exceptional sets of α in Theorems 5.5 and 5.6 may be non-empty.

We also observe there is an obvious gap between the condition

$$\text{(5.2)} \qquad \overline{D}_1\, \nu_f(z) = 0, \quad \text{a.e. } [\nu_f],$$

and the condition (5.1). We do not know if (5.1) can be replaced in Theorem 5.3 with the weaker (5.2).

__Section 6: The Exceptional Perturbations__

It is natural to ask if the exceptional sets of α mentioned in the theorems of Section 5 may be empty, and if not, can the conditions be strengthened so that they become empty.

<u>Example 6.1</u>. An example of a discontinuous function f satisfying the hypothesis of Theorem 5.2, but with absolutely continuous occupation measure is easily constructed using ternary expansions of real numbers. Let

$$t = \sum_{n=1}^{\infty} t_n(t)/3^n$$

be the ternary expansion of t with $t_n = 0, 1, 2$. Define $s_n(t)$ by $s_n(t) = 0$ if $t_n(t) = 0$, $s_n(t) = 2$ if $t_n(t) = 1$, and $s_n(t) = 1$ if $t_n(t) = 2$. Finally, set $f(t) = \sum_{n=1}^{\infty} s_n(t)/3^n$.

Except for the usual problem with triadic rationals the $\{t_n(t)\}$ are well defined and f(t) is one-to-one. For a triadic interval $I = [k/3^n,(k+1)/3^n)$ the image $f(I) = K$ is another triadic interval of length $1/3^n$. Disjoint intervals go into disjoint intervals and we may conclude that $\mu_f(dx)$ is simply the restriction of Lebesgue measure dx to [0,1].

Because f maps triadic intervals onto triadic intervals of equal length, it is clear that for each $z = (t,f(t))$

$$\overline{D}_1\ \nu_f(z) \geq \frac{1}{2\sqrt{2}} > 0,$$

from which it follows that G(f) is a one set.

Finally, to see that $|D_i| = 1$, we only need show that $f'_{ap}(t)$ is undefined for a.a. t.

First we suppose that t and n are such that $t_n(t) = 2$. Let $I = [k/3^n,(k+1)/3^n)$ be the triadic interval containing t. Then for $x \in J \cup K$ where $J = I - 1/3^n$ and $K = I + 1/3^n$, we have

$f(s) \le f(t)$. From this it follows that if $f'_{ap}(t)$ exists, and if $t_n(t) = 2$ infinitely often, then $f'_{ap}(t) = 0$.

On the other hand, if $t_n(t) = 0$ and $t \in I = [k/3^n, (k+1)/3^n)$ then for al $s \in K = I + 1/3^n$, we have $f(s) > f(t)+½(s-t)$. From this, it follows that if $f'_{ap}(t)$ exists and $t_n(t) = 0$ infinitely often, then $f'_{ap}(t) \geq ½$.

The conclusion that $f'_{ap}(t)$ does not exist for a.a. t is now clear.

We do not have an example of a continuous function satisfying the hypothesis of Theorem 5.4 for which μ_f is absolutely continuous, although we presume such functions exist. We do, however, have a strengthening of the hypothesis of Theorems 5.2 and 5.4 for which there are no exceptional sets. One such result is

<u>Theorem 6.2</u>. If f satisfies

$$\lim_{h \downarrow 0} \sup_t \frac{1}{h} |f(t+h)+f(t-h)-2f(t)| = 0, \tag{6.1}$$

and if f(t) is non-differentiable a.e., then for each C^1 function $\varphi(t)$ the function $\varphi(t)+f(t)$ has a continuous singular occupation measure.

For results of a similar sort and their relation to the perturbation of the spectrum of certain multiplication operators, we refer to [10]. In particular, we note here that the occupation mesure of $\varphi+f$ is singular for every C^1 function φ independent of its modulus of continuity.

We also observe that in [13] Sawyer has given an example of a discontinuous function f such that the range of $f+\varphi$ has measure 0 for each C^1 function φ.

Proof. Without loss of generality we may assume that φ and f both are periodic with period 1. Condition (6.1) is the definition of Zygmund's space λ^* of smooth functions. Thus, $\varphi+f \in \lambda^*$ is non-differentiable a.e. Invoking Theorem 7.1 of [2] yields the desired result.

Remark. Examples of functions in λ^* that are non-differentiable a.e. are easily given with Lacunary Fourier series, see e.g., [16], p. 47. One such is

$$f(t) = \sum_{n=1}^{\infty} (n^{1/2}\cdot 2^n)^{-1} \operatorname{Cos}(2^n \cdot 2\pi t).$$

Non-differentiable functions f in the larger Zygmund space

$$\Lambda^* = \{f: \sup_{t,h>0} \frac{1}{h}|f(t+h)+f(t-h)-2f(t)| < \infty\} \cap C,$$

satisfy the hypothesis of Theorems 5.2 and 5.4, but we do not know if they satisfy the stronger conclusion of Theorem 6.2. For more on this, see [2], Section 7.

<u>Example 6.3</u>. An example of a function $f(t)$ satisfying the hypotheses of Theorem 5.3 but with singular occupation measure will be given using known properties of Brownian motion.

We begin by constructing a continuous singular increasing function $F(x)$ satisfying

$$F(x)-F(y) \geq c|x-y|^{\alpha} \quad \text{for all } x,y \text{ with } |x-y| < 1, \tag{6.2}$$

and for some $c > 0$ and $\alpha > 1$.

For this purpose, let $x = \sum_{n=1}^{\infty} X_n(x)/2^n$ be the dyadic expansion of $x \in [0,1)$. For $p \in (½,1)$, we let $F_0(x)$ be the distribution function of x which corresponds to the $\{X_n(x)\}$ being i.i.d. random variables with $P\{X_n = 1\} = p = 1-P\{X_n = 0\}$.

We observe that F_0 is singular, so $F_0'(x) = 0$ a.e. Setting $q = 1-p$, we also note that each dyadic interval $[a,b) \subset [0,1)$ of length $1/2^n$ satisfies

$$q^n \leq F(b)-F(a), \tag{6.3}$$

and that any interval $[x,y)$ of length at least $2\cdot 2^{-n}$ will contain a dyadic interval $[a,b)$ of length 2^{-n} satisfying (6.3). Thus, for $\alpha = -\log q/\log 2 > 0$,

$$\begin{aligned} F_0(y)-F_0(x) &\geq F_0(b)-F_0(a) \\ &\geq q^n \\ &= [(\tfrac{1}{2})^n]^{\alpha} \\ &\geq \frac{1}{2^{\alpha}} |y-x|^{\alpha}. \end{aligned}$$

We now let $\lceil x \rceil$ be the integer part of $x \in \mathbb{R}^1$ and define $F(x) = \lceil x \rceil + F_0(x - \lceil x \rceil)$. We observe that (6.2) holds for all x and y with $\alpha = -\log q$ and $c = 1/2^{\alpha}$.

Let $\{B(t): 0 \le t \le 1\}$ be standard Brownian motion and introduce the function

$$f(t) = F(B(t)).$$

We claim that f(t) satisfies (5.1) with probability 1. For this, it suffices to show that

$$E \int_0^1 \int_0^1 [(s-t)^2 + (f(s)-f(t))^2]^{-\frac{1}{2}} ds\, dt.$$

Using Fubini's theorem, the stationary increments of $\{B(s)\}$ and (6.2), it will suffice to show that

$$k(t) \equiv E[t^2 + |B(t)|^{2\alpha - \frac{1}{2}}]^{-\frac{1}{2}}$$

satisfies

$$\int_0^1 k(t)dt < \infty. \tag{6.4}$$

Now

$$k(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [t^2 + (tx^2)^{\alpha}]^{-\frac{1}{2}} \exp(-x^2/2)dx$$

is not easily computed but is easily bounded. Setting $\gamma = (2-\alpha)/2\alpha$ and cutting the integral into 3 pieces at $|x| = t$ and at $|x| = 1$ gives

$$k(t) = k_1(t) + k_2(t) + k_3(t),$$

where

$$k_1(t) \le \frac{\text{const.}}{t} \int_{|x|<t^\gamma} 1\, dx$$

$$= \text{const. } t^{\gamma-1},$$

$$k_2(t) \le \text{const.} \int_{t^\gamma}^{1} t^{-\alpha/2} x^{-\alpha} dx$$

$$\le \text{const. } t^{\gamma-1},$$

and

$$k_3(t) \le \text{const. } t^{-\alpha/2} \int_1^\infty e^{-x^2/2} dx$$

$$= \text{const. } t^{-\alpha/2}.$$

For $1 < \alpha < 2$, we see that (6.4) holds and thus that (5.1) holds, almost surely.

We now claim that $f(t) = F(B(t))$ does not have local times. In fact, since $F(x)$ is singular there is a decomposition of $\mathbb{R}^1$ into two Borel sets A and B with

$$\mathbb{R}^1 = A \cup B,$$

$$|A| = 0,$$

and

$$|F^{-1}(B)| = 0.$$

Since the distribution of $B(t)$ is absolutely continuous, we know that with probability 1,

$$\int_0^1 1_{F^{-1}(A)}(B(t))dt = 1.$$

This is equivalent to the assertion that $\mu_f(A) = 1$ and hence μ_f is a singular measure.

Observing finally that $\alpha = -\log q/\log 2$ satisfies $1 < \alpha < 2$ if and only if $\frac{1}{4} < q < \frac{1}{2}$, we can formally state:

For $\frac{1}{2} < p < \frac{3}{4}$, the function $f(t) = F(B(t))$

satisfies the condition (5.1) but has with probability 1 a singular occupation measure.

Remark. This example answers in the negative the question raised by Geman and Horowitz [5], p. 16. Our function f is a Jarnik function without local times.

Section 7: Almost Sure Results for Gaussian Processes

We begin with two preliminary results.

Lemma 7.1. Let $\{X(t): t \in \mathbb{R}^1\}$ be a real mean zero measurable stationary Gaussian processes, and define

$$|D|X(t) = \operatorname{ap} \liminf_{s \to t} \frac{|X(s)-X(t)|}{|s-t|} .$$

Then

$$p(t) \equiv P\{|D|X(t) < \infty\}$$

is independent of t and equals 0 or 1.

Lemma 7.2. Under the hypothesis of Lemma 7.1, the probability $P\{\operatorname{ap} X'(t) \text{ exists}\}$ is independent of t and equals 0 or 1.

<u>Proof of Lemma 7.1</u>. By stationarity p(t) is constant.

To establish that p(t) = 0 or 1, we use the spectral representation

$$X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dW(\lambda), \tag{7.1}$$

where $\{W(A)\}$ is a mean zero complex-valued Gaussian measure on $\mathbb{R}^1$ satisfying $\overline{W}(A) \equiv W(-A)$, and with finite control measure Δ for which

$$\Delta(A \cap B) \equiv E\, W(A)\, \overline{W(B)}$$

and

$$EX(t)X(s) = \int_{-\infty}^{\infty} e^{i(t-s)x} \Delta(dx). \tag{7.2}$$

We now let

$$Z_n(t) = \int_{n-1<|\lambda|\leq n} e^{it\cdot x} dW(x).$$

The real Gaussian processes $\{Z_n(t)\}$ are stationary and independent. Moreover, if

$$X_n(t) = \int_{|\lambda|\leq n} e^{it\cdot x} dW(x), \tag{7.3}$$

and

$$Y_n(t) = \int_{|\lambda|>n} e^{it\cdot x} dW(x),$$

we note that

$$X_n(t) = \sum_{j=0}^{n} Z_j(t) \text{ and } Y_n(t) = \sum_{j=n+1}^{\infty} Z_j(t).$$

Finally, we comment that each process $X_n(t)$ is a real analytic function of t. In particular, $X_n'(t)$ exists and is finite for each t. Thus, for each n, the event $\{|D|X(t) < \infty\}$ only depends on the process $\{Y_n(s)\}$. Or, and this is equivalent, for each $n \geq 0$ the event $\{|D|X(t) < \infty\}$ only depends on the independent processes $\{Z_j(s)\}$, $j > n$. Thus, $\{|D|X(t) < \infty\}$ is a tail event for the sequence $\{Z_j(s)\}$. By Kolmogorov's 0-1 law $P\{|D|X(t) < \infty\}$ equals 0 or 1.

<u>Proof of Lemma 7.2</u>. As in the proof of Lemma 7.2, we observe that $\{\text{ap } X'(t) \text{ exists}\}$ is a tail event for the sequence $\{Z_n(s)\}$. The result then follows from Kolmogorov's 0-1 law.

<u>Lemma 7.3</u>. Let $\{X(t)\}$ be as in Lemma 7.1 and let the spectral measure Δ be as in (7.2). Then a necessary and sufficient condition for $P\{\text{ap } X'(t) \text{ exists}\} = 1$ is

$$\int_{-\infty}^{\infty} \lambda^2 \Delta(d\lambda) < \infty. \tag{7.4}$$

Condition (7.4) is known to be equivalent to the existence of L^2 derivatives. That is, (7.4) holds iff

$$\lim_{s \to t} \frac{X(s)-X(t)}{s-t} = X'(t) \tag{7.5}$$

exists in L^2.

__Proof__. Assume that $P\{\text{ap } X'(t) \text{ exists}\} = 1$. We take $t = 0$, and we set $Y = \text{ap } X'(0)$. Then the assumption

$$P\{\text{ap} \lim_{t\to 0} \frac{X(t)-X(0)}{t} = Y\} = 1$$

implies for each $\varepsilon > 0$,

$$\lim_{n\to\infty} n\int_0^{1/n} P\{|X(t)-X(0)-tY| < \varepsilon t\}\, dt = 1,$$

from which we may conclude there exists a sequence $t_n \downarrow 0$ with

$$\lim_{n\to\infty} P\{|\frac{X(t_n)-X(0)}{t_n} - Y| > \varepsilon\} = 0$$

for each $\varepsilon > 0$. Thus $(1/t_n)(X(t_n)-X(0))$ converges in probability. But this is a sequence of Gaussian variables which must converge in L^2 if it converges in probability. Thus

$$\sup_n \frac{2}{t_n^2} \int_{-\infty}^{\infty} (1-\text{Cos } t_n\lambda)\Delta(d\lambda) = \sup_n E\left[\frac{X(t_n)-X(0)}{t_n}\right]^2 < \infty.$$

Applying Fatou's inequality gives (7.4), but this implies that the convergence in (7.5) occurs in L^2.

__Remark__. It is possible to prove versions of Lemma 7.1, 7.2, and 7.3 in considerably more generality. For example: if $X(t)$ is any process with a series expansion

(7.6) $$X(t) = \sum_{n=1}^{n} X_n \varphi_n(t),$$

and if

(7.7) $\{X_n\}$ are independent.

and if

(7.8) each $\varphi_n(t)$ is continuously differentiable and $\varphi_n'(t)$ has only finitely many zeros,

then for each t, $P\{|D|X(t) < \infty\}$ and $P\{\text{ap } X'(t) \text{ exist}\}$ each will equal 0 or 1, but they will in general depend upon t.

We now turn to an idea that is originally due to Klein [8], and which enables us to convert the perturbation results in Section 5 to almost sure results on stationary Gaussian processes.

We again assume that X(t) is as in Lemma 7.1, and to avoid trivialities we assume that X(t) is not constant a.e. with probability 1. This implies that for some N the process $X_N(t)$ given in (7.3) is not constant and that

$$\rho_N(t) = EX_N(t)X_N(0) = \int_{|\lambda|\leq N} e^{it\lambda}\Delta(d\lambda)$$

is a non-constant real analytic function. Setting $\varphi(t) = \rho_N(t)/\rho_N(0)$, we observe that $X_N(0)$ is independent of $X_N(t)-\varphi(t)X_N(0)$ since they are orthogonal. If we now set

$$\Sigma(t) = Y_N(t)+[X_N(t)-\varphi(t)X_N(0)]$$

and

$$X = X_N(0),$$

we have

Lemma 7.4. A non-constant real stationary L^2 *continuous* Gaussian process $\{X(t)\}$ has a representation of the form

$$X(t) = X\varphi(t)+\Sigma(t), \tag{7.9}$$

where:

(a) $\varphi(t)$ is a C^1 function and $\varphi'(t)$ has isolated zeros,

(b) X is a real random variable with absolutely continuous distribution, and

(c) $\{\Sigma(t)\}$ is a real stochastic process that is independent of X.

Theorem 7.5. Let $\{X(t)\}$ be a real measurable stochastic process which admits a representation of the form (7.9) *satisfying* conditions a, b, and c of Lemma 7.4.

Part 1. Suppose that for a.a. $t \in [0,1]$,

$$P\{|D|X(t) < \infty\} = 1 \text{ and } P\{\text{ap } X'(t) \text{ exists}\} = 0. \tag{7.10}$$

Then, with probability 1, $\{X(t)\}$ has a singular continuous occupation measure.

Part 2. Suppose that for a.a. $t \in [0,1]$,

$$\int_0^1 ((t-s)^2+(X(t)-X(s))^2)^{-\frac{1}{2}} ds < \infty \tag{7.11}$$

holds with probability 1. Then, almost surely $\{X(t)\}$ has local times.

<u>Proof</u>. Part 1. If (7.10) holds, then Fubini's theorem implies that for almost all ω,

$$|D|\Sigma(t) < +\infty \qquad \text{for almost all } t\text{, and}$$

$$\text{ap } \Sigma'(t) \text{ exists} \qquad \text{for almost no } t.$$

For each ω, Theorem 5.5 gives that $\lambda\varphi(t)+\Sigma(t)$ has a singular continuous occupation measure. Parts (b) and (c) of Lemma 7.4 allow the conclusion

$$P\{X\varphi(t)+\Sigma(t) \text{ has a singular continuous occupation measure}\} = 1.$$

Part 2. Without loss of generality, we may assume that $\varphi'(t) > 0$ on [0,1]. With this assumption, it follows that

$$P\{\int_0^1 ((t-s)^2+(\Sigma(t)-\Sigma(s))^2)^{-1/2} ds < \infty\} = 1$$

for almost all t in [0,1]. It is now straightforward to apply Theorem 5.6 and complete the proof.

We specialize part 1 of this theorem to the case of stationary Gaussian processes.

<u>Proposition 7.6</u>. Suppose $\{X(t)\}$ is a real measurable L^2 continuous stationary Gaussian process with

$$\sup_{t>0} \frac{E(X(t)-X(0))^2}{t^2} = +\infty, \tag{7.12}$$

and

$$P\{|D|X(0) < \infty\} = 1. \tag{7.13}$$

Then with probability 1, $X(t)$ has a singular continuous occupation measure.

Proof. This follows directly from Lemmas 7.3 and 7.4 and Theorem 7.5.

Proposition 7.6 forms the basis of our main probabilistic result in this paper, Theorem 9.1.

Section 8: A Stochastic Analogue of the Zygmund Space Λ^*

The Zygmund space Λ^* of continuous real-valued functions f on $\mathbb{R}^1$ satisfying

$$\|f\|_* = \sup_{h>0,t} |\tfrac{1}{2}(f(t+h)+f(t-h))-f(t)|/h$$

naturally extends to vector valued functions. Here we introduce a stochastic version of Λ^* that we denote with $\{S\Lambda^*, \|\cdot\|_s^*\}$.

Definition 8.1. An L^2-continuous real stochastic process $\{X(t): t \in \mathbb{R}^1\}$ is in the space $S\Lambda^*$—read stochastic Λ^*—if the semi-norm

$$\|X\|_s^* \equiv \sup_{h>0,t} \|\tfrac{1}{2}(X(t+h)+X(t-h))-X(t)\|_2/h < \infty. \tag{8.1}$$

Here $\|X\|_2$ denotes the L^2 norm $(EX^2)^{1/2}$.

We will only make use of this space in the context of stationary processes.

If $\{X(t)\}$ is stationary, we set

$$\Delta_1(h) = E|X(t+h)-X(t)|^2,$$

and

$$\Delta_2(h) = E|X(t+h)+X(t-h)-2X(t)|^2.$$

Then

$$\sup_{h>0} \frac{\Delta_2(h)}{h^2} = 4(\|X\|_s^*)^2, \tag{8.2}$$

and we have

Lemma 8.2. An L^2-continuous stationary Gaussian process is in $S\Lambda^*$ iff

$$\Delta_2(h) = O(h^2) \quad \text{as } h \downarrow 0. \tag{8.3}$$

We remark that the arguments in [2] generalize to the vector valued case, and without essential change yield several useful results concerning $S\Lambda^*$. In particular we will need the following two lemmas.

Lemma 8.3 (see [2], (2.20)). For an interval $I = (a,b)$, we let

$$X_I(t) = \frac{1}{b-a}\,[(b-t)X(a)+(t-a)X(b)]$$

be the linear function interpolating $(a,X(a))$ and $(b,X(b))$. If $X \in S\Lambda^*$,

$$\sup_{a\le t\le b} \|X(t)-X_I(t)\|_2 \le 2\|X\|_s^*(b-a). \tag{8.4}$$

<u>Lemma 8.4</u> (see [2], Prop. 2.4). For $0 \leq a < b \leq 1$,

$$(8.5) \qquad \|X(b)-X(a)\|_2 \leq \left[\|X(1)-X(0)\|_2 + 6\|X\|_s^* + \frac{2\|X\|_s^*}{\log 2} \log \frac{1}{b-a}\right](b-a).$$

Finally, we derive some spectral characterizations of stationary processes in $S\Lambda^*$.

<u>Proposition 8.5</u>. Let $\{X(t)\}$ be an L^2-continuous process with covariance function

$$\rho(t) = EX(t)X(0)$$
$$= \int_{-\infty}^{\infty} e^{itx}\Delta(dx).$$

Then the following are equivalent.

(i) $\{X(t)\} \in S\Lambda^*$.

(ii) For some $K_1 < \infty$,

$$\int_{-\infty}^{\infty} (1-\operatorname{Cos} tx)^2 \Delta(dx) \leq K_1 t^2$$

holds for all $t > 0$.

(iii) $\sup_{\lambda > 0} \lambda^2 \Delta([\lambda.\infty)) \equiv K_2 < \infty$.

(iv) For some $p > 2$,

$$\sup_{\lambda > 1} \frac{1}{\lambda^{p-2}} \int_{|x| \leq \lambda} x^p \Delta(dx) \equiv C_p < \infty.$$

<u>Note</u>: If C_p in (iv) is finite for some $p > 2$, then $C_p < \infty$ for all $p > 2$.

<u>Proof</u>. By Lemma 8.2, $\{X(t)\} \in S\Lambda^*$ iff there is a $K < \infty$ with

$$\Delta_2(t) \leq Kt^2 \quad \text{for all } t.$$

But

$$\Delta_2(t) = \int_{-\infty}^{\infty} |e^{itx}+e^{-itx}-2|^2 \Delta(dx)$$

$$= 4\int_{-\infty}^{\infty} (1-\mathrm{Cos}\ tx)^2 \Delta(dx),$$

and thus (i) is equivalent to (ii).

Assuming (ii) we observe that $2^n \leq x \leq 2^{n+1}$ implies $1-\mathrm{Cos}(x/2^n) \geq 1-\mathrm{Cos}(1) > \frac{1}{3}$. Thus

$$K_1(\frac{1}{2^n})^2 \geq \int_{[2^n,2^{n+1})} \frac{1}{9}\,\Delta(dx) = \frac{\Delta([2^n,2^{n+1}))}{9}$$

and

$$\Delta([2^n,\infty)) \leq 9K_1 \sum_{m\geq n} 1/4^m$$

$$= 12\ K_1(\frac{1}{2^n})^2,$$

from which (iii) follows.

We now show (iii) implies (iv). First note that Δ is an even measure, so it suffices to consider

$$\int_1^{\lambda} x^p \Delta(dx) \leq \text{const.} + p\int_1^{\lambda} x^{p-1}\Delta([x,\infty))dx,$$

where we have used integration by parts. Bringing in (iii) gives

$$\int_1^{\lambda} x^p \Delta(dx) \leq \text{const.} + pK\int_1^{\lambda} x^{p-3}dx = O(\lambda^{p-2}),$$

which gives (iv).

Assuming (iv) we have

$$c_p \ \lambda^{p-2} \geq \int_{[\lambda/2,\lambda)} x^p \Delta(dx) \geq (\frac{\lambda}{2})^p \Delta([\lambda/2,\lambda)).$$

As in our proof that (ii) implies (iii), this gives (iii), so (iii) and (iv) are equivalent.

Finally, we show (iii) implies (ii). We write

$$\int_{-\infty}^{\infty}(1-\text{Cos } tx)^2\Delta(dx) = \int_{|x|\leq 1/|t|} + \int_{|x|>1/|t|} .$$

On $|4x| \leq 1$, we use $1-\text{Cos } tx < \frac{(tx)^2}{2}$, which gives

$$\int_{|x|\leq 1/|t|} (1-\text{Cos } tx)^2\Delta(dx) \leq \frac{t^4}{4} \int_{|x|\leq 1/|t|} x^4\Delta(dx),$$

and by (iv), which follows from (iii), this is $O(t^2)$.

For $|x| > 1/|t|$ we simply use $(1-\text{Cos } tx)^2 \leq 4$ which gives

$$\int_{|x|>1/|t|} (1-\text{Cos } tx)^2\Delta dx \leq 8 \ \Delta([1/t,\infty))$$

$$= O(t^2),$$

again by (iii).

Section 9: Proof of Theorem 9.1.

We turn now to the proof of our main result, Theorem 9.1, which is stated in the introduction. We begin with some preliminary remarks.

In the terminology of Section 8, the process $\{X(t)\}$ in Theorem 9.1 is a stationary Gaussian process in $S\Lambda^*$, but without L^2 derivatives. By Proposition 8.5 we can characterize the spectral representations of such processes: A process

$$X(t) = \int_{-\infty}^{\infty} e^{itx} W(dx) \tag{9.1}$$

with spectral measure $\Delta(dx)$ satisfies the conditions of Theorem 9.1 iff

$$\sup_{\lambda > 0} \lambda^2 \Delta([\lambda, \infty)) < \infty, \tag{9.2}$$

and

$$\int_{-\infty}^{\infty} x^2 \Delta(dx) = \infty \tag{9.3}$$

hold.

Our method is simply to apply Proposition 7.6. Since (9.3) is equivalent to (7.12), we only need show that $P\{|D|X(0) < \infty\} = 1$, and by Lemma 7.1, it will suffice to show that

$$P\{|D|X(0) < \infty\} > 0.$$

Theorem 9.1 will thus follow from

<u>Proposition 9.2</u>. If $\{X(t)\}$ is a measurable stationary Gaussian process with representation (9.1) and if conditions (9.2) and (9.3) hold, then for some constant $K < \infty$ and some sequence $\lambda_n \uparrow \infty$,

(9.4) $$P\{\sup_{0 \le t \le 1/\lambda_n} \lambda_n |X(t)-X(0)| \le 2K \text{ i.o.}\} > 0.$$

<u>Proof of Proposition 9.2</u>. Assume the sequence $\lambda_n \uparrow +\infty$ is given and introduce the two sequences of processes

$$X_n(t) = \lambda_n \int_{|x| \le \lambda_n} (e^{itx/\lambda_n} - 1)dW(x),$$

and

$$Y_n(t) = \lambda_n \int_{|x| > \lambda_n} (e^{itx/\lambda_n} - 1)dW(x).$$

For fixed K we introduce the events

$$C_n(K) = \{\sup|X_n(t)| \le K;\ t \in [0,1]\},$$

and

$$D_n(K) = \{\sup|Y_n(t)| \le K;\ t \in [0,1]\}.$$

Since

$$X(t/\lambda_n)-X(0) = X_n(t)+Y_n(t),$$

it will suffice to show that for some K,

(9.5) $$P\{C_n(K) \cap D_n(K) \text{ i.o.}\} > 0.$$

The difficult part of showing (9.5) is

<u>Proposition 9.3</u>. For $K > 0$, the sequence $\lambda_n \uparrow \infty$ may be chosen so that

$$P\{C_n(K) \text{ i.o.}\} = 1. \tag{9.6}$$

Assuming this proposition has been established, we note that the event $D_n(K)$ is independent of the σ-field

$$\mathcal{F}_n = \sigma\{X_j(s) : s \in \mathbb{R}^1;\ 1 \le j \le n\}.$$

By a variant on the Borel-Cantelli Lemma, Lemma 2 on page 86 of [1], we observe that (9.6) implies

$$P\{C_n(K) \cap D_n(K) \text{ i.o.}\} \ge \inf_n P\{D_n(K)\}. \tag{9.7}$$

(We remark here that the proof of Lemma 2 in [1] contains an unfortunate misprint. The last symbol in the proof is B_j, and it should read B_{n+1}.)

But

$$(\|Y_n\|_s^*)^2 = \sup_h \frac{\lambda_n^2}{h^2} \int_{|x|>\lambda_n} (1-\mathrm{Cos}(hx/\lambda_n))^2 \Delta(dx)$$

$$\le \sup_h \left(\frac{\lambda_n}{h}\right)^2 \int_{-\infty}^{\infty} (1-\mathrm{Cos}(hx/\lambda_n))^2 \Delta(dx)$$

$$\le (\|X\|_s^*)^2 .$$

Also,

$$E|Y_n(1)|^2 \le 4\cdot\lambda_n^2\cdot\Delta\{x: |x| > \lambda_n\}$$

which by Proposition 8.5 is bounded independently of n. Applying Lemma 8.4, we may conclude that independent of the sequence $\lambda_n \uparrow \infty$ there is a constant c_1 such that

$$E|Y_n(t)-Y_n(s)|^2 \leq c_1\left(\log \frac{1}{|t-s|} \cdot |t-s|\right)^2$$

holds for all $t,s \in [0,1]$ with $|t-s| \leq \frac{1}{2}$. Since $Y_n(0) \equiv 0$, it will follow from Garsia's inequality in the form presented, for example, in [15], p. 49, that

$$\sup_n E \sup_{0\leq t\leq 1} |Y_n(t)| = L < \infty.$$

It follows from Chebyshev's inequality that

$$\inf_n P\{D_n(K)\} > 0$$

holds for all large K. Thus by (9.7), it suffices to prove Proposition 9.3.

We first establish

Lemma 9.4. For each $K > 0$, there exists a sequence $\lambda_n \uparrow \infty$ so that

$$P\{|X'_{n-1}(0)|+|X'_n(0)| < K \quad \text{i.o.}\} = 1.$$

Proof. We will work instead with the equivalent sequence of events

$$A_n(K) = \{|X'_{n-1}(0)| < K \text{ and } |X'_n(0)| < K\}.$$

We now introduce the sequence

$$Z_n = i \int_{\lambda_{n-1}<|x|\le\lambda_n} x \, dW(x)$$

of independent mean zero normal random variables with variances

$$\sigma_n^2 = \int_{\lambda_{n-1}<|x|\le\lambda_n} x^2 \, dW(x).$$

But

$$X_n'(t) = i\int_{|x|\le\lambda_n} x \, e^{itx/\lambda_n} \, dW(x),$$

so that

$$X_n'(0) = \sum_{j=0}^{n} Z_j$$

is simply a random walk with non-identically distributed increments Z_j.

Our desired result $P\{A_n(K) \text{ i.o.}\} = 1$ is a slight twist on the usual recurrence criteria for such random walks, and our proof of Lemma 9.4 is an adaptation of a standard proof [6], p. 173.

We assume, as we may, that $\sigma_n^2\uparrow$.

Let d_n^2 be the determinant of the covariance matrix of $X_{n-1}'(0)$ and $X_n'(0)$. Because of the form of the normal density there are positive constants c_1 and c_2 that are independent of n but not of K such that

$$\frac{c_1}{d_n} < P(A_n(K)) < \frac{c_2}{d_n} . \tag{9.8}$$

But $d_n = s_{n-1}\sigma_n$ where $s_n^2 = \Sigma_1^n \sigma_j^2$, and we abbreviate the relation (9.8) with

$$P(A_n(K)) \approx \frac{1}{s_{n-1}\sigma_n}. \tag{9.9}$$

Similarly, computing the determinant of the covariance matrix of the vector $(X'_{m-1}(0), X'_m(0), X'_{n-1}(0), X'_n(0))$, and using the inequality $s_n^2 - s_m^2 \geq s_{n-m}^2$, which follows from the assumption $\sigma_n^2 \uparrow$, we find

$$P(A_m(K) \cap A_n(K)) \leq \frac{\text{Const.}}{s_{m-1}s_{n-m-1}\sigma_m\sigma_{n-m}} \tag{9.10}$$

holds for all m and n with $1 \leq m < n-1$.

If we now set

$$\mathcal{N}_n = \text{number of } j \leq n \text{ for which } A_j(K) \text{ occurs,}$$

we have

$$E\mathcal{N}_n = \sum_{j=2}^{n} P(A_j(K)) \approx \sum_{j=2}^{n} \frac{1}{s_{j-1}\sigma_j}, \tag{9.11}$$

and

$$\begin{aligned} E\mathcal{N}_n^2 &= \sum_{j,k=2}^{n} P(A_j(K) \cap A_k(K)) \\ &\leq 3\sum_{j=2}^{n} P(A_j(K)) + 2\sum_{j=2}^{n-1}\sum_{n=j+2}^{n} P(A_j(K) \cap A_k(K)) \\ &\leq \text{const. } [E\mathcal{N}_n + (E\mathcal{N}_n)^2]. \end{aligned}$$

Schwarz's inequality gives the estimate

$$E\mathcal{N}_n \leq \lambda + (P\{\mathcal{N}_n > \lambda\})^{1/2} \|\mathcal{N}_n\|_2 ,$$

which when substituted into the above inequality gives: There exist positive constants A, B, and C which depend only upon K and $\sigma_1^2 > 0$ such that

$$\|N_n\|_2^2 \leq A\lambda + B\lambda \|N_n\|_2 + CP\{N_n > \lambda\} \|N_n\|_2^2 \tag{9.12}$$

holds for all $\lambda \geq 1$.

If $\|N_n\|_2^2 \to \infty$ this implies

$$P\{A_n(K) \text{ occurs i.o.}\} \geq \frac{1}{C} > 0. \tag{9.13}$$

But, a closer look at the derivation of (9.12) reveals that if σ_1^2 is bounded away from 0 and $K \downarrow 0$, then the constant c in (9.12) may be taken arbitrarily close to 1.

From (9.11) and (9.13) we may thus conclude: If $\sigma_1^2 > 0$ and $\sigma_n^2 \uparrow$ then necessary and sufficient for $P\{A_n(K) \text{ occurs i.o.}\} = 1$ is

$$\sum_{n=2}^{\infty} \frac{1}{s_{n-1}\sigma_n} = +\infty. \tag{9.14}$$

Since the condition (9.2) and (9.3) allow us to choose $\lambda_n \uparrow \infty$ in such a manner that (9.14) holds, this completes the proof of Lemma 9.4.

Our plan now is to show that $\{C_n(K)\}$ occurs infinitely often by showing that $A_n(K)$ occurs infinitely often, and then estimating the derivatives $X_n''(t)$. We begin by considering the conditional expectation of $X_n''(t)$ given the variables $\{X_1'(0), \ldots, X_n'(0)\}$. Since $\{X(t)\}$ is Gaussian, we need only compute the orthogonal projection $P_n X_n''(t)$ onto the linear span

$sp\{X_1'(0), \ldots, X_n'(0)\} = sp\{Z_1, \ldots, Z_n\}$. Observing that the $\{Z_j\}$ are orthogonal $N(0,\sigma_j^2)$ variables

$$P_n X_n''(t) = \sum_{j=1}^{n} \frac{Z_j X_n''(t)}{\sigma_j^2} Z_j .$$

Call $E\, Z_j X_n''(t) = \alpha_{n,j}(t)$. Then the representations

$$Z_j = i \int_{\lambda_{j-1} < |x| \le \lambda_j} x \, dW(x)$$

and

$$X_n''(t) = -\frac{1}{\lambda_n} \int_{|x| \le \lambda_n} x^2 e^{ixt/\lambda_n} \, dW(x),$$

give

$$\alpha_{n,j}(t) = \frac{1}{\lambda_n} \int_{\lambda_{j-1} < |x| \le \lambda_j} x^3 \operatorname{Sin}(tx/\lambda_n) \Delta(dx),$$

where we have used here that $\alpha_{n,j}(t)$ is real and that $\Delta(dx)$ is a symmetric measure. Thus we have

$$(9.15) \qquad 0 \le \alpha_{n,j}(t) \le 2t \int_{(\lambda_{j-1}, \lambda_j]} x^4/\lambda_n^2 \Delta(dx).$$

In

$$P_n X_n''(t) = \sum_{j=1}^{n} \frac{\alpha_{n,j}(t)}{\sigma_j^2} Z_j ,$$

we now estimate the partial sum

$$E \max_{0 \le t \le 1} \left| \sum_{j=1}^{n-1} \frac{\alpha_{n,j}(t)}{\sigma_j^2} Z_j \right|$$

$$\leq E\ 2\sum_{j=1}^{n-1} \int_{(\lambda_{j-1},\lambda_j]} x^4/\lambda_n^2 \Delta(dx)\cdot\left|\frac{Z_j}{\sigma_j}\right|$$

$$\leq \frac{2}{\lambda_n^2} \int_{|x|\leq\lambda_{n-1}} x^4 \Delta(dx)$$

$$\leq 2\ C_4\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^2 \text{ by Prop. 8.5, iv.}$$

We thus have:

If $\sum_{n=2}^{\infty}\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^2 < \infty$, then with Probability 1,

$$\sup_{0\leq t\leq 1}\left|\sum_{j\neq n}\frac{\alpha_{n,j}(t)}{\sigma_j^2} Z_j\right| < K$$

holds for all large n.

Looking now at $\alpha_{n,n}(t)$, (9.15) and Proposition 8.5(iv) give $0 \leq \alpha_{n,n}(t) \leq C_4$ for all $t \in [0,1]$, so

$$\left|\frac{\alpha_{n,n}(t)}{\sigma_n^2} Z_n\right| = \left|\frac{\alpha_{n,n}(t)}{\sigma_n^2} (X_n'(0)-X_{n-1}'(0))\right|.$$

On the set $A_n(K)$, this is less than $2C_4K/\sigma_n^2$, which is always bounded if $\sigma_n^2\uparrow$ and tends to zero if $\sigma_n^2\uparrow\infty$. We formally state this result as

<u>**Lemma 9.5**</u>. If $\Sigma\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^2 < \infty$, and if $\sigma_n^2\uparrow\infty$, then for all $K > 0$,

$$\limsup_{n\to\infty} 1_{A_n(K)} \|P_n X_n''(t)\|_\infty = 0 \quad \text{a.s.}$$

As usual, $1_{A_n(K)}$ is the indicator function of the set $A_n(K)$.

Finally, we consider $Q_n X_n''(t)$, where

$$X_n''(t) = P_n X_n''(t) + Q_n X_n''(t).$$

We observe that $EQ_n X_n''(t) X_j'(0) = 0$ for $1 \le j \le n$, so the process $\{Q_n X_n''(t) : 0 \le t \le 1\}$ is independent of the sigma field $\mathcal{F}_n = \sigma\{X_1'(0), \ldots, X_n'(0)\}$.

We let

$$E_n(K) = \{\sup_{0 \le t \le 1} |Q_n X_n''(t)| \le K\},$$

and we wish to prove that for some $K > 0$,

$$\inf_n P\{E_n(K)\} > 0. \tag{9.16}$$

But Q_n is a projection in $L^2(P)$ with norm 1, so

$$\begin{aligned} E|Q_n X_n''(0)|^2 &\le E|X_n''(0)|^2 \\ &= \int_{|x| \le \lambda_n} \left|\frac{x^2}{\lambda_n} e^{itx/\lambda_n}\right|^2 \Delta(dx) \\ &\le C_4, \quad \text{by Prop. 8.5(iv).} \end{aligned}$$

Similarly,

$$\begin{aligned} E|Q_n X_n''(t) - Q_n X_n''(s)|^2 &\le \int_{|x| \le \lambda_n} \frac{x^4}{\lambda_n^2} |x/\lambda_n|^2 \Delta(dx) \\ &\le C_4 (t-s)^2. \end{aligned}$$

Applying Garsia's inequality again gives a constant $B < \infty$ with

$$\sup_n E \sup_{0\le t\le 1} |Q_n X_n''(t) - A_n X_n''(0)| < B,$$

so

$$E \sup_{0\le t\le 1} |Q_n X_n''(t)| \le B + C_4 + 1 < \infty.$$

Chebyshev's inequality now gives (9.16) for sufficiently large K and the independence of $E_n(K)$ from $\mathcal{G}_n$ together with Lemma 2 of [1] give the desired result:

$$\begin{aligned} P\{A_n(K) \cap E_n(K) \text{ i.o.}\} &\ge \inf P\{E_n(K)\} \\ &> 0, \end{aligned}$$

provided that $P\{A_n(K) \text{ i.o.}\} = 1$.

Thus we see that Proposition 9.3 will follow if we can choose $\lambda_n \uparrow$ satisfying both (9.14) and $\Sigma(\lambda_{n-1}/\lambda_n)^2 < \infty$. To see how this is possible, start with

$$\sigma_j^2 = 2 \int_{(\lambda_{j-1},\lambda_j]} x^2 \Delta(dx).$$

Set $T(x) = \Delta([x,\infty))$ and integrate by parts to obtain

$$\sigma_j^2 = -2x^2 T(x) \Big|_{\lambda_{j-1}}^{\lambda_j} + 4 \int_{(\lambda_{j-1},\lambda_j]} x\, T(x) dx.$$

But $x^2 T(x)$ is bounded by (9.2), so there exist positive constants A and B such that

$$\sigma_j^2 - A \le B \int_{\lambda_{j-1}}^{\lambda_j} \frac{dx}{x}$$

$$= B \log(\lambda_j/\lambda_{j-1}),$$

or

$$C\, e^{-D\sigma_j^2} \ge \lambda_{j-1}/\lambda_j .$$

If we now choose the σ_j^2, so that for all large j,

$$j^{1/2} \le \sigma_j^2 \le 2\, j^{1/2},$$

then

$$s_n^2 \approx n^{3/2},$$

and

$$\sum_{n\ge 2} \frac{1}{s_{n-1}\sigma_n} \approx \sum \frac{1}{n} = +\infty,$$

while

$$\sum_{n\ge 2} \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^2 \le C \sum_{n\ge 2}^{\infty} e^{-2D_n} < \infty.$$

The proofs of Proposition 9.3 and Theorem 9.1 are now complete.

Remark. This result shows that a stationary Gaussian process with spectral measure Δ satisfying

(9.17) $$\Delta([\lambda,\infty)) = O(\lambda^{-2});\ \lambda \to \infty$$

will either be differentiable or will not have local times. This result is an extension of the theorem of Lifschitz [9] where similar results are proven under the hypothesis

$$\Delta([\lambda,\infty)) = O(\lambda^{-2}/\log\log\lambda);\quad \lambda \to \infty.$$

We do not know how sharp our results are, but the example in the next section shows that (9.17) is not necessary for the singularity of the occupation measure μ_X.

Section 10. A Discontinuous Gaussian Process without Local Times

The question is raised in [5], p. 53, if every discontinuous stationary Gaussian process has analytic local times? Here we use the methods of this paper to show the answer is "no." We exhibit the existence of a stationary Gaussian process that is discontinuous, but which has singular occupation measures with probability 1.

We let $\alpha_n \geq 1$ and $\beta_n \geq 1$ be two sequences of integers satisfying the lacunary conditions,

(10.1) $$\alpha_{n+1} \geq 2\,\alpha_n, \quad \beta_{n+1} \geq 2\,\beta_n,$$

and we let $\{X_n, Y_n, U_n, V_n\}_{n\geq 1}$ be independent $N(0,1)$ random variables.

The process which we construct will have the form

$$Z(t) = X(t)+Y(t),$$

where

$$X(t) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n \sqrt{n}} [X_n \text{Cos}\, \alpha_n t - Y_n \text{Sin}\, \alpha_n t],$$

and

$$Y(t) = \sum_{n=1}^{\infty} c_n [U_n \text{Cos}\, \beta_n t - V_n \text{Sin}\, \beta_n t].$$

The sequence $\{c_n\}$ will be required to satisfy

(10.2) $$\Sigma|c_n| = \infty \quad \text{and} \quad \Sigma|c_n|^2 < \infty.$$

Other relations between $\{c_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ will arise in the presentation.

We make several observations.

(i) Since $\Sigma|c_n|^2 < \infty$ and $\Sigma\ 1/(n\alpha_n^2) < \infty$ both processes $X(t)$ and $Y(t)$ are well defined periodic stationary Gaussian processes.

(ii) A lacumary trigonometric series is bounded if and only if its coefficients are summable. Since

$$\Sigma|c_n|\{|U_n|+|V_n|\} = \infty$$

with probability 1, the process $\{Y(t)\}$ will be unbounded on each interval of positive length.

(iii) A function with a lacunary Fourier series with frequencies $\{\alpha_n\}$ and coefficients $\{f_n\}$ and $\{e_n\}$ will be differentiable on a set of positive measure if and only if both $\Sigma a_n^2|e_n|^2 < \infty$ and $\Sigma\alpha_n^2|f_n|^2 < \infty$. Since

$$P\{\Sigma(1/n)(|X_n|^2+|Y_n|^2) = +\infty\} = 1,$$

the functions $X(t)$ is almost surely non-differentiable almost everywhere.

(iv) A function with lacunary Fourier with frequencies α_n and coefficients e_n and f_n is in λ^* if and only if

$$\lim_{n\to\infty} \alpha_n(|e_n|+|f_n|) = 0.$$

Since $P\{\lim_{n\to\infty}(|X_n|+|Y_n|)/\sqrt{n}) = 0\} = 1$, the process X(t) is almost surely in λ^*.

For more detail on lacunary series, see [15], sections V.6, 7, and 8.

From (iii) and (iv), we see that for any C^1 periodic f the function f+X is a non-differentiable λ^* function, which by Theorem 7.1 of [2] has a singular occupation measure μ_{f+X}. Our idea is to separate the two sets of frequencies $\{\alpha_n\}$ and $\{\beta_n\}$ sufficiently far that from the view of the function X(t), the function Y(t) looks like a C^1 function.

More precisely, we let

$$(10.3) \qquad X_N(t) = \sum_{n=1}^{N} \frac{1}{\alpha_n \sqrt{n}} [X_n \operatorname{Cos} \alpha_n t - Y_n \operatorname{Sin} \alpha_n t]$$

$$(10.4) \qquad Y_N(t) = \sum_{n=1}^{N} c_n [U_n \operatorname{Cos} \beta_n t - V_n \operatorname{Sin} \beta_n t].$$

We will show that the α_n, β_n, and c_n may be chosen so that there exist other sequences $M_n \uparrow +\infty$, $m_n \uparrow +\infty$, and $\delta_n \downarrow 0$ such that for each n

$$(10.5) \quad P\{|\{t \in [0,1]: |X(t)-X_{M_n}(t)| > \tfrac{1}{2}\delta_n\}| < 2^{-n}\} > 1-2^{-n},$$

$$(10.6) \quad P\{|\{t \in [0,1]: |Y(t)-Y_{m_n}(t)| > \tfrac{1}{2}\delta_n\}| < 2^{-n}\} > 1-2^{-n},$$

and

$$(10.7) \quad P\{G(n)\} > 1-2^{-n},$$

where G(n) is the event that there exists a compact subset $K \subseteq [0,1]$ such that $|K| > 1-2^{-n}$ and the δ_n neighborhood of the image set $(X_{M_n}+Y_{m_n})(K)$ has total length less than 2^{-n}. Together these three conditions imply that with probability at least $1-3\cdot 2^{-n}$ there exists a measurable set $L \subseteq [0,1]$ of length at least $1-3\cdot 2^{-n}$ and such that the length $|(X+Y)(L)|$ is less than 2^{-n}.

We proceed inductively. Set $m_1 = 0$ and $Y_{m_1} \equiv 0$. Let $\alpha_n^1 = 2^n$. Replace the α_n in (10.3) with α_n^1 and call the resulting partial sum $X_N^1(t)$. Because $X^1(t) = \lim_{N\to\infty} X_N^1(t)$ has a singular occupation we can find a large M_1 and a small $\delta_1 < \frac{1}{2}$ such that the event $G(1)$ satisfies $P(G(1)) > \frac{1}{2}$.

Now we let

$$(10.7) \qquad Y_m^1(t) = \sum_{n=M_1+1}^{m} \frac{1}{m-M_1}[U_n \mathrm{Cos}\ 2^n t - V_n \mathrm{Sin}\ 2^n t].$$

Observe that $EY_m^1(t) = 0$ and $E(Y_m^1(t))^2 = \frac{1}{m-M_1} \to 0$ as $m \to 0$. Thus we can choose $m = m_2$ so large that

$$P\{|t \in [0,1]: |Y_{m_2}^1(t)| \geq \frac{\delta_1}{2}\}| < \frac{1}{4}\} > \frac{3}{4}.$$

Next we choose an integer $O_2 \geq m_2$ so that

$$P\{\sum_{n\geq O_2} 2|X_n|/2^n < \delta_1/2\} > \frac{3}{4}.$$

Now for $n = 1,\ldots,m_2-M_1$, set $\beta_n = 2^{n+M_1}$. Define the sequence $\alpha_n^2 = 2^n$ for $1 \leq n \leq m_1$ and $\alpha_{n+M_1}^2 = 2^{n+O_2}$. The process

$X_N^2(t)$ is defined by replacing the sequence $\{\alpha_n^1\}$ with $\{\alpha_n^2\}$ in the definition of $X_N^1(t)$. Again we can find a large $M_2 > O_2$ and a $\delta_2 > 0$ with $\delta_2 < \frac{1}{2}\delta_1$ for which $P(G(2)) > \frac{3}{4}$.

We define $Y_m^2(t)$ by replacing the M_1 in (10.7) with M_2. Choose an $m_3 > M_2$ with

$$P\{|\{t \in [0,1]: |Y_{m_3}^2(t)| \geq \delta_2/2\}| < \tfrac{1}{8}\} > \tfrac{7}{8},$$

and choose $O_3 \geq m_3$ so that

$$P\{\sum_{n \geq O_3} 2|X_n|/2^n < \delta_2/2\} > \tfrac{7}{8}.$$

On the interval $m_2-M_1 < n \leq (m_2-M_1)+(m_3-M_2)$ the sequence β_n is given by $\beta_{m_2-M_1+n} = 2^{M_2+n}$, and we define a new sequence α_n^3 by $\alpha_n^3 = \alpha_n^2$ for $n \leq m_1+(M_2-O_2)$ and $\alpha^3_{n+M_1+(M_2-O_2)} = 2^{n+O_3}$. The pattern should be clear now.

The sequence β_n is defined inductively on larger and larger intervals. The sequence α_n is given by $\alpha_n = \lim_{k\to\infty} \alpha_n^k$. The coefficients c_n equal $\frac{1}{m_2-M_1}$ for $1 \leq n \leq m_2-M_1$, and equal $\frac{1}{m_3-M_2}$ for $m_2-M_1 < n \leq (m_2-M_1)+(m_3-M_2)$, etc. That (10.5), (10.6), and (10.7) are satisfied follows from elementary calculations.

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