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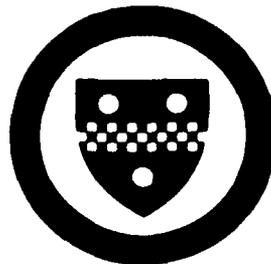
A GENERAL EXISTENCE AND UNIQUENESS

THEOREM FOR IMPLICIT DIFFERENTIAL-ALGEBRAIC EQUATIONS¹⁾

by

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* This work was supported in part by ONR-grant N-00014-90-J-1025 and NSF-grant CCR-8907654.

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A General Existence and Uniqueness
Theory for Implicit Differential-Algebraic Equations ¹⁾

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Abstract: This paper presents a general existence and uniqueness theory for differential-algebraic equations extending the well-known ODE theory. Both local and global aspects are considered, and the definition of the index for nonlinear problems is elucidated. For the case of linear problems with constant coefficients the results are shown to provide an alternate treatment equivalent to the standard approach in terms of matrix pencils. Also, it is proved that general differential-algebraic equations carry a geometric content, in that they are locally equivalent to ODEs on a "constraint" manifold. A simple example from particle dynamics is given to illustrate our approach.

1. Introduction

Differential-algebraic equations (DAEs) are frequently identified as implicit equations

$$F(t, x, x') = 0 \tag{1.1}$$

for which the derivative x' cannot be expressed explicitly as a function of t and x (see e.g. [1]). In particular, if $x \in \mathbb{R}^n$ and F maps into \mathbb{R}^n , this includes the case when the partial derivative $D_p F(t, x, p)$ of F with respect to its third variable p is not surjective. More specifically, in the setting of DAEs it is natural to require the stronger hypothesis that $D_p F(t, x, p)$ has constant rank on the domain under consideration. Indeed, the prototype for such equations is given by

$$F(t, x, p) = \begin{bmatrix} F_1(t, x) \\ F_2(t, x, p) \end{bmatrix} = 0 \tag{1.2}$$

where F_1 and F_2 map into \mathbb{R}^{n-r} and \mathbb{R}^r , respectively, and $D_p F_2(t, x, p)$ has full rank, so that, indeed, $D_p F(t, x, p)$ has constant rank $r < n$.

Many DAE-problems of practical interest do not exhibit such a convenient splitting between algebraic and differential parts as in (1.2). Moreover, even if the equations can be written in the separated form (1.2), the rank of $D_p F_2(t, x, p)$ may turn out to be less than r so that $F_2(t, x, x') = 0$ is an equation containing an implicit

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algebraic part. While these comments suggest the need for a thorough investigation of DAEs in the broad setting of (1.1), existence and uniqueness theories for these equations have not been developed in such generality and available results place more restrictive conditions on the form of F (see e.g. [4], [7],[8]).

In this paper we present existence and uniqueness results for general problems of the form (1.1) under some "generic" conditions which do not assume that F has a special form. Briefly, our technique consists in deriving a necessary condition for the existence of local solutions by differentiating the equation once followed by an application of orthogonal projections onto the range of $D_p F$ to produce a system that again contains only x and x' . For this decomposed system a sufficient condition is then obtained that guarantees the equations to define a local vector field to which the standard existence theory applies. The effectiveness of this conceptually -- but not always technically -- simple approach for providing an answer to this notoriously complex problem may be called surprising.

The sufficient condition essentially requires that the index (see e.g.[1]) of the DAE is one. But it also turns out that the theory can be applied recursively to equations of higher index provided that the resulting equations again satisfy the constant rank condition mentioned above. This is a requirement of a global nature not covered by the standard index theories. In fact, it also suggests that the constant rank condition is inherent to the definition of the index.

In Section 2 below we present the local existence and uniqueness theory sketched above. Then in Section 3 these local solutions are extended under conditions which correspond to those of the standard ODE theory. In Section 4 it is proved that when DF has full rank then the set of admissible initial points forms an r -dimensional submanifold of $\mathbf{R}^n \times \mathbf{R}^n$ and the DAE is locally equivalent to a differential equation on an r -dimensional submanifold of \mathbf{R}^n . This also shows that the geometric approach developed in [7] and [8] is conceptually valid in general. Then in Section 5 we apply our results to linear equations with constant coefficients and prove that the recursive application of the technique leads, exactly to the standard index for such linear DAEs. Finally, Section 6 concerns the generalization of this recursive application of the results to the general nonlinear case which, as mentioned before, requires the additional global assumption that the constant-rank condition remains valid. This is illustrated on the classical example of the nonlinear pendulum.

2. Local Existence and Uniqueness

For ease of notation, we shall consider (1.1) first in the autonomous form

$$F(x, x') = 0 \tag{2.1}$$

where it is assumed that

$$F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is of class } C^2 \text{ on the open set } E: \subset \mathbb{R}^n \times \mathbb{R}^n \tag{2.2a}$$

and

$$\text{rank } D_p F(x, p) = r < n, \text{ for all } (x, p) \in E. \tag{2.2b}$$

Our results will show that the differentiability assumption (2.2a) -- instead of the expected and apparently more natural, minimal C^1 -regularity of F -- turns out to be important for the theory. We note also that the character of the problem changes significantly when, instead of (2.2b), $D_p F(x, p)$ is only required to be singular on some lower dimensional sub-manifold of $\mathbb{R}^n \times \mathbb{R}^n$ (see [5]). Finally, we observe that, while the rank condition (2.2b) may suggest a transformation of the problem to some canonical form by means of a version of the "rank theorem" (see [3]), it must be noted that such a reduction mixes the variables x and p . Since in (2.1) the derivative x' occupies the position of the p -variable, such a reduction is not readily usable to transform (2.1) to an explicit ordinary differential equation.

A C^2 -solution of (2.1) shall be any function

$$x: J \rightarrow \mathbb{R}^n, (x(t), x'(t)) \in E, \text{ for } t \in J, \tag{2.3}$$

which is of class C^2 on some open interval $J \subset \mathbb{R}^n$ and satisfies $F(x(t), x'(t)) = 0$ for all $t \in J$. For any C^2 -solution (2.3) of (2.1) we obtain by differentiation

$$D_x F(x(t), x'(t))x'(t) + D_p F(x(t), x'(t))x''(t) = 0, t \in J, \tag{2.4}$$

which provides the following necessary condition:

Lemma 2.1: If (2.2a) holds for (2.1) then for a given point $(x, p) \in E$ the two conditions

$$F(x, p) = 0 \tag{2.5a}$$

$$D_x F(x, p)p \in \text{rge } D_p F(x, p) \tag{2.5b}$$

are necessary for the existence of a C^2 -solution (2.3) of (2.1) that passes through (x, p) .

The two relations (2.5a/b) define a subset of possible initial points (x, p) of E . For a closer analysis of the structure of this set we introduce the orthogonal projections

$$P: E \rightarrow L(\mathbf{R}^n, \mathbf{R}^n), \quad P(x, p) \mathbf{R}^n = \text{rge } D_p F(x, p), \quad (x, p) \in E \quad (2.6a)$$

$$Q: E \rightarrow L(\mathbf{R}^n, \mathbf{R}^n), \quad Q(x, p) = I_n - P(x, p), \quad (x, p) \in E. \quad (2.6b)$$

These projections are C^1 -functions of (x, p) . In order to see this, let $(x_0, p_0) \in E$ be given and choose an orthonormal basis $\{e_1, \dots, e_r\}$ of $\text{rge } D_p F(x_0, p_0)$. Then there exist $w_i \in \mathbf{R}^n$ such that $e_i = D_p F(x_0, p_0) w_i$ and obviously for (x, p) near (x_0, p_0) the mappings η_i , $\eta_i(x, p) = D_p F(x, p) w_i$, $i = 1, \dots, r$ are of class C^1 and $\{\eta_1(x, p), \dots, \eta_r(x, p)\}$ are linearly independent. Hence, since $\text{rank } D_p F(x, p) = r$, these vectors form a basis of $\text{rge } D_p F(x, p)$. By applying the Gram-Schmidt process we obtain now an orthonormal basis of the same space and, because the process involves only analytic operations, the vectors of this basis are again C^1 functions of (x, p) . But then the same holds for the projection $P(x, p)$ as the sum of dyadic products of the basis vectors whence also $Q = I_n - P$ is C^1 .

Evidently, the points $(x, p) \in E$ satisfying the necessary conditions (2.5a/b) are the common zeros of F and the C^1 -mapping

$$G: E \rightarrow \mathbf{R}^n, \quad G(x, p) = P(x, p)F(x, p) + Q(x, p)D_x F(x, p)p, \quad (x, p) \in E. \quad (2.7)$$

In other words,

$$E_N = \{(x, p) \in E; F(x, p) = 0, G(x, p) = 0\} \quad (2.8)$$

is the set of all points in E that satisfy the necessary conditions of Lemma 2.1. For later use, note that the set $(x(t), x'(t))$, $t \in J$ lies in E_N (and not only in $F^{-1}(0)$) for every C^2 -solution of (2.1).

A major step toward transforming (2.1) locally into an explicit ordinary differential equation will be provided by the following result about the relationship between the solutions of (2.1) and those of the equation

$$G(x, x') = 0. \quad (2.9)$$

Below and in other instances, we shall use the remark that $G(x, p) = 0$ amounts to the fact that *both* terms on the righthand side of (2.7) vanish.

Lemma 2.2: Any C^2 -solution (2.3) of (2.1) solves the equation (2.9), and conversely, any C^2 -solution (2.3) of (2.9) for which $F(x(t_1), x'(t_1)) = 0$ for some $t_1 \in J$ is a C^2 -solution of (2.1).

Proof: By Lemma 2.1 any C^2 -solution (2.3) of (2.1) satisfies $F(x(t), x'(t)) = 0$ and (2.4) for all $t \in J$ whence, because of $QD_p F \equiv 0$, we see that $Q(x, x')D_x F(x, x')x' = 0$ on all of J . This shows that x is a C^2 -solution of (2.9).

Conversely, suppose that (2.3) is a C^2 -solution of (2.9). Then, on J the identity

$$\begin{aligned} \frac{d}{dt} F(x, x') &= [P(x, x') + Q(x, x')][D_x F(x, x')x' + D_p F(x, x')x''] \\ &= P(x, x')[D_x F(x, x')x' + D_p F(x, x')x''] = P(x, x') \frac{d}{dt} F(x, x'), \end{aligned}$$

holds where we used that $P + Q = I_n$, $QD_p F \equiv 0$, and $Q(x, x')D_x F(x, x')x' = 0$. By differentiation of the identity $P(x, x')F(x, x') = 0$ (see (2.7) and (2.9)) it follows that

$$P(x, x') \frac{d}{dt} F(x, x') = - \left[\frac{d}{dt} P(x, x') \right] F(x, x').$$

Hence the function $\xi: J \rightarrow \mathbb{R}^n$, $\xi(t) = F(x(t), x'(t))$

satisfies the linear system $\frac{d}{dt} \xi = A(t)\xi$, where $A: J \rightarrow L(\mathbb{R}^n)$, $A(t) = - \left[\frac{d}{dt} P(x, x') \right](t)$ is a continuous

function since the projection P is C^1 on E and the solution x is C^2 on J . Hence the standard uniqueness theory for linear systems²⁾ together with the condition $F(x(t_1), x'(t_1)) = 0$ implies that $F(x(t), x'(t)) = 0$ for t in a neighborhood of t_1 in J . Because of the connectedness of J it follows readily that this local result holds for all $t \in J$. Thus x is a solution of (2.1) on J as claimed.

In order for the equation (2.9) to induce a unique vector field on some neighborhood of any given point $(x_0, p_0) \in E_N$, we need to guarantee that for each x near x_0 there exists only one vector p near p_0 for which $G(x, p) = 0$. Obviously, a sufficient condition for this will be that $D_p G(x_0, p_0)$ is an isomorphism on \mathbb{R}^n . Accordingly, we define the set of admissible initial points of (1.1) in E as

$$E_A = \{ (x, p) \in E_N; D_p G(x, p) \in \text{Isom}(\mathbb{R}^n) \}. \tag{2.10}$$

2) We emphasize that continuity of A with respect to t is sufficient to guarantee uniqueness, as a straightforward verification confirms.

Clearly, by the continuity of $D_p G$, the set E_A is (relatively) open in E_N . Moreover, note that when $(x_0, p_0) \in E_N$ then, by (2.5b), there exists a $q_0 \in \mathbb{R}^n$ such that

$$D_x F(x_0, p_0)p_0 + D_p F(x_0, p_0)q_0 = 0. \quad (2.11)$$

The following lemma provides a characterization of E_A in terms of F and its derivatives. For a somewhat related condition in the quasilinear case see also [2].

Lemma 2.3: For any $(x_0, p_0) \in E_N$ we have $D_p G(x_0, p_0) \in \text{Isom}(\mathbb{R}^n)$ if and only if for some q_0 which satisfies (2.11) the following condition holds

$$u \in \ker D_p F(x_0, p_0) \text{ and } D_{x^2}^2 F(x_0, p_0)(p_0, u) + D_{p^2}^2 F(x_0, p_0)(q_0, u) + D_x F(x_0, p_0)u \in \text{rge } D_p F(x_0, p_0) \quad 3) \\ \text{together imply that } u = 0. \quad (2.12)$$

This equivalence does not depend on the particular choice of q_0 satisfying (2.11).

Proof: Evidently, for any $u \in \mathbb{R}^n$, we have

$$D_p G(x, p)u = K_1(x, p, u) + K_2(x, p, u)$$

where

$$K_1(x, p, u) \equiv D_p [P(x, p)F(x, p)]u = [D_p P(x, p)u]F(x, p) \\ + P(x, p)D_p F(x, p)u = [D_p P(x, p)u]F(x, p) + D_p F(x, p)u \quad (2.13)$$

and

$$K_2(x, p, u) \equiv D_p [Q(x, p)D_x F(x, p)p]u. \quad (2.14)$$

The condition $(x_0, p_0) \in E_N$ implies that $F(x_0, p_0) = 0$ and hence by (2.13) that

$$K_1(x_0, p_0, u) = D_p F(x_0, p_0)u. \quad (2.15)$$

For the evaluation of K_2 note that for any fixed $q \in \mathbb{R}^n$ and by definition of Q it follows that

$$Q(x, p)D_x F(x, p)p = Q(x, p)[D_x F(x, p)p + D_p F(x, p)q],$$

and therefore that

3) Here $D_{x^2}^2 F(x_0, p_0)(p_0, u)$ means $D^2 F(x_0, p_0)((p_0, 0), (0, u))$ and hence p_0 and u do not play symmetric roles.

$$K_2(x, p, u) = (D_p Q(x, p)u)(D_x F(x, p)p + D_p F(x, p)q) \\ + Q(x, p)[D_{x,p}^2 F(x, p)(p, u) + D_x F(x, p)u + D_{p,p}^2 F(x, p)(q, u)].$$

Now with any point $q_0 \in \mathbb{R}^n$ for which (2.11) holds we find that

$$K_2(x_0, p_0, u) = Q(x_0, p_0)[D_{x,p}^2 F(x_0, p_0)(p_0, u) + D_{p,p}^2 F(x_0, p_0)(q_0, u) + D_x F(x_0, p_0)u]. \quad (2.16)$$

Together, (2.15) and (2.16) show that $K_1(x_0, p_0, u)$ and $K_2(x_0, p_0, u)$ are the components of the vector $D_p G(x_0, p_0)u$ along $\text{rge } D_p F(x_0, p_0)$ and its orthogonal complement, respectively. This implies that $D_p G(x_0, p_0)u = 0$ if and only if

$$K_1(x_0, p_0, u) = 0 \text{ and } K_2(x_0, p_0, u) = 0. \quad (2.17)$$

Thus $D_p G(x_0, p_0)$ is an isomorphism exactly if (2.17) holds only for $u = 0$ which by (2.15) and (2.16) is equivalent with (2.12). The last part of the lemma now is a direct consequence of the fact that the invertibility of $D_p G(x_0, p_0)$ is independent of the choice of q_0 .

As an immediate corollary of Lemma 2.3 we obtain from (2.10) and the implicit function theorem the following result:

Lemma 2.4: For any $(x_0, p_0) \in E_A$ there exists an open neighborhood $U_0 \times V_0 \subset E$ and a unique C^1 -mapping $\Phi : U_0 \rightarrow V_0$, $\Phi(x_0) = p_0$, such that $(x, p) \in U_0 \times V_0$ and $G(x, p) = 0$ if and only if $p = \Phi(x)$.

Lemma 2.4 shows that for any initial point $(x_0, p_0) \in E_A$ there exists an open neighborhood $E_0 = U_0 \times V_0$ in E where the system (2.9) can be written in the explicit form $x' = \Phi(x)$. Hence the standard theory for initial value problems ensures that, modulo translations in time, this explicit system has a unique solution x in U_0 through any given point of that set, and clearly this solution is of class C^2 since Φ is C^1 . It follows that for any given point of E_0 the system (2.9) has a unique C^2 solution x such that (x, x') passes through that point of E_0 . Now Lemma 2.2 asserts that such a solution is a solution of (2.1) if and only if (x, x') passes through some point of E_N . Thus we conclude that (2.1) has a unique C^2 solution for which (x, x') passes through any given point of $E_N \cap E_0$. As noted earlier, E_A is open in E_N and, clearly, for sufficiently small E_0 we have $E_N \cap E_0 = E_A \cap E_0$. Thus, in particular, the result applies to the given point $(x_0, p_0) \in E_A$.

We summarize this conclusion in the following form:

Theorem 2.1: Suppose that for the problem (2.1) the conditions (2.2a/b) are valid and choose any $t_0 \in \mathbf{R}$. Then, $(x_0, p_0) \in E_N$ is necessary for the existence of a C^2 -solution of the initial value problem

$$F(x, x') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = p_0. \quad (2.19)$$

Moreover, if $(x_0, p_0) \in E_A$ then there exists a unique C^2 -solution of (2.19).

Note that the theorem does remain valid when $r = n$ in (2.2b). In that case we have $\text{rge } D_p F(x, p) = \mathbf{R}^n$ for all $(x, p) \in E$ and hence the necessary condition $(x_0, p_0) \in E_N$ simply is $F(x_0, p_0) = 0$ while the sufficient condition $(x_0, p_0) \in E_A$ reduces to $F(x_0, p_0) = 0$ and $\ker D_p F(x_0, p_0) = \{0\}$. Thus we recover here the usual situation when the implicit function theorem provides that (2.1) can be written locally as an explicit ODE. In this case, it is of course sufficient that F be of class C^1 on E since the second derivatives of F are no longer involved in the definition of E_A .

Similarly, the extreme case $r = 0$ in (2.2b) is trivial since then $D_p F(x, p) = 0$ for $(x, p) \in E$; that is, F is independent of p . Here we have $(x_0, p_0) \in E_N$ if and only if $F(x_0) = 0$ and $D_x F(x_0)p_0 = 0$ while $(x_0, p_0) \in E_A$ under the additional requirement that $D_x F$ is at x_0 an isomorphism of \mathbf{R}^n to itself. But then the equation $F = 0$ has x_0 as isolated solution and the unique solution of (2.1a/b) is $x(t) = x_0$. This is consistent with the remark that if $D_x F(x_0)$ is an isomorphism, then $D_x F(x_0)p_0 = 0$ only if $p_0 = 0$.

We end this section by considering the conditions $(x_0, p_0) \in E_N$ and $(x_0, p_0) \in E_A$ of Theorem 2.1 for the general nonautonomous case (1.1); that is,

$$F(t, x, x') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = p_0, \quad F(t_0, x_0, p_0) = 0. \quad (2.20)$$

We use the standard approach to make this problem autonomous and hence introduce the mapping

$$H: (\mathbf{R} \times \mathbf{R}^n) \times (\mathbf{R} \times \mathbf{R}^n) \rightarrow \mathbf{R}^{n+1}, \quad H((t, x), (\tau, p)) = \begin{bmatrix} \tau - 1 \\ F(t, x, p) \end{bmatrix} \quad (2.21)$$

and the corresponding initial point $((t_0, x_0), (\tau_0, p_0))$, $\tau_0 = 1$. Then, under the required smoothness assumptions the condition (2.5b) for H , written in terms of F , assumes the form

$$D_t F^0 + D_x F^0 p_0 \in \text{rge } D_p F^0 \quad (2.22)$$

while the condition (2.12) becomes

$$\begin{aligned}
 & \text{for } q_0 \in \mathbf{R}^n \text{ such that } D_t F^0 + D_x F^0 p_0 + D_p F^0 q_0 = 0, \\
 & u \in \ker D_p F^0 \text{ and} \\
 & D_{t,p}^2 F^0 u + D_{x,p}^2 F^0(p_0, u) + D_{p,p}^2 F^0(q_0, u) + D_x F^0 u \in \text{rge } D_p F^0
 \end{aligned} \tag{2.23}$$

together imply that $u = 0$.

In (2.22) and (2.23) the superscript 0 indicates that the particular function is to be evaluated at (t_0, x_0, p_0) . Note also that in (2.22) and (2.23) we explicitly used $\tau_0 = 1$ and that the condition (2.23) is independent of the specific choice of q_0 .

3. Global Behavior of the Solutions

In order to determine the global behavior of the local solutions of the initial value problem (2.19) suppose again that the conditions (2.2a/b) hold. Then Theorem 2.1 guarantees the existence of a unique C^2 -solution (2.3) x for which (x, x') passes through any given point of E_A . For any such solution and any $\gamma \in J$ we introduce the sets $T_{\gamma+} = \{p \in \mathbf{R}^n; p = x'(t), \gamma \leq t < b\}$ and $T_{\gamma-} = \{p \in \mathbf{R}^n; p = x'(t), a < t \leq \gamma\}$. Then the following extendability result holds:

Theorem 3.1: Assume that for the problem (2.1) the conditions (2.2a/b) hold and that $E_A = E_N$. Then the following statements are valid:

- (i) If the set $T_{\gamma+}$ [or $T_{\gamma-}$] is bounded for some $\gamma \in J$, then $\lim_{t \rightarrow b-} x(t) = x_b$ [or $\lim_{t \rightarrow a+} x(t) = x_a$] exists.
- (ii) If $\lim_{t \rightarrow b-} x(t) = x_b$ [or $\lim_{t \rightarrow a+} x(t) = x_a$] exists and for some sequence $\{t_k\} \in J$ with $\lim_{k \rightarrow \infty} t_k = b$ [or $\lim_{k \rightarrow \infty} t_k = a$] the sequence $\{x'(t_k)\}$ has an accumulation point p^* for which $(x_b, p^*) \in E$ [or $(x_a, p^*) \in E$] then $\lim_{t \rightarrow b-} x'(t) = p^*$ [or $\lim_{t \rightarrow a+} x'(t) = p^*$] and for $b < \infty$ [or $a > -\infty$] the solution x can be continued to the right of b [or to the left of a]:

Proof: We present the proof only for the right endpoint b , for the other one it proceeds analogously. Suppose first that $T_{\gamma+}$ is bounded for some $\gamma \in J$ and hence that $\|x'(t)\| \leq M < \infty$ for $t \in J_\gamma = [\gamma, b)$. Since

$$\|x(t) - x(s)\| \leq \int_s^t \|x'(\tau)\| d\tau \leq M |t-s|, \text{ for all } s, t \in J_\gamma$$

we see that for any sequence $\{t_k\} \subset J_\gamma$ with $\lim_{k \rightarrow \infty} t_k = b$ the sequence $\{x(t_k)\}$ is a Cauchy sequence, and moreover that its limit is independent of the particular choice of $\{t_k\}$. This proves the existence of the limit point x_b .

Now assume that $\lim_{x \rightarrow b} x(t) = x_b$ exists and that for some sequence $\{t_k\} \in J$ with $\lim_{k \rightarrow \infty} t_k = b$ the sequence $\{x'(t_k)\}$ has an accumulation point p^* for which $(x_b, p^*) \in E$. Then there is a sub-sequence $\{s_k\} \subset \{t_k\}$ for which $\lim_{k \rightarrow \infty} s_k = b$ as well as $\lim_{k \rightarrow \infty} x'(s_k) = p^*$, and, because of $F(x(s_k), x'(s_k)) = 0$ and $(x_b, p^*) \in E$, we also have $F(x_b, p^*) = 0$. As in the previous section, for all (x, p) near (x_b, p^*) let $Q(x, p)$ be the orthogonal projection onto $[rge D_p F(x, p)]^\perp$. Recall that Q is continuous (even C^1). Hence, from $D_x F(x(s_k), x'(s_k))x'(s_k) \in rge D_p F(x(s_k), x'(s_k))$ (see Lemma 2.1) it follows that $Q(x(s_k), x'(s_k))D_x F(x(s_k), x'(s_k))x'(s_k) = 0$ for all k and therefore, in the limit, that $Q(x_b, p^*)D_x F(x_b, p^*)p^* = 0$. In other words, p^* is a solution of the system

$$F(x_b, p) = 0, D_x F(x_b, p)p \in rge D_p F(x_b, p). \quad (3.2)$$

Since $(x_b, p^*) \in E$ it follows that $(x_b, p^*) \in E_N = E_A$ and thus, by Lemma 2.3, that the solution $p = p^*$ of (3.2) is isolated. In other words, there exists a ball $B_\delta \subset \mathbb{R}^n$ centered at p^* with radius $\delta > 0$ such that $(x_b, p) \in E$, for all $p \in B_\delta$, and that B_δ contains no solution p of (3.2) other than p^* .

Consider now any sequence $\{\tau_k\} \in J$ with $\lim_{k \rightarrow \infty} \tau_k = b$. We show first that $x'(\tau_k)$ must have an accumulation point. Suppose that this is not true, so that $\lim_{k \rightarrow \infty} \|x'(\tau_k)\| = \infty$. Because of $\lim_{k \rightarrow \infty} x'(s_k) = p^*$ it therefore follows that for some sufficiently large k_0

$$\|x'(\tau_k) - x'(s_k)\| \geq \delta/2, \text{ for } k \geq k_0. \quad (3.3)$$

By continuity and the intermediate value theorem there exists in each interval $[\inf(\tau_k, s_k), \sup(\tau_k, s_k)]$ a value ρ_k such that $\|x'(\rho_k) - x'(s_k)\| = \delta/2$. Then, $x'(\rho_k)$ is bounded and upon extracting a subsequence, we obtain an accumulation point p^ρ of $x'(\rho_k)$, which, of course, satisfies $\|p^\rho - p^*\| = \delta/2$. But this means that $p^\rho \in B_\delta$ and $p^\rho \neq p^*$ which is a contradiction because $(x_b, p^\rho) \in E$, $\lim_{k \rightarrow \infty} \rho_k = b$ and hence p^ρ is a solution of (3.2). Therefore $x'(\tau_k)$ must have an accumulation point. Suppose there exists such an accumulation point p^{**} which is distinct from p^* . If $(x_b, p^{**}) \in E$ then, as before, p^{**} must solve (3.2) and hence cannot be in B_δ . On the other hand, if (x_b, p^{**}) does not belong to E then necessarily, p^{**} cannot be in B_δ either. There is some sub-

sequence, to be denoted again by $\{\tau_k\}$, which converges to b and for which $\lim_{k \rightarrow \infty} x'(\tau_k) = p^{**}$. As p^{**} is not in B_δ , the relation (3.3) must be valid for sufficiently large k_0 which, of course, leads once more to a contradiction. Thus, altogether, we have found that p^* is the unique accumulation point of the sequence $\{x'(\tau_k)\}$ for any sequence $\{\tau_k\} \in J$ such that $\lim_{k \rightarrow \infty} \tau_k = b$. This proves that $\lim_{t \rightarrow b-} x'(t) = p^*$.

Since p^* solves (3.2) we have $(x_b, p^*) \in E_N$ and, by hypothesis, $E_N = E_A$. Hence, for $b < \infty$ we can apply Theorem 2.1 at $(x_0, p_0) = (x_b, p^*)$ with $t_0 = b$. Thus there exists some neighborhood $E_0 \subset E$ of (x_b, p^*) and some open interval I containing b where the C^2 -solution $y: I \rightarrow \mathbb{R}^n$ $y(b) = x_b$, $y'(b) = p^*$, of (2.1a) is the only solution of an explicit system $y' = \Phi(y)$, $x(b) = x_b$, with some C^1 -function Φ on E_0 for which $\Phi(x_b) = p^*$. By the standard uniqueness theorem it follows that any "one-sided" C^1 -solution $\xi: (b - \varepsilon, b] \rightarrow \mathbb{R}^n$ with $(\xi(t), \xi'(t)) \in E_0$, $b - \varepsilon < t \leq b$, of this explicit problem necessarily has to agree with the unique C^2 -solution y of the problem on their common interval of definition. But because of $\lim_{t \rightarrow b-} x(t) = x_b$, and $\lim_{t \rightarrow b-} x'(t) = p^*$ our given solution x is such a one-sided C^1 -solution and hence agrees with y on their common domain. This shows that the original solution indeed can be continued beyond the right endpoint b of J and the proof is complete.

The result implies that any local C^2 -solution of (2.1a) can be extended to some open interval $J = (a^*, b^*)$, $b^* \leq \infty$, $a^* \geq -\infty$, which is maximal under set inclusion.

Now consider Theorem 3.1 in the case $E = \mathbb{R}^n \times \mathbb{R}^n$. If $\lim_{x \rightarrow b-} x(t) = x_b$ exists, then any accumulation point p of $x(t)$ as $t \rightarrow b-$ in \mathbb{R}^n necessarily satisfies $(x_b, p) \in E$ and we have $\lim_{t \rightarrow b-} x'(t) = p$. Clearly, if T_{γ^*} is bounded then there must be such an accumulation point; in other words, for bounded T_{γ^*} both limits $\lim_{x \rightarrow b-} x(t)$ and $\lim_{x \rightarrow b-} x'(t)$ always exist and the solution can be extended. Suppose now that $b = b^* < \infty$ which implies that the solution cannot be extended. Then T_{γ^*} must be unbounded. Moreover, if $\lim_{x \rightarrow b-} x(t) = x_b$ exists then $x'(t)$ cannot have an accumulation point as $t \rightarrow b-$ which means that $\lim_{x \rightarrow b-} \|x'(t)\| = \infty$. The analogous result holds at a^* and altogether we have the following corollary of Theorem 3.1.

Theorem 3.2: Assume that the mapping F of (2.1a) satisfies the conditions (2.2a/b) and (3.1) on $E = \mathbb{R}^n \times \mathbb{R}^n$ and let $x: J \rightarrow \mathbb{R}^n$, $(x(t), x'(t)) \in E$ for $t \in J$, be any C^2 -solution of (2.1a) where $J = (a^*, b^*)$ is a maximal interval. If $b^* < \infty$ then the set T_{γ^*} is unbounded for some, and hence all, $\gamma \in J$, and, in particular, if

$\lim_{t \rightarrow b^+} x(t)$ exists then $\lim_{t \rightarrow b^+} \|x'(t)\| = \infty$. Correspondingly, for $a^* > -\infty$ the set T_{γ} is unbounded and, in particular, if $\lim_{t \rightarrow a^*} x(t)$ exists then $\lim_{t \rightarrow a^*} \|x'(t)\| = \infty$.

4. The Structure of the Set of Admissible Initial Points

In this section we analyze the structure of the set E_A of admissible initial points of (2.1) in E as well as that of its projection onto the x -space. For this, we assume that, in addition to the properties (2.2a/b) the mapping F satisfies

$$\text{rank } DF(x, p) = n, \text{ for all } (x, p) \in E; \quad (4.1)$$

that is, that the equations (2.1) are independent.

We prove first the following preliminary lemma:

Lemma 4.1: Suppose that F satisfies the conditions (2.2a/b) and (4.1). Then the mapping $QF: E \rightarrow \mathbb{R}^n$ has at any point $(x, p) \in F^{-1}(0)$ the partial derivatives

$$D_x(QF)(x, p) = Q(x, p)D_x F(x, p), \quad D_p(QF)(x, p) = 0. \quad (4.2)$$

Moreover, the linear map $Q(x, p)D_x F(x, p) \in L(\mathbb{R}^n)$ has constant rank $n-r$ on all of $F^{-1}(0)$.

Proof: Let $(x, p) \in F^{-1}(0)$. For any $u \in \mathbb{R}^n$ we have

$$D_x[Q(x, p)F(x, p)]u = [D_x Q(x, p)u]F(x, p) + Q(x, p)D_x F(x, p)u$$

which implies the first part of (4.2) while, because of $QD_p F = 0$, the second part is a consequence of

$$D_p[Q(x, p)F(x, p)]u = [D_p Q(x, p)u]F(x, p) + Q(x, p)D_p F(x, p)u.$$

By (4.2) we see that $Q(x, p)D_x F(x, p) = Q(x, p)DF(x, p)$ and, because $DF(x, p)$ has full rank, that $\text{rank } Q(x, p)DF(x, p) = \text{rank } Q(x, p) = n-r$. This proves the second part of the assertion.

With this we obtain now the following result about the structure of E_A :

Theorem 4.1: Suppose that F satisfies the conditions (2.2a/b) and (4.1). Then the set $E_A \subset E$ of admissible initial points of (2.1) is an r -dimensional C^1 -sub-manifold of $\mathbb{R}^n \times \mathbb{R}^n$.

Proof: Let $(x_0, p_0) \in E_A$, so that $F(x_0, p_0) = G(x_0, p_0) = 0$ and $D_p G(x_0, p_0) \in \text{Isom}(\mathbb{R}^n)$. Since the condition $D_p G(x, p) \in \text{Isom}(\mathbb{R}^n)$ is fulfilled by all points $(x, p) \in E$ near (x_0, p_0) , the points of E_A in the vicinity of (x_0, p_0) are characterized by the sole conditions $F(x, p) = G(x, p) = 0$. Actually, this system may be replaced by

$$G^*(x, p) \equiv (Q(x_0, p_0)F(x, p), G(x, p)) = 0.$$

Indeed, it is obvious that $G^*(x, p) = 0$ whenever $F(x, p) = G(x, p) = 0$. Conversely, suppose that $G^*(x, p) = 0$, so that $Q(x_0, p_0)F(x, p) = 0, G(x, p) = 0$. The latter relation implies in particular that $P(x, p)F(x, p) = 0$, which, together with $Q(x_0, p_0)F(x, p) = 0$ yields $F(x, p) = 0$ because $\text{rge } Q(x, p) = [\text{rge } D_p F(x, p)]^\dagger$ and $\text{rge } P(x_0, p_0) = \text{rge } D_p F(x_0, p_0)$ remain complementary for (x, p) close enough to (x_0, p_0) (by constancy of rank $D_p F$ and continuity of P and Q on E ; see Section 2).

Identifying $\text{rge } Q(x_0, p_0) \approx \mathbb{R}^{n-r}$, we see that the mapping G^* maps a neighborhood of (x_0, p_0) in $\mathbb{R}^n \times \mathbb{R}^n$ into the fixed space $\mathbb{R}^{n-r} \times \mathbb{R}^n$, and with $F(x_0, p_0) = 0$ and $QD_p F = 0$, that

$$DG^*(x_0, p_0) = \begin{bmatrix} Q(x_0, p_0)D_x F(x_0, p_0) & 0 \\ D_x G(x_0, p_0) & D_p G(x_0, p_0) \end{bmatrix}. \quad (4.3)$$

Recalling that $D_p G(x_0, p_0) \in \text{Isom}(\mathbb{R}^n)$ and from Lemma 4.1 with $(x, p) = (x_0, p_0)$, we conclude that $D_p G^*(x_0, p_0)$ maps onto $\mathbb{R}^{n-r} \times \mathbb{R}^n$. The implicit function theorem now ensures that $(G^*)^{-1}(0)$, and hence E_A is a r -dimensional C^1 sub-manifold of $\mathbb{R}^n \times \mathbb{R}^n$ in the vicinity of (x_0, p_0) . This completes the proof since (x_0, p_0) was an arbitrary point of E_A .

Let $\Pi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the projection onto the first factor and let $(x_0, p_0) \in E_A$. It follows from the proof of Theorem 4.1 that the tangent space $T_{(x_0, p_0)} E_A$ may be identified with the null-space of the mapping $DG^*(x_0, p_0)$ in (4.3). Thus, for given $(u, q) \in T_{(x_0, p_0)} E_A$, one has $\Pi(u, q) = u$ and hence $\Pi(u, q) = 0$ if and only if $(u, q) = (0, 0)$. This means that the restriction of Π to the manifold E_A is an immersion at (x_0, p_0) . It follows that there is an open neighborhood N_0 of (x_0, p_0) in E_A such that $M_0 = \Pi(N_0)$ is an r -dimensional C^1 sub-manifold of \mathbb{R}^n and $\Pi : N_0 \rightarrow M_0$ is a C^1 -diffeomorphism. We are now in a position to prove that, locally, the DAE (2.1) is equivalent to an explicit ODE on the manifold M_0 .

Theorem 4.2: *There is a neighborhood $E_0 = U_0 \times V_0 \subset \mathbb{R}^n \times \mathbb{R}^n$ of $(x_0, p_0) \in E_A$ and a C^1 vector field*

$\Phi : U_0 \cap M_0 \rightarrow \mathbb{R}^n$, $\Phi(x_0) = p_0$, with $\Phi(x) \in T_x M_0$, for $x \in U_0 \cap M_0$, such that $x : J \rightarrow \mathbb{R}^n$ is a C^2 -solution of $F(x, x') = 0$ satisfying $(x(t), x'(t)) \in E_0$, for $t \in J$, if and only if $x(t) \in U_0 \cap M_0$, $x'(t) = \Phi(x(t))$, for all $t \in J$.

Proof: Lemma 2.4 has already established existence of a neighborhood $E_0 = U_0 \times V_0$ and C^1 -mapping $\Phi : U_0 \rightarrow \mathbb{R}^n$, $\Phi(x_0) = p_0$, such that $(x, p) \in E_0$ and $G(x, p) = 0$ if and only if $x \in U_0$ and $p = \Phi(x)$. Moreover, it was also shown that $x : J \rightarrow \mathbb{R}^n$ is a C^2 solution of (2.1) satisfying $(x(t), x'(t)) \in E_0$ if and only if $x(t) \in U_0$, $x'(t) = \Phi(x(t))$, for all $t \in J$, and (x, x') passes through one point of E_N (so that, in fact, (x, x') lies entirely in E_N).

Clearly, the neighborhood $E_0 = U_0 \times V_0$ can be shrunk to arbitrarily small size with no prejudice to the mentioned properties. One may then assume that $N_0 = E_0 \cap E_A = E_0 \cap E_N$ in the discussion preceding the theorem. In this case, $M_0 = \Pi(N_0) \subset U_0$ so that $U_0 \cap M_0 = M_0$.

Since the curve (x, x') lies in E_N whenever x is a C^2 -solution of $F(x, x') = 0$, it follows that $x(t) \in M_0$ as soon as $(x(t), x'(t)) \in E_0$. To prove the theorem, it suffices to show that

(a) $x \in M_0$ implies $(x, \Phi(x)) \in N_0$,

(b) $x \in M_0$ implies $\Phi(x) \in T_x M_0 \subset \mathbb{R}^n$.

If (b) is assumed to hold then, (a) is needed to show that for any (automatically C^2) solution $x : J \rightarrow M_0$ to $x' = \Phi(x)$ the curve (x, x') lies in $N_0 \subset E_N$ and hence is a C^2 -solution of (2.1). But (a) holds since every $x \in M_0$ has the form $x = \Pi(x, p)$ with $(x, p) \in N_0 \subset E_0$, whence $p = \Phi(x)$.

To prove (b), we shall use a characterization of the points of N_0 that is slightly different from that involved in the proof of Theorem 4.1: Note that $(x, p) \in N_0$ if and only if $Q(x, p)F(x, p) = 0$ and $G(x, p) = 0$ since $P(x, p)F(x, p) = 0$ is already ensured by the second relation and since $D_p G(x, p) \in \text{Isom}(\mathbb{R}^n)$ is guaranteed by the hypothesis $E_0 \cap E_N = E_0 \cap E_A (=N_0)$. Since $(x, p) \in E_0$, we see that $G(x, p) = 0$ is equivalent with $x \in U_0$ and $p = \Phi(x)$ and hence $(x, p) \in N_0$ if and only if $x \in U_0$ and $H(x) = 0$ where we have set

$$H(x) = Q(x, \Phi(x))F(x, \Phi(x)), \quad x \in U_0.$$

Using (a), one finds at once that $M_0 = H^{-1}(0)$. In turn, this shows that for $x \in M_0$, the space $T_x M_0$ identifies

with $\ker DH(x)$. Because of $H(x) = 0$ we have $(x, \Phi(x)) \in N_0 \subset E_N \subset F^{-1}(0)$ and, with the help of Lemma 4.1, an elementary calculation yields

$$DH(x)u = Q(x, \Phi(x))D_x F(x, \Phi(x))u ,$$

for $u \in \mathbb{R}^n$. Letting $u = \Phi(x)$, we find that $\Phi(x) \in \ker DH(x) = T_x M_0$ because $Q(x, \Phi(x))D_x F(x, \Phi(x))\Phi(x) = 0$ follows from $G(x, \Phi(x)) = 0$. This completes the proof of Theorem 4.2.

Remark: In the proof of Theorem 4.1, the projection $Q(x_0, p_0)$ cannot be replaced by $Q(x, p)$, for $\text{rge } Q(x, p)$ varies with (x, p) and the resulting mapping G^* must be viewed as taking its values in $\mathbb{R}^n \times \mathbb{R}^n$ instead of $\mathbb{R}^{n-r} \times \mathbb{R}^n$, and the implicit function theorem cannot be used. Neither can the rank theorem be applied since there is no guarantee that $\text{rank } DG^*(x, p) = 2n-r$ for (x, p) near (x_0, p_0) but not on E_A . On the other hand, $Q(x_0, p_0)$ cannot be substituted for $Q(x, p)$ in the proof of Theorem 4.2, for it would become impossible to take advantage of Lemma 4.1 with $(x, p) \neq (x_0, p_0)$.

As in [7] the local result in Theorem 4.2 may be globalized to some extent by applying the theory of covering spaces. We sketch only briefly the general approach. Clearly, the local result shows that the restriction $\Pi_A = \Pi|_{E_A}$ is a local homeomorphism between E_A and $\Pi_A E_A$. Let E_A^* be some non-empty, arc-connected subset of E_A for which (E_A^*, Π_A^*) , $\Pi_A^* = \Pi|_{E_A^*}$, is a covering space of ΠE_A^* . In other words, each point $x \in \Pi E_A^*$ is assumed to have an open, arc-connected neighborhood U such that each arc-component of $(\Pi_A^*)^{-1}(U)$ is not empty and is mapped topologically onto U by Π_A^* . Often it turns out that $E_A^* = E_A$ can be used. This is certainly the case when, for fixed $x \in \Pi E_A$ there are only finitely many p such that $(x, p) \in E_A$. In general, it is always possible to choose E_A^* as the closure of a non-empty, pre-compact, (relatively) open, and arc-connected submanifold of E_A .

For any given $(x_0, p_0) \in E_A^*$ let now M be a non-empty, (relatively) open, simply connected subset of ΠE_A that contains x_0 . For any $x \in M$ choose a path $\xi : J \rightarrow M$ which connects x_0 with x . Then there exists a unique lifting $\xi^* : J \rightarrow E_A$ with initial point (x_0, p_0) for which $\Pi_A^* \xi^* = \xi$. This lifted path has a unique endpoint (x, p) in E_A since all paths in M between x_0 and x are homotopic. Since x was arbitrary in M our above local result can now be used to prove that M indeed is an r -submanifold of \mathbb{R}^n and that the DAE (2.1) induces a tangential vector field on M for which all integral curves in M are solutions of (2.1).

5. The Linear Case With Constant Coefficients

In this section we apply our results to a non-autonomous linear problem with constant coefficients

$$Ax' + Bx = f, \quad (5.1)$$

where $f = f(t)$ is assumed to be as smooth as necessary and

$$A, B \in L(\mathbb{R}^n, \mathbb{R}^n), \text{ rank } A = r < n. \quad (5.2)$$

These linear problems probably represent the most extensively studied DAEs in the literature, (see [1],[4]).

For (5.1) the necessary condition (2.22) becomes

$$Bx - f' \in \text{rge } A \quad (5.3)$$

while the sufficient condition (2.23) has the form

$$Au = 0 \text{ and } Bu \in \text{rge } A \text{ imply } u = 0. \quad (5.4)$$

It is well-known (see e.g. [1]) that (5.1) is uniquely solvable for compatible initial data if and only if the matrix pencil (A, B) is regular; that is, exactly if there is some $\lambda \in \mathbb{R}$ such that $B + \lambda A$ is invertible. A central concept in the solvability theory of (5.1) is the index of regular pencils. For regular (A, B) choose any λ for which $B + \lambda A$ is invertible. Then, the index is the smallest integer $\kappa (\leq n)$ such that

$$\ker[(B + \lambda A)^{-1}A]^{\kappa+1} = \ker[(B + \lambda A)^{-1}A]^{\kappa}. \quad (5.5)$$

It can be shown that κ is independent of the choice of λ (see [4, App. A]), and it is also readily checked that $\kappa = 0$ if and only if A is invertible.

In order to relate our theory to this index-concept, let $P \in L(\mathbb{R}^n)$ be the orthogonal projection onto $\text{rge } A$ and $Q = I_n - P$. As in Section 2 our first step is to differentiate the DAE (5.1) and then to multiply the resulting equation $Ax'' + Bx' = f'$ by Q in order to remove again the second derivative of x . Together with the projection of the original equation onto $\text{rge } A$ this produces the system

$$PAx' + PBx = Pf, \quad QBx' = Qf. \quad (5.6)$$

Since P and Q map onto complementary spaces the two equations (5.6) can be added which results in the reduced DAE

$$A_1 x' + B_1 x = f_1 \quad (5.7a)$$

where, because of $PA = A$,

$$A_1 = PA + QB = A + QB, \quad B_1 = PB, \quad f_1 = Pf + Qf'. \quad (5.7b)$$

In Section 2, (5.7a/b) corresponds to (the non-autonomous version of) (2.9) and thus the results of that section can be applied. We concentrate here only on the effect of the sufficient condition (5.4). Even without recourse to the earlier theory, it is readily checked that (5.4) is equivalent with $A_1 \in Isom(\mathbb{R}^n)$ and, hence, that when (5.4) holds then (5.7a/b) can be transformed into an explicit ODE.

Suppose therefore that A_1 is singular. Then we may apply the same procedure repeatedly as many times as necessary, to obtain a sequence of DAEs of the form

$$A_j x' + B_j x = f_j \quad (5.8a)$$

where A_j, B_j, f_j are specified recursively by $A_0 = A, B_0 = B, f_0 = f$ and

$$A_{j+1} = P_j A_j + Q_j B_j, \quad B_{j+1} = P_j B_j, \quad f_{j+1} = P_j f_j + Q_j f'_j, \quad (5.8b)$$

and P_j is the orthogonal projection onto $rge A_j$ and $Q_j = I_n - P_j$. The process stops with the smallest integer k such that A_{k+1} is invertible; that is when the sufficient condition (5.4) holds for A_k and B_k . Explicitly, this condition has the form

$$A_k u_0 = 0 \text{ and } B_k u_0 \in rge A_k \text{ imply } u_0 = 0. \quad (5.9)$$

In terms of the original matrices A and B of (5.1) the condition (5.9) turns out to be equivalent with the condition

$$A u_0 = 0, \quad B u_j = A u_{j+1}, \quad j=0,1,\dots, k, \text{ implies } u_0 = 0. \quad (5.10)$$

The proof will follow by repeated application of the following result where, for ease of notation, the matrices A, B now stand for any A_i, B_i :

Lemma 5.1: Let A, B be any $n \times n$ matrices and, with the orthogonal projections P and $Q = I_n - P$ onto $rge A$ and $(rge A)^\perp$, respectively, set $A_1 = PA + QB = A + QB, B_1 = PB$. Then, for any $k \geq 1$ the equations

$$A u_0 = 0, \quad B u_j = A u_{j+1}, \quad j=0,\dots, k \quad (5.11)$$

have a solution u_0, \dots, u_{k+1} if and only if u_0, \dots, u_k solve the equations

$$A_1 u_0 = 0, \quad B_1 u_j = A_1 u_{j+1}, \quad j=0, \dots, k-1. \quad (5.12)$$

In particular, if for all solutions of (5.11) we have $u_0 = 0$, then the same must hold for all solutions of (5.12).

Proof: Suppose that u_0, \dots, u_{k+1} solve (5.11). Since $QA = 0$, multiplication of $Bu_j = Au_{j+1}$ by Q shows that $QB u_j = 0$ for $j = 0, \dots, k$. Hence, for $j = 0, \dots, k-1$ we obtain

$$B_1 u_j = P B u_j = P A u_{j+1} = P A u_{j+1} + Q B u_{j+1} = A_1 u_{j+1}$$

and, since $A u_0 = 0$,

$$A_1 u_0 = P A u_0 + Q B u_0 = P A u_0 = 0,$$

so that u_0, \dots, u_k solve (5.12).

Conversely, assume that the vectors u_0, \dots, u_k solve (5.12). In terms of A, B the equations (5.12) assume the form

$$A u_0 + Q B u_0 = 0 \quad (5.13a)$$

$$P B u_j = A u_{j+1} + Q B u_{j+1}, \quad j = 0, \dots, k-1 \quad (5.13b)$$

where we used that $A_1 = A + QB$. Since Q maps onto a complement of $\text{rge } A$, both terms on the left of (5.13a) have to be zero; that is, we have $A u_0 = 0$, and $Q B u_0 = 0$. By multiplication of (5.13b) with Q it follows that $Q B u_{j+1} = 0$ and hence altogether that

$$B u_{j+1} = P B u_{j+1}, \quad j = 0, \dots, k-1. \quad (5.14)$$

Now, by multiplying (5.13b) with P and using (5.14), we obtain $B u_j = P B u_j = P A u_{j+1} = A u_{j+1}$ for $j = 0, \dots, k-1$. On the other hand, (5.14) for $j = k-1$ shows that $B u_k \in \text{rge } P = \text{rge } A$ and thus that $B u_k = A u_{k+1}$ for some vector u_{k+1} . This completes the proof.

For the proof of the equivalence of (5.9) and (5.10) we begin by applying Lemma 5.1 to (5.9) which corresponds to (5.12) for $A = A_{k-1}, B = B_{k-1}$. Hence, we conclude that the validity of (5.9) is equivalent with the condition that

$$A_{k-1} u_0 = 0, \quad B_{k-1} u_0 = A_{k-1} u_1, \quad B_{k-1} u_1 = A_{k-1} u_2 \quad (5.15)$$

implies $u_0 = 0$. Evidently (5.15) is (5.12) for $A = A_{k-2}$, $B = B_{k-2}$ and thus the lemma can be applied again. In other words, by repeating the process we arrive after $k-1$ applications of Lemma 5.1 at the condition (5.10) which then completes the proof of the equivalence of (5.9) and (5.10).

As before, let now k be the smallest integer for which (5.9) holds; that is, for which A_{k+1} is invertible. If no such integer exists we set $k = \infty$. Then for $k < \infty$ the system

$$A_{k+1}x' + B_{k+1}x = f_{k+1}, \quad x(t_0) = x_0$$

has a unique solution which, from our theory, is a solution to (5.1) if and only if $A_j p_0 + B_j x_0 = f_j(t_0)$, $j = 0, \dots, k$, where p_0 is characterized by $A_{k+1}p_0 + B_{k+1}x_0 = f_{k+1}(t_0)$.

This solvability result raises, of course, the question how our condition (5.9) (or (5.10)) relate to the regularity and the index κ of the matrix pencil (A, B) . This is answered as follows:

Theorem 5.1: If the matrix pencil (A, B) is regular and $\text{rank } A < n$ (so that $\kappa \geq 1$) then $k = \kappa - 1$ and hence $k < \infty$. Conversely, if $k < \infty$ then (A, B) is regular and hence $k = \kappa - 1$ if $\text{rank } A < n$.

Proof: Let (A, B) be regular and choose λ such that $C = B + \lambda A$ is invertible. Then the index κ is defined as the smallest integer for which (5.5) holds. Let u_κ be any vector for which

$$(C^{-1}A)^{\kappa+1}u_\kappa = 0 \tag{5.16}$$

and set $u_j = C^{-1}Au_{j+1} = (C^{-1}A)^{\kappa-j}u_\kappa$, $j = \kappa-1, \dots, 0$. Since (5.16) implies that $A(C^{-1}A)^\kappa u_\kappa = 0$ it follows that $Au_0 = 0$. Hence altogether we have the equations

$$Au_0 = 0, \quad Cu_j = Au_{j+1}, \quad j = 0, \dots, \kappa-1, \tag{5.17}$$

and, conversely, (5.16) holds whenever u_0, \dots, u_κ solves (5.17). Now, by (5.6), the condition (5.16) implies that $u_0 = (C^{-1}A)^\kappa u_\kappa = 0$ for every solution u_0, \dots, u_κ of (5.17). This implication is not true when κ is replaced by any smaller integer as is easily seen when taking $l < \kappa$ and choosing $u_l \in \ker C^{l+1}$ not in $\ker C^l$. In fact, the family $u_j = C^{-1}Au_{j+1}$, $j = l-1, \dots, 0$ obviously satisfies $Au_0 = 0$, $Cu_j = Au_{j+1}$, $j = 0, \dots, l-1$, but $u_0 \neq 0$ since u is not in $\ker (C^{-1}A)^l$.

Thus, at this stage, we know that κ is the smallest integer for which existence of the solution u_0, \dots, u_κ of (5.17) implies that $u_0 = 0$. Given arbitrary vectors u_0, \dots, u_κ in \mathbb{R}^n , define

$$v_0 = u_0, v_j = \sum_{i=0}^{j-1} \binom{j-1}{i} (-\lambda)^i u_{j-i}, j = 1, \dots, \kappa. \quad (5.18)$$

With $C = B + \lambda A$, a straightforward induction argument now shows that if u_0, \dots, u_κ solve (5.17), then

$$Av_0 = 0, Bv_j = Av_{j+1}, j = 0, \dots, \kappa-1. \quad (5.19)$$

Conversely, if v_0, \dots, v_κ solve (5.19), then the relation (5.18) may be inverted to produce a solution u_0, \dots, u_κ of (5.17). In fact, we have

$$u_0 = v_0, u_j = \sum_{i=0}^{j-1} \binom{j-1}{i} \lambda^i v_{j-i}, j = 1, \dots, \kappa.$$

It then clearly follows that κ is the smallest integer for which existence of a solution v_0, \dots, v_κ of (5.19) implies that $v_0 = 0$, whence $k = \kappa-1$ by definition of k .

Conversely, suppose now that $k < \infty$. Then, our theory ensures that (5.1) has at most one C^2 -solution for each prescribed initial condition. In particular, this holds for the homogeneous problem

$$Ax' + Bx = 0, x(t_0) = 0. \quad (5.20)$$

But then the pencil (A, B) must be regular because, otherwise, the system (5.20) has infinitely many C^∞ -solutions (see e.g. [4]).

The case $\kappa = 1$ was noted already in [4], but the general result appears to be new.

6. Nonlinear Problems with Higher Index

As shown in the previous section, for the linear problems (5.1) the sufficient condition (2.23) is equivalent with the statement that the pencil (A, B) is regular with index one. The discussion in that section suggests that we may proceed analogously when, for the general (autonomous) problem

$$F(x, x') = 0, \quad (6.1)$$

the condition (2.12) does not hold but F is of class C^m , $m \geq 3$.

The first step in the construction of a sequence of problems corresponding to (5.9a/b) was already done in Section 2 when F was only C^2 . In fact, we differentiated (6.1) and then applied the projections P and Q to obtain the new problem (2.9); that is,

$$G(x, x') \equiv P(x, x')F(x, x') + Q(x, x')D_x F(x, x')x' = 0. \quad (6.2)$$

Our sufficient condition (2.12) is exactly that $D_p G(x, p)$ is invertible at the given point $(x_0, p_0) \in F^{-1}(0) \cap G^{-1}(0)$, and hence that, by the implicit function theorem, (6.2) can be transformed locally into an explicit ODE.

If this sufficient condition does not hold, then, as in Section 5, it would now be natural to set $F^0 \equiv F$, $F^1 \equiv G$ and, to construct recursively the sequence of mappings

$$F^{j+1}(x, p) = P_j(x, p)F^j(x, p) + Q_j(x, p)D_x F^j(x, p), \quad j = 0, 1, \dots \quad (6.3)$$

where P_j again is the orthogonal projection onto $\text{rge } D_p F^j$ and $Q_j = I_n - P_j$. Formally, the process is repeated until the sufficient condition (2.12) is satisfied for F^k ; that is, until $D_p F^{k+1}(x, p)$ is invertible at the point under consideration.

As before, one might consider calling the integer $k+1$ the local index of the problem at the particular point. However, the situation differs here in a critical way. Indeed, the very definition of the iterate F^{j+1} assumes some smoothness of its predecessor F^j and of the projections P_j and Q_j . But P_j and Q_j cannot even be continuous at (x_0, p_0) unless $\text{rank } D_p F^j(x, p)$ is locally constant near (x_0, p_0) (in general, continuity of a parametrized family in $L(\mathbb{R}^n)$ does not require constancy of the rank, but it does for projections). The validity of such a condition for $j = 0, \dots, k$ is then a necessary prerequisite to iterating the procedure as outlined above. This is not a restriction in the linear case with constant coefficients of the previous section because each F^j involves matrices independent of (x, p) .

Conversely, it is easily seen that the constant rank condition near (x_0, p_0) implies that only one degree of regularity is lost when passing from F^j to F^{j+1} , and hence that $F \in C^m$, $m \geq k+2$, suffices to ensure that F^{k+1} is C^1 . It thus appears that aside from sufficient smoothness, the constant rank condition near (x_0, p_0) is the crucial ingredient needed for the definition of a local index. Clearly, in general, local constancy of the rank cannot be captured by a finite list of requirements about F and its derivatives at the point (x_0, p_0) alone: even locally,

it is a condition of a global nature.

Now, consider a problem with local index $k+1$ near (x_0, p_0) . If so, a C^2 -solution to (6.1) may pass through (x_0, p_0) only if $F^j(x_0, p_0) = 0, j = 0, \dots, k+1$. As $D_p F^{k+1}(x_0, p_0) \in \text{Isom}(\mathbb{R}^n)$ by hypothesis, the problem $F^{k+1}(x, x') = 0, x(t_0) = x_0, x'(t_0) = p_0$, may be made into an explicit ODE for which the solution, is necessarily C^2 if $F \in C^m, m \geq k+2$, and is readily seen to solve $F^j(x, x') = 0, j = k, \dots, 0$, by recursive application of Lemma 2.2. In sharp contrast, nonexistence of a local index because of failure of the constant rank condition for some iterate F^j leads to a singular ODE for which the existence (and uniqueness) theory should be expected to be considerably different from standard explicit ODE theory in view of the results in [5].

In the hypothesis of the existence of a global index; that is, of the same local index $k+1$ near each point of the domain of definition E of F , the global results of Section 3, corresponding to the case $k = 0$, remain valid since any solution of (6.1) is one of $F^k(x, x') = 0$, a problem of global index 1. Finally, it should be mentioned that problems of arbitrary index $k+1$ also reduce (locally) to explicit ODE's on manifolds, at least under mild additional assumptions such as surjectivity of the total derivatives $DF^j(x, p), j = 0, \dots, k$. The effect of a higher index is merely to shrink the dimension of the underlying manifold: In fact, with $r = \text{rank } D_p F(x, p), r_j = \text{rank } D_p F^j(x, p), j = 1, \dots, k$, we can show that the relevant manifold has dimension $r_k + \dots + r_1 + r - kn$ (compare with Section 4 when $k = 0$). Since this dimension must be nonnegative and $r, r_1, \dots, r_k \leq n-1$, one infers that, generically, the index $k+1$ cannot exceed n . In the linear case with constant coefficients, this result follows from Theorem 5.1 and $\kappa \leq n$. This example (via the results in [4]) also shows that the existence and uniqueness theory for problems where the iterates F^j are defined beyond $j = n$ may differ significantly from standard ODE theory. A somewhat formal definition of the index in terms of manifolds and related in its spirit to the above remarks is given in [6].

As an example when the constant-rank condition remains valid for the iterated maps F^j we consider here the classical pendulum problem. A pendulum with mass m attached at the end of a rigid massless wire with length l attached at the origin in the plane (x_1, x_2) satisfies the second-order DAE

$$\begin{cases} x_1^2 + x_2^2 = l^2, \\ m\ddot{x}_1 = -\lambda x_1, \\ m\ddot{x}_2 = -\lambda x_2 - mg, \end{cases} \quad (6.4)$$

where λ is the (unknown) tension of the wire and g is the gravity constant.

In order to reduce the problem to a first order DAE we introduce the new variables $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$ and write $x_5 = \lambda/m$. Then, the problem assumes the form (6.1) with $x = (x_1, \dots, x_5) \in \mathbb{R}^5$, $p = (p_1, \dots, p_5) \in \mathbb{R}^5$ and

$$F(x, p) = \begin{bmatrix} x_1^2 + x_2^2 - l^2 \\ p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1 x_5 \\ p_4 + x_2 x_5 + g \end{bmatrix}. \quad (6.5)$$

Evidently, if e_1, \dots, e_5 denote the standard basis vectors of \mathbb{R}^5 then $\text{rge } D_p F(x, p) = \text{span} \{e_j, j = 2, \dots, 5\}$. Hence the conditions (2.2a/b) are satisfied and the orthogonal projections P, Q onto $\text{rge } D_p F(x, p)$ and its complement, respectively, are independent of (x, p) . A straightforward calculation shows that the mapping $F^1 = G$ of (6.2) has the form

$$F^1(x, p) = \begin{bmatrix} 2x_1 p_1 + 2x_2 p_2 \\ p_1 - x_3 \\ p_2 - x_4 \\ p_3 + x_1 x_5 \\ p_4 + x_2 x_5 + g \end{bmatrix}. \quad (6.6)$$

It is easily checked that $D_p F^1(x, p)$ has the constant rank 4 and hence is not invertible. In other words, the sufficient condition (2.12) does not hold for the pendulum problem. But since F^1 does indeed satisfy again the constant rank condition, we may proceed.

With $z(x) = (1, -2x_1, -2x_2, 0, 0)^T$ the orthogonal projection onto $[\text{rge } D_p F^1(x, p)]^\perp$ is the rank-one matrix $Q_1(x) = z(x)z(x)^T / z(x)^T z(x)$. Since with Q_1 also $P_1 = I_5 - Q_1$ depends only on x it follows that

$$D_p F^2(x, p)u = P_1(x)D_p F^1(x, p)u + Q_1(x)D_p [D_x F^1(x, p)p]u. \quad (6.7)$$

Thus we have $u \in \ker D_p F^2(x, p)$ if and only if both terms on the right side of (6.7) are zero. Since $P_1 D_p F^1 = D_p F^1$ it follows that $\dim \ker D_p F^2(x, p) \leq \dim \ker D_p F^1(x, p) = 1$. Now a short calculation shows that $D_p F^2(x, p)e_5 = 0$ and therefore that $\text{rank } D_p F^2(x, p) = 4$.

In order to check whether the sufficient condition (2.12) holds for F^2 that is, whether $D_p F^3(x, p)$ maps

onto \mathbb{R}^5 let $u = \alpha e_5 \in \ker D_p F^2(x, p)$. Then $D_{p,p}^2 F^1 = 0$, and $D_{p,p}[D_x F^1(x, p)p](u, q) = (4q_1 u_1 + 4q_2 u_2, 0, 0, 0, 0)^T$ from the explicit calculation of $D_x F^1(x, p)p$ whence, because of $\ker D_p F^2(x, p) = \text{span}\{e_5\}$ we have $D_{p,p}^2 F^2(x, p)(q, e_5) = 0$. Since $Q_1(x)e_5 = 0$, we have $DQ_1(x)e_5 = 0$ and $DP_1(x)e_5 = -DQ_1(x)e_5 = 0$. Moreover, one also finds $Q_1(x)D_{x,p}[D_x F^1(x, p)p](p, e_5) = 0$. Thus altogether,

$$D_{x,p}^2 F^2(x, p)(p, u) = P_1(x)D_{x,p}^2 F^1(x, p)(p, u) = 0,$$

because $D_{x,p}^2 F^1(x, p)(p, e_5) = 0$ as a result of $D_p F^1(x, p)e_5 = 0$. Similarly, we have $D_x F^2(x, p)u = P_1(x)D_x F^1(x, p)u = D_x F^1(x, p)u$ because $D_x F^1(x, p)e_5 = (0, 0, 0, x_1, x_2)^T = v(x)$ and hence $Q_1(x)v(x) = 0$.

Thus in this case the sufficient condition (2.12) for F^2 simply requires that when $\alpha v(x) \in \text{rge } D_p F^2(x, p)$ then $\alpha = 0$. Since the two terms on the right of (6.7) are complementary and $Q_1(x)v(x) = 0$ this means that the equations $D_p F^1(x, p)q = \alpha v(x)$ and $Q_1(x)D_p[D_x F^1(x, p)p]q = 0$ only have a solution q when $\alpha = 0$. In fact, the first of these equations amounts to $q_1 = q_2 = 0$, $q_3 = \alpha x_1$, and $q_4 = \alpha x_2$, while from the second equations it follows that $q_3 = q_4 = 0$. Hence, for $(x_1, x_2) \neq (0, 0)$ we indeed have $\alpha = 0$ as desired.

Because F^1 and F^2 satisfy the constant-rank condition, the existence and uniqueness results apply for every initial condition $(x_0, p_0) \in \mathbb{R}^5$ satisfying $F^j(x_0, p_0) = 0$, $j = 0, \dots, 3$ since $(x_{01}, x_{02}) \neq (0, 0)$ from $F(x_0, p_0) = 0$ (see (6.5)). Moreover, here we may indeed say that the problem has global index three since the condition $(x_1, x_2) \neq (0, 0)$ is not a restriction along and hence near the solutions of (6.4).

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