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Title: A Study of the Facial Structure of a Class of Polytopes Related to the Steiner Tree Problem

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ABSTRACT

Given an undirected graph $G = (N, A)$ with positive arc weights $c_a > 0$, $a \in A$ and P a subset of two or more nodes of N, the Steiner Tree problem on graphs (STG) is to find T, a connected subgraph of G such that all the nodes of P are included in T and the sum of the weights of the arcs in T is minimized. In the general case, STG is NP-hard. A method of attacking NP-hard combinatorial problems, which has not been generally exploited in the case of the Steiner tree problem, is that of polyhedral combinatorics. This method of attack revolves around studying the characteristics of the convex hull of feasible solutions. We define a class of polytopes related to the Steiner Tree problem on graphs and examine inequalities generated by placing connectivity requirements on these graphs. We define a class of facets induced by inequalities that specify the number of arcs which must flow across cuts and between the sets of a partition of the nodes.

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A Study of the Facial Structure of a Class of Polytopes Related to the Steiner Tree Problem

by

Keith A. Ware, Capt, USAF

A Dissertation submitted to the faculty of The University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Operations Research.

Chapel Hill

1988

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KEITH A. WARE, CAPT, USAF. A Study of the Facial Structure of a Class of Polytopes Related to the Steiner Tree Problem (Under the direction of David S. Rubin)

ABSTRACT

Given an undirected graph G = (N, A) with positive arc weights $c_a > 0$, $a \in A$ and P a subset of two or more nodes of N, the Steiner Tree problem on graphs (STG) is to find T, a connected subgraph of G such that all the nodes of P are included in T and the sum of the weights of the arcs in T is minimized. In the general case, STG is NP-hard. A method of attacking NP-hard combinatorial problems, which has not been generally exploited in the case of the Steiner tree problem, is that of polyhedral combinatorics. This method of attack revolves around studying the characteristics of the convex hull of feasible solutions. We define a class of polytopes related to the Steiner Tree problem on graphs and examine inequalities generated by placing connectivity requirements on these graphs. We define a class of facets induced by inequalities that specify the number of arcs which must flow across cuts and between the sets of a partition of the nodes.

Preface

The completion of a doctoral degree requires a lot of work, to say the very least. It is very rare that anyone can do this amount of work without the assistance and support of others. In my case there are many people who have helped me along the way. I would like to recognize some of them here. First, I would like to thank the US Air Force for giving me the opportunity to work for this degree. I would like to thank the faculty, staff and students of the Department of Operations Research, with a special note of thanks going to my committee: Doug Kelly, Joe Mazzola, Jon Tolle and Scott Provan. I am greatly indebted to my advisor, David Rubin, for all he has taught me, and for all the time and effort he put into the reading and rereading of this dissertation.

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Contents

Chapter I

Introduction

Given an undirected graph $G = (N, A)$ with positive arc weights $c_a > 0$, $a \in A$ and P, a subset of two or more nodes of N, the Steiner Tree problem on graphs (STG) is to find T, a connected subgraph of G such that all the nodes of P are included in T and the sum of the weights of the arcs in T is minimized. Though STG is NP-hard [16] in the general case, there are some interesting special cases which can be solved in polynomial time. The first two special cases come from placing restrictions on the size of P. If $|P| = 2$, then STG is just the shortest path problem, while if we have $|P| = |N| - 1$, then STG reduces to the minimum spanning tree problem. Both of these problems have well known polynomial-time algorithms [26]. The second class of STG problems which have polynomial-time algorithms comes from restrictions on the graph G. If G is a series-parallel or Hamlin graph, then we can solve STG in linear time [37]. There are exact solution algorithms for the general case of STG, but as Winter states in his survey paper [37] ". . . the problem seems to be extremely difficult and only small problem instances (with up to 30 points) can be solved in less than one hour."

A method of attacking NP-hard combinatorial problems, which has not been generally exploited in the case of the Steiner tree problem, is that of polyhedral combinatorics. This method of attack revolves around studying the characteristics of the convex hull of feasible solutions, trying to identify classes of inequalities that describe facets or high dimensional faces of this polytope, and then using these facets and faces in an algorithm to speed solution. (See [12] for an example.) The most common use of these facets is as cutting planes in a linear programming algorithm, although they can also be used in a Lagrangian relaxation approach (See [5, 18, 19, 28, 30] for examples of these approaches to the Travelling Salesman Problem).

The facet identification method has recently been applied to several other combinatorial problems including the set covering problem [3, 11], the three-index assignment problem [4] and the set packing and knapsack problems [28, 29].

The purpose of this research is to study the facial structure of the convex hull of solutions to STG on complete graphs. The main thrust of the research is to define classes of facets that have the potential to be used as cutting planes which can then be added to the LP relaxation of the problem or alternativly incorporated through the Lagrangian approach. The choice of working with complete graphs is made to facilitate the identification of structures that give rise to facets, and is not very restrictive, since every undirected graph is a subgraph of a complete graph.

1.1 Definitions and Notation

A graph G = (N, A) consists of a set of n nodes, N, and a set A of m unordered pairs of nodes called arcs. If i and j are nodes of the graph and $a \in A$ is an arc of G with $a = (i, j)$, then i and j are called the end nodes of a, a is said to connect or join i and j, and i and j are said to be incident to a. By convention we will always write (i, j) with $i < j$.

The cardinality, n, of the node set is called the order of the graph, and if $(i, j) \in A$ for all i and j in N, then the graph is called complete. The complete graph of order n will be denoted by K_n, and the cardinality of its arc set by

$$m_n = \frac{n(n-1)}{2}.$$

Since we deal with complete graphs, any numbering assigned to the node set is totally arbitrary, so we will always assume that $P = \{1, 2, \ldots, p\}$, where $p = |P|$.

If G = (N, A) with $W \subseteq N$ and $V \subseteq A$, then the set of arcs that have both end nodes in W is denoted by A(W) and the set of nodes incident to the arcs in V is denoted by N(V). A graph H = (V, E) is a subgraph of G if V $\subseteq$ N and E $\subseteq$ A(V). If W and V are disjoint subsets of N, then the set of arcs in A that have one end node in W and the other end node in V will be denoted by (W, V). We will denote N $-$ W by $\overline{W}$, and the set (W, $\overline{W}$) is called a cut.

A path Γ from s to t in G = (N, A), where s, t in N, is a sequence of nodes, n_i, and arcs, $a_i = (n_i, n_{i+1})$:

$$\Gamma = \{s = n_1, a_1, n_2, \ldots, n_{k-1}, a_{k-1}, n_k = t\}.$$

The nodes s and t are called the end nodes of Γ, and k is the length of Γ. If all nodes of Γ are distinct, then the path is called simple. If $s = t$, then the path is called a cycle, and if all nodes except $s = t$ are distinct, then the cycle is called simple. If there is a path between two nodes i and j in N, then i and j and said to be connected. If all pairs of nodes in a graph are connected, then the graph is said to be connected. If V $\subseteq$ N, we say that V is connected if (V, A(V)) is a connected graph.

A tree T is a connected graph that contains no cycles. Any subgraph of a graph G that contains no cycles is called a forest of G. A tree that is a subgraph of G and contains all the nodes in G is called a spanning tree. Clearly, since all arc weights are positive, the solution to STG will be a tree that connects the nodes of the set P. Such a tree is called a Steiner tree. The set P $\subseteq$ N of nodes taht we want to connect is called the set of terminal nodes. Any node in the set S = N $-$ P which is contained in T is called a Steiner node. A leaf or terminal node t of a tree is a node satisfying d(t) = 1, where d(t) denotes the degree of a node and equals the number of arcs in the tree which are incident to t. Let L be the set of leaves of a Steiner tree T. If L $\subseteq$ P, then T is called a P-Steiner tree or a P-tree.

In order to examine the facial structure of STG, we need to relate the Steiner and P-trees of K_n to $\mathbf{R}^{m_n}$. To do this let $A = \{(i_1, j_1), \ldots, (i_{m_n}, j_{m_n})\}$ Then if H = (V, E) is a subgraph of K_n, the characteristic or incidence vector of H in $\mathbf{R}^{m_n}$ will be the vector x satisfying

$$x_l = \begin{cases} 1 & \text{if } (i_l, j_l) \in E \\ 0 & \text{otherwise} \end{cases}$$

The vector x is said to describe or induce the subgraph H on G. The specific ordering of the arcs that we will use is the following. Arc a = (i, j) will correspond to component $x_{\psi(a)}$, where

$$\psi((i, j)) = \frac{(j-1)(j-2)}{2} + i.$$

This mapping corresponds to listing the arcs in the order (1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (1, 5), . . ., etc.. Throughout this dissertation, arc a = (i, j) will be denoted interchangeably as a or (i, j), and the components of characteristic vectors will be denoted x_a, $x_{i,j}$ or $x_{\psi(a)}$. The notation chosen will be clear from the context and will be used to place the emphasis on either the arc itself, the end nodes of the arc, or its order in a listing of all arcs, whichever is appropriate.

Now that we have a relation between a graph G and $\mathbf{R}^m$, we need to define some of the relevant ideas in polyhedral theory. The convex hull of a set of points $X = \{x^i\}_{i=1}^k$ in $\mathbb{R}^m$ is the set of all convex combinations of these points:

$$\text{conv}(X) = \{x \in \mathbf{R}^m \mid x = \sum_{i=1}^{k} \alpha_i x^i,\ \alpha_i \geq 0 \text{ and } \sum_{i=1}^{k} \alpha_i = 1\}.$$

A hyperplane H in $\mathbf{R}^m$ is the set $H = \{x \in \mathbf{R}^m \mid \langle x, y\rangle = \alpha\}$, where $y \in \mathbf{R}^m$, with $y \neq 0$, $\alpha \in \mathbf{R}$ and $\langle x, y\rangle$ is the inner product of x and y. A closed half space H' in $\mathbf{R}^m$ is the set $H' = \{x \in \mathbf{R}^m \mid \langle x, y\rangle \leq \alpha\}$. A polyhedron Q in $\mathbf{R}^m$ is the set of points satisfying a finite set of linear equalities and weak inequalities, i.e., the intersection of a finite number of closed half spaces. If Q is bounded, then Q is called a polytope. A polytope is both the intersection of a finite number of closed half spaces and the convex hull of a finite set of points.

A set of points $\{x^i\}_{i=1}^k$ in $\mathbf{R}^m$ are linearly independent if there is no solution to

$$\sum_{i=1}^{k} \lambda_i x^i = 0$$

other than $\lambda_i = 0$, $i = 1, 2, \ldots, k$. If $\lambda_i = 0$, $i = 1, 2, \ldots, k$ is the unique solution to the system

$$\sum_{i=1}^{k} \lambda_i x^i = 0$$

$$\sum_{i=1}^{k} \lambda_i = 0.$$

then the points $\{x^i\}_{i=1}^k$ are affinely independent. Linear independence implies affine independence. If the characteristic vectors of a set of trees of a graph are affinely/linearly independent, then we will say that the trees are affinely/linearly independent.

The dimension of a polytope Q, dim(Q) is k if the maximum number of affinely independent points in Q is $k + 1$. Let H be a hyperplane in $\mathbf{R}^m$, then if $Q \subseteq \mathbf{R}^m$ is a polytope with $Q \subseteq H'$, H is said to be valid for Q, and if $F = H \bigcap Q \neq \emptyset$, then F is a face of Q. A face of Q is also clearly a polytope in $\mathbf{R}^m$, so it is proper to speak of the dimension of a face. If dim(F) = dim(Q), then F = Q and F is called an improper face of Q. If dim(F) = 0, then F is a vertex of Q, and finally, if dim(F) = dim(Q) − 1, then F is a facet of Q. Since a half space can be represented by an inequality, the inequality describing H' will also be called a facet of Q equivalently, or will be said to induce (or define) a facet of Q.

If $X = \{x^i\}_{i=1}^k$ is a set of k vectors in $\mathbf{R}^m$, then X_l will denote the $l \times k$ matrix whose columns are the first l components of the k vectors in X. The $m \times 1$ vector of m 1's will be denoted by e_m. If A is an $m \times n$ matrix, then $\overline{A}$ will denote the $(m + 1) \times n$ matrix

$$\overline{A} = \begin{bmatrix} A \\ e_n^T \end{bmatrix}.$$

If the m x n matrix A has rank n, then the columns of A are linearly independent. If $\overline{A}$ has rank n, then the columns of A are affinely independent.

Finally, we need some definitions from matroid theory (See for example [14, 26]). Let E be a finite set, and $\mathcal{I}$ be a family of subsets of E. Then the structure M = (E, $\mathcal{I}$) is a matroid if $\mathcal{I}$ has the following properties.

1) $\emptyset \in \mathcal{I}$

2) If $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$

3) If I and J are in $\mathcal{I}$ with $|I| = |J| + 1$, then there exists an element $e \in I - J$ such that $J \bigcup \{e\} \in \mathcal{I}$.

A subset I in $\mathcal{I}$ is called an independent set of the matroid, and a maximal independent set is called a basis of the matroid. The rank r(V) of any subset $V \subseteq E$ is the cardinality of a maximal independent subset of V.

To relate matroids to the Spanning Tree Problem we will use the well known Forest matroid [17]. For a graph G = (N, A) let E = A and let $\mathcal{I}$ be the collection of all forests of G. (Recall that a forest is any subgraph of G that contains no cycles.) M = (A, $\mathcal{I}$) is a matroid. The rank of a set $B \subseteq A$ is given by

$$r(B) = |N(B)| - l$$

where l is the number of connected components in the subgraph (N(B), B). Spanning trees of G are the bases of the Forest Matroid.

1.2 Current Work

Several complete characterizations have been given for the polytope of the Spanning Tree Problem. The first comes from the work of Edmonds [14] in 1971. Edmonds proved that for any matroid on a set E, the vertices of polytope described by

$$\sum_{e \in E} x_e = r(E)$$

$$\sum_{e \in A} x_e \leq r(A), \quad \text{for all } A \subseteq E$$

$$x_e \geq 0, \quad \text{for all } e \in A$$

are the incidence vectors of the bases of M. Since the bases of the Forest Matroid are the spanning trees of G, this will give us a complete linear characterization of the spanning tree polytope. That same year, Fulkerson [15] conjectured that for a graph G = (N, A), the vertices of the polyhedron described by the inequality system

$$\sum_{a \in A-V} x_a \geq (n-1) - r(V) \quad \text{for all } V \subseteq A, \ V \text{ closed}$$

$$x_a \geq 0 \quad \text{for all } a \in A$$

are precisely the incidence vectors of the spanning trees of G, where a set $V \subseteq A$ is closed if there is no cycle of G with $|C \cap (A - V)| = 1$. Chopra [8] proved Fulkerson's conjecture in 1988. Both of these descriptions are complete, but neither is minimal. In 1977, Grötschel [17] gave a complete non-redundant linear characterization of the polytope of the bases of a general matroid, and then specialized this characterization to the polytope of spanning trees of graphs. Grötschel's linear characterization for complete graphs is

$$\sum_{a \in A} x_a = n - 1$$

$$\sum_{a \in A(W)} x_a \le |W| - 1 \text{ for all } W \subset N,\ 2 \le |W| \le n - 1$$

$$x_a \ge 0 \text{ for all } a \in A.$$

There has been very little work done on the facial structure of the convex hull of solutions of STG. In 1980, Aneja [1] formulated STG as a set covering problem in the following fashion. Let $(X, \overline{X})$ be a cut in G such that $P \cap X \neq \emptyset$ and $P \cap \overline{X} \neq \emptyset$. Then the inequality

$$\sum_{a \in (X, \overline{X})} x_a \ge 1$$

must be satisfied, since any tree connecting the nodes in P must have at least one arc in $(X, \overline{X})$. So the problem could be formulated as a set covering problem, i.e., trying to cover all the cuts with the arcs. Aneja then presented an algorithm based on the set covering algorithm. Although the number of constraints is exponential in the order of the graph, the algorithm only used these constraints implicitly. In 1984, Wong [38] worked with a version of STG on a directed graph, namely:

Given: A directed graph $G = (N, A)$, $P \subseteq N$, an arbitrary node r in $N - P$, and arc weights c_a, $a \in A$.

Find: The minimum weight set of arcs that span $P \cup \{r\}$ with every arc directed away from r. (Such a set is called a Steiner arborescence).

Wong formulated this problem as a mixed integer program and developed a dual ascent method for its solution.

In 1985, Prodon, Liebling and Groflin [32] considered the directed version of STG with the modification that the underlying graph G be strongly connected, i.e., for every pair of nodes

i and j there exists a simple path directed from i to j. They formed a strongly connected directed graph by replacing the arcs (i, j) of an undirected series-parallel graph with the directed arcs (i, j) and (j, i). They then proved that the extreme points of an unbounded polyhedron formed by some cut type inequalities from this new graph were the characteristic vectors of the P-trees (in the directed sense) of this graph. Later that year Prodon [31] extended these results by constructing a polyhedron whose extreme points were the characteristic vectors of the P-trees of an undirected series-parallel graph in the following manner. Let $F(k) = \{V^i\}_{i=1}^{k}$ be any family of connected subsets of N, such that each member of the family satisfies $V^i \bigcap P \neq \emptyset$ and $(N - \bigcup_{i=1}^{k} V^i) \bigcap P \neq \emptyset$. Then define the coefficient $r_a(F(k))$ for each arc $a = (i, j)$ in A by

$$r_a(F(k)) = \max \begin{cases} \left|\{l \mid V^l \in F(k),\ i \notin V^l,\ j \in V^l\}\right| \\ \left|\{l \mid V^l \in F(k),\ i \in V^l,\ j \notin V^l\}\right| \end{cases}$$

Then the polyhedron defined by

$$Q = \{x \in \mathbf{R}^{|A|} \mid \sum_{a \in A} r_a(F(k))\, x_a \geq k \ \text{ for any family } F(k)\}$$

has the characteristic vectors of P-trees of G as extreme points.

Finally, in 1987, Ball, Liu and Pulleyblank [7] studied two-terminal Steiner arborescences in general directed graphs. They noticed that every two-terminal Steiner arborescence consisted of a directed path from the root node r, to some node p, and directed paths from p to the two terminal nodes s and t, where p could be one of r, s or t. Defining

$d^r(p)$ = the length of the shortest path from node r to node p,

$d^s(p)$ = the length of the shortest path from node p to node s,

$d^t(p)$ = the length of the shortest path from node p to node t, for each node p of N, then the solution to the problem is to find that node p that determines

$$\min \mathrm{d}^r(p) + \mathrm{d}^s(p) + \mathrm{d}^t(p).$$

Ball, Liu and Pulleyblank take this combinatorial algorithm and derive a linear programming description of the problem from it using elimination and projection. They then turn their attention to complete directed graphs and prove, in that case, that every inequality in the formulation is a facet and therefore necessary.

1.3 Preliminaries

The approach that we have taken to this problem is similar to the general approach that has been taken for the traveling salesman problem [19], and is also similar to that used by Balas [2], Balas and Ng [3], and Balas and Saltzman [4] on other combinatorial problems. The approach is to define polytopes related to the problem, determine the dimension of those polytopes, and then study the problem for structures that might yield strong valid inequalities or facets. Before proceeding we need to prove a few lemmas which we will use throughout the dissertation.

Lemma 1.1: $m_n + n = m_{n+1}$

Proof: By definition

$$m_n + n = \frac{n\,(n-1)}{2} + n = \frac{n\,(n+1)}{2} = m_{n+1}. \quad \square$$

Lemma 1.2: Let $\{x^i\}_{i=1}^k$ be a set of points in $\mathbf{R}^m$ such that

$$Ax^i = b \qquad i = 1, 2, \ldots, k.$$

for some $l \times m$ matrix A and $l \times 1$ vector b. If y is an affine combination of the points x^i, then y also satisfies

$$Ay = b.$$

<u>Proof:</u> Let $y = X\alpha$, where X is the $m \times k$ matrix whose columns are the vectors x^i, and $e_k^T \alpha = 1$. Then

$$Ay = A(X\alpha) = (AX)\alpha = (be_k^T)\alpha = b(e_k^T\alpha) = b. \ \square$$

<u>Corollary 1.3:</u> If y is a convex combination of the points x^i, then

$$Ay = b.$$

<u>Lemma 1.4:</u> Let $X = \{x^i\}_{i=1}^k$ be a set of points in $\mathbf{R}^m$ and $Az = b$ be a system of r independent equations such that

$$Ax^i = b \qquad i = 1, \ldots, k$$

Then

$$\dim(\mathrm{conv}(X)) \leq m - r.$$

<u>Proof:</u> Let $Q = \{z \in \mathbf{R}^m | \ Az = b\}$. Then it is well known that

$$\dim(Q) = m - r \qquad [22, 27].$$

Thus, since $\mathrm{conv}(X) \subseteq Q$, by Corollary 1.3, we have that

$$\dim(\mathrm{conv}(X)) \leq m - r. \ \square$$

All of the proofs in this dissertation will establish that a set of vectors is either linearly or affinely independent. The most common way to do this is to show the matrix A or $\overline{A}$, whose columns are these vectors has full column rank. The following propositions ease this task.

<u>Proposition</u> <u>1.5:</u> Let M be the $m \times n$ matrix having upper triangular block form

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,k} \\ 0 & A_{2,2} & & \vdots \\ \vdots & & \ddots & A_{k-1,k} \\ 0 & \cdots & 0 & A_{k,k} \end{bmatrix}$$

where $A_{i,j}$ is $m_i \times n_j$. If the columns of the diagonal blocks $A_{i,i}$ are linearly independent for $i = 1, \ldots, k$ then the columns of M are linearly independent.

<u>Proof:</u> Consider the system

$$M\,x = 0 \quad (*)$$

Clearly, $x = 0$ is a solution to $(*)$. To prove that the columns of M are linearly independent we need to show that if $\alpha \in \mathbf{R}^n$ is a solution to $(*)$, then $\alpha = 0$. So assume that α is a solution to $(*)$. We see from the last m_k equations that α must satisfy

$$A_{k,k}\, \alpha_{n_k} = 0$$

where α_{n_k} is the vector consisting of the last n_k components of α. The columns of $A_{k,k}$ are linearly independent, so $\alpha_{n_k} = 0$. Substituting this partial solution into the next set of m_{k-1}

equations, we see that α must also satisfy

$$A_{k-1,k-1}\, \alpha_{n_{k-1}} = 0$$

where $\alpha_{n_{k-1}}$ is the the vector of components of α corresponding to these columns of A. Again we see that $\alpha_{n_{k-1}} = 0$ since the columns of $A_{k-1,k-1}$ are linearly independent.

Clearly, as we continue to iterate this back substitution process all the components of α will be forced to be 0. Therefore, if α is a solution to $(*)$ then $\alpha = 0$, and, hence, the columns of M are linearly independent since the only solution to $(*)$ is the trivial solution. □

Corollary 1.5.1: If M has lower triangular block form and the columns of the diagonal blocks are linearly independent, then the columns of M are linearly independent.

Proposition 1.6: Let M be the $m \times n$ matrix having the form

$$M = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

If the columns of A_1 and A_3 are affinely and linearly independent respectively, then the columns of M are affinely independent.

Proof: Consider the matrix

$$\overline{M} = \left[\begin{array}{c|c} A_1 & A_2 \\ 0 & A_3 \\ \hline e_{n_1}^T & e_{n_3}^T \end{array}\right]$$

If $\overline{\mathrm{M}}$ has linearly independent columns, then M has affinely independent columns. If we move the last row of $\overline{\mathrm{M}}$ up, we obtain the equivalent matrix

$$\overline{\mathrm{M}}' = \left[\begin{array}{c|c} A_1 & A_2 \\ e_{n_1}^{\top} & e_{n_3}^{\top} \\ \hline 0 & A_3 \end{array}\right]$$

which has linearly independent columns by Propositon 1.5 since the diagonal blocks have linearly independent columns. Thus, M has affinely independent columns. □

The next lemma relates the degree of nodes in any P-tree to $p = |\mathrm{P}|$.

<u>Lemma</u> <u>1.7:</u> Let K_n be the complete undirected graph of order n, with $\mathrm{N} = \mathrm{P} \bigcup \mathrm{S}$, $\mathrm{S} \bigcap \mathrm{P} = \emptyset$, and $|\mathrm{P}| = p$. In any P-tree if s is

1) contained in S, then $d(s) \leq p$

2) contained in P, then $d(s) \leq p - 1$.

<u>Proof:</u> First we show that if s is any non-terminal node of a tree with k leaves, then $d(s) \leq k$. Let $\deg(s) = d \geq 2$. Now delete s and all of its incident arcs from the tree. The result is a forest of d trees (some may be isolated nodes), each of which must contain at least one of the k leaves of the tree. Therefore, $d \leq k$.

Now, if T is a P-tree and $s \in \mathrm{P}$, then either s is a leaf of T, in which case $d(s) = 1 \leq p - 1$, or $d(s) \geq 2$. In the later case, there can be at most $p - 1$ leaves of T, so $d(s) \leq p - 1$. If $s \in \mathrm{S}$, then by the above argument, $d(s) \leq p$. □

1.5 Organization

Define the polytope

$$T_{p,n} = \text{conv}\{x \in \mathbf{R}^{mn} \mid x \text{ is the characteristic vector of a P-tree of } K_n\}$$

$T_{n,n}$ is the convex hull of spanning trees of K_n. In Chapter II we look at $T_{n,n}$, then in Chapter III we cover the other interesting special case, $T_{2,n}$, the shortest path polytope. After covering these two special cases, we consider the general STG polytope $T_{p,n}$ in Chapter IV. Chapter V presents presents some areas for further research, and the appendix gives some counting results.

Chapter II

The Spanning Tree Polytope $T_{n,n}$

In this chapter we will consider the polytope

$$T_{n,n} = \{x \in \mathbf{R}^{m_n} \mid x \text{ is the characteristic vector of a spanning tree of } K_n\}.$$

As we have noted, Grötschel has already developed a complete, minimal linear characterization of $T_{n,n}$. In this chapter we show that the facet-inducing inequalities of Grötschel's formulation correspond to upper and lower bounds on the variables and to certain partitions of the node set of K_n. Each of these inequalities is a facet of $T_{n,n}$, hence there exists a set of $m_n - 1$ affinely independent spanning trees that satisfy each of them at equality. We prove the stronger result that these sets of spanning trees are linearly independent. This strengthened property will be very important when we generalize to P-trees in Chapter IV.

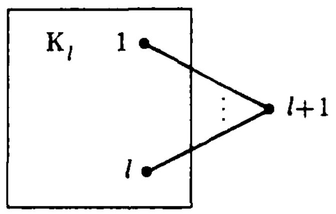
All of the proofs in this chapter either use induction or have an inductive flavor. To avoid the tedious repetition of defining the same notation in each proof, we introduce that notation now and use it throughout the chapter. Let $\{y^i\}_{i=1}^{k}$ be the set of characteristic vectors in $\mathbf{R}^{m_q}$ of k linearly independent spanning trees of K_q. Let Y be the matrix whose columns are these vectors. The values of k and q will be clear from each particular proof. For example, in order to establish a result for K_n, we look at the subgraph K_{n-1}. Either through assumption, or by invoking a previous result, we establish the existence of a set of k linear independent spanning trees of K_{n-1}. The columns of Y will be the characteristic vectors in $\mathbf{R}^{m_{n-1}}$ of these trees.

2.1 The Dimension of $T_{n,n}$

Grötschel proved that the dimension of $T_{n,n}$, $\dim(T_{n,n}) = m_n - 1$ [17]. We now show how to construct m_n linearly independent spanning trees of K_n.

Proposition 2.1: There exist m_n linearly independent spanning trees of K_n.

Proof: We prove this by induction on n. If $n = 2$, then $m_2 = 1$ and clearly there is only one spanning tree of two nodes. Now assume that there are m_l linearly independent spanning trees for some $l \geq 2$, and view K_{l+1} as:



Let Y be the matrix whose columns are the characteristic vectors of the m_l spanning trees, and for $i = 1, \ldots, m_l$ define the spanning trees of K_{l+1} as follows:

$$x_j^i = \begin{cases} y_j^i & j \in A(K_l) \\ 1 & j = (1, l+1) \\ 0 & \text{otherwise} \end{cases}$$

These vectors correspond to each of the m_l spanning trees of K_l with node $l+1$ attached as a leaf to node 1. For $i = 2, \ldots, l$ define the $l - 1$ vectors

$$x_j^{m_l+i} = \begin{cases} y_j^1 & j \in A(K_l) \\ 1 & j = (i, l+1) \\ 0 & \text{otherwise} \end{cases}$$

These vectors correspond to the spanning tree y^1 of K_l with node $l+1$ attached as a leaf to each of the nodes 2 through l respectively. Finally, define

$$x_j^{m_l+1} = \begin{cases} 0 & j \in A(K_l) \\ 1 & \text{otherwise} \end{cases}$$

which is the tree with all nodes connected to node $l+1$. By Lemma 1.1 we now have a total of $m_l + l = m_{l+1}$ vectors which describe spanning trees of K_{l+1}. To see that they are linearly independent, consider the matrix $M = (x^i)_{i=1}^{m_{l+1}}$. M has the form:

$$M = \begin{bmatrix} Y & 0 & Y^1 \\ e_{m_l}^T & 1 & 0 \\ 0 & e_{l-1} & I_{l-1} \end{bmatrix}$$

where Y^1 is the $m_l \times (l-1)$ matrix each of whose columns is the vector y^1. To see that this matrix has full rank consider the following sequence of elementary row and column operations. Subtract each of the last $l-1$ columns from column $m_l + 1$, and add $(l-1)$ times the first column to column $m_l + 1$ to get the matrix

$$M' = \begin{bmatrix} Y & 0 & Y^1 \\ e_{m_l}^T & l & 0 \\ 0 & 0 & I_{l-1} \end{bmatrix}$$

Rearranging rows and columns gives us the matrix

$$M'' = \left[\begin{array}{c|c|c} l & e_{m_l}^T & 0 \\ \hline 0 & Y & Y^1 \\ \hline 0 & 0 & I_{l-1} \end{array}\right]$$

which has linearly independent columns by Proposition 1.5. So the vectors $\{x^i\}_{i=1}^{m_l+1}$ are linearly independent for $n = l + 1$. By induction there are m_n linearly independent spanning trees of K_n for all $n \geq 2$. □

2.2 The Trivial Inequalities

We now turn our attention to the so-called "trivial" inequalities, namely the inequalities that define the boundary of the n-dimensional hypercube.

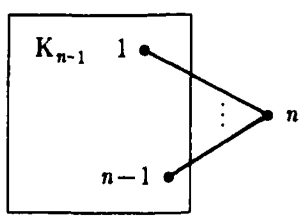
$$\left.\begin{array}{l} x_j \geq 0 \\ x_j \leq 1 \end{array}\right\} \; j = 1, \ldots, n$$

Proposition 2.2: For $n \geq 4$ the inequality $x_a \geq 0$ defines a facet of $T_{n,n}$ for every arc $a \in A$.

Proof: For $n \geq 4$ this comes directly from Grötschel's results. The lower bounds, $x_a \geq 0$, do not define facets for $n = 2$ or 3. In the case of $n = 2$, there is only one spanning tree, so no spanning tree of K_2 will satisfy $x_{1,2} = 0$. Similarly, for $n = 3$, each of the three arcs is in two of the three spanning trees, so only one spanning tree will satisfy $x_a = 0$, while $m_3 - 1 = 2$. □

Proposition 2.3: For $n \geq 4$ there exist $m_n - 1$ linearly independent spanning trees of K_n satisfying $x_a = 0$ for all arcs $a \in A$.

Proof: We assume that $a = (1, n)$ without loss of generality, and view K_n as



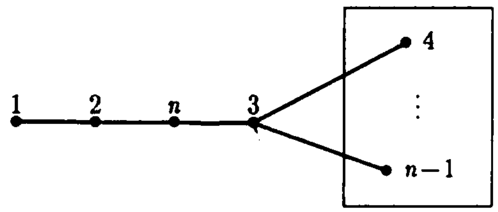
Let $\{y^i\}_{i=1}^{m_{n-1}}$ be a set of linearly independent spanning trees of K_{n-1}. The existence of this set is guarant ' by Proposition 2.1. For $i = 1, \ldots, m_{n-1}$ define m_{n-1} spanning trees of K_n by

$$x_j^i = \begin{cases} y_j^i & j \in A(K_{n-1}) \\ 1 & j = (2, n) \\ 0 & \text{otherwise} \end{cases}$$

These vectors correspond to the spanning trees of K_{n-1} with node n attached to node 2 as a leaf. Next define the tree

$$x_j^{m_{n-1}+1} = \begin{cases} 1 & j = (3, r),\ 4 \leq r \leq n-1 \\ 1 & j = (1, 2), (2, n) \text{ and } (3, n) \\ 0 & \text{otherwise} \end{cases}$$

This tree satisfies $x_{1,n} = 0$ and has the form



Now for $i = 3, \ldots, n-1$ define $n - 3$ additional spanning trees by

$$x_j^{m_{n-1}+i-1} = \begin{cases} y_j^1 & j \in A(K_{n-1}) \\ 1 & j = (i, n) \\ 0 & \text{otherwise} \end{cases}$$

These $n - 3$ trees are the spanning tree y^1 of K_{n-1} with node n attached as a leaf to each of nodes 3 through $(n-1)$, and hence satisfy $x_{1,n} = 0$. The total number of points is

$$m_{n-1} + (n - 3) + 1 = m_n - 1$$

by Lemma 1.1. To see that they are linearly independent look at $M = (x^i)_{i=1}^{m_n - 1}$.

$$M = \left[\begin{array}{cccc} Y & x_{m_{n-1}}^{m_{n-1}+1} & y^1 & Y^1 \\ 0 & 0 & 0 & 0 \\ e_{m_{n-1}}^T & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & I_{n-4} \end{array}\right] \begin{array}{l} \\ \text{row } (1, n) \\ \text{row } (2, n) \\ \\ \\ \end{array}$$

where Y^1 is the $m_{n-1} \times (n - 4)$ matrix, each of whose columns is the vector y^1. The columns of the submatrices Y and Y^1each contain $n - 2$ 1's since they represent spanning trees of K_{n-1}. The vector $x_{m_{n-1}}^{m_{n-1}+1}$ contains $n - 3$ 1's by construction. So, if we drop row $(1, n)$, multiply row $(2, n)$ by $(n - 2)$ and subtract from it each of the rows above it, we get the matrix M'.

$$M' = \left[\begin{array}{c|cc|c} Y & x_{m_{n-1}}^{m_{n-1}+1} & y^1 & Y^1 \\ \hline 0 & 1 & 2 - n & 2 - n \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & I_{n-4} \end{array}\right] \begin{array}{l} \\ \text{row } (2, n) \\ \\ \\ \end{array}$$

Which has linearly independent columns by Proposition 1.5 since $n \geq 4$. □

<u>Proposition</u> <u>2.4:</u> The inequality $x_a \leq 1$ defines a facet of $T_{n,n}$ for all arcs $a \in A$ and $n \geq 2$.

Proof: By Grötschel's work, inequalities of the form

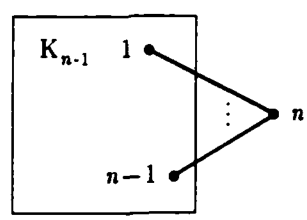
$$\sum_{a \in A(W)} x_a \leq |W| - 1$$

for all $W \subset N$, $2 \leq |W| < n$ are facets of $T_{n,n}$. Clearly, if we consider sets with $|W| = 2$, i.e., W consists of any two nodes in N, we get the set of inequalities

$$x_a \leq 1 \quad \text{for all } a \in A. \;\square$$

Proposition 2.5: For $n \geq 2$, there exist $m_n - 1$ linearly independent spanning trees of K_n satisfying $x_a = 1$ for all $a \in A$.

Proof: We lose no generality in assuming that $a = (1, n)$. Now view K_n as



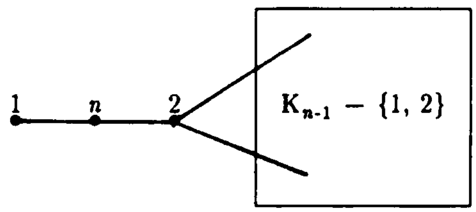
By Proposition 2.1 there are m_{n-1} linearly independent spanning trees of K_{n-1}. Let $\{y^i\}_{i=1}^{m_{n-1}}$ be their characteristic vectors. For $i = 1, \ldots, m_{n-1}$ define spanning trees of K_n as follows:

$$x_j^i = \begin{cases} y_j^i & j \in A(K_{n-1}) \\ 1 & j = (1, n) \\ 0 & \text{otherwise} \end{cases}$$

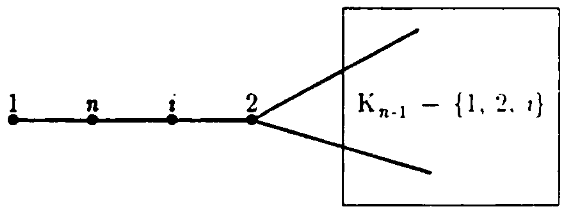
These trees correspond to the spanning trees of K_{n-1} with node n attached to node 1 as a leaf. Now, for $i = 2, \ldots, n-1$ define $n - 2$ additional spanning trees of K_n by

$$x_j^{m_{n-1}+i-1} = \begin{cases} 1 & j = (2, r),\ 3 \le r \le n-1 \\ 1 & j = (1, n) \text{ and } (i, n) \\ 0 & \text{otherwise} \end{cases}$$

The trees $x^{m_{n-1}+i-1}$ have the form shown below.



$i = 2$



$i = 3, \ldots, n-1$

We now have the correct number of points. All that remains to be shown is that they are linearly independent. Consider the matrix $M = (x^i)_{i=1}^{m_{n-1}}$. M has the form

$$M = \begin{bmatrix} \overline{Y} & \overline{X}_{m_{n-1}} \\ 0 & I_{n-2} \end{bmatrix}$$

and clearly has full column rank by Proposition 1.5. Thus, the spanning trees which we constructed are linearly independent. $\square$

2.3 Inequalities Generated by Partitions of the Node Set of K_n

Grötschel proved that inequalities of the form

$$(2.1) \qquad \sum_{a \in A(W)} x_a \le |W| - 1$$

are facets of $T_{n,n}$ for all $W \subset N$, $2 \leq |W| < n$. Further, all spanning trees of K_n must satisfy

$$(2.2) \qquad \sum_{a \in A} x_a = n - 1.$$

If we subtract (2.1) from (2.2) we get the inequality

$$(2.3) \qquad \sum_{a \in A - A(W)} x_a \geq n - |W|$$

which defines the same facet as (2.1), but has a different graphical interpretation. Let $|W| = t$, and assume without loss of generality that $W = \{n-t+1,\ n-t+2,\ \ldots,\ n\}$. Define a partition $\{V^i\}_{i=1}^{n-t+1}$ of the node set N by setting $V^i = \{i\}$ for $i = 1, \ldots, n-t$ and $V^{n-t+1} = W$. Let $k = n - t + 1$, then the inequality can be rewritten as

$$(2.4) \qquad \sum_{i=1}^{k-1} \left[\sum_{j=i+1}^{k} \left[\sum_{a \in (V^i, V^j)} x_a \right] \right] \geq k - 1.$$

Inequality (2.4) states that the number of arcs in any spanning tree crossing between the sets of the partition must be at least $k - 1$, where k is the number of sets in the partition.

<u>Proposition 2.6:</u> Let the family of sets $\{V^i\}_{i=1}^{k}$, $k < n$ be a partition of the node set of K_n such that $|V^i| = 1$, $i = 1, \ldots, k - 1$ and $|V^k| = (n - k + 1)$. Then (2.4) induces a facet of $T_{n,n}$.

<u>Proof:</u> This follows directly from Grötschel's work and the discussion above. □

<u>Corollary 2.7:</u> If $(X,\bar{X})$ is a cut in K_n, and either $|X| = 1$ or $|\bar{X}| = 1$, then the inequality

$$\sum_{a \in (X,\bar{X})} x_a \geq 1 \qquad (2.5)$$

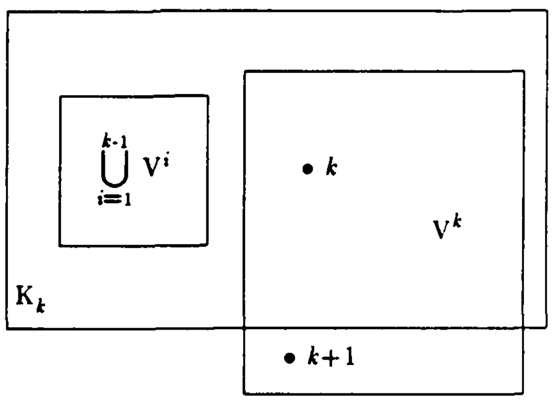
defines a facet of SP_n.

Proposition 2.8: There exist $m_n - 1$ linearly independent spanning trees of K_n satisfying

$$\sum_{i=1}^{k-1} \left[\sum_{j=i+1}^{k} \left[\sum_{a \in (V^i, V^j)} x_a \right] \right] = k - 1 \quad (2.6)$$

where $\{V^i\}_{i=1}^{k}$ is a partition of N satisfying the conditions in Proposition 2.6.

Proof: Let $k \geq 2$ be given. Without loss of generality we assume that $V^i = \{i\}$ for $i = 1, \ldots,$ $k-1$ and that $V^k = \{k, \ldots, n\}$ and proceed by induction on n. For $n = k + 1$ we can view K_{k+1} as:



By Proposition 2.1, there are m_k linearly independent spanning trees of K_k. We lose no generality by assuming that at least one tree has node k as a leaf. Let Y be the matrix whose columns are the characteristic vectors of these trees, with y^1 denoting the tree having node k as a leaf. By adding node $k+1$ to node k as a leaf we can construct spanning trees of K_{k+1} that satisfy (2.6). Namely, for $i = 1, \ldots, m_k$ define

$$x_j^i = \begin{cases} y_j^i & j \in A(K_k) \\ 1 & j = (k,\ k+1) \\ 0 & \text{otherwise} \end{cases}$$

Now, we note that $\bigcup_{i=1}^{k-1} V_i = K_{k-1}$ and define $k - 1$ additional trees as follows. For $i = 1, \ldots, k - 1$ let

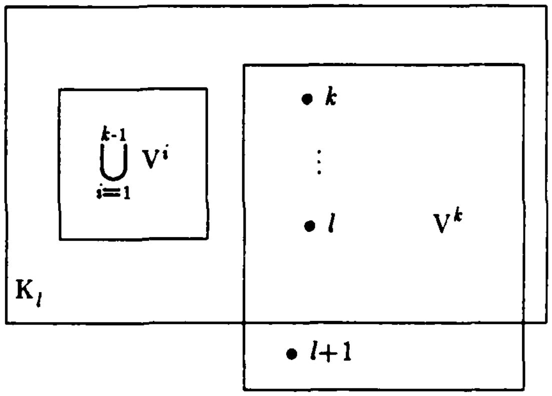
$$x_j^{m_k+i} = \begin{cases} y_j^1 & j \in A(K_{k-1}) \\ 1 & j = (k,\ k+1) \text{ and } (i,\ k+1) \\ 0 & \text{otherwise} \end{cases}$$

These trees correspond to spanning tree y^1 of K_k with leaf k sheared off, node $k+1$ is attached to node i, and node k is reattached as a leaf to node $k+1$. These trees satisfy (2.6) since they are spanning trees of nodes $1, \ldots, k-1$ plus a single arc to the set V^k. The matrix whose columns are the characteristic vectors of these $m_{k+1} - 1$ spanning trees has the form:

$$M = \begin{bmatrix} Y & Y^{1\prime} \\ 0 & I_{k-1} \\ e_{m_k}^T & e_{k-1}^T \end{bmatrix}$$

The columns of M are clearly linearly independent, hence we have constructed $m_{k+1} - 1$ linearly independent spanning trees satisfying (2.6).

Now, assume that there exist $m_l - 1$ linearly independent spanning trees of K_l satisfying (2.6) for some $l \geq k + 1$ with $V^k = \{k, \ldots, l\}$. We now look at K_{l+1}, which can be viewed as:



By the assumption, there exist $m_l - 1$ linearly independent spanning trees of K_l satisfying

$$\sum_{i=1}^{k-1} \left[\sum_{j=i+1}^{k} \left[\sum_{a \in (U^i, U^j)} x_a \right] \right] = k - 1$$

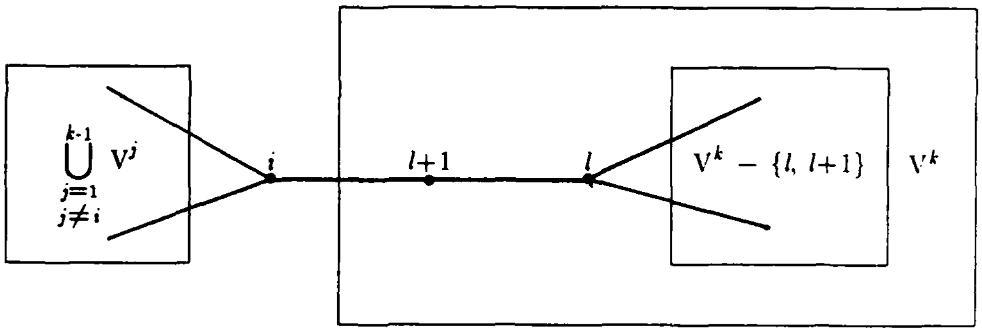
for $U^i = V^i$, $i = 1, 2, \ldots, k-1$, and $U^k = V^k - \{l+1\}$. Let Y be the matrix whose columns are the characteristic vectors of these trees. We note that if we add node $l+1$, as a leaf to one of the nodes in the set $V^k - \{l+1\}$, we obtain a spanning tree of K_{l+1} which satisfies (2.6). So define the following trees for $i = 1, \ldots, m_l - 1$.

$$x_j^i = \begin{cases} y_j^i & j \in A(K_l) \\ 1 & j = (l, l+1) \\ 0 & \text{otherwise} \end{cases}$$

Now, for $i = 1, \ldots, k-1$ define the $k - 1$ additional trees

$$
x_j^{m_l\text{-}1+i} = \begin{cases} 1 & j = (r,\ i),\ \ 1 \le r < i \\ 1 & j = (i,\ r),\ \ i < r \le k-1 \\ 1 & j = (r,\ l),\ \ k \le r \le l-1 \\ 1 & j = (i,\ l+1) \text{ and } (l,\ l+1) \\ 0 & \text{otherwise} \end{cases}
$$

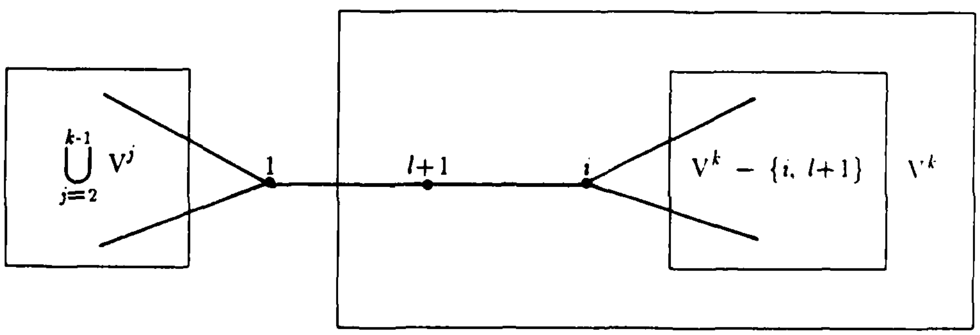
These trees have the form



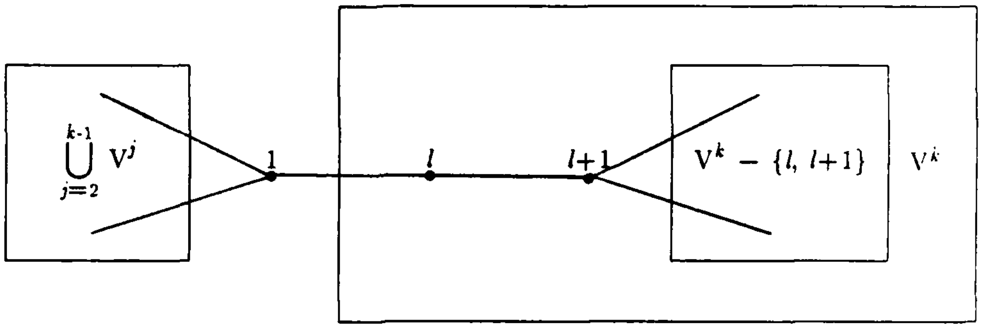
and are easily seen to satisfy (2.6). For $i = k, \ldots, l-1$ define $l - k$ more spanning trees

$$
x_j^{m_l\text{-}1+i} = \begin{cases} 1 & j = (1,\ r),\ \ 2 \le r < k-1 \\ 1 & j = (r,\ i),\ \ k \le r < i \\ 1 & j = (i,\ r),\ \ i < r \le l \\ 1 & j = (1,\ l+1) \text{ and } (i,\ i+1) \\ 0 & \text{otherwise} \end{cases}
$$

These trees have the form



and also satisfy (2.6). Finally, define the tree having the form



This tree satisfies (2.6) and can be described by

$$x_j^{m_{l+1}-1} = \begin{cases} 1 & j = (1, r), \quad 2 \le r \le k-1 \\ 1 & j = (r, l+1), \quad k \le r \le l \\ 1 & j = (1, l) \\ 0 & \text{otherwise} \end{cases}$$

The matrix whose columns are the characteristic vectors of these $m_{l+1} - 1$ trees has the form.

$$M = \left[\begin{array}{cc|cc|c} Y & x^{m_l+1} & \multicolumn{2}{c|}{X^l} & x^{m_{l+1}-1} \\ \hline 0 & 1 & 0 & e_{l-k}^T & 0 \\ 0 & 0 & I_{k-2} & 0 & 0 \\ 0 & 0 & 0 & I_{l-k} & e_{l-k} \\ e_{m_l-1}^T & 1 & e_{k-2}^T & 0 & 1 \end{array} \right]$$

Each column represents a spanning tree on $l + 1$ nodes, so it is easy to see that each column of the submatrix Y contains $l - 1$ 1's, while all the vectors x^i, except $x^{m_{l+1}-1}$, contain only $l - 2$ 1's in the first m_l rows. The last vector contains $k - 1$ 1's in the first m_l rows. So multiply the last row by $l - 1$ and subtract each of the first m_l rows. The result is

$$M' = \left[\begin{array}{cc|cc|c} Y & x^{m_l+1} & \multicolumn{2}{c|}{X'} & x^{m_{l+1}-1} \\ \hline 0 & 1 & 0 & e_{l-k}^{T} & 0 \\ 0 & 0 & I_{k-2} & 0 & 0 \\ 0 & 0 & 0 & I_{l-k} & e_{l-k} \\ 0 & 1 & e_{k-2}^{T} & 0 & l - k \end{array}\right]$$

Subtracting rows $(1,\ l+1)$ to $(k-1,\ l+1)$ from the last row, then adding rows $(k,\ l+1)$ through $(l-1,\ l+1)$ to the last row gives us the matrix

$$M'' = \left[\begin{array}{ccccc} Y & x^{m_l+1} & \multicolumn{2}{c}{X'} & x^{m_{l+1}-1} \\ 0 & 1 & 0 & e_{l-k}^{T} & 0 \\ 0 & 0 & I_{k-2} & 0 & 0 \\ 0 & 0 & 0 & I_{l-k} & e_{l-k} \\ 0 & 0 & 0 & 0 & 2(l - k) \end{array}\right].$$

which has linearly independent columns by Proposition 1.5 since $l > k + 1$. Therefore, the $m_{l+1} - 1$ spanning trees satisfying (2.6) that we constructed are linearly independent, and hence by induction, there exist $m_n - 1$ linearly independent spanning trees of K_n satisfying (2.6) at equality for all $n > k$. □

We can also establish upper bounds on the dimension of any face induced by a general partition, and give an exact dimension for faces induced by general cuts in K_n.

<u>Proposition</u> <u>2.9:</u> Let $\{V^i\}_{i=1}^{k}$ be a family of sets that partitions the node set N of K_n. Let q be the number of sets in this family having more than one member. Then (2.4) describes a face of $T_{n,n}$ of dimension at most $m_n - (q + 1)$.

Proof: Construct the following graph. For each set V^i in the partition add node v^i, and let arc (v^i, v^j) be in the graph if there is an arc in K_n connecting any two nodes in V^i and V^j. Since the family $\{V^i\}_{i=1}^k$ is a partition of the nodes of K_n it is clear that the transformation of K_n is just K_k. Any spanning tree of K_n must induce a spanning tree on K_k. Thus the number of arcs crossing between the sets V^i must be at least $k - 1$. So (2.4) is a valid inequality for all spanning trees of K_n. Now consider any spanning tree T that satisfies (2.4) at equality. If we remove the $k - 1$ arcs which are in the sets (V^i, V^j) for all i and j, we are left with k connected components, some of which may be single nodes. Further, it is easy to see that each component is a spanning tree of one of the sets V^i. Thus, for any set V^i that contains more than one node, T must satisfy

$$\sum_{a \in A(V^i)} x_a = |V^i| - 1.$$

Therefore, each spanning tree T satisfying (2.4) at equality must also satisfy q additional equalities, one for each of the q sets V^i having more than one member. Each of these equalities deals with a different set of variables, so they are linearly independent. So, by Proposition 1.4 the dimension of the face of $T_{n,n}$ described by (2.4) is at most $m_n - (q + 1)$. □

Proposition 2.10: Let $(X,\overline{X})$ be an arbitrary cut in K_n with $1 < |X| < n - 1$, then the cut-set inequality (2.5) defines a face of $T_{n,n}$ of dimension exactly $m_n - 3$.

Proof: $\{X, \overline{X}\}$ is a special partition of N, with $q = 2$. Thus, by Proposition 2.9, the dimension of the face described by the inequality is at most $m_n - 3$. Now, we construct $m_n - 2$ linearly independent spanning trees of K_n satisfying (2.5) at equality to establish that the dimension of the face induced by a general cut is exactly $m_n - 3$. Without loss of generality, let $X = \{1, 2, \ldots, k\}$ and $\overline{X} = \{k+1, \ldots, n\}$ where $1 < k < n$. By Proposition 2.1 there exist m_k linearly independent spanning trees of G(X,A(X)) and m_{n-k} linearly independent spanning trees of $G(\overline{X},A(\overline{X}))$. Let Y and Z, respectively, be the matrices whose columns are the characteristic vectors of these sets of spanning trees. For this proof, we associate the arcs of the graph with the components of the vectors in $\mathbf{R}^{m_n}$ in the following fashion. The first m_k components will correspond to the arcs in G(X,A(X)), the next $k(n - k)$ components will correspond to the arcs in $(X,\overline{X})$, and the last m_{n-k} components will correspond to the arcs in $G(\overline{X},A(\overline{X}))$. Within each of these three divisions, the arcs are ordered in the usual manner. Now, define $m_n - 2$ spanning trees of K_n as follows. For $i = 1, \ldots, m_k$ define

$$x_j^i = \begin{cases} y_j^i & j \in A(X) \\ 1 & j = (1,k+1) \\ z_j^1 & j \in A(\overline{X}) \\ 0 & \text{otherwise} \end{cases}$$

These trees have spanning tree z^1 of $G(\overline{X}, A(\overline{X}))$ attached to each spanning tree of G(X, A(X)) by arc $(1, k+1)$. For $a \in (X,\overline{X}) - \{(1, k+1)\}$ define

$$x_j^{\psi(a)-1} = \begin{cases} y_j^1 & j \in A(X) \\ 1 & j = a \\ z_j^1 & j \in A(\overline{X}) \\ 0 & \text{otherwise} \end{cases}$$

Finally, for $i = 2, \ldots, m_{n-k}$ define

$$x_j^{m_k+k(n-k)-2+i} = \begin{cases} y_j^1 & j \in A(X) \\ 1 & j = (1, k+1) \\ z_j^i & j \in A(\overline{X}) \\ 0 & \text{otherwise} \end{cases}$$

These trees consist of the spanning tree y^1 of G(X, A(X)) being connected to each of the spanning trees, except z^1, of $G(\overline{X}, A(\overline{X}))$. Each of these trees contains exactly one arc in $(X,\overline{X})$ and thus satisfies (2.5) at equality. To see that these trees are linearly independent consider the matrix whose columns are the characteristic vectors of these trees.

$$M = \begin{bmatrix} Y & Y^1 & Y^1 \\ e^T_{m_k} & 0 & e^T_{m_{n-k}-1} \\ 0 & I_{k(n-k)-1} & 0 \\ Z^1 & Z^1 & Z' \end{bmatrix}$$

Where $Z' = Z - \{z^1\}$. We can use the rows containing the submatrix $I_{k(n-k)-2}$ to clear the Y^1 from the rows above it and the Z^1 from the rows below it. Then subtract the first column from each of the last $m_{n-k} - 1$ columns to get the resulting matrix

$$M' = \left[\begin{array}{c|c|c} Y & 0 & 0 \\ e^T_{m_k} & 0 & 0 \\ \hline 0 & I_{k(n-k)-1} & 0 \\ \hline Z^1 & 0 & Z' - Z^1 \end{array}\right]$$

The columns of $Z' - Z^1$ are linearly independent [26], so M' satisfies the conditions of Corollary 1.5.1 for having linearly independent columns. Therefore, the dimension of the face of $T_{n,n}$ described by (2.5) is $m_n - 3$. $\square$

Chapter III

The (s, t)-Path Problem Polytope: $T_{2,n}$

The previous chapter presents results for the polytope $T_{n,n}$, representing the special case when STG reduces to the minimum spanning tree problem. When the order of P is two, STG reduces to another important special case, the shortest (s, t)-path problem. The results for the shortest (s, t)-path polytope are presented separately from the case of general p for two reasons. First, it is an important problem in its own right; second, and perhaps more importantly, the polytope

$$T_{2,n} = \text{conv}\{ x \in R^{m_n} \mid x \text{ is a P-tree of } K_n \}$$

is not full dimensional, whereas $T_{p,n}$ has full dimension if $p \geq 3$. The proof techniques used in this chapter lay the groundwork for the results in the next chapter, which covers the case of general p.

In Chapter II we defined a generic set $\{y^i\}_{i=1}^{k}$ of k characteristic vectors in $\mathbb{R}^{m_q}$. We will use such sets in this chapter, with Y denoting the matrix whose columns are these vectors. As before, the values of k and q will be clear from the proofs.

3.1 The Dimension of $T_{2,n}$

One factor which separates the case of $p = 2$ from that of $p \geq 3$ is that the polytope $T_{2,n}$ is not full dimensional, while for $p \geq 3$ it will be. In the case of $n = 2$, there is only one arc, and only one P-tree, so the dimension of $T_{2,2} = 0 < 1 = m_2$

Proposition 3.1: For $n > 2$, $\dim(T_{2,n}) = m_n - 2$

Proof: Since $|P| = 2$, both nodes in P must be leaves of any P-tree in K_n, so every P-tree must satisfy the two equations:

$$\sum_{j=2}^{n} x_{1,j} = 1, \qquad (3.1)$$

$$x_{1,2} + \sum_{j=3}^{n} x_{2,j} = 1. \qquad (3.2)$$

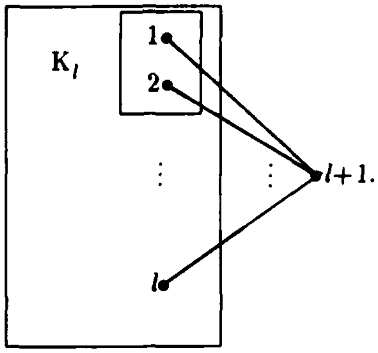
These two equations are independent, therefore, by Lemma 1.4

$$\dim(T_{2,n}) \leq m_n - 2. \qquad (3.3)$$

We now list $m_n - 1$ linearly independent points of $T_{2,n}$, which satisfy (3.3) at equality. As in the previous chapter we will proceed by induction on n. For $n = 3$, $m_3 - 1 = 2$, and there are only two P-trees, namely

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

These two trees are clearly linearly independent. Now assume that, for $n = l$ there are $m_l - 1$ linearly independent spanning trees of K_l, for some $l \geq 3$, and let's look at K_{l+1}. We can view K_{l+1} as:



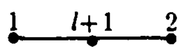
Any P-tree of K_l is also a P-tree of K_{l+1}. By the assumption there exists a set of $m_l - 1$ linearly independent P-trees of K_l. Let Y be the matrix whose columns are the characteristic vectors of these trees in $R^{m_{l+1}}$. Define l additional P-trees of K_{l+1} by the following set of characteristic vectors. Let the first two vectors be

$$x_j^1 = \begin{cases} 1 & j = (1,\ l+1) \text{ and } (2,\ l+1) \\ 0 & \text{otherwise} \end{cases}$$

and,

$$x_j^2 = \begin{cases} 1 & j = (1,\ 3),\ (2,\ l+1) \text{ and } (3,\ l+1) \\ 0 & \text{otherwise} \end{cases}$$

These vectors correspond to the P-trees:



x^1

x^2

Define the remaining $l - 2$ P-trees as follows, for $i = 3, \ldots, l$

$$x_j^i = \begin{cases} 1 & j = (2,\ i),\ (1,\ l+1) \text{ and } (i,\ l+1) \\ 0 & \text{otherwise} \end{cases}$$

This family of vectors corresponds to the family of P-trees of the following form:

x^i, $i = 3, \ldots, l$

The total number of P-trees is $(m_l - 1) + l$, which by Lemma 1.1 is $m_{l+1} - 1$. To see that these vectors are linearly independent consider the $m_{l+1} \times (m_{l+1} - 1)$ matrix M whose columns are the vectors Y and X. M has the form:

$$M = \left[\begin{array}{c|ccc|c} Y & & X_{m_l} & & \\ \hline 0 & 1 & 0 & 1 & e_{l-3}^T \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{l-3} \end{array}\right]$$

By Proposition 1.5 the columns of M are linearly independent. Therefore,

$$\dim(T_{2,n}) = m_n - 2$$

for all $n \geq 3$. $\square$

We now turn our attention to inequalities that describe facets of $T_{2,n}$. As in Chapter II, we start with the hypercube bounding inequalities.

3.2 The Trivial Inequalities

The hypercube bounding inequalities are those of the form:

$$x_a \geq 0$$
$$x_a \leq 1 \qquad (3.4)$$

for any arc a $\in$ A. They are clearly valid for $T_{2,n}$, so for each possible value of $n \geq 3$ we need only demonstrate $m_n - 2$ affinely independent points satisfying each of these inequalities at equality in order to show that it is a facet. In the case of $n = 3$, $\dim(T_{2,3}) = 1$ so the only possible facets of $T_{2,3}$ are its two vertices.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Clearly one of these two vectors satisfies each of the six bounding inequalities (3.4) at equality. Thus, the inequalities (3.4) describe facets of $T_{2,3}$. For $n \geq 4$, however, we need to consider the location of the arc a, i.e., is a = (1, 2) = A(P), or is a $\in$ (P, S) or A(S)?

3.2.1 The Lower-Bound Inequalities

Proposition 3.2: For $p = 2$ and $n \geq 4$, the inequality $x_{1,2} \geq 0$ induces a facet of $T_{2,n}$.

Proof: The proof will be by induction on n. For $n = 4$ there exactly four P-trees that satisfy $x_{1,2} = 0$, namely

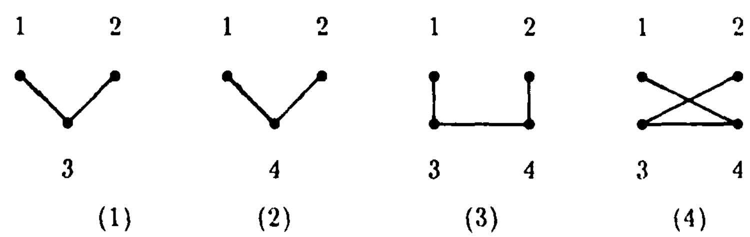


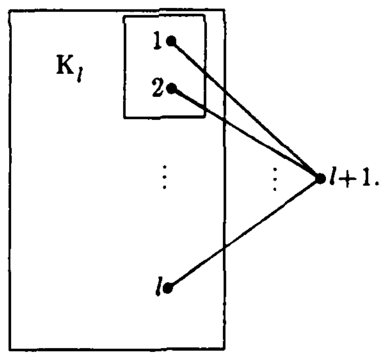
Figure 3-1

The matrix of characteristic vectors of these trees is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which is easily seen to have full column rank. Thus there are $m_4 - 2 = 4$ linearly independent points in $T_{2,4}$ satisfying $x_{1,2} = 0$, and hence $x_{1,2} \geq 0$ induces a facet of $T_{2,4}$.

Now assume that there exist $m_l - 2$ linearly independent P-trees of K_l for some $l \geq 4$ and look at K_{l+1}.



Every P-tree of K_l is also a P-tree of K_{l+1}. By the assumption there exist $m_l - 2$ linearly independent P-trees of K_l satisfying $x_{1,2} = 0$. Let $Y = \{y^i\}_{i=1}^{m_l-2}$ be a set of characteristic vectors in $\mathbf{R}^{m_{l+1}}$ of these trees. Now notice that the set X of l vectors constructed in the proof of Proposition 3.1 all satisfy $x_{1,2} = 0$. Furthermore, the arguments used in that proof to establish the linear independence of the total set of P-trees only required the columns of Y to be linearly independent. This independence follows by assumption. Therefore, we have

$$m_l - 2 + l = m_{l+1} - 2$$

linearly independent P-trees of K_{l+1} which satisfy $x_{1,2} = 0$, and hence, $x_{1,2} \geq 0$ induces a facet of $T_{2,l+1}$. By the induction principle, this inequality induces a facet of $T_{2,n}$ for all $n \geq 4$. □

Since $T_{2,n}$ is not full dimensional, there may be several representations of facets. To illustrate this, consider the following corollary.

<u>Corollary 3.3:</u> The inequality

$$\sum_{j=3}^{n} x_{1,j} + \sum_{j=3}^{n} x_{2,j} \leq 2$$

induces a facet of $T_{2,n}$.

<u>Proof:</u> Consider the facet $x_{1,2} \geq 0$. If we multiply this facet by (-2) and add to it the two equations (3.1) and (3.2) that all P-trees in this category must satisfy, we get the inequality above. □

In the case of $n = 4$, the arc (1, 2) turns out to be the only arc whose corresponding lower bound inequality induces a facet.

<u>Lemma 3.4:</u> For $n = 4$, $x_a \geq 0$ is a face of $T_{2,4}$ of dimension 2 for all arcs $a \in A$, except (1, 2).

<u>Proof:</u> There are only five P-trees in this case, the four listed in Figure 3-1 above plus the tree

The matrix whose columns are these five vectors is

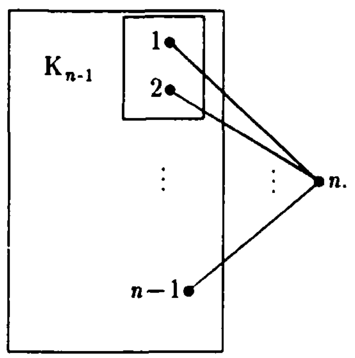
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

A quick examination shows that every arc $a \in A$, except (1, 2), is contained in exactly two trees. So for any arc $a \in A$, $a \neq (1, 2)$, there are three trees that satisfy $x_a = 0$. These trees are linearly independent since all five of the trees are linearly independent. Therefore, $x_a \geq 0$ describes a face of $T_{2,4}$ of dimension 2 for all arcs $a \in A$ except (1, 2). □

We now turn our attention to the arcs in (P, S) and $n \geq 5$.

<u>Proposition 3.5:</u> For $n \geq 5$, the inequality $x_a \geq 0$ describes a facet of $T_{2,n}$ for all arcs $a \in (P, S)$.

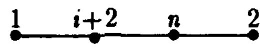
<u>Proof:</u> We lose no generality by assuming that $a = (1, n)$. We view K_n as



and notice that any P-tree of K_{n-1} is a P-tree of K_n satisfying $x_{1,n} = 0$. By Proposition 3.1, there are $m_{n-1} - 1$ linearly independent P-trees of K_{n-1}, so let Y represent these trees. Now define $n - 3$ additional P-trees. For $i = 1, \ldots, n-3$ define

$$x_j^i = \begin{cases} 1 & j = (2, n), (i+2, n) \text{ and } (1, i+2) \\ 0 & \text{otherwise} \end{cases}$$

These trees satisfy $x_{1,n} = 0$, and have the form

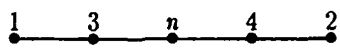


x^i, $i = 1, \ldots, n-3$

Define one last P-tree by

$$x_j^{n-2} = \begin{cases} 1 & j = (1, 3), (2, 4), (3, n) \text{ and } (4, n) \\ 0 & \text{otherwise} \end{cases}$$

x^{n-2} represents the tree



x^{n-2}

which satisfies $x_{1,n} = 0$. The total number of trees is

$$m_{n-1} - 1 + (n-2) = m_n - 2$$

by Lemma 1.1, so all that remains to be shown is that they are linearly independent. Let M be the matrix (Y, X). Then

$$
M = \left[\begin{array}{c|cccc}
Y & \multicolumn{4}{c}{X_{m_{n-1}}} \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & e^{\top}_{n-5} & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & I_{n-5} & 0
\end{array}\right]
\begin{array}{l}
\\
\text{row } (1,\ n) \\
\\
\\
\\
\\
\end{array}
$$

We can rearrange the columns of M and drop row $(1, n)$ to get

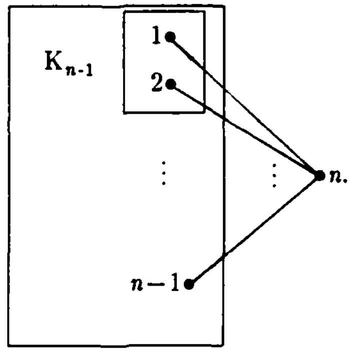
$$
M' = \left[\begin{array}{c|ccc|c}
Y & \multicolumn{3}{c}{X_{m_{n-1}}} & \\
\hline
0 & 1 & 1 & 0 & e^{\top}_{n-5} \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & I_{n-5}
\end{array}\right]
$$

The columns of M' are linearly independent by Proposition 1.5. Therefore, $x_a \geq 0$ induces a facet of $T_{2,n}$ for all arcs a $\in$ (P, S). □

As in the case of arcs in (P, S), the arcs in A(S) all have facet inducing lower bound inequalities.

Proposition 3.6: For $n \geq 5$, the inequality $x_a \geq 0$ induces a facet of $T_{2,n}$ for all arcs a $\in$ A(S).

Proof: Without loss of generality, we can assume $a = (n-1, n)$. Look at K_n as



We will list $m_n - 2$ linearly independent P-trees satisfying $x_{n-1,n} \geq 0$ at equality to prove that the inequality is a facet of $T_{2,n}$. As in the previous proof, there are $m_{n-1} - 1$ linearly independent P-trees of K_{n-1}, all of which satisfy $x_{n-1,n} = 0$. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now define $n - 2$ additional trees in the following manner. First, let

$$x_j^1 = \begin{cases} 1 & j = (1, 3),\ (2, n) \text{ and } (3, n) \\ 0 & \text{otherwise} \end{cases}$$

which satisfies $x_a = 0$ and has the form

x^1

Now define x^2 as

$$x_j^2 = \begin{cases} 1 & j = (1, n) \text{ and } (2, n) \\ 0 & \text{otherwise} \end{cases}$$

which has the form

x^2

And finally, for $i = 3, \ldots, n - 2$ define the remaining $n - 4$ trees by

$$x_j^i = \begin{cases} 1 & j = (1,\ n),\ (2,\ i) \text{ and } (i,\ n) \\ 0 & \text{otherwise} \end{cases}$$

These trees all satisfy $x_{n\text{-}1,n} = 0$ and they represent the family of P-trees

$x^i,\ i = 3, \ldots, n - 2$

As in the proof of Proposition 3.3 we have $m_n - 2$ vectors, and the matrix M whose columns are these vectors has the last row whose elements are all 0's. If we drop this row we have the matrix

$$M' = \left[\begin{array}{c|cccc} Y & & X_{m_{n-1}} & & \\ \hline 0 & 0 & 1 & 1 & e_{n\text{-}5}^T \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n\text{-}5} \end{array}\right]$$

The columns of M' are linearly independent by Proposition 1.5. Therefore, the columns of M are linearly independent. □

3.2.2 The Upper-Bound Inequalities

We now turn to the upper bound inequalities. Recall, that for $T_{2,3}$, both the upper- and lower-bound inequalities define facets, so we need only look at the case of $n \geq 4$.

Lemma 3.7: For $n \geq 4$, the inequality $x_{1,2} \leq 1$ describes a vertex of $T_{2,n}$.

<u>Proof:</u> For any $n \geq 2$ there is exactly one P-tree satisfying $x_{1,2} = 1$, namely

Thus, for any n, $x_{1,2} \leq 1$ can only describe a vertex of $T_{2,n}$. □

<u>Proposition 3.8:</u> For $n = 4$, the inequality $x_a \leq 1$ describes a face of $T_{2,n}$ of dimension 1 for all $a \in A$, except (1, 2).

<u>Proof:</u> As was noted in the proof of Proposition 3.2, there are only two P-trees containing any arc $a \in A$, except (1, 2). These two trees are linearly independent, so the dimension of the face described by $x_a \leq 1$ is 1. □

<u>Proposition 3.9:</u> For $n \geq 5$, the inequality $x_a \leq 1$ describes a face of $T_{2,n}$ of dimension at most $m_{n-1} - 2$ for any arc $a \in (P, S)$.

<u>Proof:</u> Without loss of generality, let $a = (1, n)$. Since $|P| = 2$, all P-trees are contained in an equality space of dimension at least 2, because they must satisfy (3.1) and (3.2). But since any P-tree satisfying $x_{1,n} = 1$ has node 1 attached to node n, and node 1 must be a leaf of the tree, we can replace (3.1) with the $n - 1$ independent equalities

$$x_{1,n} = 1$$

$$x_{1,j} = 0, \;\; j = 2, 3, \ldots, n - 1.$$

Also, the degree of node n must be 2. Thus the equality

$$\sum_{i=2}^{n-1} x_{i,n} = 1$$

must be satisfied. Each of these $n + 1$ equations involves different variables, so they are independent. Therefore, by Lemma 1.4, the face described by $x_{1,n} \leq 1$ has dimension at most

$$m_n - (n + 1) = m_n - (n - 1) - 2 = m_{n-1} - 2. \ \square$$

Finally, we turn our attention to arcs in A(S).

Proposition 3.10: For $n \geq 5$, the inequality $x_a \leq 1$ describes a face of $T_{2,n}$ of dimension at most $m_n - 6$ for any arc $a \in A(S)$.

Proof: Without loss of generality, let $a = (n-1, n)$. As in the proof of the previous proposition, any P-tree satisfying $x_{n-1,n} = 1$ must also satisfy the two equations (3.1) and (3.2) as well as:

$$1)\ x_{1,2} = 0,$$

$$2) \sum_{k=1}^{n-2} x_{k,n} = 1,$$

$$3) \sum_{k=1}^{n-2} x_{k,n-1} = 1.$$

Equation (1) tells us that arc (1, 2) cannot be in the tree, while equations (2) and (3) force the degree of nodes $n-1$ and n to be 2. All six equations are clearly independent, so by Lemma 1.4 the face of $T_{2,n}$ described by $x_{n-1,n} \leq 1$ is at most $m_n - 6$. $\square$

3.3 Cut-Set Inequalities

In Chapter II we examined inequalities generated from cuts in K_n and from partitions of the node set. In the case of $p = 2$, it is easy to see that the only partitions that generate valid inequalities are cuts, because the P-tree

will violate any $\geq$ inequality with 0, 1 coefficients whose right hand side is greater than 1, and the P-tree which is a (1, 2) path spanning all the nodes of K_n will violate any $\leq$ inequality with 0, 1 coefficients whose right hand side is less than $n - 1$. First, we establish which cut-set inequalities are valid.

<u>Lemma</u> <u>3.11:</u> Let $(X, \overline{X})$ be a cut in K_n. The inequality

$$\sum_{a \in (X, \overline{X})} x_a \geq 1 \qquad (3.5)$$

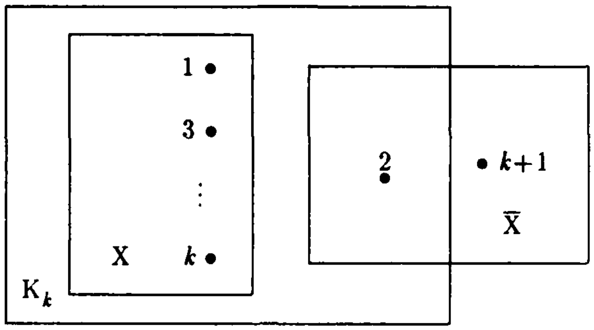
is valid for $T_{2,n}$ if and only if $X \cap P \neq \emptyset$ and $\overline{X} \cap P \neq \emptyset$.

<u>Proof:</u> Assume that (3.5) is valid. This implies that every P-tree has at least one arc in $(X, \overline{X})$. In particular, the arc (1, 2) is in $(X, \overline{X})$ which implies that $X \cap P \neq \emptyset$ and $\overline{X} \cap P \neq \emptyset$. Conversely, if this latter condition is satisfied, then at least one arc in any P-tree must cross the cut. Thus, (3.5) is valid. □

If $(X, \overline{X})$ generates a valid inequality and has the property that either X or $\overline{X}$ contains a single node, then (3.5) reduces to one of the equalities (3.1) or (3.2). So, if either X or $\overline{X}$ is a singleton, then (3.5) is an improper face of $T_{2,n}$. All other valid cuts are facets of $T_{2,n}$.

<u>Proposition</u> <u>3.12:</u> For $n \geq 4$, if $(X, \overline{X})$ is a cut in K_n such that $X \cap P \neq \emptyset$, $\overline{X} \cap P \neq \emptyset$, and $1 < |X| < n - 1$, the inequality (3.5) induces a facet of $T_{2,n}$.

Proof: The inequality is valid by Lemma 3.11, so we need only demonstrate that we can find $m_n - 2$ affinely independent P-trees satisfying $(*)$ at equality for any such cut $(X, \overline{X})$. To do this we let $|X| = k - 1 \geq 2$ be given and proceed by induction on n. Without loss of generality, let $X = \{1, 3, 4, \ldots, k\}$ and $\overline{X} = N - X$. Proceeding by induction, first let $n = k + 1$. View K_{k+1} as



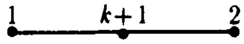
If we restrict $(*)$ to K_k we get the inequality

$$x_{1,2} + \sum_{i=3}^{k} x_{2,i} \geq 1,$$

which must be satisfied at equality by every P-tree of K_k, since it is one of the equalities which defines the space. By Proposition 3.1 there exist $m_k - 1$ linearly independent P-trees of K_k. These trees clearly satisfy the inequality $(*)$ at equality. We now construct $k - 1$ additional trees as follows. First, define

$$x_j^1 = \begin{cases} 1 & j = (1, k+1) \text{ and } (2, k+1) \\ 0 & \text{otherwise,} \end{cases}$$

which has the form



and satisfies (3.5) at equality. Now, for $i = 3, \ldots, k$ define

$$x_j^i = \begin{cases} 1 & j = (1, i),\ (2, k+1) \text{ and } (i, k+1) \\ 0 & \text{otherwise} \end{cases}$$

These trees have the form

and satisfy (3.5) at equality since the only arc in $(X, \overline{X})$ is $(i, k+1)$. We now have

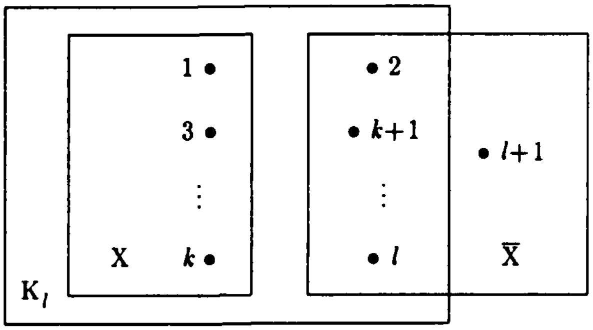
$$m_k - 1 + (k - 1) = m_{k+1} - 2$$

P-trees by Lemma 1.1. To see that these P-trees are linearly independent consider the matrix M whose columns are their characteristic vectors.

$$M = \left[\begin{array}{c|cc} Y & & X_{m_k} \\ \hline 0 & 1 & 0 \\ 0 & 1 & e_{k-2}^{T} \\ 0 & 0 & I_{k-2} \end{array}\right]$$

M clearly has linearly independent columns by Proposition 1.5. Thus, the P-trees we constructed satisfying (3.5) are linearly independent, and the proposition holds for $n = k + 1$.

Now, assume that there exist $m_l - 2$ linearly independent P-trees satisfying (3.5) at equality for some $l \geq k + 1$. We can view K_{l+1} as



Every P-tree of K_l is also a P-tree of K_{l+1}, and further, every P-tree of K_l satisfying

$$\sum_{a \in (X, \overline{X} - \{l+1\})} x_a = 1$$

will satisfy (3.5) at equality. But, by the assumption, there exist $m_l - 2$ linearly independent P-trees of K_l satisfying the above equation. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now construct l additional trees. First, let

$$x_j^1 = \begin{cases} 1 & j = (1,\ l+1) \text{ and } (2,\ l+1) \\ 0 & \text{otherwise} \end{cases}$$

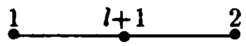
and

$$x_j^2 = \begin{cases} 1 & j = (1,\ l+1),\ (2,\ k+1) \text{ and } (k+1,\ l+1) \\ 0 & \text{otherwise} \end{cases}$$

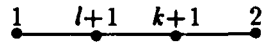
Then, for $i = 3, \ldots, l$ define

$$x_j^i = \begin{cases} 1 & j = (1,\ i),\ (2,\ l+1) \text{ and } (i,\ l+1) \\ 0 & \text{otherwise} \end{cases}$$

These l trees all clearly satisfy (3.5) at equality and they have the forms shown below.



x^1



x^2

$\mathrm{x}^i \; i = 3, \ldots, l$

Now consider the matrix whose columns are the characteristic vectors of these $m_{l+1} - 2$ trees.

$$
M = \begin{array}{c}
\begin{array}{cccccc} Y & \mathrm{x}^1_{m_l} & \mathrm{x}^2_{m_l} & \mathrm{X}_{m_l}{}' & \mathrm{x}^{k+1}_{m_l} & \mathrm{X}_{m_l}{}'' \end{array} \\
\left[\begin{array}{cccccc}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & e^T_{k-2} & 1 & e^T_{l-k-1} \\
0 & 0 & 0 & I_{k-2} & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{l-k-1}
\end{array}\right]
\end{array}
\begin{array}{l} \\ \\ \\ \\ \\ \text{row } (k+1,\, l+1) \\ \\ \end{array}
$$

Subtracting the column corresponding to x^{k+1} from the column corresponding to x^2 gives us the matrix

$$
M = \left[\begin{array}{c|cc|c}
Y & & \mathrm{X}'_{m_l} & \\
\hline
0 & 1 & 1 & 0 \\
0 & 1 & -1 & e^T_{l-2} \\
\hline
0 & 0 & 0 & I_{l-2}
\end{array}\right]
$$

which clearly has linearly independent columns by Proposition 1.5. Therefore, the P-trees that we constructed are linearly independent, and by the principle of induction (3.5) induces a facet of $T_{2,n}$ for any $n \geq 4$. □

3.4 Other Facet-Inducing Inequalities

In any P-tree, every node in S will either have degree 0, or degree 2. The inequality that expresses this fact induces a facet of $T_{2,n}$.

Proposition 3.13: Let $n \geq 4$. Then for each node $s \in S$, the inequality

$$2\, x_{1,2} + \sum_{k \in N-\{s\}} x_{s,k} \leq 2 \qquad (3.6)$$

induces a facet of $T_{2,n}$.

Proof: Any P-tree falls into eactly one of the following three cases.

1) The P-tree consists of the arc (1, 2) in which case (3.6) is satisfied at equality.

2) The arc (1, 2) is not in the P-tree and

   a) node s is in the tree having degree 2. In this case (3.6) is satisfied at equality.

   b) node s is not in the tree, so the left hand side of (3.6) is 0.

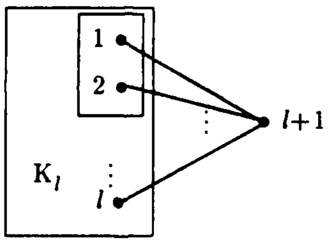
Thus, the inequality is valid. Now, we must show that for any value of n $\geq$ 4 and any node $s \in S$, there exist $m_n - 2$ affinely independent P-trees satisfying (3.6) at equality. We show, in fact, that the required number of P-trees are linearly independent. Without loss of generality, let s =3. We proceed by induction on n. For $n = 4$, there are exactly $m_4 - 2 = 4$ P-trees that satisfy (3.6) at equality, namely:

The matrix whose columns are the characteristic vectors of these trees is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which is easily seen to have full column rank.

Now, assume that there exist $m_l - 2$ linearly independent P-trees of K_l that satisfy (3.6) at equality for some $l \geq 4$. We can view K_{l+1} as:



The assumed $m_l - 2$ linearly independent P-trees of K_l satisfying

$$2\,x_{1,2} + \sum_{k \in N\text{-}\{3,l+1\}} x_{s,k} \leq 2$$

are also P-trees of K_{l+1} satisfying

$$2\,x_{1,2} + \sum_{k \in N\text{-}\{3\}} x_{s,k} \leq 2.$$

Let Y the matrix whose columns are the characteristic vectors of these trees. Now construct l additional P-trees as follows. First, let

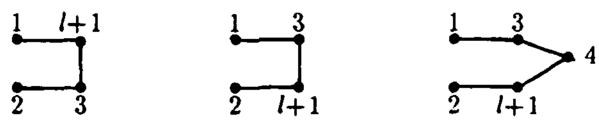
$$x_j^1 = \begin{cases} 1 & j = (2, 3), (3, l+1) \text{ and } (1, l+1) \\ 0 & \text{otherwise} \end{cases}$$

$$x_j^2 = \begin{cases} 1 & j = (1, 3), (2, l+1) \text{ and } (3, l+1) \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_j^3 = \begin{cases} 1 & j = (1, 3), (3, 4), (4, l+1) \text{ and } (2, l+1) \\ 0 & \text{otherwise} \end{cases}$$

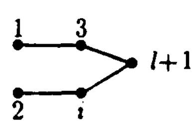
These trees have the forms



Finally, for $i = 4, \ldots, l$ define the trees

$$x_j^i = \begin{cases} 1 & j = (1, 3), (2, i), (3, l+1) \text{ and } (i, l+1) \\ 0 & \text{otherwise} \end{cases}$$

which have the form



All of these trees satisfy (3.6) at equality. The matrix whose columns are the characteristic vectors of these trees has the form

$$
M = \begin{array}{c}
\begin{array}{cccccc} \mathrm{Y} & \mathrm{x}^1_{m_l} & \mathrm{x}^2_{m_l} & \mathrm{x}^3_{m_l} & \mathrm{x}^4_{m_l} & \mathrm{X}_{m_l}{}' \end{array} \\
\left[\begin{array}{c|cccc|c}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & \mathrm{e}^{\mathsf{T}}_{l\text{-}4} \\
0 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \mathrm{I}_{l\text{-}4}
\end{array}\right]
\end{array}
$$

This matrix has linearly independent columns by Proposition 1.5. Therefore, the $m_{l+1} - 2$ P-trees which we constructed that satisfied (3.6) at equality are linearly independent, and by induction this proves that (3.6) induces a facet of $\mathrm{T}_{2,n}$ for $n \geq 4$ and any node $s \in \mathrm{S}$. $\square$

Chapter IV

The P-tree Polytope $T_{p,n}$

In this chapter we consider the case of $3 \leq p \leq n-1$, and we look at the polytope

$$T_{p,n} = \text{conv}\{\, x \in \mathbf{R}^{m_n} \mid x \text{ is the characteristic vector of a P-tree on } K_n\}.$$

Prior to proving that the polytope is full dimensional, we set up some tools that will facilitate all the proofs in this chapter.

Lemma 4.1: If a matrix has one of the following forms

$$Q_k^1 = \left[I_k \; e_k \right] \quad k \geq 2,$$

$$Q_k^2 = \begin{bmatrix} 1 & e_{k-1}^\top \\ e_{k-1} & I_{k-1} \end{bmatrix} \quad k \geq 3,$$

$$Q_k^3 = \begin{bmatrix} 1 & 0 & e_{k-1}^\top \\ e_{k-1} & e_{k-1} & I_{k-1} \end{bmatrix} \quad k \geq 3$$

then the columns of that matrix are affinely independent. Furthermore, the columns are linearly independent if the matrix is equivalent to Q_k^2.

Proof: Q_k^1: Consider the $(k+1) \times (k+1)$ matrix

$$\overline{Q_k^1} = \begin{bmatrix} I_k & e_k \\ e_k^T & 1 \end{bmatrix}$$

Subtracting the first k rows from the last row gives the equivalent matrix

$$\begin{bmatrix} I_k & e_k \\ 0 & 1 - k \end{bmatrix},$$

which has linearly independent columns for $k \geq 2$ by Proposition 1.5. Thus, Q_k^1 has affinely independent columns for $k \geq 2$.

Q_k^2: Rearranging the rows and columns of Q_k^2 gives us the equivalent matrix

$$\begin{bmatrix} I_{k-1} & e_{k-1} \\ e_{k-1}^T & 1 \end{bmatrix}.$$

This matrix, however, is identical to $\overline{Q_{k-1}^1}$, which has full rank if $k - 1 \geq 2$. Therefore, for $k \geq 3$ the columns of Q_k^2 are linearly independent, and hence affinely independent.

Q_k^3: Consider the $(k+1) \times (k+1)$ matrix

$$\overline{Q_k^3} = \begin{bmatrix} 1 & 0 & e_{k-1}^T \\ e_{k-1} & e_{k-1} & I_{k-1} \\ 1 & 1 & e_{k-1}^T \end{bmatrix}$$

subtracting the second column from the first yields the equivalent matrix

$$\left[\begin{array}{c|cc} 1 & 0 & e_{k-1}^T \\ \hline 0 & e_{k-1} & I_{k-1} \\ 0 & 1 & e_{k-1}^T \end{array}\right]$$

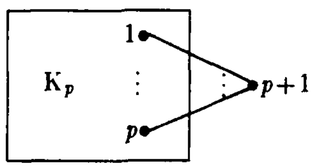
The lower right $k \times k$ submatrix is equivalent to Q_k^2, which has linearly independent columns for $k \geq 3$. Thus, by Proposition 1.5, the columns of $\overline{Q_k^3}$ are linearly independent. Thus $\overline{Q_k^3}$ has full rank, and the columns of Q_k^3 are affinely independent. □

As in Chapters II and III, many of the proofs will be inductive, and will either through assumption or by invoking a previous result, establish the existence of a set of k affinely independent P-trees of K_q for $q = p$, $(n - 1)$, or some generic l. As in the previous chapters we define $\{y^i\}_{i=1}^{k}$ to be the set of characteristic vectors in $\mathbf{R}^{m_q}$of these k P-trees, and Y to be the matrix whose columns are these vectors. The values of q and k will be clear from the context of the proof. Now we proceed to prove that the polytope $T_{p,n}$ has full dimension.

4.1 The Dimension of $T_{p,n}$

Proposition 4.2: For any $p \geq 3$, there are $m_{p+1} + 1$ affinely independent P-trees in K_{p+1}.

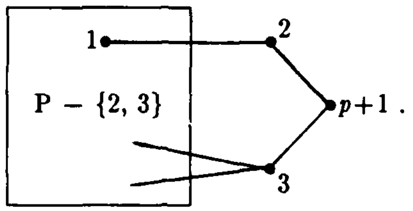
Proof: Let $p \geq 3$ be given. Then K_{p+1} has the form



Clearly, every spanning tree of P is a P-tree of K_{p+1}, and by Lemma 2.1 there are m_p linearly independent spanning trees of K_p. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now define $p + 1$ additional P-trees as follows. First

$$x_j^1 = \begin{cases} 1 & j = (1, 2), (2, p+1) \text{ and } (3, p+1) \\ 1 & j = (3, r) \quad 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

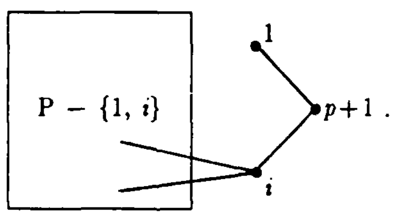
x^1 is the spanning tree of K_{p+1} shown below.



Next, for $i = 2, \ldots, p$ define

$$x_j^i = \begin{cases} 1 & j = (1,\ p+1) \text{ and } (i,\ p+1) \\ 1 & j = (r,\ i) \quad 2 \leq r < i \\ 1 & j = (i,\ r) \quad i < k \leq p \\ 0 & \text{otherwise} \end{cases}$$

This family of $p - 1$ spanning trees of K_{p+1} has the form shown below.



Finally, define the P-tree x^{p+1} by

$$x_j^{p+1} = \begin{cases} 1 & j = (r,\ p+1) \quad 1 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

which has all the nodes of P connected to node $p+1$ as leaves. The number of P-trees is

$$m_p + p + 1 = m_{p+1} + 1$$

from Lemma 1.1. As before we demonstrate their affine independence by letting M be the matrix whose columns are the vectors X and Y and looking at

$$\overline{\mathrm{M}} = \begin{bmatrix} \mathrm{Y} & \mathrm{x}^1_{m_p} & \mathrm{x}^2_{m_p} & \mathrm{x}^3_{m_p} & \mathrm{X}'_{m_p} & 0 \\ 0 & 0 & 1 & 1 & \mathrm{e}^{\top}_{p\text{-}3} & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \mathrm{I}_{p\text{-}3} & \mathrm{e}_{p\text{-}3} \\ \mathrm{e}^{\top}_{m_p} & 1 & 1 & 1 & \mathrm{e}^{\top}_{p\text{-}3} & 1 \end{bmatrix}$$

The columns of Y are spanning trees of K_p and thus contain $p - 1$ 1's. By their construction the columns of X_{m_p} each contain $p - 2$ 1's. Thus, if we multiply the last row of $\overline{\mathrm{M}}$ by $p - 1$ and subtract each of the first m_p rows we get the matrix

$$\overline{\mathrm{M}}' = \begin{bmatrix} \mathrm{Y} & \mathrm{x}^1_{m_p} & \mathrm{x}^2_{m_p} & \mathrm{x}^3_{m_p} & \mathrm{X}'_{m_p} & 0 \\ 0 & 0 & 1 & 1 & \mathrm{e}^{\top}_{p\text{-}3} & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \mathrm{I}_{p\text{-}3} & \mathrm{e}_{p\text{-}3} \\ 0 & 1 & 1 & 1 & \mathrm{e}^{\top}_{p\text{-}3} & p - 1 \end{bmatrix}$$

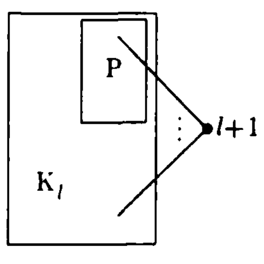
Now multiply the last row by 2 and subtract each of the p rows immediately above it. The final result is the matrix

$$\overline{M}'' = \left[\begin{array}{c|cccc|c}
Y & x^1_{m_p} & x^2_{m_p} & x^3_{m_p} & X'_{m_p} & 0 \\
\hline
0 & 0 & 1 & 1 & e^T_{p\text{-}3} & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & I_{p\text{-}3} & e_{p\text{-}3} \\
0 & 0 & 0 & 0 & 0 & p-2
\end{array}\right]$$

which is upper block triangular with diagonal blocks that have linearly independent columns. Thus, by Proposition 1.5, the columns of $\overline{M}$ are linearly independent and, hence, the P-trees we constructed are affinely independent. □

Proposition 4.3: For $3 \leq p \leq n - 1$, $\dim(T_{p,n}) = m_n$.

Proof: Let $p \geq 3$ be given. We need to show that there are $m_n + 1$ affinely independent P-trees of K_n for all values of $n \geq p + 1$. We proceed by induction on n. First, for $n = p + 1$, we have $m_p + 1$ affinely independent points in $T_{p,n}$ by Lemma 4.2. Now assume that there are $m_l + 1$ affinely independent points in $T_{p,l}$ for some $l \geq p + 1$. This implies that there are $m_l + 1$ affinely independent P-trees in K_l. Look at K_{l+1}.



All P-trees of K_l are also P-trees of K_{l+1}. Let Y be the matrix whose columns are the characteristic vectors of a set of $m_l + 1$ affinely independent P-trees of K_l. Now form l

additional P-trees as follows. First, define x^1 by

$$x_j^1 = \begin{cases} 1 & j = (r,\ l+1) \quad 1 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

This corresponds to the P-tree with all nodes in P adjacent to node $l+1$, and is illustrated in (1) of Figure 4-1. For $i = 2, 3, \ldots, p$ define

$$x_j^i = \begin{cases} 1 & j = (1,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (r,\ i) \quad 2 \le r < i \\ 1 & j = (i,\ r) \quad i < k \le p \\ 0 & \text{otherwise} \end{cases}$$

which corresponds to the family of trees shown in (2) of Figure 4-1. Finally, for $i = p + 1$ to l define

$$x_j^i = \begin{cases} 1 & j = (1,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (r,\ i) \quad 2 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

giving us $l - p$ P-trees of form (3) in Figure 4-1.

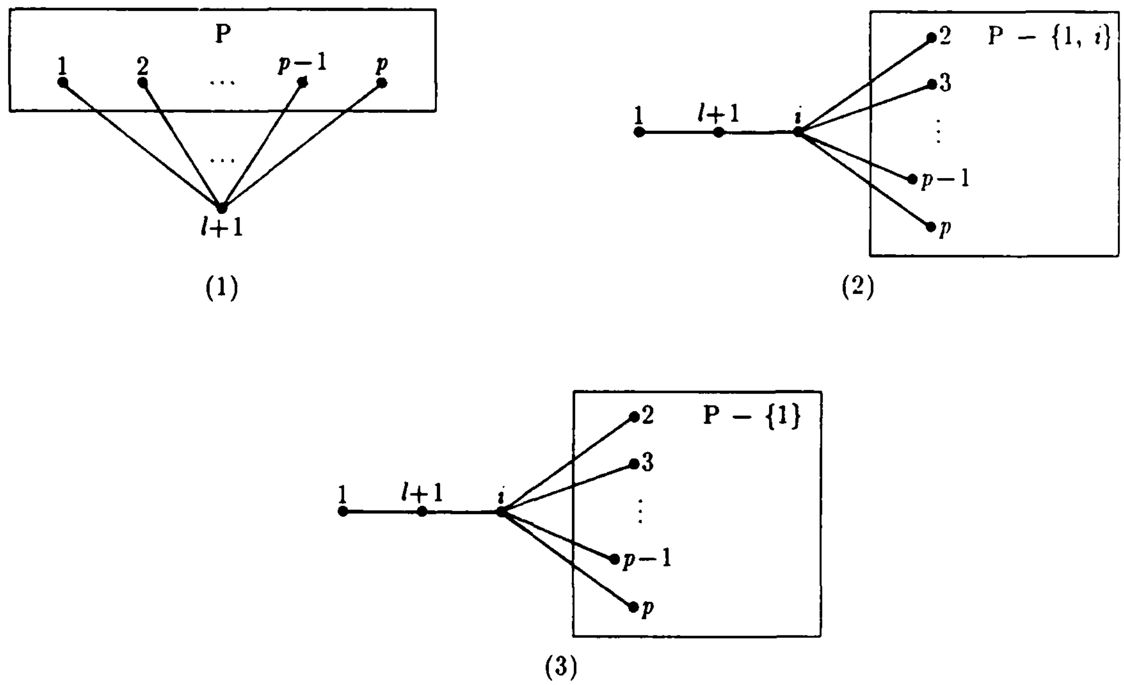


Figure 4-1

By Lemma 1.1 we now have $m_{l+1} + 1$ P-trees, and all that remains to be shown is that they are affinely independent. So consider the matrix whose columns are the vectors X and Y. This matrix has the form

$$M = \left[\begin{array}{c|ccc} Y & & X_{m_l} & \\ \hline 0 & 1 & e_{p-1}^T & e_{l-p}^T \\ 0 & e_{p-1} & I_{p-1} & 0 \\ 0 & 0 & 0 & I_{l-p} \end{array}\right]$$

The center diagonal submatrix is equivalent to Q_p^2, and since $p \geq 3$, this submatrix has linearly independent columns by Lemma 4.1. Thus, M has affinely independent columns by Proposition 1.6. Therefore,

$$\dim(T_{p,l+1}) = m_{l+1}.$$

And so by induction,

$$\dim(T_{p,n}) = m_n$$

for all values of n with $3 \leq p \leq n - 1$. □

Now that we have established the dimension of $T_{p,n}$, we turn our attention to the hypercube bounding inequalities.

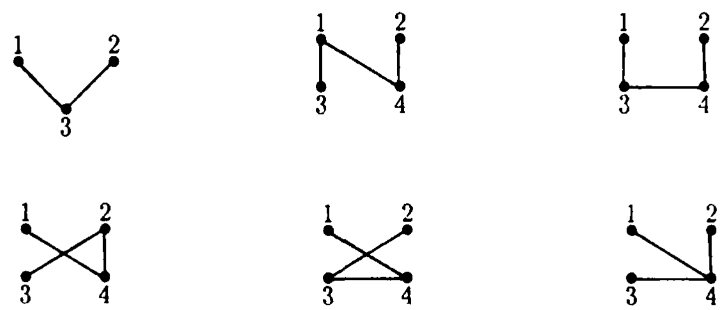
4.2 The Trivial Inequalities

4.2.1 The Lower-Bound Inequalities

As in Chapter III we need to consider the location of the arc. First we consider the special case of $p = 3$.

Proposition 4.4: For $p = 3$ and $n \geq 4$, $x_a \geq 0$ describes a facet of $T_{3,n}$ for all arcs $a \in A(P)$.

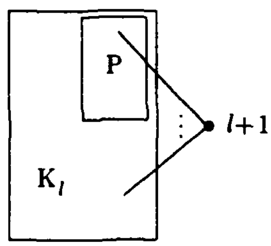
Proof: Without loss of generality, assume that $a = (1, 2)$. We need to list m_n affinely independent points satisfy-ing $x_{1,2} = 0$ for all values of $n \geq 4$. We proceed by induction on n. For $n = 4$ there are six arcs ($m_4 = 6$) and there are only six P-trees satisfying $x_{1,2} = 0$. They are



Let M be the matrix of characteristic vectors of these trees, then

$$\overline{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which is easily seen to have rank 6. So for $n = 4$ there are $m_4 = 6$ affinely independent P-trees, and $x_{1,2} \geq 0$ describes a facet of $T_{3,4}$. Now assume that there are m_l affinely independent P-trees of K_l satisfying $x_{1,2} = 0$ for some $l \geq 4$. View K_{l+1} as



and again note, that every P-tree of K_l is also a P-tree of K_{l+1}. Let Y be the matrix whose columns are a set of m_l affinely independent characteristic vectors of P-trees satisfying $x_{1,2} = 0$, and define l additional trees in the following manner. Let

$$x_j^1 = \begin{cases} 1 & (1, l+1), (2, l+1) \text{ and } (3, l+1) \\ 0 & \text{otherwise} \end{cases}$$

$$x_j^2 = \begin{cases} 1 & (1, l+1), (2, l+1) \text{ and } (2, 3) \\ 0 & \text{otherwise} \end{cases}$$

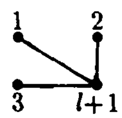
and

$$x_j^3 = \begin{cases} 1 & (1,\ l+1),\ (3,\ l+1) \text{ and } (2,\ 3) \\ 0 & \text{otherwise} \end{cases}$$

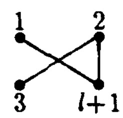
For the last $l - 3$ trees define for $i = 4, \ldots, l$

$$x_j^i = \begin{cases} 1 & j = (1,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (2,\ i) \text{ and } (3,\ i) \\ 0 & \text{otherwise} \end{cases}$$

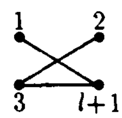
All of these trees satisfy $x_{1,2} = 0$. They have the forms shown below.



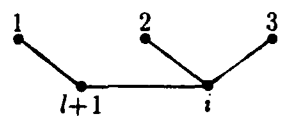
x^1



x^2



x^3



$x^i,\ i = 4, \ldots, l$

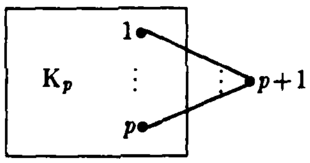
We now have $m_l + l = m_{l+1}$ P-trees. If they are affinely independent, then $x_{1,2} \geq 0$ induces a facet of $T_{3,l+1}$. The matrix whose columns are the columns of Y and X is

$$M = \left[\begin{array}{c|ccc|c} Y & & X_{m_l} & & \\ \hline 0 & 1 & 1 & 1 & e_{l-3}^T \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{l-3} \end{array}\right]$$

Clearly by Proposition 1.6, the columns of M are affinely independent. Thus, the m_{l+1} columns of Y and X are affinely independent, and $x_{1,2} \geq 0$ induces a facet of $T_{3,l+1}$. So by the principle of induction, the lower bound inequality $x_a \geq 0$ defines a facet of $T_{3,n}$ for all n and any arc $a \in A(P)$. □

<u>Proposition 4.5:</u> For $p \geq 4$ and $n \geq p + 1$, the inequality $x_a \geq 0$ describes a facet of $T_{p,n}$ for all arcs $a \in A(P)$.

<u>Proof:</u> Let $p \geq 4$ be given. Then, without loss of generality, we assume that $a = (1, 2)$. We now demonstrate m_n affinely independent P-trees satisfying $x_{1,2} = 0$ for all $n \geq p + 1$. Working by induction, let $n = p + 1$ and view K_{p+1} as:



Every spanning tree of K_p is a P-tree of K_{p+1}. By Proposition 2.3 there exists a set of $m_p - 1$ linearly independent spanning trees of K_p satisfying $x_{1,2} = 0$. Let Y be the matrix whose columns are characteristic vectors of these spanning trees. Now construct $p + 1$ additional P-trees as follows. For $i = 2, \ldots, p$ define

$$x_j^{i-1} = \begin{cases} 1 & j = (1,\ p+1) \text{ and } (i,\ p+1) \\ 1 & j = (r,\ i) \quad 2 \leq r < i \\ 1 & j = (i,\ r) \quad i < k \leq p \\ 0 & \text{otherwise} \end{cases}$$

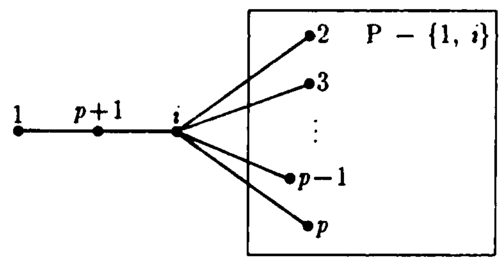
Then define

$$x_j^p = \begin{cases} 1 & j = (r, p+1) \;\; 1 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

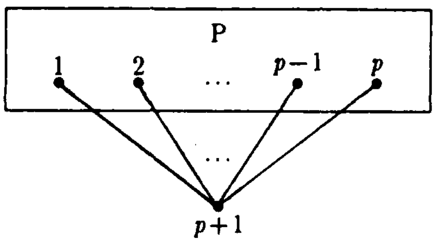
and

$$x_j^{p+1} = \begin{cases} 1 & j = (r, p+1) \;\; 2 \le r \le p \\ 1 & j = (1, 3) \\ 0 & \text{otherwise} \end{cases} \quad .$$

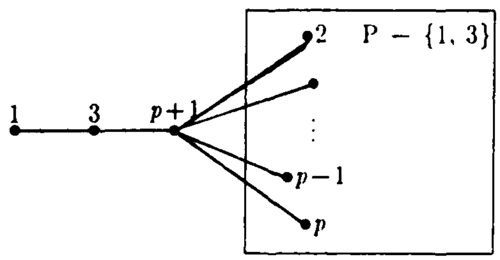
These trees have the forms



$x^{i\text{-}1}$, $i = 2, \ldots, p$



x^p



x^{p+1}

We now have m_{p+1} P-trees by Lemma 1.1. Let M be the matrix (Y, X), and look at

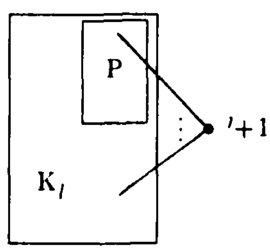
$$\overline{\mathrm{M}} = \begin{bmatrix} \mathrm{Y} & \mathrm{X'}_{m_p} & 0 & \mathrm{x}_{m_p}^{p+1} \\ 0 & \mathrm{e}_{p\text{-}1}^{\mathrm{T}} & 1 & 0 \\ 0 & \mathrm{I}_{p\text{-}1} & \mathrm{e}_{p\text{-}1} & \mathrm{e}_{p\text{-}1} \\ \mathrm{e}_{m_p\text{-}1}^{\mathrm{T}} & \mathrm{e}_{p\text{-}1}^{\mathrm{T}} & 1 & 1 \end{bmatrix}$$

The columns of Y are spanning trees of K_p and hence contain $p - 1$ 1's. The columns of $\mathrm{X'}_{m_p}$ each contain $p - 2$ 1's by construction, and $\mathrm{x}_{m_p}^{p+1}$ contains exactly one 1. So multiply the last row by $p - 1$ and subtract the first m_p rows and the rows containing the submatrix $\mathrm{I}_{p\text{-}1}$ from it. The resulting matrix is

$$\overline{\mathrm{M}}' = \left[\begin{array}{c|ccc|c} \mathrm{Y} & \mathrm{X'}_{m_p} & 0 & & \mathrm{x}_{m_p}^{p+1} \\ \hline 0 & \mathrm{e}_{p\text{-}1}^{\mathrm{T}} & 1 & & 0 \\ 0 & \mathrm{I}_{p\text{-}1} & \mathrm{e}_{p\text{-}1} & & \mathrm{e}_{p\text{-}1} \\ \hline 0 & 0 & 0 & & -1 \end{array}\right]$$

which, by lemma 4.1, satisfies the conditions of Proposition 1.5 for having linearly independent columns. Thus the m_{p+1} P-trees that we constructed are affinely independent. So there are m_{p+1} affinely independent P-trees of K_{p+1} satisfying $\mathrm{x}_{1,2} = 0$.

Now assume that there are m_l affinely independent P-trees of K_l for some $l \geq p + 1$, and look at K_{l+1}.



Any P-tree of K_l is also a P-tree of K_{l+1}. Let Y be the matrix whose columns are the characteristic vectors of a set of m_l affinely independent P-trees of K_l which satisfy $x_{1,2} = 0$. Using the same construction as in the induction step of the proof of Proposition 4.3 gives a matrix M with the same form as that shown on page 64, but in the present case the columns of Y are affinely independent and not linearly independent. Since all columns of X satisfy $x_{1,2} = 0$ by construction, Proposition 1.6 enables us to conclude that the columns of (Y, X) are the characteristic vectors of $m_l + l = m_{l+1}$ affinely independent P-trees of K_{l+1}, all of which satisfy $x_{1,2} = 0$. Thus, by the induction principle, the inequality $x_{1,2} \geq 0$, and hence any inequality of the form $x_a \geq 0$, $a \in A(P)$, is a facet of $T_{p,n}$ for any $n \geq p + 1$. □

We turn now to the arcs in (P, S), and immediately need to treat a special case.

<u>Lemma</u> <u>4.6:</u> For $p = 3$ and $n = 4$, the inequality $x_a \geq 0$ defines a face of $T_{3,4}$ of dimension 3 for all arcs $a \in (P, S)$.

<u>Proof:</u> If an arc $a \in (P, S)$, say $a = (1, 4)$ is not allowed to be in a P-tree, then the only possible P-trees are the three spanning trees of P and the two trees of the form:

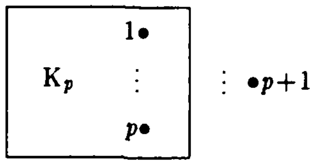
If M is a matrix with these five trees as columns, then

$$\overline{M} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

can quickly be seen to have rank 4. So the dimension of the face described by the inequality $x_a \geq 0$ is 3. □

<u>Proposition</u> <u>4.7:</u> For $p \geq 4$ and $n = p + 1$, the inequality $x_a \geq 0$ induces a facet of $T_{p,p+1}$ for any arc $a \in (P, S)$.

<u>Proof:</u> Let $p \geq 4$ be given. Without loss of generality, let $a = (1, p+1)$. The inequality $x_{1,p+1} \geq 0$ is clearly valid for $T_{p,p+1}$, so we need only show that there are m_{p+1} affinely independent P-trees in K_{p+1} satisfying $x_{1,p+1} = 0$. K_{p+1} can be viewed as



Any spanning tree of P is a P-tree of K_{p+1} satisfying $x_{1,p+1} = 0$. By Proposition 2.1 there are m_p linearly independent spanning trees of P. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now define p additional P-trees. First, for $i = 3, \ldots, p$ define

$$x_j^{i-2} = \begin{cases} 1 & j = (1, 2), (2, p+1) \text{ and } (i, p+1) \\ 1 & j = (r, i) \quad 3 \leq r < i \\ 1 & j = (i, r) \quad i < k \leq p \\ 0 & \text{otherwise} \end{cases}$$

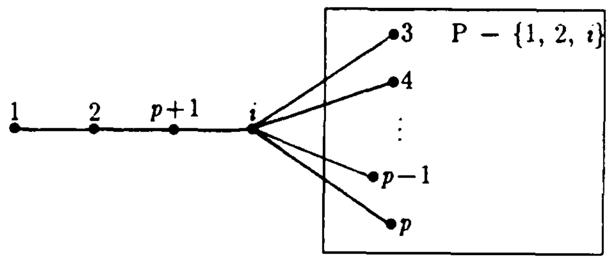
Then define the two trees

$$x_j^{p-1} = \begin{cases} 1 & j = (1, 2) \\ 1 & j = (r, p+1) \quad 2 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

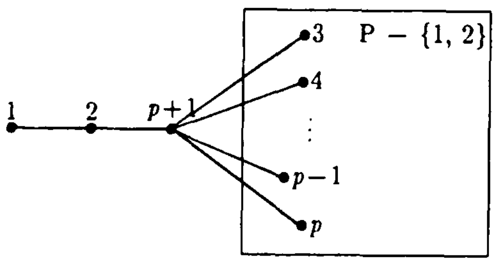
and

$$x_j^p = \begin{cases} 1 & j = (1, 3) \text{ and } (2, 3) \\ 1 & j = (r, p+1) \quad 3 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

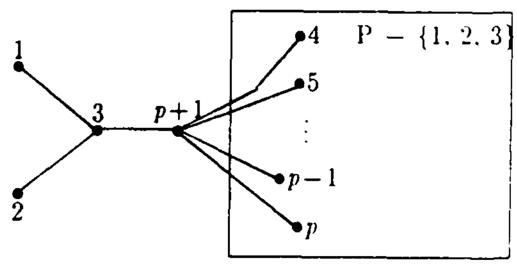
These trees satisfy $x_{1,p+1} = 0$, and are shown below.



$x^{i-2}, \quad i = 3, \ldots, p$



x^{p-1}



x^{p}

Let M be the matrix (Y, X). Since row $(1, p+1)$ is all 0's, drop it and look at

$$\overline{M}' = \begin{array}{cl} \begin{array}{cccc} Y & X'_{m_p} & x^{p-1}_{m_p} & x^{p}_{m_p} \end{array} & \\ \left[\begin{array}{cccc} 0 & e^T_{p-2} & 1 & 0 \\ 0 & I_{p-2} & e_{p-2} & e_{p-2} \\ e^T_{m_p} & e^T_{p-2} & 1 & 1 \end{array}\right] & \begin{array}{l} \text{row } (2, p+1) \\ \\ \\ \end{array} \end{array}$$

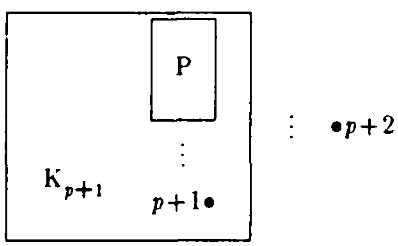
The columns of Y are spanning trees of K_p and hence contain $p - 1$ 1's. The columns of X'_{m_p} each contain $p - 2$ 1's by construction. Finally, vector $x^{p-1}_{m_p}$ contains exactly one 1 and vector $x^{p+1}_{m_p}$ contains two 1's. So multiply the last row by $p - 1$ and subtract the first m_p rows and the rows containing the submatrix I_{p-2} from it. the resulting matrix is

$$\overline{M}'' = \left[\begin{array}{c|ccc} Y & X'_{m_p} & x^{p-1}_{m_p} & x^{p}_{m_p} \\ \hline 0 & e^T_{p-2} & 1 & 0 \\ 0 & I_{p-2} & e_{p-2} & e_{p-2} \\ \hline 0 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} \\ \text{row } (2, p+1) \\ \\ \\ \end{array}$$

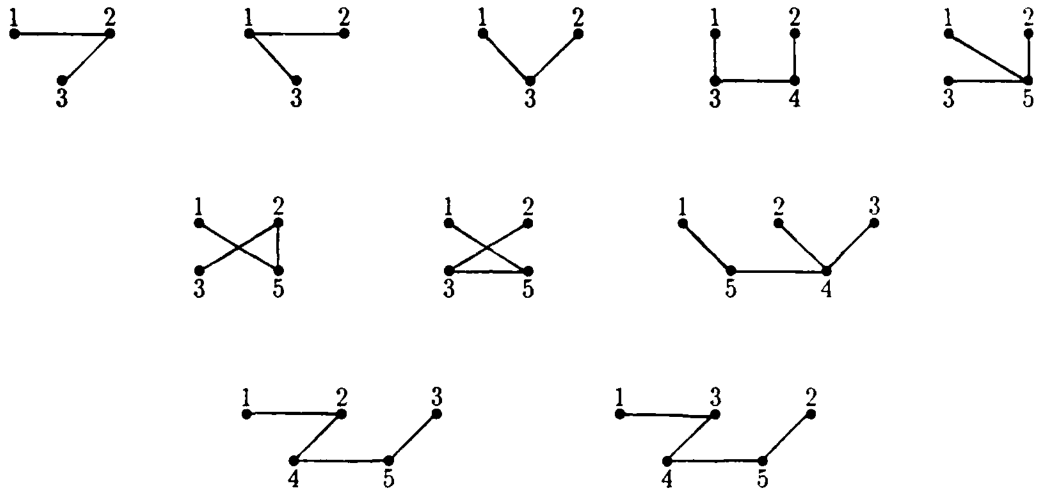
which has linearly independent columns by Propositions 4.1 and 1.5 . Thus, the m_{p+1} P-trees that we constructed are affinely independent and $x_a \geq 0$ induces a facet of $T_{p,p+1}$ for all arcs $a \in A$. □

<u>Proposition 4.8:</u> For $p \geq 3$ and $n \geq p + 2$, the inequality $x_a \geq 0$ induces a facet of $T_{p,n}$ for all arcs $a \in (P, S)$.

<u>Proof:</u> Let $p \geq 3$ be given. Without loss of generality, assume $a = (1, p+1)$. The inequality is clearly valid for $T_{p,n}$, so we need to find m_n affinely independent P-trees that satisfy $x_a = 0$. We proceed by induction on n. For $n = p + 2$, we view K_{p+2} as



and have two cases to consider. If $p \geq 4$, then by Proposition 4.7, there are m_{p+1} affinely independent P-trees of K_{p+1} satisfying $x_{1,p+1} = 0$. We then define $p + 1$ additional P-trees exactly as was done in the proof of Proposition 4.3 with $l = p + 1$. By their construction these trees satisfy $x_{1,p+1} = 0$, and by Proposition 1.6, the total set of m_{p+2} P-trees are affinely independent, so we are done. If $p = 3$, however, we need to look at the following 10 P-trees that satisfy $x_{1,4} = 0$.

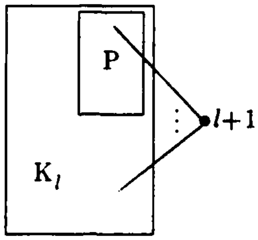


Let M be the matrix of characteristic vectors of these trees, then

$$\overline{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which has rank 10. So these trees form the set we need.

Now assume that there are m_l affinely independent P-trees of K_l, which do not use arc $a = (1, p + 1)$, for some $l \geq p + 2$, and look at K_{l+1}.

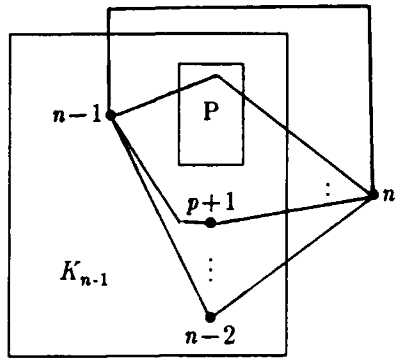


By the assumption, there are m_l affinely independent P-trees of K_l satisfying $x_{1,p+1} = 0$, and every P-tree of K_l is also a P-tree of K_{l+1}. Let Y be the matrix whose columns are the characteristic vectors of these trees, and X be the matrix whose columns are the characteristic vectors of the l P-trees constructed in the proof of Proposition 4.3. We see that we now have a set of m_{l+1} P-trees that satisfy $x_{1,p+1} = 0$ and are affinely independent by Proposition 1.6. Therefore $x_{1,p+1} \geq 0$ describes a facet of $T_{p,l+1}$. By the induction principle $x_a \geq 0$ will describe a facet of $T_{p,n}$ for $p \geq 3$ and $n \geq p + 2$ and all arcs $a \in (P, S)$. □

Last, we turn to the lower-bound inequalities involving arcs in A(S).

Proposition 4.9: For $p \geq 3$ and $n \geq p + 2$, the inequality $x_a \geq 0$ induces a facet of $T_{p,n}$ for any arc $a \in A(S)$.

Proof: Let $p \geq 3$ be given. Without loss of generality, let $a = (n-1, n)$. The inequality is clearly valid, so we need only demonstrate m_n affinely independent P-trees satisfying $x_{n-1,n} = 0$. View K_n as



The diagram is drawn for $n \geq p + 4$, for the purpose of clarity, but as will become clear by the construction of the trees satisfying $x_{n\text{-}1,n} = 0$, we can prove the result for $n \geq p + 2$. By Proposition 4.2, there are $m_{n\text{-}1} + 1$ affinely independent P-trees of $K_{n\text{-}1}$. None of these trees uses arc $(n-1,\ n)$. Let Y be the matrix whose columns are the characteristic vectors of these trees. We can now construct $n - 2$ additional P-trees by letting

$$x_j^1 = \begin{cases} 1 & j = (r,\ n) \quad 1 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

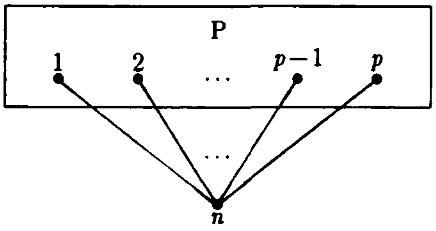
For $i = 2, \ldots, p$ define

$$x_j^i = \begin{cases} 1 & j = (1,\ n) \text{ and } (i,\ n) \\ 1 & j = (r,\ i) \quad 2 \leq r < i \\ 1 & j = (i,\ r) \quad i < k \leq p \\ 0 & \text{otherwise} \end{cases}$$

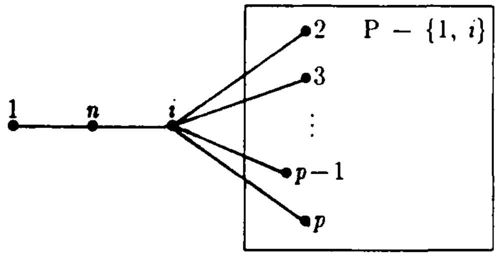
and if $n > p + 2$, then for $i = p + 1$ to $n - 2$ define

$$x_j^i = \begin{cases} 1 & j = (1,\ n) \text{ and } (i,\ n) \\ 1 & j = (r,\ i) \quad 2 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

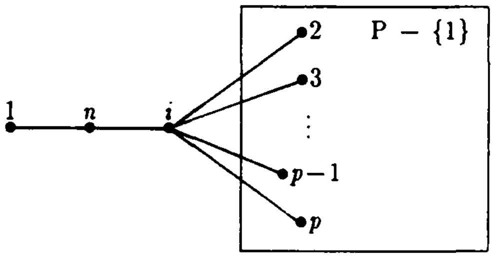
These $n - 2$ vectors satisfy $x_{n\text{-}1,n} = 0$ since they do not use arc $(n-1,\ n)$, and correspond to the trees



x^1



x^2



$x^i, \; i = p + 1, \ldots, n - 2$

The total number of trees is

$$m_{n-1} + 1 + (n - 2) = m_{n-1} + (n - 1) = m_n .$$

To see that these vectors are affinely independent let M be the matrix (Y, X).

$$M = \begin{array}{cl} \left[\begin{array}{c|cc|c} Y & & X_{m_{n-1}} & \\ \hline 0 & 1 & e_{p-1}^{\top} & e_{n-2-p}^{\top} \\ 0 & e_{p-1} & I_{p-1} & 0 \\ \hline 0 & 0 & 0 & I_{n-2-p} \\ \hline 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \text{row } (n-1,\, n) \end{array} \end{array}$$

Dropping row $(n-1, n)$ from M, we see that the matrix has affinely independent columns by

Proposition 1.6. Thus the inequality $x_a \geq 0$ induces a facet of $T_{p,n}$ for $p \geq 3$, $n \geq p + 2$ and all arcs $a \in A(S)$. □

In summary, except in the case of $p = 3$, $n = 4$ with $a \in (P, S)$, the inequality $x_a \geq 0$ induces a facet of $T_{p,n}$, for $p \geq 3$, $n \geq p + 1$ and all arcs $a \in A$. This will not prove to be the case for the upper-bound inequalities.

4.2.2 The Upper-Bound Inequalities

As in the case of the lower bounds, we will consider the inequalities by the location of the arc a, starting with $a \in A(P)$. We immediately have a special case.

Proposition 4.10: For $p = 3$, $n \geq p + 1$, and $a \in A(P)$, the inequality $x_a \leq 1$ induces a face of $T_{3,n}$ of dimension at most $m_n - 3$.

Proof: Without loss of generality, let $a = (1, 2)$. To establish this upper bound on the dimension of the face we demonstrate that every P-tree that satisfies $x_{1,2} = 1$ must also satisfy two other independent equations. Since nodes 1 and 2 are connected, node 3 must be a leaf, so we have the equation

$$x_{1,3} + x_{2,3} + \sum_{j=4}^{n} x_{3,j} = 1.$$

Also, either node 1 or node 2 must be the other leaf, so only one other arc can be incident to either of nodes 1 and 2. This can be expressed in the equation

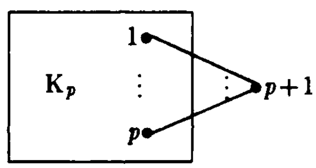
$$\sum_{j=3}^{n} (x_{1,j} + x_{2,j}) = 1.$$

Each of the three equations deals with different variables, so they are independent. Therefore, there are three independent equations that all P-trees which include arc (1, 2) must satisfy. Thus, by Lemma 1.3, the dimension of the face described by $x_{1,2} \leq 1$ is at most $m_n - 3$. □

For $p \geq 4$, however, we have the following result.

<u>Proposition 4.11:</u> For $p \geq 4$ and $n \geq p + 1$, the inequality $x_a \leq 1$ describes a facet of $T_{p,n}$ for each arc $a \in A(P)$.

<u>Proof:</u> Let $p \geq 4$ be given. Without loss of generality, let $a = (1, 2)$. The inequality is valid for $T_{p,n}$, so we list m_n linearly independent P-trees of $T_{p,n}$ satisfying $x_{1,2} = 1$ for any $n \geq p + 1$. By induction, let $n = p + 1$ and view K_{p+1} as



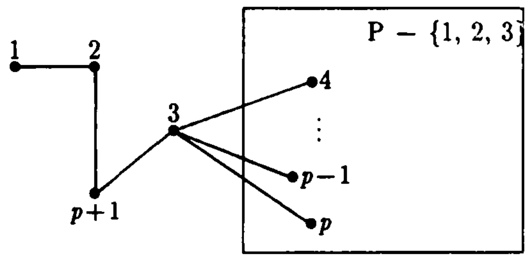
Every spanning tree of P is a P-tree of K_{p+1}, and by Proposition 2.2, there exists a set of $m_p - 1$ linearly independent spanning trees of P satisfying $x_{1,2} = 1$. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now define two P-trees by

$$x^1_j = \begin{cases} 1 & j = (1,\ 2),\ (2,\ p+1) \text{ and } (3,\ p+1) \\ 1 & j = (3,\ r) \quad 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

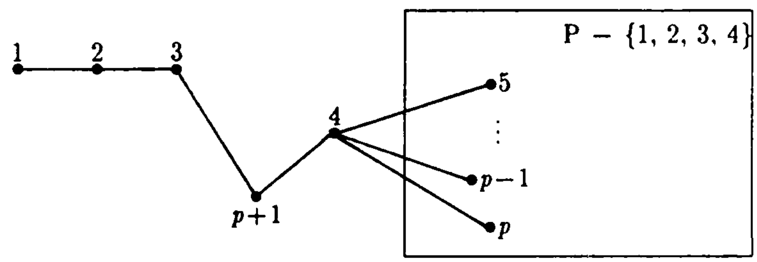
and

$$x^2_j = \begin{cases} 1 & j = (1,\ 2),\ (2,\ 3),\ (3,\ p+1) \text{ and } (4,\ p+1) \\ 1 & j = (4,\ r) \quad 5 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

These two trees satisfy $x_{1,2} = 1$ and have the forms shown below. Please note that the trees are illustrated for a value of $p \geq 7$ for clarity sake, but they apply for the case of $p \geq 4$.



x^1

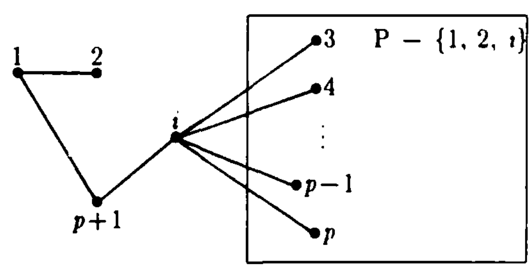


x^2

Now for $i = 3, \ldots, p$ define the $p - 2$ additional P-trees

$$x_j^i = \begin{cases} 1 & j = (1,2),\ (1, p+1) \text{ and } (i, p+1) \\ 1 & j = (r, i) \quad 3 \leq r < i \\ 1 & j = (i, r) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

These trees also satisfy $x_{1,2} = 1$ and have the form



$x^{i+1} \quad i = 3, \ldots, p$

Finally, define

$$x_j^{p+1} = \begin{cases} 1 & j = (1, 2) \\ 1 & j = (r, p+1) \quad 2 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

which has all nodes except node 1 attached to node $p+1$. Node 1 is attached to node 2 as a leaf. We now have

$$m_p - 1 + 3 + (p - 2) = m_p + p = m_{p+1}$$

P-trees, all satisfying $x_{1,2} = 1$. Let M be the $m_{p+1} \times m_{p+1}$ matrix whose columns are the characteristic vectors of these trees. Then

$$M = \begin{bmatrix} Y & x^1_{m_p} & x^2_{m_p} & & X_{m_p}{}' & & x^{p+1}_{m_p} \\ 0 & 0 & 0 & 1 & 1 & e^T_{p-4} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & I_{p-4} & e_{p-4} \end{bmatrix}$$

where row 1 is all 1's. The columns of Y are spanning trees of K_p containing $p - 1$ 1's. The vectors $x^1_{m_p}$ and $x^2_{m_p}$ each contain $p - 2$ 1's by construction, the columns of $X_{m_p}{}'$ also contain $p - 2$ 1's. $x^{p+1}_{m_p}$ contains exactly one 1. Multiply row 1 by $p - 2$, subtract rows 2 to m_p from it and append it as an extra row at the bottom of the matrix. Note that this new matrix, M', has the same column rank as M.

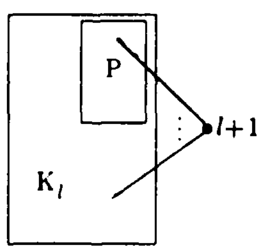
$$
M' = \begin{array}{c}
\begin{array}{ccccccc} Y & x^1_{m_p} & x^2_{m_p} & & X_{m_p}{}' & & x^{p+1}_{m_p} \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & e^T_{p\text{-}4} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & I_{p\text{-}4} & e_{p\text{-}4} \\
0 & 1 & 1 & 1 & 1 & e^T_{p\text{-}4} & p-2
\end{bmatrix}
\end{array}
$$

Multiply the last row by 2 and subtract rows $(1,\ p+1)$ through $(p,\ p+1)$ from it to get

$$
M'' = \begin{array}{c}
\begin{array}{c|cccc|cc} Y & x^1_{m_p} & x^2_{m_p} & & X_{m_p} & & x^{p+1}_{m_p} \end{array} \\
\left[\begin{array}{c|cccc|cc}
\hline
0 & 0 & 0 & 1 & 1 & e^T_{p\text{-}4} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & I_{p\text{-}4} & e_{p\text{-}4} \\
0 & 0 & 0 & 0 & 0 & 0 & p-3
\end{array}\right]
\end{array}
$$

which satisfies the conditions of Proposition 1.5 for having linearly independent columns. Therefore, we constructed m_{p+1} linearly independent P-trees satisfying $x_{1,2} = 1$.

Now assume that there are m_l linearly independent P-trees of K_l satisfying $x_{1,2} = 1$ for some $l \geq p + 1$ and look at K_{l+1}.



We will find m_{l+1} linearly independent P-trees of K_{l+1} satisfying $x_{1,2} = 1$. Since every P-tree of K_l is also a P-tree of K_{l+1}, let Y be the matrix whose columns are the characteristic vectors of m_l linearly indeppendent P-trees of K_l that satisfy $x_{1,2} = 1$. Now define the P-trees

$$x_j^1 = \begin{cases} 1 & j = (1, 2), (2, l+1) \text{ and } (3, l+1) \\ 1 & j = (3, r) \;\; 4 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_j^2 = \begin{cases} 1 & j = (1, 2), (2, 3), (3, l+1) \text{ and } (4, l+1) \\ 1 & j = (4, r) \;\; 5 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

For $i = 3, \ldots, p$ define the trees

$$x_j^i = \begin{cases} 1 & j = (1, 2), (1, l+1) \text{ and } (i, l+1) \\ 1 & j = (r, i) \;\; 3 \le r < i \\ 1 & j = (i, r) \;\; i < r \le p \\ 0 & \text{otherwise} \end{cases}$$

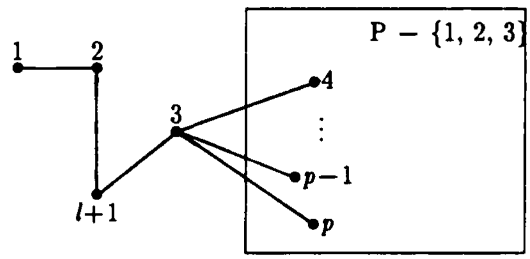
and for $i = p + 1, \ldots, l$ define

$$x_j^i = \begin{cases} 1 & j = (1, 2), (1, l+1) \text{ and } (i, l+1) \\ 1 & j = (i, r) \;\; 3 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

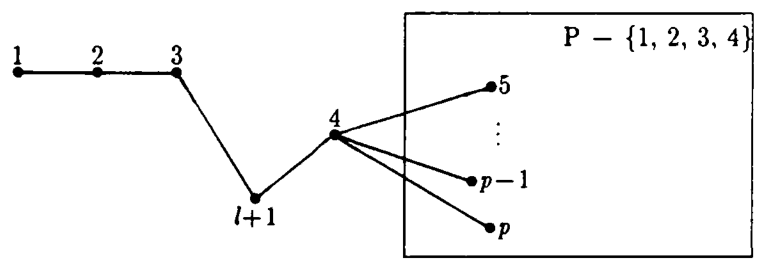
The total number of P-trees is

$$m_l + (l - 2) + 2 = m_{l+1}$$

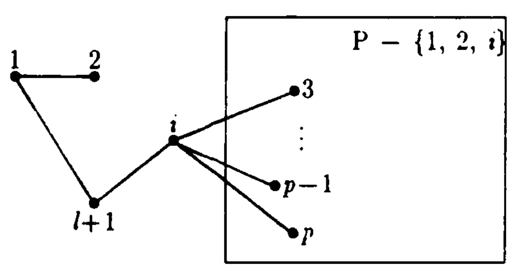
and they all satisfy $x_{1,2} = 1$. The trees that the vectors correspond to are listed below.



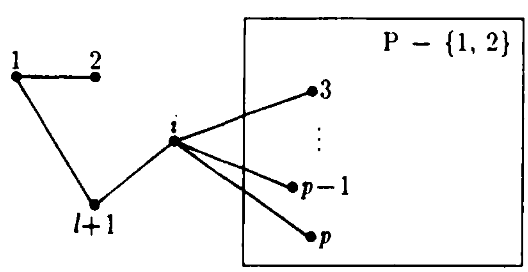
x^1



x^2



x^i, $i = 3, \ldots, p$



x^i, $i = p + 1, \ldots, l$

All that remains to be shown is that these P-trees are linearly independent, so look at the matrix M = (Y, X).

$$
M = \left[\begin{array}{c|cccc|c}
Y & & & X_{m_p} & & \\
\hline
0 & 0 & 0 & 1 & 1 & e^T_{p-4} \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & I_{l-4}
\end{array}\right]
$$

The columns of M are linearly independent by Proposition 1.5. Thus, there exist m_{l+1} linearly independent P-trees of K_{l+1} satisfying $x_{1,2} = 1$, and hence, by the principle of induction, $x_{1,2} \leq 1$ defines a facet of $T_{p,n}$ for $p \geq 3$ and $n \geq p + 1$. □

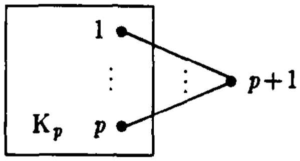
When we consider the arcs in (P, S) we see that the upper-bound inequalities do not induce facets unless $n \geq p + 2$. Similarly, for arcs in A(S) the upper-bound inequalities do not induce facets unless $n \geq p + 3$.

<u>Proposition</u> <u>4.12:</u> For $p \geq 3$ and $n = p + 1$, the inequality $x_a \leq 1$ describes a face of $T_{p,p+1}$ of dimension $m_{p+1} - 2$ for all arcs $a \in$ (P, S).

<u>Proof:</u> Since $n = p + 1$, and arc $a \in$ (P, S) is forced to be in the P-tree, the P-tree is also a spanning tree of K_{p+1}. Therefore, any P-tree satisfying $x_a = 1$ must also satisfy

$$\sum_{i=1}^{m_{p+1}} x_i = p.$$

These two equations are clearly independent, so by Lemma 1.4, the dimension of the face described by $x_a \leq 1$ can be at most $m_{p+1} - 2$ for all arcs $a \in (P, S)$. We will now demonstrate $m_{p+1} - 1$ linearly independent P-trees satisfying $x_a = 1$ to prove that the dimension is exactly $m_{p+1} - 2$. Let $p \geq 3$ be given. As always, view K_{p+1} as:



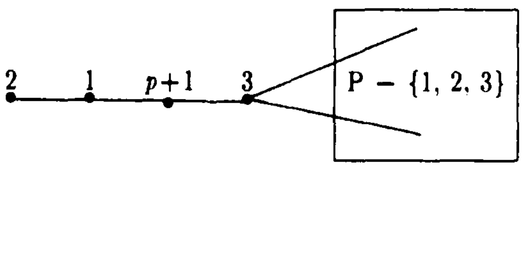
Without loss of generality, let $a = (1, p+1)$. By Proposition 2.5 there exist $m_p - 1$ linearly independent spanning trees of K_p satisfying $x_{1,2} = 1$. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now, define m_p P-trees of K_{p+1} as follows. First, define

$$x_j^1 = \begin{cases} 1 & j = (1, 2), (1, p+1) \text{ and } (3, p+1) \\ 1 & j = (3, r)\ 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

and then, for $i = 1, \ldots, m_p - 1$ define

$$x_j^{i+1} = \begin{cases} y_j^i & j \in A(P) - \{(1, 2)\} \\ 1 & j = (1, p+1) \text{ and } (2, p+1) \\ 0 & \text{otherwise} \end{cases}$$

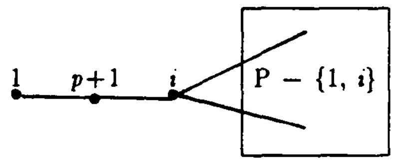
The tree x^1 has the form:



and the $m_p - 1$ trees x^i, $i = 2, \ldots, m_p$ correspond to the spanning trees Y of K_p with arc (1, 2) replaced by the two arcs $(1, p+1)$ and $(2, p+1)$. Note that these trees do not use arc (1, 2). We now construct $p - 2$ additional trees in the following manner. For $i = 3, \ldots, p$ define

$$x_j^{m_p+i-2} = \begin{cases} 1 & j = (1, p+1) \text{ and } (i, p+1) \\ 1 & j = (r, i), \quad 2 \leq r < i \\ 1 & j = (i, r), \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

These trees satisfy $x_{1,p+1} = 1$ and have the form

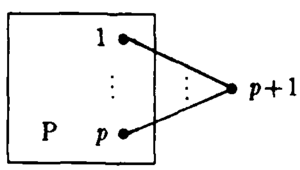


$i = 3, \ldots, p$

All of these trees use the arc $(1, p+1)$, and do not use arc (1, 2). Finally, define the tree

$$x_j^{m_{p+1}-1} = \begin{cases} 1 & j = (r, p+1) \ \ 1 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

This tree satisfies $x_{1,p+1} = 1$, does not use arc (1, 2), and has the form



We now have constructed $m_{p+1} - 1$ P-trees satisfying $x_{1,p+1} = 1$. To see that they are linearly independent, consider the matrix whose columns are the characteristic vectors of these trees. This matrix has the form

$$
M = \begin{array}{c}
\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
v & Y' & x^{m_p+1} & X'_{m_p} & 0 \\
1 & e^{T}_{m_p\text{-}1} & 1 & e^{T}_{p\text{-}3} & 1 \\
0 & e^{T}_{m_p\text{-}1} & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & I_{p\text{-}3} & e_{p\text{-}3}
\end{array}\right]
\end{array}
\begin{array}{l}
\text{row } (1, 2) \\
\\
\text{row } (1, p+1) \\
\text{row } (2, p+1) \\
\text{row } (3, p+1) \\
\\
\end{array}
$$

Where Y' is composed of the last $m_p - 1$ rows of Y. Use the first row to clear the first column. Then multiply row $(1, p+1)$ by $(p - 2)$. The resulting matrix is

$$
M' = \left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & Y' & x^{m_p+1} & X'_{m_p} & 0 \\
0 & (p-2)\, e^{T}_{m_p\text{-}1} & p-2 & (p-2)\, e^{T}_{p\text{-}3} & p-2 \\
0 & e^{T}_{m_p\text{-}1} & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & I_{p\text{-}3} & e_{p\text{-}3}
\end{array}\right]
\begin{array}{l}
\text{row } (1, 2) \\
\\
\text{row } (1, p+1) \\
\text{row } (2, p+1) \\
\text{row } (3, p+1) \\
\\
\end{array}
$$

Each of the columns of the submatrices Y' and X_{m_p}, and the vector x^{m_p+1} contain $(p - 2)$ 1's by construction, so subtract the rows containing these submatrices from row $(1, p+1)$. Then move the resulting row to the bottom of the matrix. The result is

$$M'' = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & Y' & x^{m_p+1} & X'_{m_p} & 0 \\ 0 & e^T_{m_p-1} & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & I_{p-3} & e_{p-3} \\ 0 & 0 & 0 & 0 & p-2 \end{array}\right] \begin{array}{l} \text{row } (1,\ 2) \\ \\ \text{row } (2,\ p+1) \\ \text{row } (3,\ p+1) \\ \\ \text{row } (1,\ p+1) \end{array}$$

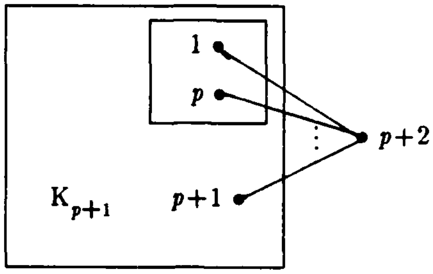
The matrix $\begin{bmatrix} Y' \\ e^T_{m_p-1} \end{bmatrix}$ is Y with its first row moved to the bottom, and has full column rank since Y has full column rank. Hence M'' satisfies the requirement of Proposition 1.5 to have linearly independent columns. Therefore, the dimension of the face of $T_{p,p+1}$ described by $x_a \leq 1$ is $m_{p+1} - 2$ for $a \in (P, S)$. □

<u>Proposition</u> <u>4.13:</u> For $p \geq 3$ and $n = p + 2$, the inequality $x_a \leq 1$ describes a face of $T_{p,p+2}$ of dimension $m_{p+2} - 2$ for all arcs in A(S).

<u>Proof:</u> As above, any P-tree satisfying $x_a = 1$ for the arc $a \in A(S)$, must be a spanning tree of K_{p+2} and must, therefore, satisfy

$$\sum_{i=1}^{m_{p+2}} x_i = p + 1.$$

making the dimension of the face that it describes at most $m_{p+2} - 2$. We can now construct $m_{p+2} - 1$ linearly idependent P-trees satisfying $x_a = 1$, proving that the dimension of the face is exactly $m_{p+2} - 2$. Let $p \geq 3$ be given. As before, view K_{p+2} as:



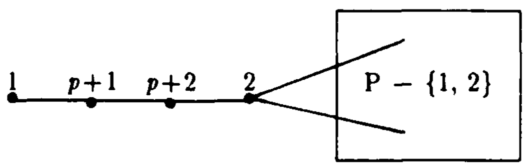
The only arc in A(S) is $(p+1,\ p+2)$. Therefore, we need to construct $m_{p+2} - 1$ linearly independent P-trees satisfying $x_{p+1,p+2} = 1$. By the proof of Proposition 4.12, there exists a set of $m_{p+1} - 1$ linearly independent P-trees of K_{p+1} satisfying $x_{1,p+1} = 1$. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now, construct m_{p+1} P-trees of K_{p+2} as follows. First, let

$$
x_j^1 = \begin{cases} 1 & j = (p+1,\ p+2),\ (1,\ p+1) \text{ and } (2,\ p+2) \\ 1 & j = (2,\ r) \quad 3 \le r \le p \\ 0 & \text{otherwise} \end{cases}
$$

and then, for $i = 1, \ldots, m_{p+1} - 1$ define

$$
x_j^{i+1} = \begin{cases} y_j^i & j \in A(K_{p+1}) - \{(1,\ p+1)\} \\ 1 & j = (1,\ p+2) \text{ and } (p+1,\ p+2) \\ 0 & \text{otherwise} \end{cases}
$$

Each of these trees satisfies $x_{p+1,p+2} = 1$. The tree x^1 has the form:



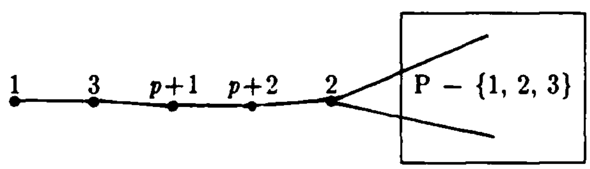
As in the proof of Proposition 4.12, the $m_{p+1} - 1$ trees x^i, $i = 2, \ldots, m_p$ correspond to the P-trees Y of K_{p+1} with arc $(1, p+1)$ replaced by the two arcs $(1, p+2)$ and $(p+1, p+2)$. We now construct p additional trees in the following manner. First, define

$$x_j^{m_{p+1}+1} = \begin{cases} 1 & j = (1, 3), (3, p+1), (2, p+2) \text{ and } (p+1, p+2) \\ 1 & j = (2, r) \quad 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

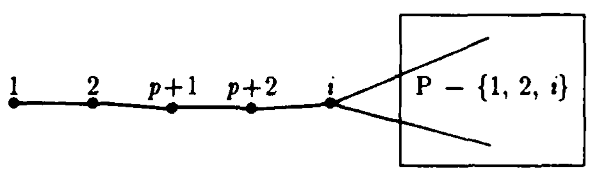
and for $i = 3, \ldots, p$ define

$$x_j^{m_p+i-1} = \begin{cases} 1 & j = (1, 2), (2, p+1), (i, p+2) \text{ and } (p+1, p+2) \\ 1 & j = (r, i) \quad 3 \leq r < i \\ 1 & j = (i, r) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

These trees satisfy $x_{p+1,p+2} = 1$ and have the forms



$x^{m_{p+1}+1}$

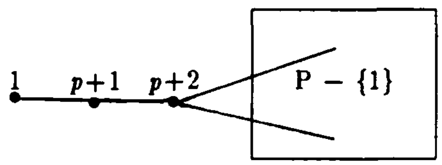


$x^{m_{p+1}+i-1}$ $\quad i = 3, \ldots, p$

Finally, define the tree

$$x_j^{m_{p+2}-1} = \begin{cases} 1 & j = (1,\ p+1) \\ 1 & j = (r,\ p+2) \quad 2 \le r \le p+1 \\ 0 & \text{otherwise} \end{cases}$$

This tree satisfies $x_{p+1,p+2} = 1$ and has the form



We now have constructed $m_{p+2} - 1$ P-trees. They all satisfy $x_{p+1,p+2} = 1$, and none of them, except for the first and last, use arc $(1,\ p+1)$. To see that they are linearly independent consider the matrix whose columns are the characteristic vectors of these trees. If we move rows $(1,\ p+1)$ and $(p+1,\ p+2)$ to the positions shown, this matrix has the form

$$M = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ v & Y' & x^{m_p+1} & X'_{m_p} & 0 \\ 1 & e^{\top}_{m_p-1} & 1 & e^{\top}_{p-2} & 1 \\ 0 & e^{\top}_{m_p-1} & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & I_{p-2} & e_{p-2} \end{array}\right] \begin{array}{l} \text{row } (1,\ p+1) \\ \\ \text{row } (p+1,\ p+2) \\ \text{row } (1,\ p+2) \\ \text{row } (2,\ p+2) \\ \\ \end{array}$$

where Y' is Y with row $(1,\ p+1)$ deleted. The structure is similar to that of the matrix in the proof of Proposition 4.12. As before, use the first row to clear the first column. Then subtract row $(p+1,\ p+2)$ from row $(1,\ p+2)$. The resulting matrix is

$$
M' = \begin{array}{c}
\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 1 \\
0 & Y' & x^{m_p+1} & X'_{m_p\text{-}1} & -v \\
0 & e^{\top}_{m_p\text{-}1} & 1 & e^{\top}_{p\text{-}2} & 0 \\
0 & 0 & -1 & -e^{\top}_{p\text{-}2} & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & I_{p\text{-}2} & e_{p\text{-}2}
\end{array}\right]
\end{array}
\begin{array}{l}
\text{row } (1,\ p+1) \\
\\
\text{row } (p+1,\ p+2) \\
\text{row } (1,\ p+2) \\
\text{row } (2,\ p+2) \\
\\
\end{array}
$$

Add each of the last $p - 1$ rows to row (1, $p+2$). Then move the resulting row to the bottom of the matrix. The result,

$$
M'' = \left[\begin{array}{c|cc|ccc}
1 & 0 & & 0 & 0 & 1 \\
\hline
0 & Y' & & x^{m_p+1} & X'_{m_p\text{-}1} & -v \\
0 & e^{\top}_{m_p\text{-}1} & & 1 & e^{\top}_{p\text{-}2} & 0 \\
\hline
0 & 0 & & 1 & 0 & 0 \\
0 & 0 & & 0 & I_{p\text{-}2} & e_{p\text{-}2} \\
0 & 0 & & 0 & 0 & p - 2
\end{array}\right]
\begin{array}{l}
\text{row } (1,\ p+1) \\
\\
\text{row } (p+1,\ p+2) \\
\text{row } (2,\ p+2) \\
\\
\text{row } (1,\ p+2)
\end{array}
$$

satisfies the conditions of Proposition 1.5 for have linearly independent columns since

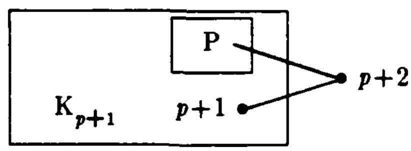
$$
Y = \begin{bmatrix} Y' \\ e^{\top}_{m_p\text{-}1} \end{bmatrix}.
$$

Therefore, the $m_{p+2} - 1$ P-trees we constructed are linearly independent, and the dimension of the face of $T_{p,p+2}$ described by $x_a \le 1$, $a \in A(S)$ is $m_{p+2} - 2$. $\square$

For $n \ge p + 2$, $a \in (P, S)$ and $n \ge p + 3$, $a \in A(S)$, the upper-bound inequalities do define facets of $T_{p,n}$.

Proposition 4.14: For $p \geq 3$ and $n \geq p + 2$, the inequality $x_a \leq 1$ induces a facet of $T_{p,n}$ for all arcs $a \in (P, S)$.

Proof: Let $p \geq 3$ be given. We will proceed by induction on n. First, for $n = p + 2$, look at K_{p+2} as



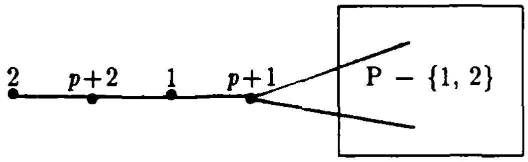
Without loss of generality, let $a = (1, p+1)$. Now, by the proof of Proposition 4.12, there exists a set of $m_{p+1} - 1$ linearly independent P-trees of K_{p+1} satisfying $x_{1,p+1} = 1$. These trees are also P-trees of K_{p+2}. Let Y be the matrix whose columns are the characteristic vectors of these trees, and define the following two P-trees.

$$x_j^1 = \begin{cases} 1 & j = (1,\ p+1),\ (1,\ p+2) \text{ and } (2,\ p+2) \\ 1 & j = (r,\ p+1) \quad 3 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

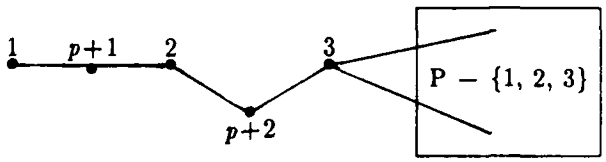
and

$$x_j^2 = \begin{cases} 1 & j = (1,\ p+1),\ (2,\ p+1),\ (2,\ p+2) \text{ and } (3,\ p+2) \\ 1 & j = (3,\ r) \quad 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

Both of these trees satisfy $x_{1,p+1} = 1$. They have the forms



x^1

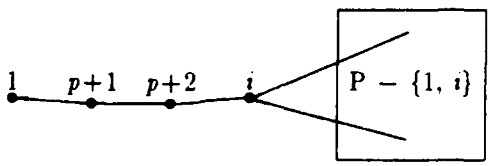


x^2

Now, for $i = 2, \ldots, p$ define the $p - 1$ P-trees

$$x_j^{i+1} = \begin{cases} 1 & j = (1,\ p+1),\ (i,\ p+2) \text{ and } (p+1,\ p+2) \\ 1 & j = (r,\ i) \quad 1 \leq r < i \\ 1 & j = (r,\ i) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

Each of the trees in this family satisfies $x_{1,p+1} = 1$, and has the form



$i = 2, \ldots, p$

Finally, define the tree

$$x_j^{p+2} = \begin{cases} 1 & j = (1,\ p+1) \\ 1 & j = (r,\ p+2) \quad 2 \leq r \leq p+1 \\ 0 & \text{otherwise} \end{cases}$$

which also satisfies $x_{1,p+1} = 1$. This tree has all nodes except node 1 attached to node $p+2$. Node 1 is attached to node $p+1$ as a leaf. By Lemma 1.1, the total number of P-trees we have is

$$m_{p+1} - 1 + (p + 2) = m_{p+2}$$

To see that they are linearly independent, let M be the matrix whose columns are the characteristic vectors of these trees. Let M' be M with row $(1, p+1)$ repeated as row $(1, p+1)^{\bigstar}$ in the position shown below:

$$M' = \begin{array}{c} \left[\begin{array}{ccccccc} Y & & & X_{m_{p+1}} & & & 0 \\ e^{T}_{m_{p+1}-1} & 1 & 1 & 1 & 1 & e^{T}_{p-3} & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & I_{p-3} & e_{p-3} \\ 0 & 0 & 0 & 1 & 1 & e^{T}_{p-3} & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \text{row } (1, p+1)^{\bigstar} \\ \text{row } (1, p+2) \\ \text{row } (2, p+2) \\ \text{row } (3, p+2) \\ \\ \text{row } (p+1, p+2) \end{array}$$

Note that M' and M have the same column rank. Each of the columns of Y is a P-tree of K_{p+1} which contains node $p+1$, thus it is also a spanning tree of $p + 1$ nodes and hence the columns of Y contain p 1's. By construction, each of the columns of the submatrix $X_{m_{p+1}}$ contains $p - 1$ 1's. So, if we multiply row $(1, p+1)^{\bigstar}$ by p and subtract from it each of the first m_{p+1} rows, we get the resulting matrix

$$\begin{array}{cl}
\left[\begin{array}{ccccccc}
Y & & & X_{m_{p+1}} & & & 0 \\
0 & 1 & 1 & 1 & 1 & e_{p-3}^T & p \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & I_{p-3} & e_{p-3} \\
0 & 0 & 0 & 1 & 1 & e_{p-3}^T & 1
\end{array}\right]
&
\begin{array}{l}
\\
\text{row } (1,\ p+1)^{\star} \\
\text{row } (1,\ p+2) \\
\text{row } (2,\ p+2) \\
\text{row } (3,\ p+2) \\
\\
\text{row } (p+1,\ p+2)
\end{array}
\end{array}$$

Subtracting row $(1,\ p+2)$ from rows $(1,\ p+1)^{\star}$ and $(2,\ p+2)$, and subtracting row $(p+1,\ p+2)$ from row $(1,\ p+1)^{\star}$ gives us the matrix

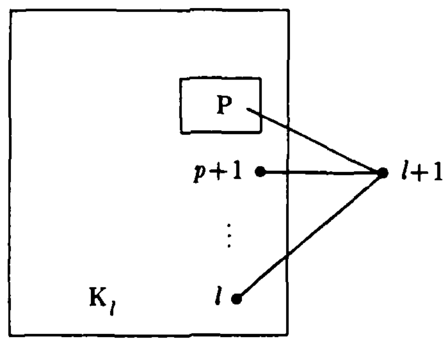
$$\begin{array}{cl}
\left[\begin{array}{ccccccc}
Y & & & X_{m_{p+1}} & & & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & p-1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & I_{p-3} & e_{p-3} \\
0 & 0 & 0 & 1 & 1 & e_{p-3}^T & 1
\end{array}\right]
&
\begin{array}{l}
\\
\text{row } (1,\ p+1)^{\star} \\
\text{row } (1,\ p+2) \\
\text{row } (2,\ p+2) \\
\text{row } (3,\ p+2) \\
\\
\text{row } (p+1,\ p+2)
\end{array}
\end{array}$$

Finally, subtract row $(1,\ p+1)^{\star}$ from rows $(2,\ p+2)$ and $(3,\ p+2)$ and then subtract these two rows and the rows containing the submatrix I_{p-3} from row $(p+1,\ p+2)$ to get

$$
\begin{array}{cl}
\left[\begin{array}{c|cccccc}
Y & & & X_{m_{p+1}} & & & 0 \\
\hdashline
0 & 0 & 1 & 0 & 0 & 0 & p-1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2-p \\
0 & 0 & 0 & 0 & 1 & 0 & 2-p \\
0 & 0 & 0 & 0 & 0 & I_{p\text{-}3} & e_{p\text{-}3} \\
0 & 0 & 0 & 0 & 0 & 0 & p
\end{array}\right]
&
\begin{array}{l}
\\
\text{row } (1,\ p+1)^{\star} \\
\text{row } (1,\ p+2) \\
\text{row } (2,\ p+2) \\
\text{row } (3,\ p+2) \\
\\
\text{row } (p+1,\ p+2)
\end{array}
\end{array}
$$

Thus, M has linearly independent columns by Proposition 1.5. and the m_{p+2} P-trees that we constructed are linearly independent.

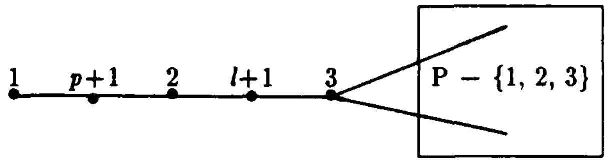
Now, assume that there exist m_l linearly independent P-trees satisfying $x_{1,p+1} = 1$ for some $l \geq p + 2$, and look at K_{l+1}.



Each P-tree of K_l is also a P-tree of K_{l+1}, so let Y be the matrix whose columns are the characteristic vectors of m_l linearly independent P-trees of K_l satisfying $x_{1,l+1} = 1$. Next, define l additional trees of K_{l+1} beginning with

$$
x_j^1 = \begin{cases} 1 & j = (1,\ p+1),\ (2,\ p+1),\ (2,\ l+1) \text{ and } (3,\ l+1) \\ 1 & j = (3,\ r) \quad 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}
$$

This tree satisfies $x_{1,l+1} = 1$ and has the form



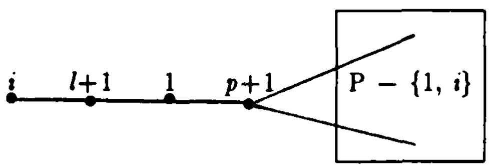
For $i = 2, \ldots, p$ define the trees

$$x_j^i = \begin{cases} 1 & j = (1, p+1), (1, l+1) \text{ and } (i, l+1) \\ 1 & j = (r, p+1) \quad 2 \leq r < i \\ 1 & j = (r, p+1) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

and for $i = p + 2, \ldots, l$ define

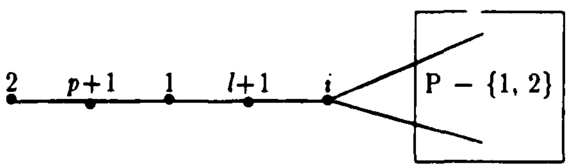
$$x_j^{i-1} = \begin{cases} 1 & j = (1, p+1), (1, l+1), (2, p+1) \text{ and } (i, l+1) \\ 1 & j = (r, i) \quad 3 \leq r < i \\ 1 & j = (i, r) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

These two families of trees satisfy the equality and have the forms



$i = 2, \ldots, p$

and



$i = p + 2, \ldots, l$

Finally, define

$$x_j^l = \begin{cases} 1 & j = (1,\ p+1) \text{ and } (p+1,\ l+1) \\ 1 & j = (r,\ l+1) \quad 2 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

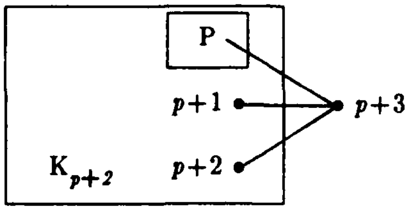
which has nodes 2, ..., $p+1$ connected to node $l+1$ and node 1 attached to node $p+1$ as a leaf. This gives us a total of m_{l+1} P-trees satisfying $x_{1,p+1} = 1$. To see that these trees are linearly independent consider the matrix whose columns are the characteristic vectors of these trees.

$$M = \begin{array}{cl} \left[\begin{array}{c|ccc|ccc} Y & & X_{m_l} & & & & v \\ \hline 0 & 0 & 1 & 1 & e_{p-3}^{T} & e_{l-(p+1)}^{T} & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & I_{p-3} & 0 & e_{p-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & I_{l-(p+1)} & 0 \end{array}\right] & \begin{array}{l} \\ \text{row } (1,\ l+1) \\ \\ \\ \\ \text{row } (p+1,\ l+1) \\ \\ \end{array} \end{array}$$

M has linearly independent columns by Proposition 1.5. Thus, by the principle of induction, $x_a \le 1$ induces a facet of $T_{p,n}$ for $p \ge 3$ and $n \ge p + 2$ and all arcs $a \in (P, S)$. ☐

<u>Proposition</u> <u>4.15:</u> For $p \ge 3$ and $n \ge p + 3$, the inequality $x_a \le 1$ induces a facet of $T_{p,n}$ for all arcs $a \in A(S)$.

<u>Proof:</u> Let $p \ge 3$ be given. Without loss of generality, let $a = (p+1,\ p+2)$. We will proceed by induction on n. For $n = p + 3$, look at K_{p+3}.



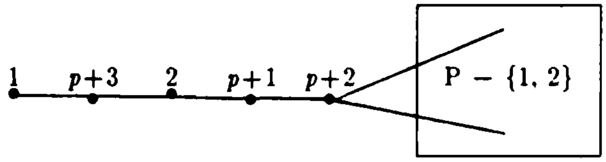
By Proposition 4.13, there exist $m_{p+2} - 1$ linearly independent P-trees of K_{p+2} satisfying $x_{p+1,p+2} = 1$. These trees are also P-trees of K_{p+3}. Let Y be the matrix whose columns are the characteristic vectors of these trees. Define the following two P-trees.

$$x_j^1 = \begin{cases} 1 & j = (1,\ p+3) \text{ and } (p+1,\ p+2) \\ 1 & j = (2,\ p+3) \text{ and } (2,\ p+1) \\ 1 & j = (r,\ p+2) \quad 3 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

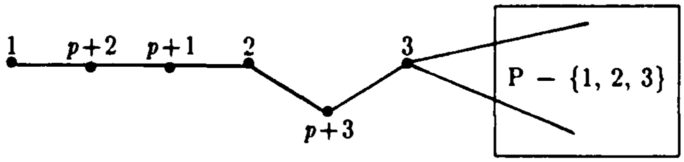
and

$$x_j^2 = \begin{cases} 1 & j = (1,\ p+2),\ (2,\ p+1) \text{ and } (p+1,\ p+2) \\ 1 & j = (2,\ p+3) \text{ and } (3,\ p+3) \\ 1 & j = (3,\ r) \quad 4 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

Both of these trees satisfy $x_{p+1,p+2} = 1$. They have the forms



x^1

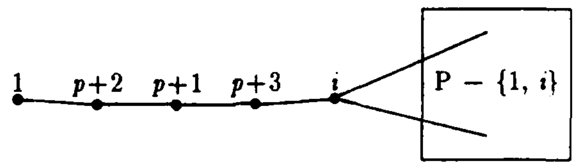


x^2

Now, for $i = 2, \ldots, p$ define the $p - 1$ P-trees

$$x_j^{i+1} = \begin{cases} 1 & j = (1,\ p+2) \text{ and } (p+1,\ p+2) \\ 1 & j = (i,\ p+3) \text{ and } (p+1,\ p+3) \\ 1 & j = (r,\ i) \quad 2 \leq r < i \\ 1 & j = (i,\ r) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

Each of the trees in this family satisfies $x_{p+1,p+2} = 1$, and has the form



$i = 2, \ldots, p$

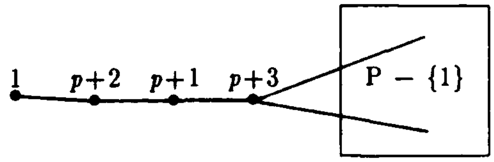
Finally, define the two trees

$$x_j^{p+2} = \begin{cases} 1 & j = (1,\ p+2) \text{ and } (p+1,\ p+2) \\ 1 & j = (r,\ p+3) \quad 2 \leq r \leq p+1 \\ 0 & \text{otherwise} \end{cases}$$

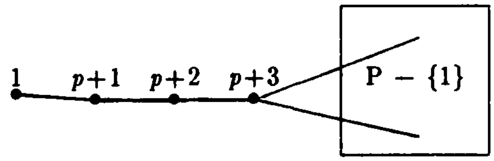
and

$$x_j^{p+3} = \begin{cases} 1 & j = (1,\ p+1)\ (p+1,\ p+2) \text{ and } (p+2,\ p+3) \\ 1 & j = (r,\ p+3) \quad 2 \leq r \leq p+1 \\ 0 & \text{otherwise} \end{cases}$$

which also satisfy $x_{1,p+1} = 1$. These trees have the forms



x^{p+2}



x^{p+3}

The total number of P-trees we have is

$$m_{p+2} - 1 + (p + 3) = m_{p+3}$$

To see that they are linearly independent, let M be the matrix whose columns are the characteristic vectors of these trees, and let M$'$ be M with row m_{p+2} repeated as row $(p+1,\ p+2)^{\star}$. M$'$ has the form

$$\left[\begin{array}{c|ccccccc}
Y & & & & X_{m_{p+2}} & & & \\
\hline
e^{T}_{m_{p+2}-1} & 1 & 1 & 1 & 1 & e^{T}_{p-3} & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & I_{p-3} & e_{p-3} & e_{p-3} \\
0 & 0 & 0 & 1 & 1 & e^{T}_{p-3} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{array}{l}
\\
(p+1,\ p+2)^{\star} \\
(1,\ p+3) \\
(2,\ p+3) \\
(3,\ p+3) \\
\\
(p+1,\ p+3) \\
(p+2,\ p+3)
\end{array}$$

and has the same column rank as M. Each of the columns of Y is a P-tree of K_{p+2} which contains nodes $p+1$ and $p+2$, thus it is also a spanning tree of $p + 2$ nodes and hence the columns of Y contain $p + 1$ 1's. By construction, each of the columns of the submatrix $X_{m_{p+2}}$, except the last two, contain p 1's. The last two columns of $X_{m_{p+2}}$ contain exactly two 1's. So, multiply row $(p+1,\ p+2)^\star$ by $p + 1$ and subtract from it each of the rows above it. The resulting matrix is

$$\begin{array}{cl}
\left[\begin{array}{c|ccccccc}
Y & \multicolumn{7}{c}{X_{m_{p+2}}} \\ \hline
0 & 1 & 1 & 1 & 1 & e_{p-3}^T & p-1 & p-1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & I_{p-3} & e_{p-3} & e_{p-3} \\
0 & 0 & 0 & 1 & 1 & e_{p-3}^T & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] &
\begin{array}{l}
\\
(p+1,\ p+2)^\star \\
(1,\ p+3) \\
(2,\ p+3) \\
(3,\ p+3) \\
\\
(p+1,\ p+3) \\
(p+2,\ p+3)
\end{array}
\end{array}$$

Subtract row $(1,\ p+3)$ from rows $(p+1,\ p+2)^\star$ and $(2,\ p+3)$, and subtract row $(p+1,\ p+3)$ from row $(p+1,\ p+2)^\star$ to get the matrix

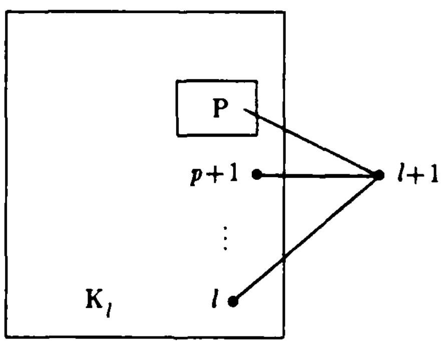
$$\begin{array}{cl}
\left[\begin{array}{c|ccccccc}
Y & \multicolumn{7}{c}{X_{m_{p+2}}} \\ \hline
0 & 0 & 1 & 0 & 0 & 0 & p-2 & p-1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & I_{p-3} & e_{p-3} & e_{p-3} \\
0 & 0 & 0 & 1 & 1 & e_{p-3}^T & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] &
\begin{array}{l}
\\
(p+1,\ p+2)^\star \\
(1,\ p+3) \\
(2,\ p+3) \\
(3,\ p+3) \\
\\
(p+1,\ p+3) \\
(p+2,\ p+3)
\end{array}
\end{array}$$

Finally, subtract row $(p+1,\ p+2)^{\bigstar}$ from rows $(2,\ p+3)$ and $(3,\ p+3)$, and then subtract these two rows and the rows containing the submatrix $I_{p\text{-}3}$ from row $(p+1,\ p+3)$ to get

$$\left[\begin{array}{c|ccccccc}
Y & & & & X_{m_{p+2}} & & & \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & p-2 & p-1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 3-p & 2-p \\
0 & 0 & 0 & 0 & 1 & 0 & 3-p & 2-p \\
0 & 0 & 0 & 0 & 0 & I_{p\text{-}3} & e_{p\text{-}3} & e_{p\text{-}3} \\
0 & 0 & 0 & 0 & 0 & 0 & p-2 & p-1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{array}{l}
\\
(p+1,\ p+2)^{\bigstar} \\
(1,\ p+3) \\
(2,\ p+3) \\
(3,\ p+3) \\
\\
(p+1,\ p+3) \\
(p+2,\ p+3)
\end{array}$$

By Proposition 1.5, M' has linearly independent columns. Thus, the m_{p+3} P-trees that we constructed are linearly independent.

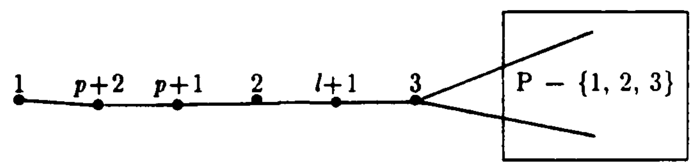
Now, assume that there exist m_l linearly independent P-trees satisfying $x_{p+1,p+2} = 1$ for some $l \geq p + 3$, and look at K_{l+1}.



Each P-tree of K_l is also a P-tree of K_{l+1}. Let Y be the matrix whose columns are the characteristic vectors of these m_l trees. Next, define l additional trees of K_{l+1} beginning with

$$x_j^1 = \begin{cases} 1 & j = (1,\ p+2),\ (2,\ l+1) \text{ and } (3,\ l+1) \\ 1 & j = (p+1,\ p+2) \text{ and } (2,\ p+1) \\ 1 & j = (3,\ r) \quad 4 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

This tree satisfies the desired equality and has the form



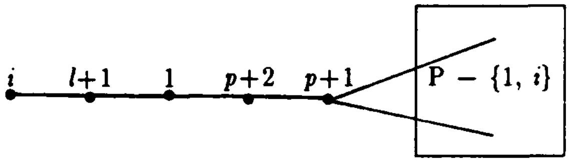
For $i = 2, \ldots, p$ define the trees

$$x_j^i = \begin{cases} 1 & j = (1,\ p+2) \text{ and } (p+1,\ p+2) \\ 1 & j = (1,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (r,\ p+1) \quad 2 \le r < i \\ 1 & j = (r,\ p+1) \quad i < r \le p \\ 0 & \text{otherwise} \end{cases}$$

and for $i = p + 3, \ldots, l$ define

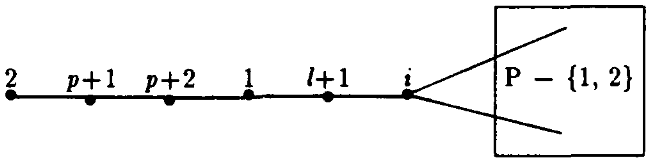
$$x_j^{i-2} = \begin{cases} 1 & j = (1,\ p+2),\ (1,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (p+1,\ p+2) \text{ and } (2,\ p+1) \\ 1 & j = (r,\ i) \quad 3 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

These two families of trees satisfy the desired equality and have the forms



$i = 2, \ldots, p$

and



$i = p + 3, \ldots, l$

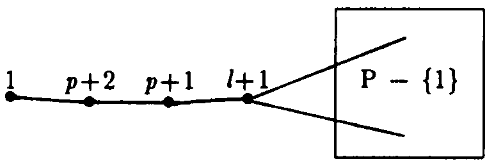
Finally, define the two trees

$$x_j^{l-1} = \begin{cases} 1 & j = (1, p+2) \text{ and } (p+1, p+2) \\ 1 & j = (r, l+1) \quad 2 \le r \le p+1 \\ 0 & \text{otherwise} \end{cases}$$

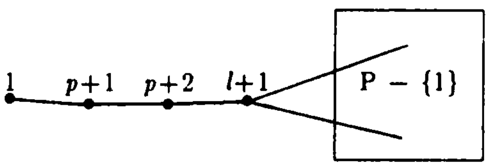
and

$$x_j^{l} = \begin{cases} 1 & j = (1, p+1), (p+1, p+2), \text{ and } (p+2, l+1) \\ 1 & j = (r, l+1) \quad 2 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

These trees satisfy $x_{p+1,p+2} = 1$, and have the forms



x^{l-1}



x^{l}

To see that these trees are linearly independent, consider the matrix whose columns are the characteristic vectors of these trees. This matrix has the form

$$
\begin{array}{cc}
\begin{array}{cccccccc} Y & & & & X_{m_l} & & & \end{array} & \\
\left[\begin{array}{c|ccc|cccc}
0 & 0 & 1 & 1 & e^T_{p\text{-}3} & e^T_{l\text{-}(p+2)} & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & I_{p\text{-}3} & 0 & e_{p\text{-}3} & e_{p\text{-}3} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & I_{l\text{-}(p+2)} & 0 & 0
\end{array}\right]
&
\begin{array}{l}
(1,\ l+1) \\
(2,\ l+1) \\
(3,\ l+1) \\
\\
(p+1,\ l+1) \\
(p+2,\ l+1) \\
\\
\end{array}
\end{array}
$$

Clearly the matrix has linearly independent columns by Proposition 1.5. Thus, by the principle of induction, the upper-bound $x_a \leq 1$ induces a facet of $T_{p,n}$ for $p \geq 3$ and $n \geq p + 2$ and all arcs $a \in (P, S)$. □

4.3 Cut-Set Inequalities

If $(X, \overline{X})$ is a cut in K_n which separates P, i.e. $X \cap P \neq \emptyset$ and $\overline{X} \cap P \neq \emptyset$, then the inequality

$$\sum_{a \in (X, \overline{X})} x_a \geq 1 \qquad (4.1)$$

is clearly valid for $T_{p,n}$. In fact, if $(X, \overline{X})$ does not separate P, the inequality is not valid, since any spanning tree of P (which uses no Steiner nodes) will violate it. As in the case of spanning trees, general cuts will not induce facets of $T_{p,n}$. Only cuts which satisfy the conditions of the next two propositions induce facets.

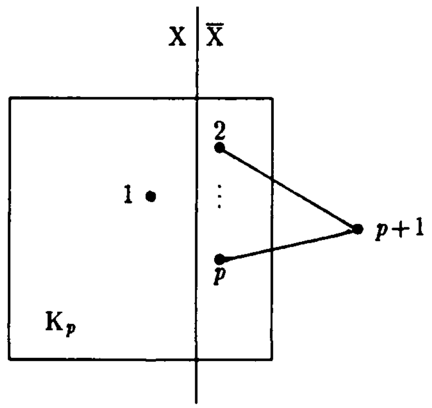
Proposition 4.16: If $p \geq 3$ and $n \geq p+1$, and if $(X,\overline{X})$ is a cut in K_n satisfying

$$1)\ |X| = 1 \text{ or } |\overline{X}| = 1$$

$$2)\ X \cap P \neq \emptyset \text{ and } \overline{X} \cap P \neq \emptyset$$

then inequality (4.1) induces a facet of $T_{p,n}$.

Proof: Let $p \geq 3$ be given. Without loss of generality, let $X = \{1\}$. We will proceed by induction on n. For $n = p + 1$ we can view K_{p+1} as:



By Proposition 2.8, there exists a set of $m_p - 1$ linearly independent spanning trees of K_p satisfying

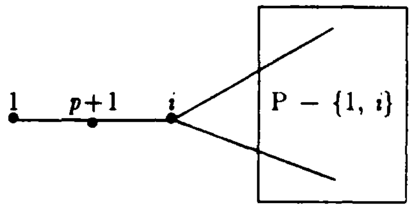
$$\sum_{a \in (X,\overline{X}-\{p+1\})} x_a = 1.$$

Each of these spanning trees is a P-tree of K_{p+1} which satisfies (4.1) at equality. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now define $p - 1$ additional

P-trees as follows. For $i = 2, \ldots, p$ define

$$x_j^{i-1} = \begin{cases} 1 & j = (1,\ p+1) \text{ and } (i,\ p+1) \\ 1 & j = (r,\ i) \quad 2 \leq r \leq i \\ 1 & j = (i,\ r) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

These trees satisfy (4.1) at equality and have the form



$i = 2, \ldots, p$

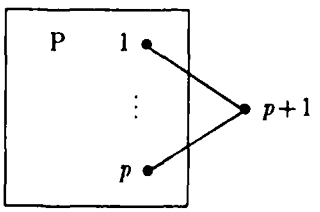
Define the two trees

$$x_j^p = \begin{cases} 1 & j = (r,\ p+1) \quad 1 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

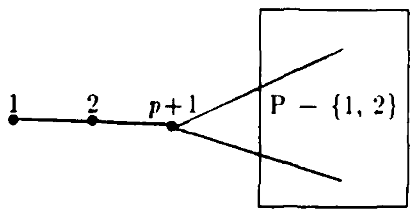
and

$$x_j^{p+1} = \begin{cases} 1 & j = (1,\ 2) \\ 1 & j = (r,\ p+1) \quad 2 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

Clearly, these two trees also satisfy (4.1) at equality and have the form



x^p



x^{p+1}

We now have

$$m_p - 1 + (p + 1) = m_{p+1}$$

P-trees satisfying (4.1) at equality. All that remains to be shown is that these trees are affinely independent. Let M be the matrix whose columns are the characteristic vectors of these trees and look at

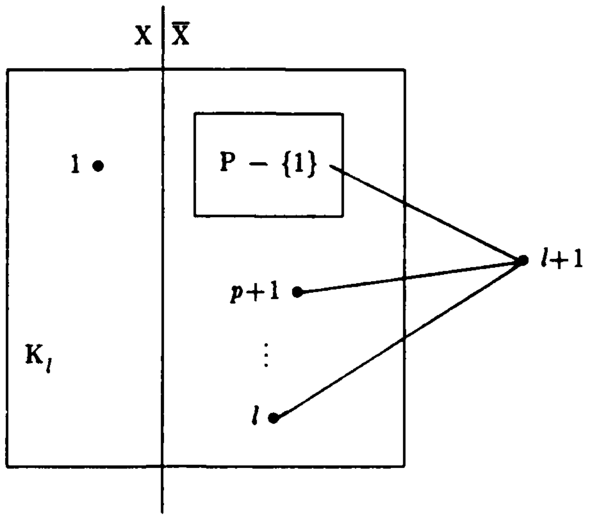
$$\begin{array}{cl} \overline{\mathrm{M}} = \begin{bmatrix} \mathrm{Y} & \mathrm{X}_{m_p} & 0 & \mathrm{v} \\ 0 & \mathrm{e}_{p\text{-}1}^{\top} & 1 & 0 \\ 0 & \mathrm{I}_{p\text{-}1} & \mathrm{e}_{p\text{-}1} & \mathrm{e}_{p\text{-}1} \\ \mathrm{e}_{m_p\text{-}1}^{\top} & \mathrm{e}_{p\text{-}1}^{\top} & 1 & 1 \end{bmatrix} & \begin{array}{l} \\ \text{row } (1,\ p+1) \\ \\ \\ \end{array} \end{array}$$

The columns of Y each contain $p - 1$ 1's, while the columns of X_{m_p} each contain $p - 2$ 1's. The vector v contains exactly one 1. If we subtract each of the rows of the matrix, except row $(1,\ p+1)$, from $(p - 1)$ times the last row, we get the matrix

$$\overline{\mathrm{M}}' = \left[\begin{array}{c|ccc|c} \mathrm{Y} & \mathrm{X}_{m_p} & 0 & & \mathrm{v} \\ \hline 0 & \mathrm{e}_{p\text{-}1}^{\top} & 1 & & 0 \\ 0 & \mathrm{I}_{p\text{-}1} & \mathrm{e}_{p\text{-}1} & & \mathrm{e}_{p\text{-}1} \\ \hline 0 & 0 & 0 & & -1 \end{array}\right]$$

Since $p \geq 3$, the middle diagonal block has linearly independent columns by Lemma 4.1. Thus, $\overline{\mathrm{M}}$ has linearly independent columns by Proposition 1.5 and, hence, the P-trees which we constructed are affinely independent.

We can view K_{l+1} as:



Assume that there exist m_l affinely independent P-trees of K_l satisfying

$$\sum_{a \in (X, \overline{X}-\{l+1\})} x_a = 1.$$

Each of these trees is a P-tree of K_{l+1} satisfying (4.1) at equality. Let Y be the matrix whose columns are the characteristic vectors of these trees. Now, define l additional P-trees as follows. First, define

$$x_j^1 = \begin{cases} 1 & j = (r, l+1) \;\; 1 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

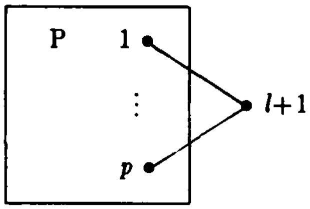
Then, for $i = 2, \ldots, p$ define

$$x_j^i = \begin{cases} 1 & j = (1, l+1) \text{ and } (i, l+1) \\ 1 & j = (r, i) \;\; 2 \le r < i \\ 1 & j = (i, r) \;\; i < r \le p \\ 0 & \text{otherwise} \end{cases}$$

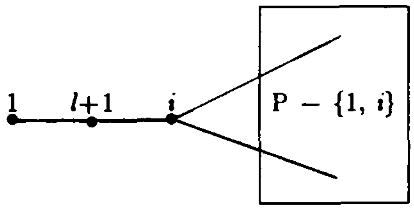
Finally, for $i = p + 1, \ldots, l$ define

$$x_j^i = \begin{cases} 1 & j = (1,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (r,\ i) \quad 2 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

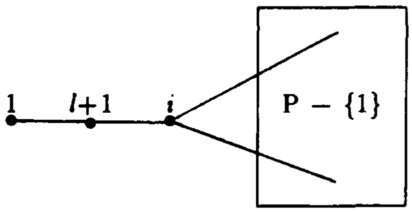
Each of these l trees satisfies (4.1) at equality. The forms of these trees are shown below.



x^1



$x^i, \ i = 2, \ldots, p$



$x^i, \ i = p + 1, \ldots, l$

We can see that these P-trees are affinely independent by looking at the matrix whose columns are their characteristic vectors.

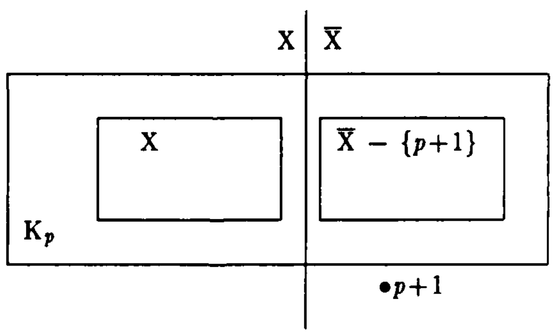
$$M = \left[\begin{array}{c|cc|c} Y & & X_{m_l} & \\ \hline 0 & 1 & e_{p-1}^{\top} & e_{l-p}^{\top} \\ 0 & e_{p-1} & I_{p-1} & 0 \\ \hline 0 & 0 & 0 & I_{l-p} \end{array}\right]$$

Since $p \geq 3$, this matrix has affinely independent columns by Lemma 4.1 and Proposition 1.6. Thus, the m_{l+1} P-trees that we constructed are affinely independent, and by the principle of induction, (4.1) induces a facet of $T_{p,n}$ for $p \geq 3$, $n \geq p + 1$, and all cuts $(X, \bar{X})$ satisfying properties 1 and 2 of the theorem. □

To prove the next proposition we will need the following lemma.

Lemma 4.17: For $p \geq 4$ and $n = p + 1$ there exist $m_{p+1} - 1$ linearly independent P-trees satisfying (4.1) at equality for any cut $(X,\bar{X})$ satisfying $1 < |X \cap P| < p - 1$.

Proof: Let $p \geq 4$ be given. Without loss of generality, let $X = \{1, \ldots, k\}$ and $\bar{X} = \{k+1, \ldots, p, p+1\}$ for some k satisfying $2 \leq k \leq p - 2$. We can view K_{p+1} as:



By the proof of Proposition 2.10 we know that there exist $m_p - 2$ linearly independent spanning trees of K_p satisfying

$$\sum_{a \in (X, \bar{X} - \{p+1\})} x_a = 1.$$

Each of these trees is a P-tree of K_{p+1} satisfying (4.1) at equality. If we rearrange the components of the vectors so that the first m_k components correspond to the arcs in $A(X)$, the next $k(p-k)$ to the arcs in $(X,\bar{X})$, and the final m_{p-k} to the arcs in $A(\bar{X}-\{p+1\})$, then by the proof of Proposition 2.10 we know that the matrix whose columns are the $m_p - 2$ spanning trees of K_p has the form:

$$\begin{bmatrix} Y & Y^1 & Y^1 \\ e_{m_k}^T & 0 & e_{m_{p-k}-1}^T \\ 0 & I_{k(p-k)-1} & 0 \\ Z^1 & Z^1 & Z' \end{bmatrix}$$

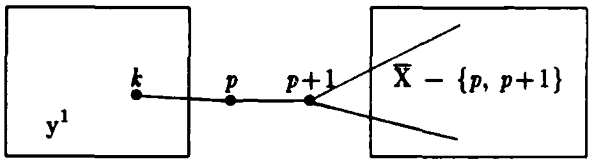
where the columns of Y are the characteristic vectors of a set of m_k spanning trees of $G(X, A(X))$ and the columns of Z are the characteristic vectors of a set of spanning trees of $G(\bar{X}-\{p+1\}, A(\bar{X}-\{p+1\}))$. Z' is Z with the first column deleted and Z^1 has all columns equal to z^1. By the proof of Proposition 2.1 there exists a set of m_{p-k} linearly independent spanning trees of $G(\bar{X}-\{p+1\}, A(\bar{X}-\{p+1\}))$ the matrix of whose characteristic vectors has the form

$$Z = \begin{bmatrix} 0 & Q & Q^1 \\ 1 & e_{m_{p-k-1}}^T & 0 \\ e_{p-k-2} & 0 & I_{p-k-2} \end{bmatrix}.$$

Q is the matrix whose columns are the characteristic vectors of m_{p-k-1} linearly indepenent spanning trees of $G(\bar{X}-\{p, p+1\}, A(\bar{X}-\{p, p+1\}))$ and Q^1 is the matrix that that has every column equal to q^1. The first vector z^1 represents the tree that has nodes $k+1$ thru $p-1$ connected to node p as leaves. Now define $p+1$ additional P-trees of K_{p+1} as follows. First, define

$$x_j^1 = \begin{cases} y_j^1 & j \in A(X) \\ 1 & j = (k,\ p) \\ 1 & j = (r,\ p+1) \quad k+1 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

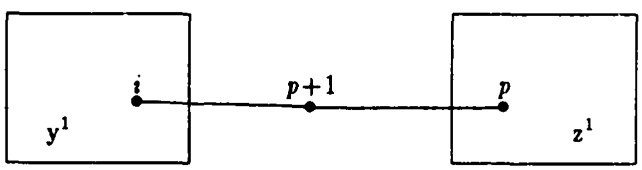
Tree x^1 satisfies (4.1) at equality and has the form



For $i = 1, \ldots, k$ define

$$x_j^{i+1} = \begin{cases} y_j^1 & j \in A(X) \\ 1 & j = (i,\ p+1) \text{ and } (p,\ p+1) \\ z_j^1 & j \in A(\overline{X} - \{p+1\}) \\ 0 & \text{otherwise} \end{cases}$$

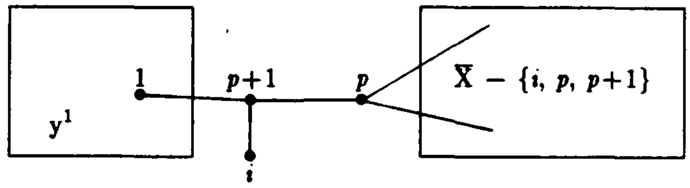
These k P-trees also satisfy (4.1) at equality, and have the forms:



For $i = k+1, \ldots, p-1$ define

$$x_j^{i+1} = \begin{cases} y_j^1 & j \in A(X) \\ 1 & j = (1,\ p+1),\ (i,\ p+1) \text{ and } (p,\ p+1) \\ 1 & j = (r,\ p) \quad k+1 \le r \le p-1,\ r \ne i \\ 0 & \text{otherwise} \end{cases}$$

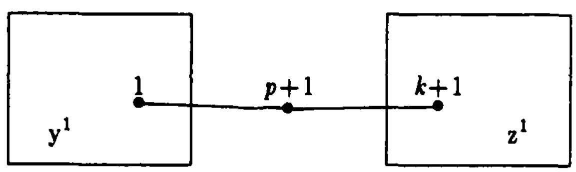
These trees have the form



and satisfy (4.1) at equality. Note that in the components corresponding to the arcs in $A(\overline{X} - \{p+1\})$ the characteristic vectors of these trees are z^1 with the component representing arc (i, p) set to zero. Let v^i be the vector composed of the components of the characteristic vector of x^i corresponding to these arcs. Finally, define the tree

$$x_j^{p+1} = \begin{cases} y_j^1 & j \in A(X) \\ 1 & j = (1, p+1) \text{ and } (k+1, p+1) \\ z_j^1 & j \in A(\overline{X} - \{p+1\}) \\ 0 & \text{otherwise} \end{cases}$$

This tree has the form



and clearly satisfies (4.1) at equality. To show that our $(m_p - 2) + (p+1) = (m_{p+1} - 1)$ trees are linearly independent, consider the matrix whose columns are their characteristic vectors.

$$\left[\begin{array}{cccc:cccccc}
Y & Y^1 & y^1 & Y^1 & y^1 & y^1 & Y^1 & y^1 & Y^1 & y^1 \\
e^T_{m_k} & 0 & 0 & e^T_{m_{p-k-1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{k(p-k)-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z^1 & Z^1 & z^1 & Z' & 0 & z^1 & Z^1 & v^1 & V' & z^1 \\
\hdashline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & e^T_{p-k-2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{k-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & e_{p-k-2} & 0 & 0 & 0 & I_{p-k-2} & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & e^T_{k-1} & 1 & e^T_{p-k-2} & 0
\end{array}\right]
\begin{array}{l}
\\
(1,\ k+1) \\
\\
\\
(k,\ p) \\
(1,\ p+1) \\
(k+1,\ p+1) \\
\\
\\
(p,\ p+1)
\end{array}$$

Note that we have moved row (k, p) below the rows corresponding to the other arcs in $A(\bar{X} - \{p+1\})$. Subtract row $(1, p+1)$ and the rows containing the submatrix I_{k-1} from row $(p, p+1)$. This gives us the matrix

$$\left[\begin{array}{cccc:cccccc}
Y & Y^1 & y^1 & Y^1 & y^1 & y^1 & Y^1 & y^1 & Y^1 & y^1 \\
e^T_{m_k} & 0 & 0 & e^T_{m_{p-k-1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{k(p-k)-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z^1 & Z^1 & z^1 & Z' & 0 & z^1 & Z^1 & v^1 & V' & z^1 \\
\hdashline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & e^T_{p-k-2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{k-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & e_{p-k-2} & 0 & 0 & 0 & I_{p-k-2} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
\begin{array}{l}
\\
(1,\ k+1) \\
\\
\\
(k,\ p) \\
(1,\ p+1) \\
(k+1,\ p+1) \\
\\
\\
(p,\ p+1)
\end{array}$$

Add the last column to the column corresponding to P-tree x^1, then subtract the columns containing the submatrix I_{p-k-2} from column x^1. The resulting matrix is

$$
\begin{array}{cc}
\left[\begin{array}{cccc:cccccc}
Y & Y^1 & y^1 & Y^1 & \alpha y^1 & y^1 & Y^1 & y^1 & Y^1 & y^1 \\
e^{\top}_{m_k} & 0 & 0 & e^{\top}_{m_{p-k}-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{k(p-k)-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z^1 & Z^1 & z^1 & Z' & u & z^1 & Z^1 & v^1 & V' & z^1 \\
\hdashline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3+k-p & 1 & 0 & 1 & e^{\top}_{p-k-2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{k-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p-k-2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
&
\begin{array}{l}
\\
(1,\ k+1) \\
\\
\\
(k,\ p) \\
(1,\ p+1) \\
(k+1,\ p+1) \\
\\
\\
(p,\ p+1)
\end{array}
\end{array}
$$

where $\alpha = 4 + k - p$ and $u = z^1 - V'e_{p-k-2}$. Subtract 2 times the column containing v^1 from x^1, then add $(p - k - 1)$ times x^2 to x^1. Finally, subtract column x^1 from the other column which contains a 1 in row $(k,\ p)$. The resulting matrix is

$$
\begin{array}{cl}
\left[\begin{array}{cccc|cccccc}
Y & Y^1 & 0 & Y^1 & y^1 & y^1 & Y^1 & y^1 & Y^1 & y^1 \\
e_{m_k}^T & 0 & 0 & e_{m_{p-k-1}}^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{k(p-k)-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z^1 & Z^1 & z^1 - w & Z' & w & z^1 & Z^1 & v^1 & V' & z^1 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & e_{p-k-2}^T & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{k-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p-k-2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
&
\begin{array}{l}
\\
(1,\ k+1) \\
\\
\\
(k,\ p) \\
(1,\ p+1) \\
(k+1,\ p+1) \\
\\
\\
(p,\ p+1)
\end{array}
\end{array}
$$

where $w = (p - k)z^1 - V'e_{p-k-2} - 2v^1$. This matrix will have linearly independent columns by Proposition 1.5 if we can show that the submatrix

$$
\left[\begin{array}{c|ccc}
Y & Y^1 & 0 & Y^1 \\
e_{m_k}^T & 0 & 0 & e_{m_{p-k-1}}^T \\
\hline
0 & I_{k(p-k)-2} & 0 & 0 \\
Z^1 & Z^1 & z^1 - w & Z'
\end{array}\right]
$$

has full column rank. Use the submatrix $I_{k(p-k)-2}$ to clear the submatrices Y^1 and Z^1. Then subtract the first column from the columns containing the submatrix Y^1. The resulting matrix

$$\left[\begin{array}{c|ccc} Y & 0 & 0 & 0 \\ e_{m_k}^{\top} & 0 & 0 & 0 \\ \hline 0 & I_{k(p-k)-2} & 0 & 0 \\ Z^1 & 0 & z^1 - w & Z' - Z^1 \end{array}\right]$$

has linearly independent columns by Corollary 1.5.1 if the submatrix $\begin{bmatrix} z^1 - w & Z' - Z^1 \end{bmatrix}$ has linearly independent columns. The first column of the submatrix is

$$z^1 - w = (1 - p + k)z^1 + V'e_{p-k-2} + 2v^1$$

Thus,

$$z^1 - w = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} (k+1,\ p)$$

So,

$$\begin{bmatrix} z^1 - w & Z' - Z^1 \end{bmatrix} = \begin{bmatrix} 0 & Q & Q^1 \\ -1 & 0 & -e_{p-k-2}^{\top} \\ 0 & -E^1 & I_{p-k-2} - E^2 \end{bmatrix} (k+1,\ p)$$

where E^1 is the $(p - k - 2) \times m_{p-k-1}$ matrix of 1's and E^2 is the $(p - k - 2) \times (p - k - 2)$ matrix of 1's. Subtract the first and second columns from each of the columns in the submatrix Q^1, and rearrange the columns to get the matrix

$$\begin{bmatrix} Q & 0 & 0 \\ 0 & -1 & 0 \\ -E^1 & 0 & I_{p\text{-}k\text{-}2} \end{bmatrix}$$

which has linearly independent columns by Corollary 1.5.1. Therefore, the $m_{p+1} - 1$ P-trees satisfying (4.1) at equality are linearly independent. □

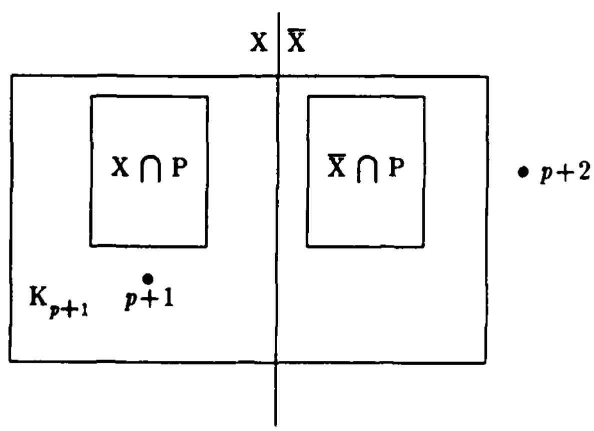
Proposition 4.18: For $p \geq 4$ and $n \geq p + 2$, if $(X, \overline{X})$ is a cut in K_n satisfying

1) $|X \cap S| \geq 1$ and $|\overline{X} \cap S| \geq 1$

2) $|X \cap P| \geq 2$ and $|\overline{X} \cap P| \geq 2$

then inequality (4.1) induces a facet of $T_{p,n}$.

Proof: Let $p \geq 4$ be given. Without loss of generality, let $X \cap P = \{1, \ldots, k\}$ and $\overline{X} \cap P = \{k+1, \ldots, p\}$ for some k satisfying $2 \leq k \leq p - 2$. We will proceed by induction on n. For $n = p + 2$, we can view K_{p+2} as:



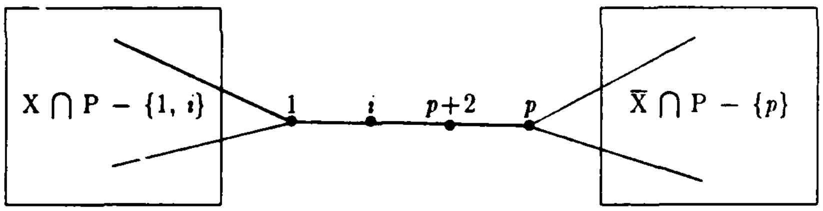
By Lemma 4.17, there exists a set of $m_{p+1} - 1$ linearly independent P-trees of K_{p+1} satisfying

$$\sum_{a \in (X, \overline{X} - \{p+2\})} x_a = 1$$

Each of these trees is a P-tree of K_{p+2} satisfying (4.1) at equality. Let Y be the matrix whose columns are the characteristic vectors of these P-trees and define $p + 2$ additional P-trees as follows. For $i = 1, \ldots, k$ define

$$x_j^i = \begin{cases} 1 & j = (p,\ p+2) \text{ and } (i,\ p+2) \\ 1 & j = (1,\ r) \quad 2 \leq r \leq k \\ 1 & j = (r,\ p) \quad k+1 \leq r \leq p-1 \\ 0 & \text{otherwise} \end{cases}$$

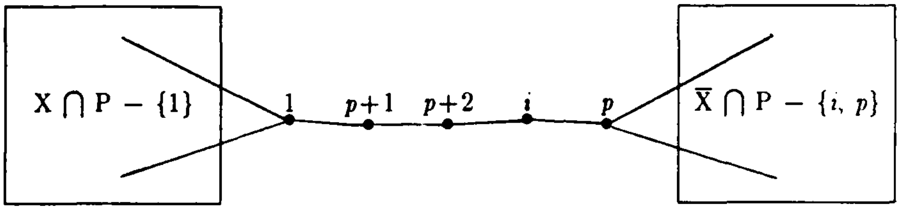
These trees have exactly one arc, namely $(1,\ p+2)$, in $(X, \overline{X})$ and thus satisfy (4.1) at equality. They have the form



For $i = k + 1, \ldots, p - 1$ define

$$x_j^i = \begin{cases} 1 & j = (i,\ p+2),\ (1,\ p+1) \text{ and } (p+1,\ p+2) \\ 1 & j = (1,\ r) \quad 2 \leq r \leq k \\ 1 & j = (r,\ p) \quad k+1 \leq r \leq p-1 \\ 0 & \text{otherwise} \end{cases}$$

Again, only the arc $(p+1,\ p+2)$ is in the cut $(X, \overline{X})$, so these trees satisfy (4.1) at equality. Their form is shown below.



Finally, define the three P-trees

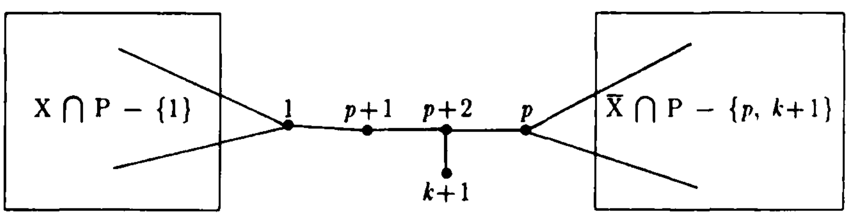
$$x_j^p = \begin{cases} 1 & j = (1,\ p+1) \text{ and } (k+1,\ p+2) \\ 1 & j = (p,\ p+2) \text{ and } (p+1,\ p+2) \\ 1 & j = (1,\ r) \quad 2 \le r \le k \\ 1 & j = (r,\ p) \quad k+2 \le r \le p-1 \\ 0 & \text{otherwise} \end{cases}$$

$$x_j^{p+1} = \begin{cases} 1 & j = (1,\ p+2),\ (p,\ p+2) \text{ and } (k+1,\ p+2) \\ 1 & j = (1,\ r) \quad 2 \le r \le k \\ 1 & j = (r,\ p) \quad k+2 \le r \le p-1 \\ 0 & \text{otherwise} \end{cases}$$

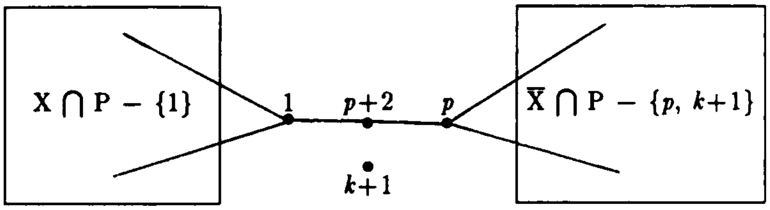
and

$$x_j^{p+2} = \begin{cases} 1 & j = (1,\ k+1),\ (p,\ p+2) \text{ and } (k+1,\ p+2) \\ 1 & j = (1,\ r) \quad 2 \le r \le k \\ 1 & j = (r,\ p) \quad k+2 \le r \le p-1 \\ 0 & \text{otherwise} \end{cases}$$

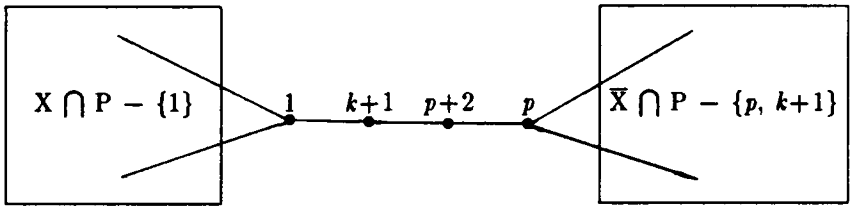
Since each of these trees contains only one arc in the cut $(X, \bar{X})$ it is clear that they also satisfy (4.1) at equality. The forms of these trees is shown below.



x^p



x^{p+1}



x^{p+2}

We now have a total of

$$m_{p+1} - 1 + (p + 2) = m_{p+2}$$

P-trees that satisfy (4.1) at equality. Let M be the matrix whose columns are the characteristic vectors of these trees. To see that these trees are affinely independent look at the matrix

$$\overline{M} = \left[\begin{array}{c|cccc|ccc} Y & & X'_{m_{p+1}} & & & x^{p}_{m_{p+1}} & x^{p+1}_{m_{p+1}} & x^{p+2}_{m_{p+1}} \\ \hdashline 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I_{k-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & I_{p-k-2} & 0 & 0 & 0 \\ 0 & 1 & e^{\top}_{k-1} & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & e^{\top}_{p-k-2} & 1 & 0 & 0 \\ e^{\top}_{m_{p+1}-1} & 1 & e^{\top}_{k-1} & 1 & e^{\top}_{p-k-2} & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ (p,\ p+2) \\ (p+1,\ p+2) \\ \\ \end{array}$$

Now we note that every column of Y and of $X'_{m_{p+1}}$ represents a P-tree of K_{p+2} with exactly one arc in the cut $(X, \overline{X})$. If that arc is removed we are left with two trees, one of which spans the $p - k$ nodes of $A(\overline{X} - \{p+2\})$. This means that each column of the submatrices Y and $X'_{m_{p+1}}$ contain exactly $p - k - 1$ 1's in the rows corresponding to the arcs in this set. The last three columns of $\overline{M}$ each contain $p - k - 2$ 1's in these rows. Therefore, multiply the last row by $p - k - 1$ and subtract each of the rows corresponding the arcs in $A(\overline{X} - \{p+2\})$. The resulting matrix is

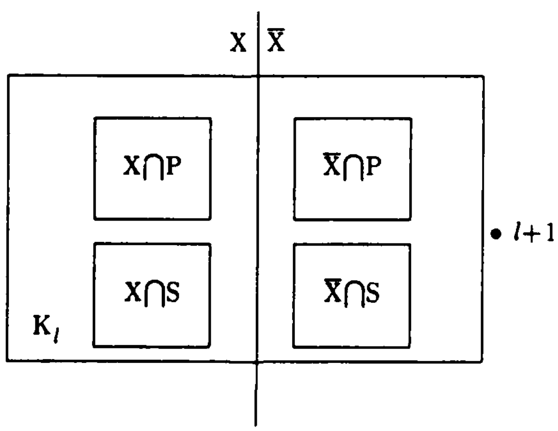
$$
\begin{array}{cc}
\left[\begin{array}{c|cccc|ccc}
Y & & X'_{m_{p+1}} & & & x^{p}_{m_{p+1}} & x^{p+1}_{m_{p+1}} & x^{p+2}_{m_{p+1}} \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & I_{k-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & I_{p-k-2} & 0 & 0 & 0 \\
0 & 1 & e^{T}_{k-1} & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & e^{T}_{p-k-2} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]
&
\begin{array}{c}
\\ \\ \\ \\ \\ (p,\ p+2) \\ (p+1,\ p+2) \\ \\
\end{array}
\end{array}
$$

Now subtract row $(1, p+2)$ and the rows containing the submatrix I_{k-1} from row $(p, p+2)$ and subtract row $(k+1, p+1)$ and the rows containing the submatrix I_{p-k-2} from row $(p+1, p+2)$. The resulting matrix

$$
\left[\begin{array}{c|cccc|ccc}
Y & & X'_{m_{p+1}} & & & x^{p}_{m_{p+1}} & x^{p+1}_{m_{p+1}} & x^{p+2}_{m_{p+1}} \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & I_{k-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & I_{p-k-2} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \\
(p,\ p+2) \\
(p+1,\ p+2) \\
\\
\end{array}
$$

satisfies the conditions of Proposition 1.5. Therefore, the m_{p+2} P-trees which we constructed satisfying (4.1) at equality are affinely independent.

Now, assume that there exist m_l affinely independent P-trees of K_l satisfying (4.1) at equality for some cut $(X, \overline{X})$ satisfying the requirements of the proposition and some $l \geq p + 2$. Without loss of generality assume that $X \cap S = \{p+1, \ldots, q\}$ for some q satisfying $p + 1 \leq q \leq l - 1$ and place node $l+1$ in $\overline{X}$. We now look at K_{l+1} in the following manner



By the assumption there exists a set of m_l affinely independent P-trees of K_l satisfying

$$\sum_{a \in (X, \overline{X} - \{l+1\})} x_a = 1$$

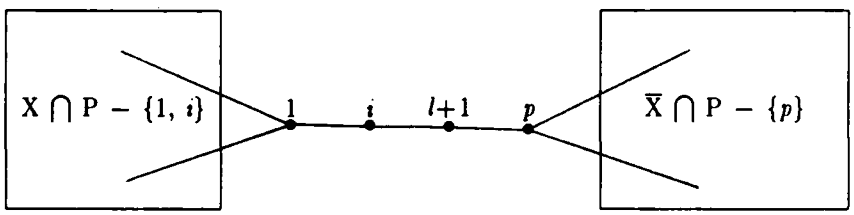
Clearly, these trees are P-trees of K_{l+1} satisfying (4.1) at equality. Let Y be the matrix whose columns are the characteristic vectors of these trees and construct l additional trees as follows. First, we associate trees with the nodes in X. For $i = 1, \ldots, k$ define

$$x_j^i = \begin{cases} 1 & j = (p,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (1,\ r)\quad 2 \le r \le k \\ 1 & j = (r,\ p)\quad k+1 \le r \le p-1 \\ 0 & \text{otherwise} \end{cases}$$

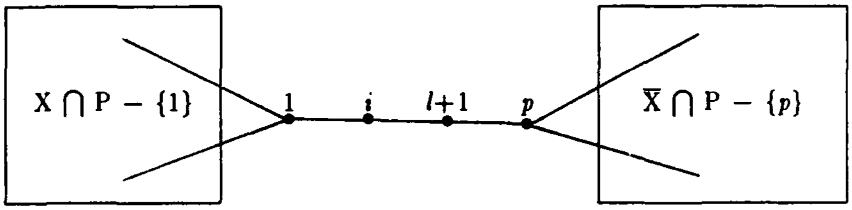
and for $i = p + 1, \ldots, q$ define

$$x_j^i = \begin{cases} 1 & j = (1,\ i),\ (p,\ l+1) \text{ and } (i,\ l+1) \\ 1 & j = (1,\ r)\quad 2 \le r \le k \\ 1 & j = (r,\ p)\quad k+1 \le r \le p-1 \\ 0 & \text{otherwise} \end{cases}$$

Each of these trees has only arc $(i,\ l+1)$ in $(X, \overline{X})$ and hence satisfies (4.1) at equality. These trees have the forms



$x^i \ i = 1, \ldots, k$



$x^i \; i = p + 1, \ldots, q$

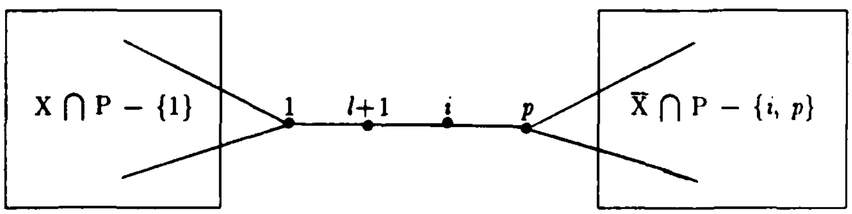
Next, we construct trees corresponding to the nodes in $\overline{\mathrm{X}}$. For $i = k + 1, \ldots, p - 1$ define

$$x_j^i = \begin{cases} 1 & j = (1, l{+}1) \text{ and } (i, l{+}1) \\ 1 & j = (1, r) \;\; 2 \leq r \leq k \\ 1 & j = (r, p) \;\; k + 1 \leq r \leq p - 1 \\ 0 & \text{otherwise} \end{cases}$$

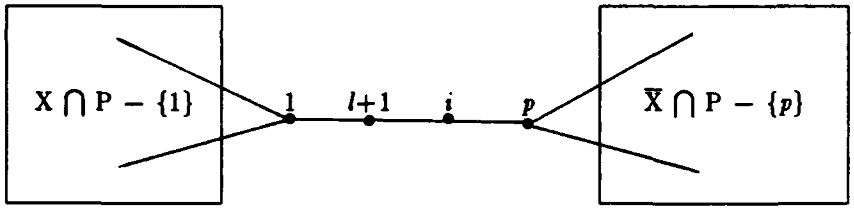
and for $i = q + 1, \ldots, l$ define

$$x_j^i = \begin{cases} 1 & j = (p, i), (1, l{+}1) \text{ and } (i, l{+}1) \\ 1 & j = (1, r) \;\; 2 \leq r \leq k \\ 1 & j = (r, p) \;\; k + 1 \leq r \leq p - 1 \\ 0 & \text{otherwise} \end{cases}$$

Again each of these trees only have one arc in the cut $(\mathrm{X}, \overline{\mathrm{X}})$ and thus, satisfy (4.1) at equality. These two groups of trees have the forms



$x^i \; i = k + 1, \ldots, p - 1$

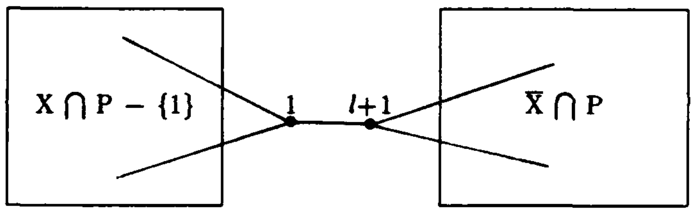


$x^i \; i = q + 1, \ldots, l$

Finally, define

$$x_j^p = \begin{cases} 1 & j = (1, l{+}1) \\ 1 & j = (1, r) \;\; 2 \le r \le k \\ 1 & j = (r, l{+}1) \;\; k + 1 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

Arc $(1, l{+}1)$ is the only arc in $(X, \overline{X})$, so x^p satisfies (4.1) at equality.. This tree has the form



x^p

We now have a total of m_{l+1} P-trees which satisfy (4.1) at equality. Look at the matrix whose columns are the characteristic vectors of these trees.

$$M = \begin{array}{cc} \begin{array}{ccccccc} Y & & & X_{m_l} & & & \end{array} & \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & e_{p-k-1}^T & 1 & 0 & e_{l-q}^T \\ 0 & 0 & I_{k-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{p-k-1} & e_{p-k-1} & 0 & 0 \\ 0 & 1 & e_{k-1}^T & 0 & 1 & e_{q-p}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{q-p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{l-q} \end{array}\right] & \begin{array}{c} (1, l{+}1) \\ \\ \\ (p, l{+}1) \\ \\ \\ \end{array} \end{array}$$

Subtract row $(1,\ l+1)$ and the rows containing the submatrix I_{k-1} from row $(p,\ l+1)$, then add the rows containing the submatrix I_{p-k-1} to this same row. The result is the matrix

$$M' = \left[\begin{array}{ccccccc}
Y & & & X_{m_l} & & & \\
0 & 1 & 0 & e^{\top}_{p-k-1} & 1 & 0 & e^{\top}_{l-q} \\
0 & 0 & I_{k-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{p-k-1} & e_{p-k-1} & 0 & 0 \\
0 & 0 & 0 & 0 & p-k-1 & e^{\top}_{q-p} & -e^{\top}_{l-q} \\
0 & 0 & 0 & 0 & 0 & I_{q-p} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{l-q}
\end{array}\right]
\begin{array}{c}
\\ (1,\ l+1) \\ \\ \\ (p,\ l+1) \\ \\ \\
\end{array}$$

Since $2 \leq k \leq p - 2$, $p - k - 1 > 0$ and M' has affinely independent columns by Proposition 1.6. Thus, by the principle of induction, the inequality (4.1) will induce a facet of $T_{p,n}$ for $p \geq 4$ and $n \geq p + 2$. □

Before moving on to general partitions of the node set, we prove that these cuts are indeed the only cuts that generate facets.

<u>Proposition 4.19:</u> If $(X, \bar{X})$ separates P and does not satisfy the conditions in either Proposition 4.16 or 4.18, then inequality (4.1) induces a face of $T_{p,n}$ of dimension at most $m_n - 2$.

<u>Proof:</u> If $(X, \bar{X})$ is a cut in K_n which separates P, then it must satisfy one of the following conditions

1) X or $\bar{X}$ is strictly contained in P, but not equal to P.
2) Both X and $\bar{X}$ contain elements of P and S.

<u>Case 1:</u> Without loss of generality assume $X \subset P$. Since $(X, \bar{X})$ does not satisfy the requirements of Proposition 4.16, $|X| \geq 2$. Thus, any P-tree satisfying (4.1) at equality must

also satisfy the independent equation

$$\sum_{a \in A(X)} x_a = |X| - 1$$

since the removal of the arc in $(X, \bar{X})$ must leave two connected components. Hence, the arcs within $A(X)$ must span X. Thus, the maximum dimension of a face induced by (4.1) is $m_n - 2$.

Case 2: Since $(X, \bar{X})$ does not satisfy the requirements of Proposition 4.18 then without loss of generality assume $|X \cap P| = 1$. Any P-tree satisfying (4.1) at equality must have the node in $X \cap P$ as a leaf. Thus, if we assume that node $1 \in X$, any P-tree must satisfy the independent equation

$$\sum_{j \in N-\{1\}} x_{1,j} = 1.$$

Thus, the face induced by (4.1) has dimension at most $m_n - 2$. □

4.4 Inequalities Generated By Partitions of the Node Set of K_n

The partitions of N, discussed in Chapter II, that yield facets of $T_{n,n}$ can be generalized in the following manner.

Proposition 4.20: Let $\{V^i\}_{i=1}^{k}$ be a partition of the node set N of K_n for some k such that $2 \leq k < p$ satisfying

1) $|V^i| = 1, \;\; i = 1, \ldots, k - 1$

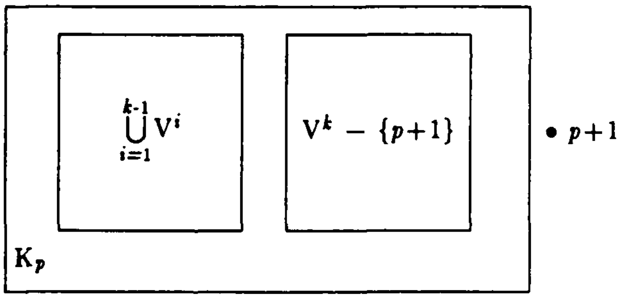
2) $V^i \cap P \neq \emptyset, \;\; i = 1, \ldots, k$

Then the inequality

$$\sum_{i=1}^{k-1}\left[\sum_{j=i+1}^{k}\left[\sum_{a\in(V^i,V^j)} x_a\right]\right] \geq k-1 \quad (4.2)$$

induces a facet of $T_{p,n}$.

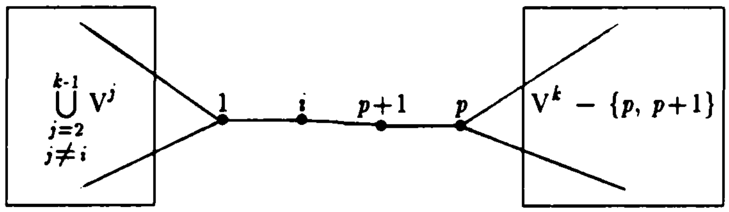
Proof: Let k satisfying $2 \leq k < p$ be given. Without loss of generality, let $V^i = \{i\}$ for $i = 1, \ldots, k-1$ and $V^k = N - \bigcup_{i=1}^{k-1} V^i$. Consider any P-tree T of K_n. By contracting all arcs in the set $A(V^k)$ we obtain a spanning tree of K_k, which must have $k-1$ arcs. Therefore, T must have at least $k-1$ arcs in the sets (V^i, V^j). Thus, we see that (4.2) is valid for $T_{p,n}$, and all that remains to be shown is that we can construct m_n affinely independent P-trees satisfying (4.2) at equality. To that end we proceed by induction on n. For $n = p + 1$ we can view K_{p+1} as



From this picture we see that we can apply Proposition 2.8 to get that there exist $m_p - 1$ lineary independent spanning trees of K_p satisfying (4.2) at equality if we replace V^k with $V^k - \{p+1\}$. Each of these trees is clearly a P-tree of K_{p+1} satisfying (4.2) at equality. Let Y be the matrix whose columns are the characteristic vectors of these trees and define $p + 1$ additional P-trees as follows. For $i = 1, \ldots, k-1$ define

$$x_j^i = \begin{cases} 1 & j = (i,\ p+1) \text{ and } (p,\ p+1) \\ 1 & j = (1,\ r) \quad 2 \leq r \leq k-1 \\ 1 & j = (r,\ p) \quad k \leq r \leq p-1 \\ 0 & \text{otherwise} \end{cases}$$

These trees have $k - 2$ arcs spanning the $k - 1$ nodes in $\bigcup_{i=1}^{k-1} V^i$ and a single arc connecting these "spanning subtrees" to V^k, so they satisfy (4.2) at equality. These trees have the form

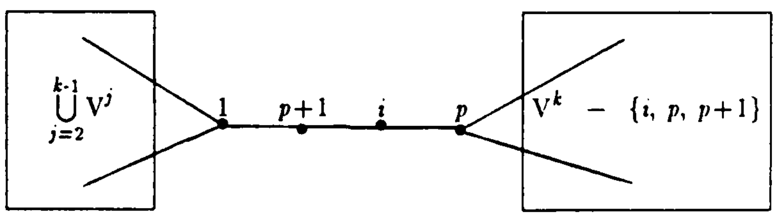


$i = 1, \ldots, k - 1$

Similarly, for $i = k, \ldots, p - 1$ define the trees

$$x_j^i = \begin{cases} 1 & j = (i, p+1) \text{ and } (1, p+1) \\ 1 & j = (1, r) \quad 2 \leq r \leq k - 1 \\ 1 & j = (r, p) \quad k \leq r \leq p - 1 \\ 0 & \text{otherwise} \end{cases}$$

By looking at the form of these trees we can see that they also satisfy (4.2) at equality by having the structure mentioned above.



$i = k, \ldots, p - 1$

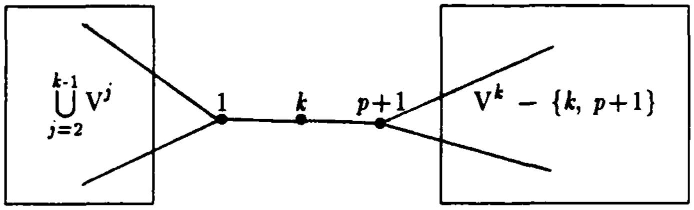
Finally, define the two trees

$$x_j^p = \begin{cases} 1 & j = (1, k) \\ 1 & j = (1, r) \quad 2 \leq r \leq k - 1 \\ 1 & j = (r, p+1) \quad k \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

and

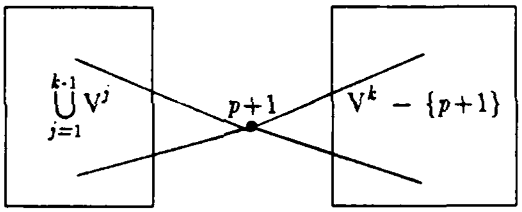
$$x_j^{p+1} = \begin{cases} 1 & j = (r,\, p+1) \;\; 1 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

By their construction these two trees satisfy (4.2) at equality and have the forms



x^p

and



x^{p+1}

To see that these m_{p+1} trees are affinely independent let M be the matrix whose columns are the characteristic vectors of these trees and consider the matrix

$$\overline{M} = \begin{array}{cl} \left[\begin{array}{cccccc} Y & & X'_{m_p} & & x^p_{m_p} & 0 \\ 0 & 1 & 0 & e^{\top}_{p-k} & 0 & 1 \\ 0 & 0 & I_{k-2} & 0 & 0 & e_{k-2} \\ 0 & 0 & 0 & I_{p-k} & e_{p-k} & e_{p-k} \\ 0 & 1 & e^{\top}_{k-2} & 0 & 1 & 1 \\ e^{\top}_{m_p-1} & 1 & e^{\top}_{k-2} & e^{\top}_{p-k} & 1 & 1 \end{array} \right] & \begin{array}{l} \\ \\ \\ \\ \text{row } (p,\, p+1) \\ \\ \end{array} \end{array}$$

Each column of Y is a spanning tree of K_p and thus contains $p - 1$ 1's. Each column of X'_{m_p} contains $p - 2$ 1's, and the vector $x^p_{m_p}$ contains $k - 1$ 1's. So we will multiply the last row of $\overline{M}$ by $-(p - 1)$ and add each of the first m_p rows, the rows containing the submatrix I_{p-k} and row $(p, p+1)$ to it. The resulting matrix is

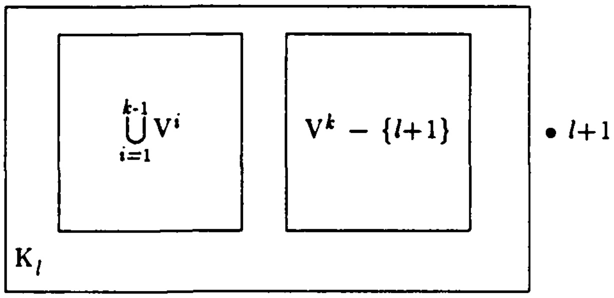
$$\overline{M}' = \begin{array}{c} \left[\begin{array}{cccccc} Y & & X'_{m_p} & & x^p_{m_p} & 0 \\ 0 & 1 & 0 & e^T_{p-k} & 0 & 1 \\ 0 & 0 & I_{k-2} & 0 & 0 & e_{k-2} \\ 0 & 0 & 0 & I_{p-k} & e_{p-k} & e_{p-k} \\ 0 & 1 & e^T_{k-2} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2-k \end{array}\right] \end{array} \begin{array}{l} \\ \text{row } (1, p+1) \\ \\ \\ \text{row } (p, p+1) \\ \\ \end{array}$$

Now subtract row $(1, p+1)$ and the rows containing the submatrix I_{k-1} from row $(p, p+1)$ and then add the rows containing the submatrix I_{p-k} to row $(p, p+1)$. The resulting matrix is

$$\overline{M}'' = \left[\begin{array}{c|cccc|cc} Y & & X'_{m_p} & & & x^p_{m_p} & 0 \\ \hline 0 & 1 & 0 & e^T_{p-k} & & 0 & 1 \\ 0 & 0 & I_{k-2} & 0 & & 0 & e_{k-2} \\ 0 & 0 & 0 & I_{p-k} & & e_{p-k} & e_{p-k} \\ \hline 0 & 0 & 0 & 0 & & p-k+1 & p-2k+2 \\ 0 & 0 & 0 & 0 & & 1 & 2-k \end{array}\right] \begin{array}{l} \\ \\ \text{row } (1, p+1) \\ \\ \\ \text{row } (p, p+1) \\ \\ \end{array}$$

The determinant of the 2×2 submatrix in the lower-right corner of $\overline{M}''$ is $(p - k)(1 - k)$, which is nonzero for $2 \leq k < p$. Hence $\overline{M}''$ satisfies the requirements of Proposition 1.5 for having linearly independent columns. Thus, we have constructed m_{p+1} P-trees satisfying (4.2) at equality which are affinely independent.

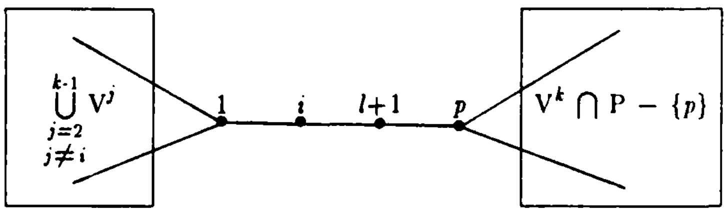
Now assume that there exist m_l affinely independent P-trees of K_l satisfying (4.2) at equality for some $l \geq p + 1$. We can view K_{l+1} as:



P-trees of K_l are also P-trees of K_{l+1}, so let Y be the matrix whose columns are the characteristic vectors of a set of affinely independent P-trees of K_l satisfying (4.2) at equality. For $i = 1, \ldots, k - 1$ define

$$x_j^i = \begin{cases} 1 & j = (i,\ l+1) \text{ and } (p,\ l+1) \\ 1 & j = (1,\ r) \quad 2 \leq r \leq k - 1 \\ 1 & j = (r,\ p) \quad k \leq r \leq p - 1 \\ 0 & \text{otherwise} \end{cases}$$

These trees have the form



$i = 1, \ldots, k - 1$

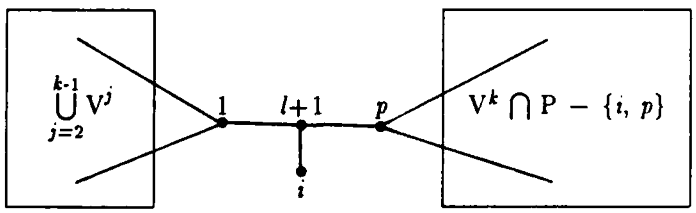
Clearly, these trees satisfy (4.2) at equality since their construction is identical that of the trees in the first part of this proof. Now, in a manner which is also similar to the first part of this proof, we construct trees for $i = k, \ldots, p-1$.

$$x_j^i = \begin{cases} 1 & j = (1, l+1), (p, l+1) \text{ and } (i, l+1) \\ 1 & j = (1, r) \quad 2 \le r \le k-1 \\ 1 & j = (r, p) \quad k \le r \le p-1 \\ 0 & \text{otherwise} \end{cases}$$

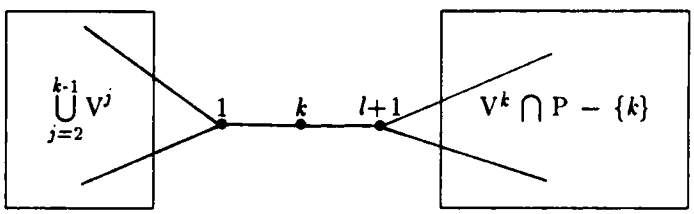
and for $i = p$

$$x_j^p = \begin{cases} 1 & j = (1, k) \\ 1 & j = (1, r) \quad 2 \le r \le k-1 \\ 1 & j = (r, l+1) \quad k \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

These trees have the required structure to satisfy (4.2) at equality and they have the forms



$i = k, \ldots, p-1$

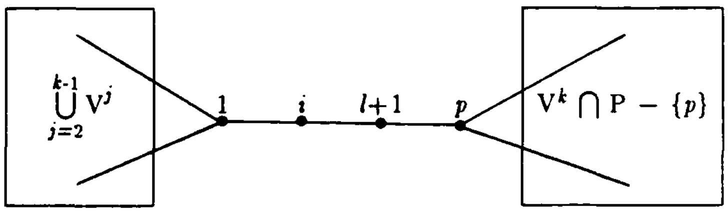


x^p

Finally, for $i = p+1, \ldots, l$ define the trees

$$
x_j^i = \begin{cases} 1 & j = (1,\, i),\ \ (p,\, l{+}1) \text{ and } (i,\, l{+}1) \\ 1 & j = (1,\, r) \ \ 2 \le r \le k-1 \\ 1 & j = (r,\, p) \ \ k \le r \le p-1 \\ 0 & \text{otherwise} \end{cases}
$$

which have the form



$i = p + 1, \ldots, l$

and satisfy (4.2) at equality. We now have m_{l+1} P-trees that satisfy (4.2) at equality. To see that they are affinely independent look at the matrix whose columns are the characteristic vectors of these trees.

$$
M = \begin{array}{c} \left[\begin{array}{ccccc} Y & & X_{m_l} & & \\ 0 & I_{k-1} & A & 0 & 0 \\ 0 & 0 & I_{p-k} & e_{p-k} & 0 \\ 0 & e_{k-1}^{\top} & e_{p-k}^{\top} & 1 & e_{l-p}^{\top} \\ 0 & 0 & 0 & 0 & I_{l-p} \end{array} \right] \end{array} \begin{array}{l} \\ \\ \\ \text{row } (p,\, l{+}1) \\ \\ \end{array}
$$

where A is the $(k - 1) \times (p - k)$ matrix whose first row is $e_{p-k}^{\top}$ and all other entries are 0's. If we subtract the rows containing the submatrix I_{k-1} from row $(p,\, l+1)$ we get the matrix

$$
\begin{array}{l}
M' = \left[\begin{array}{ccccc}
Y & & & X_{m_l} & \\
0 & I_{k-1} & A & 0 & 0 \\
0 & 0 & I_{p-k} & e_{p-k} & 0 \\
0 & 0 & 0 & 1 & e_{l-p}^T \\
0 & 0 & 0 & 0 & I_{l-p}
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \text{row } (p,\ l+1) \\ \\
\end{array}
\end{array}
$$

which satisfies the conditions of Proposition 1.6. Thus, the m_{l+1} P-trees we constructed are affinely independent, and by the principle of induction, the inequality (4.2) induces a facet of $T_{p,n}$ for $p \geq 3$ and $n \geq p + 1$ and partitions of the type specified. □

4.5 Other Facet-Inducing Inequalities

In Proposition 3.13 we proved that an inequality derived from considering the degrees of the nodes in any P-tree induced a facet of $T_{2,n}$. We can generalize this result.

Proposition 4.21: For $p \geq 3$ and $n \geq p + 1$, the inequality

$$2\sum_{a \in A(P)} x_a + \sum_{a \in (P,S)} x_a \leq 2(p - 1) \quad (4.3)$$

induces a facet of $T_{p,n}$.

Proof: Let $p \geq 3$ be given. If $d(i)$ is the degree of node i in a P-tree, then we see that

$$\sum_{i \in P} d(i) = 2\sum_{a \in A(P)} x_a + \sum_{a \in (P,S)} x_a$$

Now, let T be any P-tree, and let q be the number of nodes in the tree. Then for T we know that

$$2(q-1) = \sum_{i \in P} d(i) + \sum_{i \in S} d(i)$$

and

$$\sum_{i \in S} d(i) \geq 2(q-p)$$

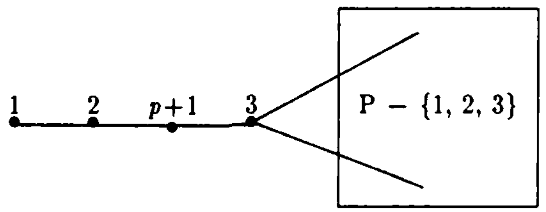
since the $q - p$ nodes in S cannot be terminal nodes. Therefore,

$$2\sum_{a \in A(P)} x_a + \sum_{a \in (P,S)} x_a = \sum_{i \in P} d(i) = 2(q-1) - \sum_{i \in S} d(i) \leq 2(q-1) - 2(q-p) = 2(p-1).$$

Thus, any P-tree T must satisfy (4.3), hence it is valid for $T_{p,n}$. All that we need to prove now is that we can find m_n affinely independent P-trees satisfying (4.3) at equality for any $n \geq p + 1$. We proceed by induction on n. For $n = p + 1$ we recall that every spanning tree of P is a P-tree of K_{p+1}. Further, a spanning tree of P is easily seen to satisfy (4.3) at equality. By Proposition 2.1, there are m_p linearly independent spanning trees of K_p. Let Y be the matrix whose columns are the characteristic vectors of these trees and define p additional P-trees as follows. First define

$$x_j^1 = \begin{cases} 1 & j = (1, 2),\ (2, p+1) \text{ and } (3, p+1) \\ 1 & j = (3, r) \quad 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

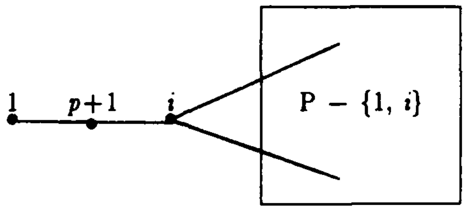
which has the form



Then, for $i = 2, \ldots, p$ define

$$
x_j^i = \begin{cases} 1 & j = (1, p+1) \text{ and } (i, p+1) \\ 1 & j = (r, i) \quad 2 \leq r < i \\ 1 & j = (i, r) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}
$$

This family of trees has the form



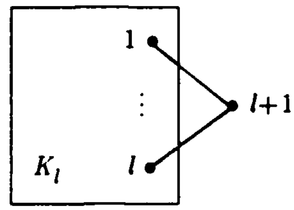
$i = 2, \ldots, p$

In each of these p P-trees there are two arcs in (P, S), namely those incident to node $p+1$, and $p - 2$ arcs in A(P). Thus these trees satisfy (4.3) at equality. Now consider the matrix whose columns are the characteristic vectors of these m_{p+1} trees. This matrix has the form

$$
M = \left[\begin{array}{c|ccc|c}
Y & & X_{m_p} & & \\
\hline
0 & 0 & 1 & 1 & e_{p-3}^T \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & I_{p-3}
\end{array}\right]
$$

This matrix clearly has linearly independent columns by Propositon 1.5. Therefore, the P-trees we constructed are linearly independent.

Now, assume that there exist m_l linearly independent P-trees satisfying (4.3) at equality for some $l \geq p + 1$. We can view K_{l+1} as



By the assumption there exists a set of m_l linearly independent P-trees of K_l satisfying

$$2\sum_{a \in A(P)} x_a + \sum_{a \in (P, S-\{l+1\})} x_a = 2(p-1)$$

These trees are P-trees of K_{l+1} and satisfy (4.3) at equality. Let Y be the matrix whose columns are the characteristic vectors of these trees and define l additional P-trees in the following manner. First, let

$$x_j^1 = \begin{cases} 1 & j = (1, 2), (2, l+1) \text{ and } (3, l+1) \\ 1 & j = (3, r) \quad 4 \leq r \leq p \\ 0 & \text{otherwise} \end{cases}$$

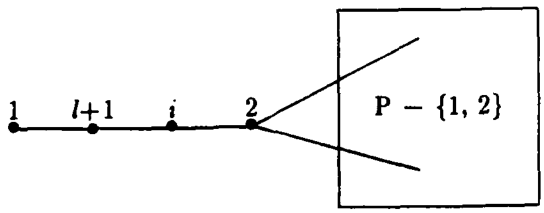
Then for $i = 2. \ldots, p$ define

$$x_j^i = \begin{cases} 1 & j = (1, l+1) \text{ and } (i, l+1) \\ 1 & j = (r, i) \quad 2 \leq r < i \\ 1 & j = (i, r) \quad i < r \leq p \\ 0 & \text{otherwise} \end{cases}$$

These trees are the same as the first two sets of trees defined in the first part of the proof, with node $p+1$ replaced by node $l+1$. Thus, they satisfy (4.3) at equality. Finally, for $i = p + 1, \ldots, l$ define

$$x_j^i = \begin{cases} 1 & j = (2, i),\ (1, l+1) \text{ and } (i, l+1) \\ 1 & j = (2, r) \quad 3 \le r \le p \\ 0 & \text{otherwise} \end{cases}$$

which have the form



$i = p + 1, \ldots, l$

These trees contain two arcs in (P, S) and $p - 2$ arcs in A(P) and thus satisfy (4.3) at equality. The matrix whose columns are the characteristic vectors of these trees has the form

$$M = \left[\begin{array}{c|ccc|c} Y & & X_{m_l} & & \\ \hline 0 & 0 & 1 & 1 & e_{l-3}^T \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{l-3} \end{array}\right]$$

Hence by Proposition 1.5 the m_{l+1} P-trees satisfying (4.3) at equality are linearly independent, and by the principle of induction, (4.3) induces a facet of $T_{p,n}$ for $p \ge 3$. □

Chapter V

Conclusions and Areas for Further Research

5.1 Conclusions

In this dissertation we defined a class of polytopes related to the Steiner Tree Problem on undirected graphs. We then explored inequalities that bounded the components of the characteristic vectors of P-trees and that placed conditions on the number of arcs that pass between sets of nodes of the graph. In general we found that if n and p are sufficiently large, the upper- and lower-bound inequalities and inequalities derived from partitions of the node set which separate P and have only one set containing more that one member induce facets of $T_{p,n}$. We also found an inequality that was derived from the degrees of the nodes in any P-tree that induced a facet of our polytope.

5.2 Areas for Further Research

5.2.1 The Set Covering Problem

One of the ways of attacking a combinatorial problem such as the STG is to develop a hierarchy of polytopes that properly contain the polytope of interest. Aneja's formulation [1], as was shown in Chapter I, is that of a set covering problem, and it is easy to see that the polytope $T_{p,n}$ is a proper subset of the set covering polytope. It may be possible to lift or otherwise strengthen the known facets of the set covering polytope in order to generate new facets of the Steiner tree polytope. This method would parallel the method Balas uses in deriving facets of the prize-collecting traveling salesman problem [2].

5.2.2 Development of a Solution Algorithm

Now that we have a general class of facets we may be able to develop an algorithm to solve the STG that exploits this knowledge.

5.2.3 Generalization to Matroids

Edmonds' complete characterization of $T_{n,n}$ [14] and Grötschel's complete, non-redundant characterization [17] were both derived from results on more general matroids. It would be worthwhile to attempt to generalize our current results to general matroids and then build on this foundation using the tools of matroid theory.

Appendix

Some Counting Results

One method we used to identify possible strong valid inequalities and facets was to generate the convex hull of the set of feasible solutions to some small cases of STG. This required that we generate sets of spanning and P-trees for these problem instances. We developed the following formulas for counting P-trees in K_n as a check on the number of P-trees we were generating.

First we define some new notation and prove a few elementary lemmas. Let

(p, k) = the number of P-trees containing at most k Steiner nodes

$[p, k]$ = the number of P-trees containing exactly k Steiner nodes.

The first important observation is that if $k = 0$, then we are counting spanning trees of K_p. To do this we use Cayley's Theorem [21].

Lemma A.1: $(p, 0) = [p, 0] = p^{p-2}$.

The next relation follows from our notation and gives us a method of counting the P-trees. Clearly, to count all the possible P-trees which may have as many as k Steiner nodes, we need to count all possible trees that have exactly i Steiner nodes for each value of $i \le k$. Since there are $\binom{k}{i}$ ways to chose the i Steiner nodes from the k possibilities, the total number of P-trees containing exactly i of k possible Steiner nodes is $\binom{k}{i}[p, i]$. The total number of P-trees is given by the next result.

Lemma A.2: $(p, k) = \sum_{i=0}^{k} \binom{k}{i}[p, i]$.

Most of our counting will be done inductively. We will determine a number for $[p, k]$ by determining a number for $[p, k - 1]$ and then adding another Steiner node to the tree. We look at the types of trees that can result, and then find some convenient way to break the tree into a series of chains of nodes. To that end, the following observation is quite helpful.

__Lemma A.3:__ The number of ways to divide k Steiner nodes into d chains, some of which may be empty, is

$$k! \binom{k + d - 1}{d - 1}.$$

__Proof:__ Consider the following combinatorial problem: find the number of distinct ways to put k numbered white balls and $d - 1$ identical black balls into a row of $k + d - 1$ slots. This problem is identical to the problem at hand. The k numbered white balls correspond to the Steiner nodes, and the black balls serve to form the d partitions. There are $\binom{k + d - 1}{d - 1}$ ways to choose the slots for the black balls, and then k! ways to place the numbered white balls into the remaining k slots. Hence, the number of ways to divide k Steiner nodes into d chains is

$$k! \binom{k + d - 1}{d - 1}. \quad \square$$

We next consider some counting results for $p = 2$, 3 and 4.

__A.1 Results for $p = 2$__

When $|P| = 2$, every P-tree in K_n is a path from node 1 to node 2. (Recall, if $|P| = 2$, then by assumption $P = \{1, 2\}$) There is a single P-tree if $k = 0$, so $(2, 0) = [2, 0] = 1$. For $k \geq 1$, all the Steiner nodes are in a single chain between the two leaves. There are $k!$ ways to do this, which leads to our first result.

Lemma A.4: $[2, k] = k!$.

We now use Lemma A.2 to develop a relation between $(2, k)$ and $(2, k-1)$.

Proposition A.5: $(2, k) = k\,(2, k-1) + 1$.

Proof: By Lemma A.2

$$(2, k) = \sum_{i=0}^{k} \binom{k}{i} [2, i].$$

Applying Lemma A.4 gives us the relation

$$\sum_{i=0}^{k} \binom{k}{i} [2, i] = \sum_{i=0}^{k} \binom{k}{i}\, i! = k! \sum_{i=0}^{k} \frac{1}{(k-i)!} = k! \sum_{j=0}^{k} \frac{1}{j!} = k \left[(k-1)! \sum_{j=0}^{k-1} \frac{1}{j!} \right] + 1.$$

Similarly, we can get

$$(2, k-1) = (k-1)! \sum_{j=0}^{k-1} \frac{1}{j!}.$$

Thus, by substitution

$$(2, k) = k\,(2, k-1) + 1. \quad \square$$

As it turns out, there is a closed form expression for $(2, k)$.

Proposition A.6: For $k \geq 1$, $(2, k) = \lfloor k!\, e \rfloor$.

Proof: Consider

$$\frac{(2,\, k)}{k!} = \sum_{i=0}^{k} \frac{1}{i!} = e - \sum_{i=k+1}^{\infty} \frac{1}{i!}.$$

Hence

$$0 < k!e - (2,\, k) = \sum_{i=k+1}^{\infty} \frac{k!}{i!} < \sum_{i=1}^{\infty} \left(\frac{1}{k+1}\right)^{i} = \frac{1}{k}.$$

Since $k \geq 1$, we have that

$$0 < k!\, e - (2,\, k) < 1,$$

which means

$$k!\, e - 1 < (2,\, k) < k!\, e,$$

so

$$(2,\, k) = \lfloor k!\, e \rfloor. \quad \square$$

A.2 Results for $p = 3$

When we move to $p = 3$, the situation is a little more complex. When $p = 2$, there is only one type of P-tree, a path between the leaves. In this case, however, there are two types of trees, paths between two of the terminal nodes, and trees that have all three terminal nodes as leaves. If $k = 0$, then we have $(3,\, 0) = [3,\, 0] = 3$, and there are no trees having three leaves. For $k \geq 1$, however, we have the following results

Proposition A.7: $[3,\, k] = (k + 1)\, [3,\, k - 1] + \frac{1}{2}\, (k + 1)!$

<u>Proof:</u> Consider an arbitrary Steiner node, say the kth Steiner node, in a P-tree containing exactly k Steiner nodes. From Proposition 1.7, $d(k) = 2$ or $d(k) = 3$. In the first case, node k must have been inserted in the middle of an arc of some P-tree containing exactly $k - 1$ Steiner nodes. There are [3, k-1] trees of this type, each having $3 + (k - 1) - 1 = k + 1$ arcs. So there are $(k + 1)[3, k - 1]$ P-trees containing k Steiner nodes with $d(k) = 2$. If $d(k) = 3$, then delete node k and consider the three components that remain. Each consists of a chain, possibly empty, of Steiner nodes and a terminal node. There are $k - 1$ Steiner nodes in these three chains, so by Lemma A.3, there are

$$(k - 1)! \binom{3 + (k - 1) - 1}{3 - 1} = (k - 1)! \binom{k + 1}{2} = \frac{1}{2}(k + 1)!$$

possible P-trees in which $d(k) = 3$. The total number of P-trees containing exactly k Steiner nodes is

$$(k + 1)\,[3, k - 1] + \frac{1}{2}(k + 1)!. \quad \square$$

Our next result gives a closed form for $[3, k]$.

<u>Proposition</u> <u>A.8:</u> $[3, k] = \binom{k + 6}{2} (k + 1)!.$

<u>Proof:</u> We prove the result using induction on k. For $k = 1$, Proposition A.7 tells us

$$[3, 1] = 2\,[3, 0] + \frac{1}{2}(2) = 7,$$

while the formula gives

$$\binom{7}{2}\, 2! = 7.$$

Thus, the formula holds for $k = 1$. Now assume that

$$[3,\ k-1] = \binom{k+5}{2}\,k!$$

and look at $[3,\ k]$. By Proposition A.7

$$[3,\ k] = (k+1)\,[3,\ k-1] + \tfrac{1}{2}(k+1)!.$$

Substituting, we get

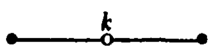
$$[3,\ k] = (k+1)\left[\binom{k+5}{2}\,k!\right] + \tfrac{1}{2}(k+1)! = (k+1)!\,\binom{k+6}{2}.$$

Thus, the formula holds for all $k \geq 1$ by induction. □

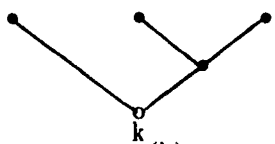
By using the formulas from Propositions A.8 and A.2 we calculate $(3,\ k)$ for any value of k.

A.3 Results for $p = 4$

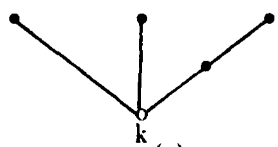
As when we moved from two terminals to three, we again increase the number of types of trees that can occur by moving to four terminals. For $p = 4$, there are five types of P-trees which are shown below.



(a)



(b)



(c)

Terminals are designated by a (•), while Steiner nodes are shown as an (o). The lines connecting the nodes indicate chains of Steiner nodes. For $k = 0$, we have $(4, 0) = [4, 0] = 16$, while for $k \geq 1$ we have the following result.

<u>Proposition A.9:</u>

$$[4, k] = (k + 2)\,[4, k - 1] + \frac{6(k - 1)(k + 2)!}{4!} + 2(k + 2)! + \frac{(k + 2)!}{3!}$$

$$= (k + 2)\,[4, k - 1] + \frac{(k + 2)!}{4!}\,(6k + 46).$$

<u>Proof:</u> As in the proof of Proposition A.7, each of the terms in the expression comes from considering the degree of the "kth" Steiner node in a P-tree. These nodes are shown in the figure above.

If $d(k) = 2$ (tree types a and b), then the node was inserted in the middle of an arc of a P-tree containing exactly $k - 1$ Steiner nodes. There are $[4, k - 1]$ such trees with $(k - 1) + 4 - 1 = k + 2$ arcs each. Thus, the total number of trees for which $d(k) = 2$ is

$$(k + 2)\,[4, k - 1].$$

If $d(k) = 3$, then there are two type of P-trees that could contain k. If the tree is of type (c), then deleting k will leave us with $k - 1$ Steiner nodes divided into three chains, one of which contains two terminal nodes. The number of these trees is determined both by possible arrangements of Steiner nodes and by arrangements of terminal nodes. The terminal node arrangements are determined by which two of the four terminals are in the chain containing two terminals, and their order. There are $\binom{4}{2} = 6$ ways to choose the two terminals, and then 2 ways to arrange them. For each of these 12 situations there are $k - 1$ Steiner nodes divided among four chains, so the total number of trees of type (c) is

$$12\,(k-1)!\binom{k+2}{4} = 2(k+2)!.$$

If $d(k) = 3$ and the tree is of type (d), then deleting node k leaves three components, one of which is a "Y" containing a "splitting" Steiner node h and two terminal nodes. There are $(k - 1)$ ways to choose h and $\binom{4}{2} = 6$ ways to choose the two terminal nodes in the "Y". The remaining $k - 2$ Steiner nodes are divided among five chains. (Three in the "Y", and one in each of the other two components) Thus, there are

$$6(k-1)(k-2)!\binom{k+2}{4} = \frac{6(k-1)(k+2)!}{4!}$$

of this type of P-tree.

Finally, if $d(k) = 4$ (tree type e), then deleting k from the tree leaves us with $k - 1$ Steiner nodes divided among 4 chains. The number of trees of this type is

$$(k-1)\binom{k+2}{4} = \frac{(k+2)!}{3!}.$$

Adding these four expressions together gives us the desired total of P-trees containing exactly k Steiner nodes. $\square$

As in the case of $p = 3$ we can obtain a closed form expression for $[4, k]$.

<u>Proposition</u> <u>A.10:</u> $[4, k] = \frac{(k+2)!}{4!}(192 + 49k + 3k^2)$.

Proof: As in the proof of Proposition A.8, we will proceed by induction on k. For $k = 1$, we can use Proposition A.9 to get

$$[4, 1] = 3[4, 0] + \frac{3!}{4!}(52) = 61.$$

By the above formula we get

$$[4, 1] = \frac{3!}{4!}(192 + 49 + 3) = 61,$$

showing that the result holds for $k = 1$. Now assume that the formula holds for $k - 1$, and look at $[4, k]$. By proposition A.9

$$[4, k] = (k + 2)[4, k - 1] + \frac{(k + 2)!}{4!}(6k + 46)$$

Substituting the formula for $[4, k - 1]$ we get

$$[4, k] = (k + 2)\left[\frac{(k + 1)!}{4!}\left(192 + 49(k - 1) + 3(k - 1)^2\right)\right] + \frac{(k + 2)!}{4!}(6k + 46)$$

$$= \frac{(k + 2)!}{4!}\left[(192 + 49k + 3k^2) - (6k + 46) + (6k + 46)\right]$$

$$= \frac{(k + 2)!}{4!}(192 + 49k + 3k^2) \quad \square$$

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