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On Calculating Analytic Centers

A. A. Goldstein*

This note was motivated by papers of Renegar and Shub(88) and by Ye(89). We apply Smale's(86) estimates at one point for Newton's method to the problem of finding the analytic center of a polytope. The method converges globally in the appropriate norm. The ideas are then applied to obtain a possible benchmark for path following methods.

When Smale's method is tractable its power stems not only from the fact that the information is concentrated at one point. There are 2 norms to estimate, not 3 as in the Kantorovich estimate. Moreover no estimate of the inverse of the derivative operator by itself is needed. The need for the norm of the inverse by itself often makes for coarse estimates.

1. Setting

Let A denote an m by n orthonormal matrix of rank n and b an m by 1 matrix. We assume that $m > n$. Denote by e a m by 1 matrix whose components are all ones. Transposes of matrices will be denoted by an asterisk, rows of a matrix by superscripts, and columns by subscripts. The euclidean space of real m-tuples will be denoted by E_m. If u $\in E_m$ we mean by diag(u) the diagonal matrix with entries u_{ii}. The dot product corresponding to the usual norm will be denoted by [,]. The usual norm will be written as $\| \|_2$. E_n will be also be renormed under a dot product that will be denoted by $< , >$. The norm arising from this dot product will be written as $\| \|$. Let P be a polytope with non-empty interior given by the inequalities

$$b - Ax \geq 0.$$

Given x_0 in the interior of P and $\epsilon > 0$, we seek the analytic center ξ of P to within a tolerance of ϵ. Let $R_i(x) = b_i - A^i x$.

Claim 1. Let N be the smallest integer exceeding

$$1 + \log_2 \left[\log_2 (4.95\, m^{\frac{1}{4}}\, max R_i(x_0)) \; + \; log_2 \left(\frac{1}{\epsilon} \right) \right]$$

Then if N steps of the Newton sequence are generated using the gradient of the potential function below

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$$\|x_N - \xi\|_2 \leq \epsilon$$

The proof of the claim depends on the following ingredients.

2. Some ingredients.

The *potential* of P is defined by the expression

$$\pi(P) = max \prod_{1=1}^{m} R_i(x) : x \in P\}$$

The maximum is achieved at an unique point called the *analytic center* of P. (Ye 87). We shall find this point by seeking a zero for the gradient of the logarithmic potential

$$\phi(x) = \sum_{i=1}^{m} \log(R_i(x)$$

Let $D(x) = diag(1/R_i(x))$, thus $D_{ii}(x) = 1/R_i(x)$.

We apply Newton's method to the gradient of ϕ which we denote by F. F(x) may be represented by the matrix $A^* D(x)e$, and $F(x) \in E_n$. The kth Frechet differential of F at x can be identified with a multi-linear mapping from $(E_N)^k$ to E_n. A representation of these differentials as matrices follows.

$$F'(x)h_1 = -A^* D^2(x)\, diag(Ah_1)e = -A^* D(x) D(x) Ah_1$$

$$F''(x, h_1, h_2) = -2! A^* D(x) D^2(x)\, diag(Ah_2)\, Ah_1$$

$$F'''(x, h_1, h_2, h_3) = -3! A^* D(x) D^3(x)\, diag(Ah_3)\, diag(Ah_2)\, Ah_1$$

and

$$F^{(k)}(x, h_1, h_2, \ldots, h_k) = -k! A^* D(x) D^k(x)\, diag(Ah_k), \ldots\, diag(Ah_2)\, Ah_1$$

$$= -k! A^* D(x) Q(x, h_1, \ldots, h_k)$$

Here

$$\|Q(x)\|_2 = sup\{\|Q(x, h_1, \ldots, h_k)\|_2 : \|h_1\|_2 = \|h_2\|_2, =, \ldots = \|h_k\|_2 = 1\} \leq 1$$

Theorem 1. (Smale 86) Assume F is an analytic map between real Banach spaces X and Y, that is the Frechet derivatives $F^{(k)}(x)$ exist for all x $\in$ X and k=1,2,3......Given $x_0 \in$ X, assume that the inverse of $F'(x)$ which we denote by $F'_{-1}(x)$ exists. Set

$$\beta(x_0) = \|F'_{-1}(x_0)F(x_0)\| \qquad and$$

$$\gamma(x_0) = sup\left\{ \|\frac{1}{k!}F'_{-1}(x_0)F^{(k)}(x_0)\|^{\frac{1}{k-1}} : k \geq 2 \right\}$$

If

$$\beta(x_0)\gamma(x_0) \ < \ .130707$$

then x_0 is an approximate root of F. That is, the Newton sequence

$$x_{k+1} = x_k - F'_{-1}(x_k)F(x_k)$$

is well defined and $\{x_k\}$ converges to say ξ, a root of F at the rate:

$$\|x_{k+1} - x_k\| \ \leq \ 2\left(\frac{1}{2}\right)^{2^k}\beta(x_0)$$

Moreover

$$\|x_k - \xi\| \ \leq \ \frac{7}{4}\left(\frac{1}{2}\right)^{2^{k-1}}\beta(x_0) \qquad\qquad (A)$$

3. **Proof of Claim 1.**

Assume x_0 is given in P(M). The matrix

$$P(x_0) = D(x_0)A(A^*D(x_0)D(x_0)A)^{-1}A^*D(x_0)$$

maps each point in E_m to its closest point in the range of the matrix $D(x_0)A$. Hence $\|P(x_0)\|_2 = 1$. We renorm E_n by

$$\|x\| = \|D_C Ax\|_2$$

Here $D_C = CD(x_0)$ with $C = 1/8^{\frac{1}{2}}m^{\frac{1}{4}}$. With this definition we get:

$$\beta(x_0) = C\|P(x_0)e\|_2 = m^{\frac{1}{4}}/8^{\frac{1}{2}}$$

Also

$$\gamma(x_0) \leq C\,sup(\|P(x_0)\|_2^{\frac{1}{k-1}})\,sup(\|Q^k Ah_1\|_2^{\frac{1}{k-1}}) \ \leq \ C$$

$$\text{Thus} \qquad \beta(x_0)\gamma(x_0) \ \leq \ \frac{1}{8} \ < \ .130707$$

Hence by Smale's theorem the sequence generated by the Newton algorithm converges to the analytical center ξ with a rate given by (A) in Theorem 3.1 above.

Since $< x, x >= [D_C Ax, D_C Ax] \geq C^2 \|x\|_2^2 / max R_i(x_0)^2$, then

$$\|x\|_2 \leq C R_i(x_0) \|x\|$$

Now choose N so that

$$C R_i(x_0) \|x_N - \xi\| \leq \epsilon$$

4. Application to programming

By a theorem of Ye (89), if one of the hyperplanes of P is translated to pass thru ξ then the resulting polytope P^+ satisfies

$$\frac{\pi(P^+)}{\pi(P)} \leq \frac{1}{\epsilon}$$

.

Consider the following algorithm for linear inequalities. We wish to solve the system $b - Ax \geq 0$ if this is possible. Given an arbitrary x_0 choose M so that $b + M - Ax > 0$. Find the center of this polytope P(M). Take the smallest component of $R(\xi)$, say $R_q(\xi)$. Begin anew with the polytope $P(M - R_q(\xi))$. This algorithm has a worst case iteration count of $O(m)$ times our cost of getting to the center.

For linear programming let the polytope P be given by b - Ax $\geq$ 0 and P(M) the polytope define by the inequalities for P together with the inequality M - [c , x] $\geq$ 0. We seek the smallest M for which P(M) is non-empty. We first find the center ξ of the polytope P. We then find the intersection of the ray $\{x = \xi - tc : t > 0\}$ with P Translate the cost hyperplane to pass thru this point. Then find the center of the new polytope P(M).

5. Benchmark

We now consider the possibility of starting from a point in a polytope P(M) and moving to the center of a neighboring polytope $P(M - 1/2\sqrt{m})$ by Newton steps.

Assume that at (x_0, M_0), $R_i(x, M) = b_i + M_0 - A^i x > 0$. We seek a point (x_1, M_1) such that

$$\frac{\partial \phi(x, M)}{\partial x_j} = 0, \qquad 1 \leq j \leq n \qquad (1a)$$

$$\frac{\partial \phi(x, M)}{\partial M} - \frac{\partial \phi(x_1, M_1)}{\partial M} = 0 \qquad (1b)$$

and such that

$$R_i(b_i + M_1 - A^i x_1) \ > \ 0 \qquad (2)$$

Let $M_1 = M_0 - 1/2\sqrt{m}$. Assume that the value of x_1 is well defined and given. Otherwise $P(M_1)$ is empty and M_0 is within $1/2\sqrt{m}$ of M^* the optimal value of M. We show that (x_0, M_0) is an "approximate root" for system (I).

Remark The matrix (A e) has rank n+1.

Proof Because of our boundedness assumption on the polytopes, the system of inequalities $Ax \ > \ 0$ is inconsistent. If u is in the null space of (A e) then $Au = -u_{n+1}e \neq 0$, a contradiction.

In matrix notation the system (1) (after scaling the second entry) is

$$F(x, M) = \left(-A^* D(x, M)e \qquad \frac{1}{2\sqrt{m}} e^* (D(x, M)e - D(x_1, M_1)e) \right)^* \qquad (I)$$

Thus we see that - $F'(x, M)$ may be generated from the matrix

$$B = (A_1, A_2, ..., A_n, A_{n+1}) \qquad where \ A_{n+1} = \frac{-e}{2\sqrt{m}}$$

Assume that $(A_1, A_2, ..., A_n)$ is rescaled if necessary so that $||B|| \ \leq \ 1$. By the Remark we see that B has rank n+1. Thus Claim 1 holds for this case as well. If we are satisfied with a reduction of $1/3\sqrt{m}$ this will happen in N steps by the claim with ϵ set to $1/6\sqrt{m}$. We have then the following result: (not an algorithm but a benchmark)

Claim 2. We are given a point (x_k, M_k),. Let $M_{k+1} = M_k - 1/2\sqrt{m}$. If $P(M_{k+1})$ is not empty, take x_{k+1} for its center. Let the system (I) be run with Newtons' method. Otherwise, stop. In N steps M_k will be reduced by at least $1/3\sqrt{m}$. This value updates M_{k+1} and the corresponding iterate for x updates x_{k+1}. Assume the optimal M say M^* known. Then the global Newton process can be terminated in no more than Q steps, where

$$Q \geq 3\sqrt{m}(M_0 - M^*)(1 + \log_2 \left[\log_2 (4.95\, m^{\frac{1}{4}}\, max R_i(x_0)) \ + \ log_2 \left(6\sqrt{m} \right) \right]$$

At termination M_N is within $1/2\sqrt{m}$ of M^* and x_N is an approximate root for system (I) with M^* replacing M_1 and ξ replacing x_1, respectively.

A similar result holds for linear programming.

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