ON CALCULATING ANALYTIC CENTERS

Allen Goldstein

August 1989

Approved for public release; distribution unlimited
Prepared for:
Naval Postgraduate School
Monterey, CA 93943
This report was prepared in conjunction with research conducted for the Naval Postgraduate School and funded by the Naval Postgraduate School. Reproduction of all or part of this report is authorized.

Prepared by:

ALLEN A. GOLDSTEIN
Adjunct Professor of Mathematics

Reviewed by:

HAROLD M. FREDRICKSEN
Chairman
Department of Mathematics

Released by:

KNEALE T. MARSHALL
Dean of Information and Policy Sciences
ON CALCULATING ANALYTIC CENTERS

The analytic center of a polytope can be calculated in polynomial time by Newton's method.
On Calculating Analytic Centers
A. A. Goldstein

This note was motivated by papers of Renegar and Shub(88) and by Ye(89). We apply Smale’s(86) estimates at one point for Newton’s method to the problem of finding the analytic center of a polytope. The method converges globally in the appropriate norm. The ideas are then applied to obtain a possible benchmark for path following methods.

When Smale’s method is tractable its power stems not only from the fact that the information is concentrated at one point. There are 2 norms to estimate, not 3 as in the Kantorovich estimate. Moreover no estimate of the inverse of the derivative operator by itself is needed. The need for the norm of the inverse by itself often makes for coarse estimates.

1. Setting

Let $A$ denote an $m \times n$ orthonormal matrix of rank $n$ and $b$ an $m \times 1$ matrix. We assume that $m > n$. Denote by $e$ a $m \times 1$ matrix whose components are all ones. Transposes of matrices will be denoted by an asterisk, rows of a matrix by superscripts, and columns by subscripts. The euclidean space of real $m$-tuples will be denoted by $E_m$. If $u \in E_m$ we mean by $\text{diag}(u)$ the diagonal matrix with entries $u_{ii}$. The dot product corresponding to the usual norm will be denoted by $[,]$. The usual norm will be written as $\| \|_2$. $E_n$ will be also be renormed under a dot product that will be denoted by $<,>$. The norm arising from this dot product will be written as $\| \|$. Let $P$ be a polytope with non-empty interior given by the inequalities

$$b - Ax \geq 0.$$ 

Given $x_0$ in the interior of $P$ and $\epsilon > 0$, we seek the analytic center $\xi$ of $P$ to within a tolerance of $\epsilon$. Let $R_t(x) = b - A^t x$.

Claim 1. Let $N$ be the smallest integer exceeding

$$1 + \log_2 \left[ \log_2 (4.95 m^\frac{1}{2} \max R_t(x_0)) + \log_2 \left( \frac{1}{\epsilon} \right) \right]$$

Then if $N$ steps of the Newton sequence are generated using the gradient of the potential function below

* supported by grants NIH RR01243-05 AND NPS LMC-M4E1
The proof of the claim depends on the following ingredients.

2. Some ingredients.

The potential of \( P \) is defined by the expression

\[
\pi(P) = \max \prod_{i=1}^{m} R_i(x) : x \in P
\]

The maximum is achieved at an unique point called the analytic center of \( P \). (Ye 87). We shall find this point by seeking a zero for the gradient of the logarithmic potential

\[
\phi(x) = \sum_{i=1}^{m} \log(R_i(x))
\]

Let \( D_i(x) = \text{diag}(1/R_i(x)) \), thus \( D_{ii}(x) = 1/R_i(x) \).

We apply Newton's method to the gradient of \( \phi \) which we denote by \( F \). \( F(x) \) may be represented by the matrix \( A^*D(x)e \), and \( F(x) \in E_n \). The \( k \)th Frechet differential of \( F \) at \( x \) can be identified with a multi-linear mapping from \( (E_N)^k \) to \( E_n \). A representation of these differentials as matrices follows.

\[
F'(x)h_1 = -A^*D^2(x)\text{diag}(Ah_1)e = -A^*D(x)D(x)Ah_1
\]

\[
F''(x, h_1, h_2) = -2!A^*D(x)D^2(x)\text{diag}(Ah_2)Ah_1
\]

\[
F'''(x, h_1, h_2, h_3) = -3!A^*D(x)D^3(x)\text{diag}(Ah_3)\text{diag}(Ah_2)Ah;
\]

and

\[
F^{(k)}(x, h_1, h_2, \ldots, h_k) = -k!A^*D(x)D^k(x)\text{diag}(Ah_k), \ldots, \text{diag}(Ah_2)Ah_1
\]

\[
= -k!A^*D(x)Q(x, h_1, \ldots, h_k)
\]

Here

\[
\|Q(x)\|_2 = \sup\{\|Q(x, h_1, \ldots, h_k)\|_2 : \|h_1\|_2 = \|h_2\|_2 = \ldots = \|h_k\|_2 = 1\} \leq 1
\]
Theorem 1. (Smale 86) Assume $F$ is an analytic map between real Banach spaces $X$ and $Y$, that is the Frechet derivatives $F^{(k)}(x)$ exist for all $x \in X$ and $k=1,2,3,\ldots$. Given $x_0 \in X$, assume that the inverse of $F'(x)$ which we denote by $F'^{-1}(x)$ exists. Set

$$
\beta(x_0) = \|F'^{-1}(x_0)F(x_0)\| \quad \text{and} \quad \gamma(x_0) = \sup\left\{ \left\| \frac{1}{k!} F'^{(k)}(x_0)F^{(k)}(x_0) \right\|^{\frac{1}{k+1}} : k \geq 2 \right\}
$$

If

$$
\beta(x_0)\gamma(x_0) < .130707
$$

then $x_0$ is an approximate root of $F$. That is, the Newton sequence

$$
x_{k+1} = x_k - F'^{-1}(x_k)F(x_k)
$$

is well defined and $\{x_k\}$ converges to say $\xi$, a root of $F$ at the rate:

$$
\|x_{k+1} - x_k\| \leq 2\left(\frac{1}{2}\right)^k \beta(x_0)
$$

Moreover

$$
\|x_k - \xi\| \leq \frac{7}{4}\left(\frac{1}{2}\right)^{k-1} \beta(x_0) \quad (A)
$$


Assume $x_0$ is given in $P(M)$. The matrix

$$
P(x_0) = D(x_0)A_1A^*D(x_0)D(x_0)A^{-1}A^*D(x_0)
$$

maps each point in $E_m$ to its closest point in the range of the matrix $D(x_0)A$. Hence $\|P(x_0)\|_2 = 1$. We renorm $E_m$ by

$$
\|x\| = \|DCAx\|_2
$$

Here $DCA = CD(x_0)$ with $C = 1/8^{\frac{1}{2}}m^{\frac{1}{4}}$. With this definition we get:

$$
\beta(x_0) = C\|P(x_0)x\|_2 = m^{\frac{1}{4}}/8^{\frac{1}{2}}
$$

Also

$$
\gamma(x_0) \leq C\sup\left\{ \|P(x_0)\|_2^{\frac{1}{k+1}} \right\} \sup\left\{ \|Q^kAh_1\|_2^{\frac{1}{k+1}} \right\} \leq C
$$

Thus

$$
\beta(x_0)\gamma(x_0) \leq \frac{1}{8} < .130707
$$
Hence by Smale's theorem the sequence generated by the Newton algorithm converges to the analytical center $\xi$ with a rate given by (A) in Theorem 3.1 above.

Since $< x, x > = [D_C A x, D_C A x] \geq C^2 \| x \|^2_2 / \max R_i(x_0)^2$, then

$$\| x \|_2 \leq CR_i(x_0)\| x \|$$

Now choose $N$ so that

$$CR_i(x_0)\| x_N - \xi \| \leq \epsilon$$

4. Application to programming

By a theorem of Ye (89), if one of the hyperplanes of $P$ is translated to pass thru $\xi$ then the resulting polytope $P^+$ satisfies

$$\frac{\pi(P^+)}{\pi(P)} \leq \frac{1}{\epsilon}$$

Consider the following algorithm for linear inequalities. We wish to solve the system $b - A x \geq 0$ if this is possible. Given an arbitrary $x_0$ choose $M$ so that $b + M - A x > 0$. Find the center of this polytope $P(M)$. Take the smallest component of $R(\xi)$, say $R_q(\xi)$. Begin anew with the polytope $P(M - R_q(\xi))$. This algorithm has a worst case iteration count of $O(m)$ times our cost of getting to the center.

For linear programming let the polytope $P$ be given by $b - A x \geq 0$ and $P(M)$ the polytope define by the inequalities for $P$ together with the inequality $M - [c, x] \geq 0$. We seek the smallest $M$ for which $P(M)$ is non-empty. We first find the center $\xi$ of the polytope $P$. We then find the intersection of the ray $\{ x = \xi - t c : t > 0 \}$ with $P$ Translate the cost hyperplane to pass thru this point. Then find the center of the new polytope $P(M)$.

5. Benchmark

We now consider the possibility of starting from a point in a polytope $P(M)$ and moving to the center of a neighboring polytope $P(M - 1/2\sqrt{m})$ by Newton steps.

Assume that at $(x_0, M_0)$. $R_i(x, M) = b_i + M_0 - A^t x > 0$. We seek a point $(x_1, M_1)$ such that

$$\frac{\partial \phi(x, M)}{\partial x_j} = 0, \quad 1 \leq j \leq n \quad (1a)$$

$$\frac{\partial \phi(x, M)}{\partial M} - \frac{\partial \phi(x_1, M_1)}{\partial M} = 0 \quad (1b)$$
and such that

\[ R_i(b_i + M_1 - A'x_1) > 0 \quad (2) \]

Let \( M_1 = M_0 - 1/2\sqrt{m} \). Assume that the value of \( x_1 \) is well defined and given. Otherwise \( P(M_1) \) is empty and \( M_0 \) is within \( 1/2\sqrt{m} \) of \( M^* \) the optimal value of M. We show that \((x_0, M_0)\) is an "approximate root" for system (I).

Remark The matrix \((A, e)\) has rank \( n+1 \).

Proof Because of our boundedness assumption on the polytopes, the system of inequalities \( Ax > 0 \) is inconsistent. If \( u \) is in the null space of \((A, e)\) then \( Au = -\varepsilon_{n+1} \neq 0 \). a contradiction.

In matrix notation the system (1) (after scaling the second entry) is

\[
F(x, M) = \left( -A^*D(x, M)e \quad \frac{1}{2\sqrt{m}}\varepsilon^*(D(x, M)e - D(x_1, M)e) \right)^* \quad (I)
\]

Thus we see that \( F'(x, M) \) may be generated from the matrix

\[ B = (A_1, A_2, \ldots, A_n, A_{n+1}) \quad \text{where} \quad A_{n+1} = \frac{-\varepsilon}{2\sqrt{m}} \]

Assume that \((A_1, A_2, \ldots, A_n)\) is rescaled if necessary so that \( \|B\| \leq 1 \). By the Remark we see that \( B \) has rank \( n+1 \). Thus Claim 1 holds for this case as well. If we are satisfied with a reduction of \( 1/3\sqrt{m} \) this will happen in \( N \) steps by the claim with \( \varepsilon \) set to \( 1/6\sqrt{m} \). We have then the following result: (not an algorithm but a benchmark)

Claim 2. We are given a point \((x_k, M_k)\). Let \( M_{k+1} = M_k - 1/2\sqrt{m} \). If \( P(M_{k+1}) \) is not empty, take \( x_{k+1} \) for its center. Let the system (I) be run with Newton's method. Otherwise, stop. In \( N \) steps \( M_k \) will be reduced by at least \( 1/3\sqrt{m} \). This value updates \( M_{k+1} \) and the corresponding iterate for \( x \) updates \( x_{k+1} \). Assume the optimal \( M \) say \( M^* \) known. Then the global Newton process can be terminated in no more than \( Q \) steps, where

\[
Q \geq 3\sqrt{m}(M_0 - M^*)/(1 + \log_2 \left[ \log_2 (4.95m^4 \max R_i(x_0)) + \log_2 (6\sqrt{m}) \right])
\]
At termination $M_N$ is within $1/2\sqrt{m}$ of $M^*$ and $x_N$ is an approximate root for system (1) with $M^*$ replacing $M_1$ and $\xi$ replacing $x_1$, respectively.

A similar result holds for linear programming.

**Bibliography**

James Renegar and Michael Shub  Simplified complexity analysis for Newton lp methods Cornell OR Report no. 807 June 88

S. Smale  Newton's method estimates from data at one point. The Merging of Disciplines... Springer-Verlag 1986 185-196.


Y. Ye  A combinatorial Property of Analytic Centers of Polytopes Dept of Management Science U of Iowa May 1989
INITIAL DISTRIBUTION LIST

DIRECTOR (2)
DEFENSE TECH. INFORMATION CENTER
CAMERON STATION
ALEXANDRIA, VA  22314

DIRECTOR OF RESEARCH ADMINISTRATION
CODE 012
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA  93943

OFFICE OF NAVAL RESEARCH
CODE 422AT
ARLINGTON, VA  22217

PROFESSOR ALLEN GOLDSTEIN (12)
CODE 53GO
DEPARTMENT OF MATHEMATICS
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA  93943

LIBRARY (2)
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA  93943

DEPARTMENT OF MATHEMATICS
CODE 53
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA  93943

CENTER FOR NAVAL ANALYSIS
4401 FORD AVENUE
ALEXANDRIA, VA  22302-0268