DECONVOLUTION METHODS FOR MULTI-DETECTORS

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Deconvolution Methods for Multi-Detectors

The feasibility of building super-resolution systems using multiple detectors has been demonstrated. The deconvolution method provides a real time linear implementation of the reconstruction problem which is robust with respect to noise and perturbations of the overall system.
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DECONVOLUTION METHODS FOR MULTIPLE DETECTORS

INTRODUCTION

Deconvolution of a single convolution equation is usually an ill-posed problem. This has been sufficiently illustrated in the literature. The shortcomings of linear and of non-linear deconvolution methods can be found, for instance, in the very clear review paper [1]. Advances in the theory of holomorphic functions of several complex variables led Berenstein, Taylor and Yger to realize that systems of convolution equations could be deconvolved exactly, thus avoiding the above ill-posedness. Their preliminary papers [2,3] eventually led to this project. The practical interest of this observation is that whenever such a set of convolution equations represents a set of physically realizable devices (e.g. transducers, sensors) then one has, by use of a digitally implemented inverse, essentially an arbitrary bandwidth device. Let us make these ideas exact by recalling the following definitions and deconvolution scheme.

Let \( \mu_1, \ldots, \mu_m \) be a collection of distributions of compact support in \( \mathbb{R}^n \) ("convolvers"). Their Fourier transforms \( \hat{\mu}_j(z) \) are holomorphic functions of \( n \) complex variable in the Paley-Wiener class. We say that \( \mu_1, \ldots, \mu_m \) are strongly coprime if there are constants \( \varepsilon > 0, C > 0 \) such that for every \( z \in \mathbb{C}^n \):
In that case, one can prove the existence of "deconvolvers" \( v_1, \ldots, v_m \). That is, distributions of compact support such that

\[
\sum_{j=1}^{m} |\hat{u}_j(z)| \geq \varepsilon \exp(-C \{ \log(1+|z|) + |\text{Im} z| \}).
\]

Once this \( v_j \) have been explicitly found, and herein lies the difficulty we set out to solve, one has the following scheme to determine an unknown signal \( \phi \):

A very thorough discussion of the meaning of this scheme, and its implementation in the 1-d case, can be found in the final report to a previous ARO supported project [4].

In this project we set ourselves the following tasks:

(i) find explicit and, relatively easy, formulas for the deconvolvers \( v_1, \ldots, v_m \) in the 1-d and 2-d situations.

(ii) show that in the above deconvolution scheme the overall system, in the presence of noise, is entirely dependent on the sampling rate and noise characteristics, i.e. no "inherent" bandwidth
limitations. This means to show that the above block diagram behaves like

\[ \text{average over small detector} = \bar{\Phi}, \]

(iii) construct 1-d simulations of the deconvolution scheme to show its feasibility beyond purely theoretical considerations.

These three objectives have all been accomplished and documented as it will be explained now. Furthermore, the payoff of the powerful techniques developed for these questions has also taken place in two other areas: explicit solution of algebraic equations and their complexity, and the local Pompeiu problem, these will be discussed below.

First of all, we have two schemes to find deconvolvers. An analytic one is presented in Chapter 1 where everything depends on the following identity illustrated for the 2-d case with three convolvers:

\[ \hat{u}(z) = \sum_{\zeta \in \mathbb{Z}} \frac{u(\zeta)}{J(\zeta)\mu_j(\zeta)} \]

\[ J(z) = \begin{vmatrix} g_1^1(z,\zeta) \cdots g_3^3(z,\zeta) \\ g_2^1(z,\zeta) \cdots g_3^3(z,\zeta) \\ \hat{\mu}_1(\zeta)\hat{\mu}_2(\zeta)\hat{\mu}_3(\zeta) \end{vmatrix} = \hat{\nu}_1(z)\mu_1u(z) + \cdots + \hat{\nu}_3(z)\mu_3u(z) \]

\[ \hat{\mu}_j(z,\zeta) = \frac{\hat{\mu}_j(z_1,\zeta_2) - \hat{\mu}_j(z_1,\zeta_2)}{z_1 - \zeta_1}, \quad \hat{\mu}_j(z_1,z_2) = \frac{\hat{\mu}_j(z_1,z_2) - \hat{\mu}_j(z_1,z_2)}{z_2 - \zeta_2} \]

\[ J(\zeta) = \text{Jacobian determinant of } \hat{\mu}_1, \hat{\mu}_2 \text{ at } \zeta \]
Here \( u \) can be taken as a mollifier of small support and the estimate \( \hat{\phi} \) of the signal \( \phi \) will then be

\[
\hat{\phi} = \psi_1 \ast \phi + \ldots + \psi_3 \ast \phi
\]

where \( \psi_j = (\nu_j \ast \phi + \text{noise}) \ast u = \text{"smoothened" data.} \) The resolution level will just correspond to the sampling rate. This chapter has appeared in print in [5].

In other words, we can say we construct analytically deconvolvers \( \nu_j \), once for all, and then we implement a numerical integration scheme (choice of \( u \) and sampling).

There is a second method to find numerical deconvolvers directly, explained by Dr. Taylor in the recent West Point ARO Conference on Computational and Applied Mathematics. In this second method, for a fixed grid (resolution) we compute the deconvolvers numerically and the existence of the analytic deconvolvers guarantees that when the mesh of the grid decreases and the noise decrease the corresponding estimates \( \hat{\phi} \) converge to \( \phi \). We do not yet have sufficient numerical data to compare these two methods.

In Chapter 2, we present an evaluation of the effect of noise on deconvolution, background limited detectors, and amplifier limited detectors are considered, a criterion is given to evaluate and design different systems of convolvers. We prove that deconvolution of strongly coprime systems behave well with respect to noise. This is crucial for practicality and for the actual engineering design of such systems. This chapter constitutes the paper [6] which has
been submitted by invitation and accepted for publication in the forthcoming IEEE Proceedings in Multidimensional Signal Processing. It is an expansion of part of the Ph.D. thesis of E. Vincent Patrick written under the direction of Dr. Berenstein.

In some applications, we find ourselves with occluded near regions where data could be collected by sensors but no deconvolution in the above sense performed. This is the so-called Local Pompeiu problem. Let us say a signal \( f \) is collected by averaging its values over any square of side \( a \), but that this can be done only inside some disk of radius \( R \). What is the precise relation between \( R \) and \( a \) (if there is any) such that the data collected determines \( f \) (this would allow to decide whether \( f \) is what we are looking for or not). Furthermore, can we reconstruct \( f \) from this data? Surprisingly, the answer to both questions is yes, at least when \( R > \sqrt{a} \) for the first problem and \( R > \frac{3}{2} \sqrt{a} \) for the second. This is the content of Chapters 3 and 4, respectively. It is not necessary to emphasize the importance of this work in the problem of Automated Target Recognition. Chapter 3 has been expanded and appeared in print as [8] and Chapter 4, [9], has been accepted for publication.

Let us conclude this introduction with the observation that the formulas from Chapter 1 have had an unexpected impact in problems about systems of algebraic equations, their complexity and properties of the solutions of the
algebraic Bezout equation. In this sense, this paper and its follow ups [10,11] have been cited in numerous papers on complexity theory and related algebraic problems. See [12] for an (already slightly outdated) introduction to this subject.

REFERENCES FOR THE INTRODUCTION


CHAPTER 1

ANALYTIC BEZOUT IDENTITIES

by

Carlos A. Berenstein* and Alain Yger*
1. Introduction

In a number of papers [2,3,4,6,7,12] the problem of finding explicit solutions $h_1,\ldots,h_m$ for the Bezout equation:

$$f_1h_1+\cdots+f mh_m = 1$$

has been considered. If $f_1,\ldots,f_m$ are complex polynomials in $n$ variables and they have no common zeros in $\mathbb{C}^n$, the existence of explicit analytic expressions for the corresponding polynomials $h_1,\ldots,h_m$ has a number of applications to control theory and commutative algebra. The most notable application being that of Brownawell [11] where the essentially best possible estimate for the degrees of the $h_1,\ldots,h_m$ was obtained using such explicit analytic expressions. Up to date, no purely algebraic proof of these bounds has been found.

Similarly, in the deconvolution problem, this time the functions $f_1,\ldots,f_m$ being the Fourier transforms of a strongly coprime family $\mu_1,\ldots,\mu_m$ of distributions of compact support, one searches for a procedure to compute explicitly distributions of compact support $\nu_1,\ldots,\nu_m$ such that $\mu_1^*\nu_1+\cdots+\mu_m^*\nu_m = \delta$. (Here $\hat{\nu}_j$ play the role of $h_j$ in the equivalent formulation $\hat{\mu}_1^*\hat{\nu}_1+\cdots+\hat{\mu}_m^*\hat{\nu}_m = 1$). This question arises in problems of robust filtering, image processing, etc. [30]. In [7] we wrote down formulas for a solution $\nu_1,\ldots,\nu_m$ of the deconvolution problem in terms of interpolation series. The problem we have faced recently is that, while for the 1-d case these formulas can be easily implemented, in the higher dimensional case they are far too cumbersome, some of them seem to be beyond the range of symbolic languages like MACSYMA on which we had, perhaps too
optimistically, relied upon to carry through the computations involved. For that reason we present here a new version of our original deconvolution formulas which assumes extra conditions on the family \( \mu_1, \ldots, \mu_m \) but has as a payoff a very simple formula for the deconvolutors \( \nu_1, \ldots, \nu_m \). We give herein simple examples where these extra conditions are satisfied.

The problem of finding an efficient algorithm to compute the above mentioned solutions to the algebraic Bezout equation being still open, we also analyze here the particular case in which those polynomials can be computed in terms of interpolation formulas. Finding an algorithm with a low complexity for this problem will have many important applications in the theory of distributed parameter systems and in robotics.

We have also found that a language barrier prevented our work [7] to be more easily available to some engineers, and hope that the present paper will overcome those shortcomings.

The first author would like to express his gratitude to the Université de Bordeaux I for its hospitality while this work was carried on. His work on the algebraic aspects of this paper are inspired on the questions raised by the APOS-URI project at the University of Maryland. We will also like to thank our friends Dale Brownawell and B. Alan Taylor for many helpful remarks.
2. **Analytic case**

We will consider only entire functions \( f \) of \( n \) complex variables satisfying inequalities of the form

\[
|f(z)| \leq A(1+|z|)^m e^{H(\text{Im } z)} , \quad z \in \mathbb{C}^n ,
\]

\( \text{Im } z = (\text{Im } z_1, \ldots, \text{Im } z_n) \in \mathbb{R}^n \) where \( H \) is a convex continuous function in \( \mathbb{R}^n \), homogeneous of degree 1 (i.e. \( H(\lambda x) = \lambda H(x) \) when \( \lambda > 0 \)). We call such a function \( H \) a **supporting function**. By the Paley-Wiener theorem [18] there is a distribution \( \mu \) of compact support in \( \mathbb{R}^n \) such that \( f = \hat{\mu} \), the Fourier transform of \( \mu \). Furthermore, the supporting function \( H_0 \) of \( \text{cv supp } \mu \) will satisfy \( H_0 \leq H \) (Here \( \text{cv} \) denotes the convex hull).

Conversely, if \( f = \hat{\mu} \) we can take \( H = H_0 \) in (1) and \( m \) is related to the order of \( \mu \) in a simple manner. Hereafter we will just write \( f \in \mathcal{C}' (= \mathcal{C}'(\mathbb{R}^n)) \) if \( f \) satisfies (1).

For simplicity denote \( \rho(z) := \log(2+|z|) + |\text{Im } z| \). A family \( f_1, \ldots, f_m \) of functions in \( \mathcal{C}' \) is said to be **strongly coprime** if there is a constant \( c \) such that

\[
\sum_{j=1}^{m} |f_j(z)|^2 \geq e^{-c\rho(z)} , \quad z \in \mathbb{C}^n .
\]

It is well known [14] that (2) is a necessary and sufficient condition for the existence of functions \( h_1, \ldots, h_m \in \mathcal{C}' \) such that

\[
\sum_{j=1}^{m} f_j h_j = 1 .
\]

In other words, a strongly coprime family is precisely a family for which the analytic Bezout equation (3) has a solution. If we
consider (3) in terms of the distributions \( \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_m \) such that \( \dot{\mu}_j = f_j \) and \( \dot{\nu}_j = h_j \) then we have the identity

\[
\mu_1 \ast \nu_1 + \cdots + \mu_m \ast \nu_m = \delta,
\]

i.e. \( \nu_1, \ldots, \nu_m \) solve the deconvolution problem stated in [7]. We will say sometimes that the family of distributions \( \mu_1, \ldots, \mu_m \) is strongly coprime.

It might be useful to explain why is (4) called a deconvolution problem. If we have an unknown signal (function or even distribution or random process) \( \varphi \) then the usual data one measures would be \( \varphi_1, \ldots, \varphi_m \) given by

\[
\varphi_i := \mu_i \ast \varphi, \quad \varphi_m := \mu_m \ast \varphi.
\]

The way to recover \( \varphi \) is by deconvolution (which is still given here by convolution with distributions of compact support).

\[
\varphi = \nu_1 \ast \varphi_1 + \cdots + \nu_m \ast \varphi_m.
\]

As we have mentioned in the Introduction our problem is to find easily computable functions \( g_j \) and corresponding distributions \( \nu_j \) solving (3) and (4) respectively. We note that under the strongly coprime condition (2), or even under the weaker assumption that (2) is only satisfied for real values of \( z (z \in \mathbb{R}^n) \), there are readily available tempered distributions \( a_j \) solving the deconvolution problem, namely let

\[
a_j(z) = \frac{\tilde{f}_j(z)}{\sum_{1}^{m} |f_j(z)|^2}, \quad z \in \mathbb{R}^n.
\]
The problem is that the $\sigma_j$ do not have compact support and furthermore, the $\sigma_j$ themselves are not so readily computable (except by inverting the Fourier transform). Nevertheless there are many situations where these $\sigma_j$ are still very useful, among other reasons because they minimize the noise amplification of the deconvolution process (6) (see [17] for an example of implementation in 2-d). On the other hand, in many applications it is often not necessary to obtain an exact solution to (4) but one is allowed to replace the Dirac $\delta$ in (4) by a (sufficiently) smooth function $\nu$ with small support, i.e. an approximation to $\delta$. It is this approximate deconvolution problem that is more readily solvable, even with very good knowledge on the support of the distributions $\nu_j$, which will turn out to be (reasonably) smooth functions.

Since the method we use relies on Koppelman type formulas, like those developed in [1,9], we need the following explicit relation whose proof is an immediate verification.

**Lemma 1.** Let $\mu$ be a distribution of compact support in $\mathbb{R}^n$, $1 \leq k \leq n$ and $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$. The holomorphic function of $2n$ complex variables $g_k(z, \zeta)$ defined by

$$g_k(z, \zeta) := \frac{\tilde{\mu}(z_1, \ldots, z_k, \zeta_{k+1}, \ldots, \zeta_n) - \tilde{\mu}(z_1, \ldots, z_{k-1}, \zeta_k, \ldots, \zeta_n)}{z_k - \zeta_k}$$

is the Fourier transform (for $z$ fixed) of the distribution denoted $I = I(\mu, \zeta, k)$, which evaluated at $\phi \in C_0^\infty(\mathbb{R}^n)$ has the value 1.6.
(9) \[ \langle I, \phi \rangle = -i \int \left( \int_0^t \phi(t_1, \ldots, t_{k-1}, u, 0, \ldots, 0) e^{i \zeta_k(t-u_t_k)} du \right) e^{-i(t_{k+1} \zeta_{k+1} + \cdots + t_n \zeta_n)} d\mu(t). \]

(By abuse of language we have written \[ \int \phi(t)d\mu(t) \] to denote \[ \langle \mu, \phi \rangle \].)

We note that for the distribution \( I \) we have \( c_v \text{ supp } I \leq c_v \text{ supp } \mu \). Furthermore, the collection of functions \( g_1, \ldots, g_n \) satisfies

\[ g_1(z, \zeta)(z_1-\zeta_1) + \cdots + g_n(z, \zeta)(z_n-\zeta_n) = \hat{\mu}(z) - \hat{\mu}(\zeta). \]

Associated to these holomorphic functions we have a \((1,0)\) differential form \( g \) in the variable \( \zeta \) given by

\[ g(z, \zeta, \mu) = \sum_{k=1}^{n} g_k(z, \zeta) d\zeta_k. \]

Given a family of \( m \) entire holomorphic functions \( f_1, \ldots, f_m \), its zero set \( Z \) is defined as

\[ Z := \{ z \in \mathbb{C}^n : f_1(z) = \cdots = f_m(z) = 0 \}. \]

In our applications, we will only consider the case where the set \( Z \) is discrete. We say that \( Z \) is almost real if there is a constant \( A > 0 \) such that

\[ Z \subseteq \{ z \in \mathbb{C}^n : |\text{Im } z| \leq A \log(2+|z|) \}. \]

It is well-known that an almost real zero set \( Z \) is discrete \([8], [15]\). For a discrete set \( Z \), \( r > 0 \), we can define a counting function \( n(Z, r) := \#(Z \cap B_r) \), \( B_r = \{ z \in \mathbb{C}^n : |z| < r \} = \text{Euclidean ball of center } 0 \text{ and radius } r \). The distance function is \( d(z, Z) = \ldots \)
Given a family of \( n \) distributions of compact support in \( \mathbb{R}^n \), \( \mu_1, \ldots, \mu_n \), let us denote by \( H_1 \) the supporting function of \( \bigcup_{j=1}^{n} \text{supp} \mu_j \), that is
\[
H_1(\theta) := \max_{1 \leq j \leq n} \max_{x \in \text{supp} \mu_j} (x \cdot \theta) , \quad (\theta \in \mathbb{R}^n),
\]
\[x \cdot \theta = x_1 \theta_1 + \cdots + x_n \theta_n .\]

**Definition 1.** A family of \( n \) distributions \( \mu_1, \ldots, \mu_n \) of compact support in \( \mathbb{R}^n \) is well-behaved if there exists positive constants \( A, B, N, x \) and a supporting function \( H_0 \) such that \( 0 \leq H_0 \leq H_1 \), such that the zero set \( Z \) of the functions \( f_1 = \mu_1, \ldots, f_n = \mu_n \), is almost real,
\[
n(Z, r) = O(r^A),
\]
and, denoting
\[
|f(z)| := \left[ \sum_{j=1}^{n} |f_j(z)|^2 \right]^{1/2},
\]
the following inequality holds:
\[
|f(z)| \leq \frac{B \text{d}(z, Z)^{x}}{e^{H_0(\text{Im} z)}} (1 + |z|)^{-N} .
\]

**Definition 2.** A well-behaved family \( \mu_1, \ldots, \mu_n \) is very well-behaved if there are constants \( c_1, M, c_1 > 0 \), such that for every \( \zeta \in Z \) we have
\[
|J(\zeta)| := \left| \det \left[ \frac{\partial f_j}{\partial z_i}(\zeta) \right]_{i,j} \right| \geq c_1 (1 + |\zeta|)^{-M} .
\]
This condition implies that the common zeros of \( f_1, \ldots, f_n \) are simple, that we can take \( s = 1 \) in (16) and that if \( \zeta, \zeta' \in \mathbb{Z}, \zeta \neq \zeta' \) then \( |\zeta - \zeta'| \geq c_2(1 + |\zeta|)^{-M'} \) for some positive constants \( c_2, M' \).

We will say also that functions \( f_1, \ldots, f_n \) are (very) well-behaved if the above properties hold.

Given a family \( f_1, \ldots, f_m \) in \( \mathcal{E} \mathcal{C}^{\infty}(\mathbb{R}^n) \), \( m > n \), with no common zeros we introduce the following functions and differential forms. First, let \( g_j = g_j(z, \zeta, \mu_j) \), \( f_j = \mu_j \), be the \((1,0)\) differential forms in \( \zeta \) given by (11), we write \( g_j = \sum_{k=1}^{n} g_{j,k} \zeta_k \). Recall the coefficients \( g_{j,k} \) are holomorphic in both \( z \) and \( \zeta \). Let \( F \) be the vector valued holomorphic function \( F: = (f_1, \ldots, f_m) \), we write

\[
|F(\zeta)| = \left[ \sum_{j=1}^{m} |f_j(\zeta)|^2 \right]^{1/2},
\]

which is a nowhere vanishing \( \mathcal{C}^{\infty} \) function of \( \zeta \). Let

\[
(18) \quad \phi = \phi(z, \zeta) = \sum_{j=1}^{m} f_j(\zeta) f_j(z)/|F(\zeta)|^2,
\]

\[
(19) \quad Q = Q(z, \zeta) = \sum_{j=1}^{m} f_j(\zeta) g_j(z, \zeta)/|F(\zeta)|^2.
\]

Therefore \( \phi \) is a \( \mathcal{C}^{\infty} \) function of \((z, \zeta)\), \( \phi(\zeta, \zeta) = 1 \) and, as a function of \( z \), \( \phi \) is a linear combination of the \( f_j \). \( Q \) is a \((1,0)\) differential form in \( \zeta \), its coefficients are \( \mathcal{C}^{\infty} \) in \((z, \zeta)\) and holomorphic in \( z \). Finally, the \( n+1 \) functions \( \delta_j, C^{\infty} \) in \((z, \zeta)\) and holomorphic in \( z \) are defined by the identities
(20) \[ g^1 \cdots g^j \cdots g^n \wedge Q = A_j \delta_{1} \cdots \delta_{n}, \quad 1 \leq j \leq n \]

(21) \[ g^1 \cdots \wedge g^n = A_{n+1} \delta_1 \cdots \delta_n. \]

It is clear that the \( A_j \) are simply \( n \times n \) determinants whose entries are obtained from the coefficients of \( g^1, \ldots, g^n, Q \).

Therefore, as functions of \( z \), they are finite linear combinations of products of \( n \) among the functions \( g^j_k, 1 \leq k \leq m \). Note that these products are just Fourier transforms of convolutions of \( n \) distributions of the form \( I(\mu_j, k, k) \). (see (9)).

In order to obtain simple and easily computable deconvolution formulas we need to assume that a strongly coprime family of distributions \( \mu_1, \ldots, \mu_m \) contains a (very) well-behaved sub-family \( \mu_1, \ldots, \mu_m \). Furthermore, we need some control on the relation between the support of all the \( \mu_j \) versus the supports of the first \( n \). Let

\[ H_2(\theta) = \max_{1 \leq j \leq n} \max_{x \in \text{supp } \mu_j} (x \cdot \theta), \quad (\theta \in \mathbb{R}^n). \]

One such relation between the supporting functions \( H_0, H_1, H_2 \) is given by

\[ H_2 \leq 2H_1, \quad \text{and} \]

\[ 2(n-1)H_1(\theta) + H_2(\theta) < 2n H_0(\theta), \quad \text{if } \theta \neq 0. \]

The last condition is equivalent to

\[ 3r_0 > 0 \text{ such that } r_0 |\theta| < 2nH_0(\theta) - 2(n-1)H_1(\theta) - H_2(\theta). \]

With all this notation in place we are now ready to state the first deconvolution formula.
Theorem 1 Let \( \mu_1, \cdots, \mu_m \) be a strongly coprime family of distributions such that \( \mu_1, \cdots, \mu_n \) is a very well-behaved subfamily. Assume further that (23) and (25) hold. For any \( u \in C_0^\infty(\mathbb{R}^n) \) with \( \text{supp}\ u \subseteq (x \in \mathbb{R}^n : |x| \leq r_0) \) one can write

\[
(26) \quad u(z) = \sum_{\zeta \in \mathbb{Z}} \hat{u}(\zeta) \frac{\Delta_{n+1}(z, \zeta)}{J(\zeta)} \varphi(z, \zeta)
\]

\[
+ \sum_{j=1}^{n} (-1)^{N+1-j} f_j(z) \sum_{\zeta \in \mathbb{Z}} \frac{\Delta_j(z, \zeta)}{J(\zeta)} \hat{u}(\zeta).
\]

Formula (26) can be rewritten as

\[
\hat{u}(z) = \sum_{j=1}^{m} h_j(z) f_j(z),
\]

were the \( h_j \) are given by explicit interpolation formulas and they are Fourier transforms of a series of distributions which are computable in terms of the original \( \mu_1, \cdots, \mu_m \). In the particular case where \( m = n+1 \) then formula (26) can be also written as

\[
(27) \quad \hat{u}(z) = \sum_{\zeta \in \mathbb{Z}} \frac{\hat{u}(\zeta)}{J(\zeta) f_{n+1}(\zeta)} \begin{vmatrix}
\varphi_{n+1}(\zeta, z) \\
\cdots \\
\varphi_1(\zeta, z)
\end{vmatrix}
\]

Proof of Theorem 1. It follows the lines of Theorem 3 from our paper [7]. It uses the Koppelman type generalization of the Cauchy integral representation formula, specially in the version due to Berndtsson-Andersson [1]. One introduces first a parameter \( \epsilon > 0 \), a function \( \varphi_\epsilon \) and two \((1,0)\) differential forms in \( \zeta \) as follows:
\[
\phi_\varepsilon(z, \zeta) = \frac{1}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\bar{t}_j(\zeta) f_j(z) + \varepsilon}{|f(\zeta)|^2 + \varepsilon}
\]

\[
s(z, \zeta) = \sum_{j=1}^{n} (\zeta - z_j) d\zeta_j
\]

\[
Q_\varepsilon(z, \zeta) = \frac{1}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\bar{t}_j(\zeta) g_j(z, \zeta)}{|f(\zeta)|^2 + \varepsilon}
\]

where as before \( f = (f_1, \ldots, f_n) \), \(|f(\zeta)|^2 = \sum_{j=1}^{n} |f_j(\zeta)|^2 \). The procedure from ([9], p.402 and p.409) gives two kernels \( K_\varepsilon, P_\varepsilon \) (i.e. differential forms in the variable \( \zeta \) of type \((n,n-1)\) and \((n,n)\) respectively) such that if \( v \) is a holomorphic function in a neighborhood of \( \bar{B}_R', z \in B_R \) then

\[
v(z) = \frac{1}{(2\pi i)^n} \left\{ \int_{\partial B_R} v(\zeta) K_\varepsilon(z, \zeta) + \int_{B_R} v(\zeta) P_\varepsilon(z, \zeta) \right\}.
\]

These two kernels are defined as follows. Let \( G_1(t) = t^n \) and \( G_2(t) = t \), we denote for any \( \alpha \in \mathbb{N} \)

\[
G_1^{(\alpha)} = G_1^{(\alpha)}(z, \zeta) := \left. \frac{d^\alpha}{dt^\alpha} G_1 \right|_{t=\varphi(z, \zeta)}
\]

\[
G_2^{(\alpha)} = G_2^{(\alpha)}(z, \zeta) := \left. \frac{d^\alpha}{dt^\alpha} G_2 \right|_{t=\varphi(z, \zeta)}
\]

where \( \varphi \) is given by (18) and \( \varphi_\varepsilon \) by (28). With \( Q_\varepsilon \) defined by (30) and \( Q \) by (19), we define
\[ K_\epsilon(z,\zeta) = \sum_{\alpha_0 + \alpha_1 + \alpha_2 = n-1} \frac{1}{a_1!a_2!} \frac{g_1^{(a_1)} g_2^{(a_2)}}{|z-\zeta|^{2(a_0+a_1)}}. \]

\[ P_\epsilon(z,\zeta) = \sum_{\alpha_1 + \alpha_2 = n} \frac{1}{a_1!a_2!} g_1^{(a_1)} g_2^{(a_2)} (\delta_{\zeta})^{a_1} (\delta Q)^{a_2}, \]

where \( \alpha_0, \alpha_1, \alpha_2 \in \mathbb{N} \). Everywhere the variable \( z \) is considered as a parameter and the \( \overline{\partial} \) derivative is taken with respect to \( \zeta \). Due to our choice of function \( G_2 \), the index \( \alpha_2 \) can only take the values 0 and 1. For this reason the expression for \( P_\epsilon \) becomes particularly simple

\[ P_\epsilon = \varphi(\delta_{\zeta})^n + n\varphi(\delta_{\zeta})^{n-1} \wedge \delta Q. \]

The terms \((\delta_{\zeta})^{n-1}\) and \((\delta_{\zeta})^n\) must be computed, for instance

\[ (\delta_{\zeta})^n = \delta_{\zeta} \wedge \cdots \wedge \delta_{\zeta} \text{ (n times)} = (2i)^n n! \frac{\overline{\partial}(\zeta)}{\nabla f(\zeta)^{n+2}} n+1 d\lambda, \]

where \( d\lambda = d\lambda(\zeta) = \text{Lebesgue measure in } \mathbb{C}^n \). (We have eliminated the variables \((z,\zeta)\) where they were evident, we will use this convention freely in the rest of the paper.)

It is clear that \( \varphi_\epsilon \) and \( Q_\epsilon \) are singular when \( \epsilon = 0 \) precisely at the points \( \zeta \in Z \). The expression (37) shows what is the strength of this singularity in one of the terms of \( P_\epsilon \). The strategy of the proof is to try to get very singular terms so that when \( \epsilon \to 0 \) the volume integrals in (31) become sums, while the boundary integrals tend to zero when we set \( v = \tilde{u} \) and let \( R \to \infty \) over a conveniently chosen sequence. The reason this idea works is the following lemma ([7], Corollary 4.1.1):
Lemma 2 Let $\sigma_0$ be the measure which is the sum of Dirac masses at the points of $Z$, i.e. for $v \in C^\infty_0(C^n)$ we have $\int v d\sigma_0 = \sum_{\zeta \in Z} v(\zeta)$.

Then, the family of measures $\sigma_\varepsilon$ given by

$$d\sigma_\varepsilon(\zeta) = \frac{\varepsilon}{(|f(\zeta)| + \varepsilon)^{n+1}} d\lambda(\zeta)$$

converges, when $\varepsilon \to 0$, to the measure

$$\frac{n}{n!} \frac{d\sigma_0}{|J(\zeta)|^2},$$

where, as always, $J$ denotes the determinant Jacobian of $f_1, \ldots, f_n$.

From (37) we see that the first term in (36) is amenable to Lemma 2. The second term is not singular enough, therefore it will be transformed using Stokes' formula in the corresponding integral of (31). Namely, due to type considerations, one obtains the first part of the following identity

$$d_\varepsilon(v(\zeta)\phi_\varepsilon(z,\zeta)(\bar{\partial}Q_\varepsilon(z,\zeta))^n - Q(z,\zeta))$$

$$= \bar{\partial}(v\phi_\varepsilon(\bar{\partial}Q_\varepsilon)^n - Q) = v\phi_\varepsilon(\bar{\partial}Q_\varepsilon)^n - \bar{\partial}Q + v \bar{\partial}\phi_\varepsilon(\bar{\partial}Q_\varepsilon)^n - Q.$$

The last identity follows from the fact that $v$ is a holomorphic function in $\zeta$ and the $(2n-2)$ form $(\bar{\partial}Q_\varepsilon)^n$ is $\bar{\partial}$-closed. Using this identity the representation formula (31) becomes

$$v(z) = \frac{1}{(2\pi i)^n} \int_{\partial B_R} v(\zeta)(K_\varepsilon + n\phi_\varepsilon(\bar{\partial}Q_\varepsilon)^n - Q)$$

$$+ \frac{1}{(2\pi i)^n} \int_{B_R} v(\zeta)(\varphi(\bar{\partial}Q_\varepsilon)^n - n \bar{\partial}\phi_\varepsilon(\bar{\partial}Q_\varepsilon)^n - Q).$$
where the integration is in the variable $\zeta$ and we have suppressed the dependency on $(z, \zeta)$ of the kernels.

Lemma 3 The following identity holds

\begin{equation}
\delta \phi_\epsilon \wedge (\delta Q_\epsilon)^{n-1} = (n-1)!(-1) \frac{(n+1)n}{2} \frac{c}{(|f(\zeta)|^2 + \epsilon)^{n+1}} \\
\left[ \sum_{j=1}^{n} (-1)^j (f_j(z) - f_j(\zeta)) \wedge g^k \right] \wedge \wedge_{k=1}^{n} \delta f_j(\zeta),
\end{equation}

where the wedge products in (41) are to be taken in their natural order, e.g. $\wedge_{k=1}^{n} g^k = g^2 \wedge \cdots \wedge g^n$.

Proof of Lemma 3 We start by rewriting $\phi_\epsilon$

\[
\phi_\epsilon = \frac{\sum_{j=1}^{n} f_j(z) \bar{f}_j(\zeta) + \epsilon}{|f(\zeta)|^2 + \epsilon} = \frac{\sum_{j=1}^{n} \bar{f}_j(\zeta) (f_j(z) - f_j(\zeta)) + |f_j(\zeta)|^2 + \epsilon}{|f(\zeta)|^2 + \epsilon}
\]

\[
= 1 + \sum_{j=1}^{n} (f_j(z) - f_j(\zeta)) \frac{\bar{f}_j(\zeta)}{|f(\zeta)|^2 + \epsilon}.
\]

Denote $f_j = f_j(\zeta)$ and $\psi_j = \psi_j(\zeta) := (|f(\zeta)|^2 + \epsilon)^{-1} \bar{f}_j$. Then we have

\[
\phi_\epsilon = 1 + \sum_{j=1}^{n} (f_j(z) - f_j) \psi_j, \quad Q_\epsilon = \sum_{j=1}^{n} \psi_j g^j.
\]

Therefore,

\[
\delta \phi_\epsilon \wedge (\delta Q_\epsilon)^{n-1} = \left[ \sum (f_j(z) - f_j) \delta \psi_j \right] \wedge \left[ \sum \delta \psi_j \wedge g^j \right]^{n-1}.
\]
\[
= \sum_{j=1}^{n} (f_j(z) - f_j) \left[ \bar{\delta} \psi_j \wedge \left\{ \sum_{k \neq j} \bar{\delta} \nu_k \wedge g^k \right\}^{n-1} \right],
\]

since \( \bar{\delta} \psi_j \wedge \bar{\delta} \psi_j = 0 \). Using that the 2-forms \( \bar{\delta} \nu_k \wedge g^k \) commute and that the product of two of them with the same index vanishes, we have

\[
\left( \sum_{k \neq j} \bar{\delta} \nu_k \wedge g^k \right)^{n-1} = (n-1)! \bigwedge_{k \neq j} (\bar{\delta} \nu_k \wedge g^k)
\]

\[
= (n-1)! (-1)^{\frac{n(n+1)}{2}} \left( \bigwedge_{k \neq j} g^k \right) \left( \bigwedge_{k \neq j} \bar{\delta} \nu_k \right).
\]

Hence

\[
\bar{\delta} \psi_j \wedge \left( \sum_{k \neq j} \bar{\delta} \nu_k \wedge g^k \right)^{n-1} = (n-1)! (-1)^{\frac{n(n+1)}{2}} \left( \bigwedge_{k \neq j} g^k \right) \left( \bigwedge_{k \neq j} \bar{\delta} \nu_k \right).
\]

Now, we have \( \bar{\delta} \nu_k = (|f|^2 + \epsilon)^{-1} \bar{\delta} \overline{f}_k - (|f|^2 + \epsilon)^{-2} \bar{\nu}_k \bar{\delta} |f|^2 \). Therefore we can use that \( \bar{\delta} |f|^2 \bar{\delta} |f|^2 = 0 \) and obtain

\[
\bigwedge_{k=1}^{n} \bar{\delta} \nu_k = \sum_{j=1}^{n} \left( \bigwedge_{1 \leq k < j} \bar{\delta} \overline{f}_k \wedge \bar{\delta} |f|^2 \wedge \bigwedge_{j < k \leq n} \bar{\delta} \nu_k \right) \frac{1}{(|f|^2 + \epsilon)^{n+1}}.
\]

If we now expand \( \bar{\delta} |f|^2 = \sum_k |f_k | \bar{\delta} \overline{f}_k \) we see that only the term \( f_j \bar{\delta} f_j \) remains in the triple product above. Hence

\[
\bigwedge_{k=1}^{n} \bar{\delta} \nu_k = \frac{1}{(|f|^2 + \epsilon)^{n+1}} \left[ \left( \sum_{j=1}^{n} |f_j|^2 \right) \bigwedge_{k=1}^{n} \bar{\delta} \overline{f}_k \right] = \frac{\varepsilon}{(|f|^2 + \epsilon)^{n+1}} \bigwedge_{k=1}^{n} \bar{\delta} \overline{f}_k.
\]

This concludes the proof of Lemma 3. \( \Box \)
Lemma 3 tells us that \( \delta \varphi_{\varepsilon}^{(\delta Q_{\varepsilon})^{n-1}} \) is the product of a measure with a smooth density, independent of \( \varepsilon \), and the function \( \varepsilon(|f(\zeta)|^{2+\varepsilon})^{-n-1} \). Lemma 2 can now be invoked to see that the volume integral in (40) reduces to a sum when \( \varepsilon \to 0 \). In fact, let us choose \( R \) so that \( |f(\zeta)|^2 = |f(\zeta)|^2 + \cdots + |f(\zeta)|^2 = 0 \) when \( |\zeta| = R \). This choice is always possible since \( Z \) is a discrete set by assumption. In this case none of \( \varphi_{\varepsilon}, K_{\varepsilon}, Q_{\varepsilon} \) have singularities when \( \varepsilon = 0 \) and \( \zeta \in \partial B_R \). We set \( \varphi_o, K_o, Q_o \) to be the correspondent quantities. Therefore

\[
v(\zeta) = \frac{1}{(2\pi i)^n} \int_{\partial B_R} v(\zeta)(K_o + n\varphi_o((\delta Q_o)^{n-1} \wedge Q))
\]

\[
+ \frac{1}{(2\pi i)^n} \lim_{\varepsilon \to 0^+} \int_{B_{\varepsilon}} v(\zeta) \left[ \varphi(\delta Q_{\varepsilon})^n - n \delta \varphi_{\varepsilon}^{(\delta Q_{\varepsilon})^{n-1}} \wedge Q \right]
\]

Recall that \( f_j(\zeta) = 0 \) if \( \zeta \in Z \) and \( 1 \leq j \leq n \). Using Lemma 2 and the definition (20) and (21) of the \( \Delta_j \) we can compute explicitly the limit and obtain

\[
(42) \quad v(\zeta) = \frac{1}{(2\pi i)^n} \int_{\partial B_R} v(\zeta)(K_o + n\varphi_o((\delta Q_o)^{n-1} \wedge Q))
\]

\[
+ \sum_{\zeta \in Z \cap B_R} v(\zeta) \frac{\Delta_{n+1}(z,\zeta)\phi(z,\zeta)}{J(\zeta)} + \sum_{j=1}^n (-1)^{n+1-j} f_j(z) \sum_{\zeta \in Z \cap B_R} v(\zeta) \frac{\Delta_j(z,\zeta)}{J(\zeta)}.
\]

Up to this moment we have only used that \( Z \cap \partial B_R = \emptyset \) and that \( v \) is holomorphic in a neighborhood of \( \overline{B}_R \). To let \( R \to \infty \) we have to choose a sequence \( R \to \infty \) judiciously. Recall that \( u(z,r) = \#(Z \cap B_r) \leq Cr^A \) for some positive constants \( A, C \) and all \( r \geq 1 \). Let \( M \) be the smallest integer \( z \leq C(R+1)^A + 1 \), divide the
shell $B_{R+1} \setminus B_R$ into $M$ concentric subshells by choosing the boundaries to be $B_{(R+j/M)}$, $0 \leq j \leq M$. There is at least one such subshell that is free from points of $Z$. Choose $R'$ to be the mid-radius of this subshell, then $d(\zeta, Z) \geq (2M)^{-1}$ if $|\zeta| = R'$. Starting from the sequence $R = q = 1, 2, \ldots$ we construct a sequence $R_q$, $q < R_q < q + 1$, such that for some positive constants $A_1, N_1$

$$|f(\zeta)| \geq A_1 q^{-N_1} e^{-N_1 H_0(\Im \zeta)} \quad \text{if } |\zeta| = R_q. \quad (43)$$

This follows from (16) and the choice of $R_q$.

We are now ready to estimate the terms in the boundary integral of (42) for $R = R_q$. We will assume $|z| \leq C_0 < \infty$ and consider those $q$ such that $R_q \geq C_0 + 1$.

First, let us observe that the functions $g^j_k$, $1 \leq j \leq m$, $1 \leq k \leq n$, satisfy an estimate of the form

$$|g^j_k(z, \zeta)| \leq C_1 (1 + |z|)^{M_1} H_2(\Im z) + H_2(\Im \zeta), \quad (44)$$

for some constants $M_1, C_1 > 0$. If $1 \leq j \leq n$ we can replace $H_2$ by $H_1$. We can now estimate the coefficients of differential from Q. Denote $\|Q(z, \zeta)\|$ the largest absolute value of the coefficients of $d\zeta_k$ at the point $(z, \zeta)$. We proceed as follows. First,

$$|F(\zeta)| \geq |f(\zeta)| \geq A_1 q^{-N_1} e^{-N_1 H_0(\Im \zeta)} \quad \text{if } |\zeta| = R_q. \quad (45)$$

Therefore

$$1.18$$
\[
\|Q(z, \xi)\| \leq \left( \frac{1}{F(\xi)} \left( \sum_{j=1}^{n} \|g^j(z, \xi)\| \right)^2 \right)^{1/2} \leq \frac{M_1 H_2(\text{Im } z)}{C_1 (1+|z|)^{1/2}} \frac{M_1 H_2(\text{Im } \xi)}{(1+R_q)^{1/2}} \frac{e^{-N_1 e^{H_0(\text{Im } \xi)}}}{A_1 q N_2 e^{H_0(\text{Im } \xi)}}
\]

which leads to

\[
\|Q(z, \xi)\| \leq C_3 q N_2 e^{H_2(\text{Im } \xi) - H_0(\text{Im } \xi)}
\]

(46)

The constant \( C_3 \) depends in fact on \( z \), but \( |z| \leq C_0 \) and \( N_2 = M_1 + N_1 \) (In fact \( C_3 \) can be estimated in terms of \( e^{\text{Im } z} \) and polynomials in \( |z| \)). Similarly, with possibly different values for the constants \( C_3, N_2 \) appearing below we have

\[
\|\delta Q(z, \xi)\| \leq C_3 q N_2 e^{H_2(\text{Im } \xi) - 2 H_0(\text{Im } \xi)}
\]

(47)

\[
\|\delta Q_0(z, \xi)\| \leq C_3 q N_2 e^{2H_1(\text{Im } \xi) - 2 H_0(\text{Im } \xi)}
\]

(48)

\[
|\varphi_0(z, \xi)| \leq C_3 q N_1 e^{H_0(\text{Im } \xi)}
\]

(49)

\[
|\varphi(z, \xi)| \leq \frac{|F(z)|}{|F(\xi)|} \leq C_3 q N_1 e^{H_0(\text{Im } \xi)}
\]

(50)

To estimate \( K_0 \) we recall that \( \alpha_2 \) can only take the values 0 and 1 in (34). In case \( \alpha_2 = 0 \), we have to estimate terms of the form \( \varphi_0^{-\alpha_1} \varphi(\delta Q_0)^{\alpha_1} \), with \( 0 \leq \alpha_1 \leq n - 1 \). There are powers of \( q \) that we will disregard, the estimate is then the following functions of \( \text{Im } \xi \)
\[ \| \varphi_0 \| \varphi (\bar{\delta} Q_0) 1 \| \ll e^{-\| \varphi_0 \|} \].

Since \( H_1 \geq H_0 \), the worst case estimate occurs when \( a_1 = n - 1 \). Hence the terms corresponding to \( a_2 = 0, a_0 + a_1 = n - 1 \), in the definition of \( K_0 \) can all be estimated by

\[ N_3 - 2nH_0 (\text{Im } \zeta) + 2(n-1)H_1 (\text{Im } \zeta) \]

(51) \[ C_3 q^3 e^{-2nH_0 (\text{Im } \zeta)} \qquad |\zeta| = R_q. \]

The terms with \( a_1 = 1, a_0 + a_1 = n - 2 \) correspond to the estimate of \( \| \varphi_0 \| \varphi (\bar{\delta} Q_0)^{a_1} \). The worst case occurs this time when \( a_1 = n - 2 \) and we obtain an estimate of the form

\[ N_3 - 2nH_0 (\text{Im } \zeta) + 2(n-2)H_1 (\text{Im } \zeta) + 2H_2 (\text{Im } \zeta) \]

(52) \[ C_3 q^3 e^{-2nH_0 (\text{Im } \zeta)} + 2(n-2)H_1 (\text{Im } \zeta) + 2H_2 (\text{Im } \zeta) \qquad |\zeta| = R_q. \]

In (42) we have one more term to estimate for \( |\zeta| = R_q \),

\[ \| \varphi_0 (\bar{\delta} Q_0)^{n-1} \| \leq C_3 q^3 e^{-2nH_0 (\text{Im } \zeta)} + 2(n-1)H_1 (\text{Im } \zeta) + 2H_2 (\text{Im } \zeta). \]

(53)

The conditions (23) and (25) imply that the largest exponential factor in (51), (52) and (53) is the one in (53) and it satisfies

\[ -2nH_0 (\text{Im } \zeta) + 2(n-1)H_1 (\text{Im } \zeta) + 2H_2 (\text{Im } \zeta) \leq -r_0 |\text{Im } \zeta| \]

since we have assumed that \( u \in C_\infty (\overline{B_{R_0}}) \) we have

\[ |\tilde{u}(\zeta)| \leq C_4 (1 + |\zeta|)^{-n-2n} r_0 |\text{Im } \zeta| \qquad , \zeta \in \mathbb{C}^n, \]

(55)

which allows us to conclude that, if \( v = \tilde{u} \),

\[ \lim_{q \to \infty} \int_{\partial B_{R_q}} \tilde{u}(\zeta) \left[ K_0 (z, \zeta) + n \varphi_0 (z, \zeta) (\bar{\delta} Q_0 (z, \zeta))^n \right] = 0. \]

(56)

1.20
To conclude the proof of Theorem 1 we need to show that the series appearing in the representation formula (26) converge absolutely and uniformly in compact subsets of $\mathbb{C}^n$ to functions in $\mathcal{E}'$. Since we have assumed that $Z$ is almost real, the estimates of all the terms $\Delta_j, \varphi, J^{-1}$ are in terms of powers of $|\zeta|$. Using that $n(Z, r) = O(r^A)$ and $|\hat{u}(\xi)|$ decreases as fast as $|\zeta|^{-N_4}$ for $\zeta \in Z$ we have the desired convergence once $N_4$ is chosen sufficiently large. The support of all the distributions thus obtained is contained in the convex set $K$ whose supporting function is $(n+1)H_2$.

**Remarks**

1. One can see that the condition $2H_1 \geq H_2$ cannot be relaxed if the other conditions of theorem remain in the same, otherwise the exponent in (52) would become positive and we would not be able to prove (56).

2. A way to weaken the conditions in Theorem 1 is to impose some better lower bound on $|F|$ than (45) that depends only on the first $n$ functions. We will do so in Theorem 2 below.

3. It is clear that one needs only $u \in C_0^N(\overline{B}_r)$ for $N$ sufficiently large to obtain (26).

The following example shows how Theorem 1 simplifies enormously the computation of the deconvolution formula proposed in [7].

Let $\mu_1, \mu_2, \mu_3$ be the characteristic functions of the squares centered at 0, of sides parallel to the axes and of length $2\sqrt{3}$, $2\sqrt{2}$, 2 respectively. One can easily show [7], [17] that
"1.2" is a very well-behaved family with
\[ H_0(\text{Im} \, \xi) = \sqrt{2}(|\text{Im} \, \xi_1| + |\text{Im} \, \xi_2|) \]

Here \( H_1(\text{Im} \, \xi) = H_2(\text{Im} \, \xi) = \sqrt{3}(|\text{Im} \, \xi_1| + |\text{Im} \, \xi_2|) \). In this case the main hypotheses (24) reduces to verify that \( 4\sqrt{2} - 3\sqrt{3} > 0 \). Since \(|x_1| + |x_2| = \sqrt{x_1^2 + x_2^2}\) for \( x \in \mathbb{R}^2 \), we have \( r_0 = 4\sqrt{2} - 3\sqrt{3} \geq 0.2 \). The variety \( Z \) in this case is given by
\[ Z = \left\{ \left( \frac{j\pi}{\sqrt{3}}, \frac{k\pi}{\sqrt{2}} \right) : j, k \in \mathbb{Z}^* \right\} \cup \left\{ \left( \frac{j\pi}{\sqrt{2}}, \frac{k\pi}{\sqrt{3}} \right) : j, k \in \mathbb{Z}^* \right\} \]

(There were about forty different types of terms in [7] compute.).

Before we proceed to state Theorem 2 we need to point out that the representation formula (31) does not depend on the particular choice of the differential forms \( g^j \) we have chosen, rather on the fact that (10) is satisfied. That is,
\[ \sum_{k=1}^{n} g_k^j(z, \xi)(z_k - \xi_k) = f_j(z) - f_j(\xi). \]

Now, let \( f = \mu \) and \( h(z) = \sin Bz_1 \) for some \( B > 0 \), and denote by \( g \) the differential form associated to \( f \) by (8) and (11). Let us define a differential form \( \gamma \) by
\[ \gamma(z, \xi) := f(\xi) \frac{\sin Bz_1 - \sin B_1^1}{z_1 - \xi_1} d\xi_1 + h(z) g(z, \xi). \]

writing \( \gamma = \sum \gamma_k d\xi_k \) we have
\[ \gamma_1(z, \xi)(z_1 - \xi_1) + \cdots + \gamma_n(z, \xi)(z_n - \xi_n) = f(z) h(z) - f(\xi) h(\xi) \]
therefore we can associate \( \gamma \) to the product \( f \cdot h \). It is also
clear that as a function of \( z \) the \( \gamma_k \) are Fourier transforms of distribution of compact support easily computable terms of \( \mu \) and \( B \). Obviously we can replace \( \sin Bz_1 \) by \( \sin Bz_j \) without any problems, hence, given a family \( f_1, \ldots, f_m \) we can construct an augmented family \( f_1, \ldots, f_m, f_{m+1} = f_1 \sin Bz_1, \ldots, f_{2m} = f_m \sin Bz_1, \ldots, f_{(n+1)m} = f_m \sin Bz_n \). The corresponding \( g^j \) for \( j \geq m+1 \) are computed following the procedure (57). It is clear that if \( f_1, \ldots, f_m \) was strongly coprime, the augmented family remains strongly coprime. If \( f_1, \ldots, f_n \) form a very well-behaved family we will keep the notation \( H_0, H_1, H_2 \) to indicate the support functions corresponding to the \( m \) original members of the augmented family \( f_1, \ldots, f_{(n+1)m} \).

**Theorem 2** Let \( f_1, \ldots, f_m \) be a strongly coprime family such that the subfamily \( f_1, \ldots, f_n \) is very well-behaved. There are constants \( B_0 \geq 0, r_o > 0 \) such that for any \( B \geq B_0 \), and any \( u \in C_0^\infty(B_{r_0}) \) the representation formula (26) is valid for the augmented family \( f_1, \ldots, f_m, \ldots, f_{(n+1)m} \) defined above if either of the following two conditions holds:

\[
(58) \quad H_2 \leq 2H_1 \quad \text{and} \quad 2(n-1)H_1 < (2n-1)H_0
\]

\[
(59) \quad 2H_1 < H_2 \quad \text{and} \quad 2(n-2)H_1 + H_2 < (2n-1)H_0.
\]

**Proof** The proof is exactly the same as that of Theorem 1 except for improvements on the estimates (46), (47) and (50) for the new \( Q \) and \( \varphi \). Recall that it is there where all the functions \( f_1, \ldots, f_{(n+1)m} \) appear. Let \( F_1 = (f_1, \ldots, f_{(n+1)m}) \) and keep the
notation \( F = (f_1, \ldots, f_m) \) as before. We have

\[
|F_1(z)| = |F(z)| \sqrt{(1 + |\sin Bz_1|^2 + \cdots + |\sin Bz_n|^2)^{1/2}} = |F(z)| \phi(z)
\]

It is clear that for some positive constants \( c_n, c'_n \) we have

\[
\phi(z) = (1 + |\sin Bz_1|^2 + \cdots + |\sin Bz_n|^2)^{1/2} \geq c'_n \left( \sum_{j=1}^{2B} |\text{Im } \zeta_1| + \cdots + \sum_{j=1}^{2B} |\text{Im } \zeta_n| \right)^{1/2} \geq c_n \left( \frac{2B}{n} |\text{Im } \zeta_1| + \frac{2B}{n} |\text{Im } \zeta_n| \right)
\]

where \( |\text{Im } \zeta_1| = \sum_{j=1}^{n} |\text{Im } \zeta_j|, |\text{Im } \zeta| = \left( \sum_{j=1}^{n} |\text{Im } \zeta_j|^2 \right)^{1/2} \).

We estimate first \( g^j, j \geq m+1, \) since that is the only case that appears in the proof of (56). As it follows from (57) we have, for some \( i, k (1 \leq i \leq n, 1 \leq k \leq m) \), the estimate

\[
\|g^j(z, \zeta)\| \leq C e^{B |\text{Im } z_1|} \left( |f_k(\zeta)| \sqrt{(1 + |\sin Bz_1|^2)^{1/2}} + \|g^k(z, \zeta)\| \right)
\]

\[
\leq C (1 + |z|) (1 + |\zeta|) e^{B |\text{Im } z_1|} \left( |f(\zeta)| \phi(\zeta) + e^{H_2(\text{Im } z) + H_2(\text{Im } \zeta)} \right)
\]

It follows that

\[
\|Q(z, \zeta)\| \leq \frac{1}{|F_1(z)|} \left( \sum_{j=1}^{n+1} \|g^j(z, \zeta)\|^2 \right)^{1/2}
\]

\[
\leq C_1 (1 + |z|) (1 + |\zeta|) e^{H_2(\text{Im } \zeta)} \cdot \left\{ \frac{e^{H_2(\text{Im } \zeta)}}{|F_1(\zeta)|} + \frac{|F(\zeta)| \phi(\zeta)}{|F_1(\zeta)|} \right\}
\]

1.24
Therefore, for a positive constant \( C_2 \) depending on \( z \), we obtain

\[
\|Q(z, \xi)\| \leq C_2 (1+|\xi|) N_1 \left[ 1 + \frac{H_2(\text{Im } \xi) - H_0(\text{Im } \xi)}{B} |\text{Im } \xi| \right].
\]

Similarly,

\[
|\varphi(z, \xi)| \leq \frac{|F_1(z)|}{|F_1(\xi)|} C_2 (1+|\xi|) e^{N_1 H_0(\text{Im } \xi) - \frac{B}{n} |\text{Im } \xi|}.
\]

Finally,

\[
\|\delta Q(z, \xi)\| \leq C \max_{1 \leq j, k \leq (n+1)m} \frac{\|\delta f_j(\xi)\|}{|F_1(\xi)|^2} \left| \frac{\theta(\xi)}{\varphi(\xi)} \right|^{1/2}.
\]

Now, for \( j = m+1 \) we have for some \( 1 \leq i \leq n, 1 \leq \ell \leq m \)

\[
\|\varphi_j(\xi)\| = \|\sin B x \cdot \varphi_j(\xi) + f_\xi(\xi) B \cdot \cos B \varphi_j(\xi)\|
\]

\[
\leq C_2 (1+|\xi|) \left[ \frac{H_2(\text{Im } \xi)}{\varphi(\xi)} + |\varphi(\xi)| \delta(\xi) \right].
\]

It follows that

\[
\|\delta Q(z, \xi)\| \leq C_2 \left( \frac{1+|\xi|}{|F_1(\xi)|^2} \right)^{N_2} \left[ \frac{|F(\xi)|^2 \theta(\xi)^2 + 2|F(\xi)| \theta(\xi)^2 e^{H_2(\text{Im } \xi)} + \delta(\xi) e^{2H_2(\text{Im } \xi)}}{1 + 2e} \right]^{N_2}
\]

\[
\leq C_2 (1+|\xi|) \left[ \frac{H_2(\text{Im } \xi) - H_0(\text{Im } \xi)}{\varphi(\xi)} + \delta(\xi) \right]^{N_2}
\]

Choose \( B_0 > 0 \) such that

\[
2H_2(\text{Im } \xi) - 2H_0(\text{Im } \xi) - \frac{B_0}{n} |\text{Im } \xi| \leq 0.
\]

When \( B \geq B_0 \) we will have \( \|Q(z, \xi)\| \leq C_2 (1+|\xi|)^{N_1} \) and

\[
\|\delta Q(z, \xi)\| \leq C_2 (1+|\xi|)^{N_2}.
\]

where \( C_2 \) still denotes a constant depending on \( z \) of the form

\[1.25\]
We can now return to the proof of Theorem 1 at the point where we obtained the estimates (51), (52), (53). Ignoring powers of \( q \) the exponential factors are:

\[
\begin{align*}
(51') & \quad \exp(-2nH_0(\Im \xi) + 2(n-1)H_1(\Im \xi) - \frac{B}{n} |\Im \xi|) \\
(52') & \quad \exp(-(2n-1)H_0(\Im \xi) + 2(n-2)H_1(\Im \xi) + H_2(\Im \xi)) \\
(53') & \quad \exp(-(2n-1)H_0(\Im \xi) + 2(n-2)H_1(\Im \xi)).
\end{align*}
\]

Under the hypothesis (58) the largest of these three is (53') and its exponent satisfies

\[
(53'') \quad -(2n-1)H_0(\Im \xi) + 2(n-1)H_1(\Im \xi) \leq - r_0 |\Im \xi|,
\]

for some \( r_0 > 0 \). If the hypothesis (59) holds, then the largest exponent is (52') and we define \( r_0 > 0 \) by

\[
(52'') \quad -(2n-1)H_0(\Im \xi) + 2(n-2)H_1(\Im \xi) + H_2(\Im \xi) \leq - r_0 |\Im \xi|.
\]

In either case the rest of the proof is the same as that of Theorem 1.

\[ \square \]

**Example** As shown in [7] the family \( \mu_1, \mu_2, \mu_3 \) obtained by taking \( \mu_1 = \text{characteristic function of the unit square} = \chi_{[-1,1] \times [-1,1]} \), \( \mu_2 \) a rotation of \( \mu_1 \) by 36°, and \( \mu_3 \) a rotation of \( \mu_1 \) by 45°, satisfies the first conditions of Theorem 1 and Theorem 2 with \( H_0(\theta) = |\theta| \) since the squares contain the unit disk. One can easily convince oneself that the hypothesis (24) does not hold (e.g. take \( \theta = (t,0), t > 0 \)). On the other hand one can take

\[ 1.26 \]
\( H_1(\theta) = H_2(\theta) = \sqrt{2} |\theta| \) since all the squares are contained in the disk of radius \( \sqrt{2} \). We are in the situation of hypothesis (58) and its verification reduces to the fact that

\[ r_o = 3 - 2\sqrt{2} > 0 \]

Furthermore \( B_o = 4(\sqrt{2} - 1) \) works in this case.
3. **Polynomial case**

The conditions on Theorem 1 and 2 imply that the convex set defined by $H_0$ contains a ball. If we want to prove an algebraic version of (26), the fact that this condition is not satisfied plays a role. Such a representation was stated in [3,4] without proof. We analyze here the conditions under which it is valid.

**Theorem 3.** Let $p_1, \ldots, p_m$ be a family of polynomials in $C^n$ without common zeros, suppose further that:

a) $D: = \max_{1 \leq j \leq m} \deg p_j = \deg p_i$ for $1 \leq i \leq n$.

b) $Z = \{z \in C^n: p_i(z) = 0; 1 \leq i \leq n\}$ is discrete.

c) $J(z): = \text{Jacobian determinant of } p_1, \ldots, p_n \text{ at } z \neq 0$ for all $z \in Z$.

d) $p_1, \ldots, p_n$ have no zeros at infinity, i.e. $\#Z = D^n$.

Then

$$1 = \sum_{\xi \in Z} \frac{\Delta_{n+1}(z, \xi)}{J(\xi)} \varphi(z, \xi) + \sum_{j=1}^{n} \frac{(-1)^{n+1-j} p_j(z)}{J(\xi)} \sum_{\xi \in Z} \frac{\Delta_j(z, \xi)}{J(\xi)},$$

where $\varphi, \Delta_j$ are defined as in (18) - (21) with respect to the polynomials $p_1, \ldots, p_m$.

**Remarks 1.** The functions $g^j_k$ defined by (8) are obviously polynomials of degree $D-1$. It follows that (66) has the form

$$p_1(z)A_1(z) + \cdots + p_m(z)A_m(z) = 1$$

for some polynomials $A_j \in C[z_1, \ldots, z_n]$ of degree at most $n(D-1)$. This follows from the fact they are given as $n \times n$ determinants involving the $g^j_k$. 

1.28
2. As before, the case \( a = n + 1 \) leads to a particularly pleasing form of (66)

\[
(67) \quad 1 = \sum_{j=1}^{n+1} \frac{1}{J(z_j)} \frac{1}{p_{n+1}(z_j)} \begin{vmatrix}
g_1(z_j) & \ldots & g_{n+1}(z_j) \\
\vdots & \ddots & \vdots \\
g_n(z_j) & \ldots & g_{n+1}(z_j) \\
p_1(z) & \ldots & p_{n+1}(z)
\end{vmatrix}.
\]

3. The two statements in condition (d) above are really a form of Bezout's theorem [19]. The meaning of the expression "\( P_1, \ldots, P_n \) have no common zeros at infinity" is that if we introduce homogeneous polynomials \( H_j(z_0, \ldots, z_n) = z_0^D P_j\left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) \) then the subset of \( \mathbb{C}^{n+1} \) defined by \( (z_0 = 0, H_1 = \cdots = H_n = 0) \) is (0). This is equivalent to the statement:

\[
(68) \quad |p(z)| = \left(\sum_{j=1}^{n} |P_j(z)|^2\right)^{1/2} \geq C|z|^D \quad \text{if} \quad |z| \geq R_\alpha^1.
\]

If we call \( P_j^0 \) the leading homogeneous polynomial of \( P_j \), then the statement (68) is also equivalent to:

\[
(69) \quad (z \in \mathbb{C}^n: P_1^0(z) = \cdots = P_n^0(z) = 0) = (0).
\]

We also note that (69) implies (b) above. That is, condition (d) above implies condition (b).

Proof of Theorem 3. The proof of the same as that of Theorem 1. This time we take \( v=1 \) and \( R \) arbitrary \( \geq R_\alpha \) (cf. (68)). One can estimate \( \|Q(z, z)\| \leq C/|z|, \|\tilde{Q}(z, z)\| \leq C/|z|^2, |\varphi_0(z, z)| \leq C/|z|^D, \|\tilde{Q}_0(z, z)\| \leq C/|z|^2 \) if \( |z| = R \) and \( |z| \leq K \leq R-1, \) with
C = C(k) > 0. These estimates imply that the boundary integral in (40) tends to zero when \( R \to \infty \).

If \( p_1, \ldots, p_n \) are such that their leading terms \( p_j^0 \) satisfy (69) but their degrees \( \deg p_j = D_j \) are not all equal or \( \max(D_j:1 \leq j \leq n) \) is smaller than \( D = \max(D_j:1 \leq j \leq m) \), then we can still prove a version of Theorem 3. That corresponds to the analytic counterpart of Theorem 1, that is to Theorem 2. For the moment we continue to assume that \( J(z) = 0 \ \forall z \in Z = \{ p_1 = \cdots = p_n = 0 \} \).

Let \( L_1(z) = u_1 z_1 + \cdots + u_n z_n \) be a linear homogeneous polynomial with generic coefficients. The condition that \( \{ z \in \mathbb{C}^n: L_1 = p_2^0 = \cdots = p_n^0 = 0 \} = \{ 0 \} \) is an algebraic condition on the coefficients of \( L_1 \). Therefore we can choose \( L_1 \) such that for any integer \( d_1 \geq 0 \) we have \( \{ p_1^{d_1} L_1 = p_2^0 = \cdots = p_n^0 = 0 \} = \{ 0 \} \). Continuing in this fashion we can choose \( L_1, \ldots, L_n, d_1, \ldots, d_n \) such that for any choice of constants \( \varepsilon_1, \ldots, \varepsilon_n \), if we define \( \tilde{p}_j = p_j(L_j + \varepsilon_j) \), then \( \tilde{p}_1^0 = \cdots = \tilde{p}_n^0 = 0 \) = \{ 0 \} and \( \deg \tilde{p}_j = D \) for \( 1 \leq j \leq n \). It is clear now that for most choices of \( \varepsilon_j \) we still have that all common zeros of \( \tilde{p}_1^0, \ldots, \tilde{p}_n^0 \) are simple and \( \tilde{p}_1^0, \ldots, \tilde{p}_n^0, \tilde{p}_{n+1}, \ldots, \tilde{p}_m \) have no common zeros. Theorem 3 can now be applied to this new family, one obtains polynomials \( A_j \)

\[
\sum_{j=1}^{m} A_j \tilde{p}_j = 1, \quad \deg A_j \leq n(D-1)
\]

and such that they have a representation of the type (65).
We remark that a representation such as (65) cannot be valid if the $p_1, \ldots, p_n$ have common zeros at infinity. For instance, in the example of Masser-Philippon [11]:

$$p_1 = z_1^D, p_2 = z_1^{-1}z_2^D, \ldots, p_{n-1} = z_{n-2}^{-1}z_{n-1}^D, p_n = 1 - z_n^{-1}z_n^D$$

one knows that $\delta = D^n - D^{n-1}$ is the best estimate possible for the degrees of $A_j$ solving the polynomial Bezout equation. The polynomials $A_1 = z_n^\delta, A_2 = -z_n^\delta \frac{z_1^D - z_2^D}{z_1 - z_2}, A_3 = -z_n^\delta \frac{z_2^D - z_3^D}{z_2 - z_3}, \ldots,$

$$A_{n-1} = -z_n^\delta \frac{z_{n-2}^D - z_{n-1}^D}{z_{n-2} - z_{n-1}}, A_n = \frac{1 - z_n^{-1}z_n^D}{1 - z_n^{-1}z_n^D}$$

have exactly this degree. On the other hand if we had a representation like (65) we could conclude that there are solutions $A_j$ of the polynomial Bezout equation with $\deg A_j \leq n(D-1)$ like in (70).

We would like now to show that the condition (c) of the simplicity of the zeros in Theorem 3 is not necessary. Regrettably, we only know how to do this in the case where $m = n + 1.$

**Theorem 4.** Let $p_1, \ldots, p_{n+1}$ be a family of polynomials in $C^n$ without any common zeros, $D = \deg p_1 = \cdots = \deg p_n = \deg p_{n+1}$ and $\{p_1^0 = \cdots = p_n^0 = 0\} = (0).$ Then we can find polynomials $A_j$ of degree $\leq n(D-1)$ satisfying the identity $\sum_{j=1}^{n+1} A_j p_j = 1.$ The coefficients of the $A_j$ can be written in terms of the values of $p_{n+1}$ and values of derivatives of $p_{n+1},$ and the coefficients of the $g_k^{n+1}(z, \xi)$ (when considered as polynomials in $z$), all of
these evaluated at the points of $Z = \{\zeta : p_1(\zeta) = \cdots = p_n(\zeta) = 0\}$.

**Proof** For $0 < \epsilon_j << 1$ the function $1/p_{n+1}$ is holomorphic in a neighborhood of $\Pi = \{|p_1| \leq \epsilon_1, \ldots, |p_n| \leq \epsilon_n\}$. This set is a compact polynomially convex set. (The compactness follows from the condition (68)). By Sard's theorem one can choose the $\epsilon_j$ so that the sets $\{|p_j| = \epsilon_j\}$ are real analytic submanifolds of $\mathbb{C}^n$. (In fact, we only need it in a neighborhood of $\Pi$.) For any $v \in \mathcal{A}(\Pi)$ we have that integral

$$\text{Res}_Z(vd\zeta_1 \wedge \cdots \wedge d\zeta_n) := \frac{1}{(2\pi i)^n} \int_{\{p_1 = \sigma_1, \ldots, p_n = \sigma_n\}} v(\zeta) \frac{d\zeta_1 \wedge \cdots \wedge d\zeta_n}{p_1(\zeta) \cdots p_n(\zeta)}$$

is independent of choice of $\sigma_1, \ldots, \sigma_n$ as long as $0 < \sigma_j \leq \epsilon_j$ and the $\{|p_j| = \sigma_j\}$ are smooth. Furthermore, if $v$ is in the ideal generated by $p_1, \ldots, p_n$ in $\mathcal{A}(\Pi)$ then this residue is zero. Therefore, it depends only the values of $v$ at $Z$ and a certain number of derivatives of $v$ at $Z$ (as it follows from the Nullstellensatz as presented e.g. in [13], [16]). In other words, the integral (71) can be considered as an operator defined by a certain linear combination of the Dirac masses $\delta_\zeta$ and their derivatives $\frac{\partial^{\alpha}}{\partial \zeta^\alpha} \delta_\zeta$, $\zeta \in Z$, applied to the holomorphic function $v$. This operator is very hard to compute explicitly except in very simple cases but it is perfectly defined as the common value of all the integrals (71). It is called the residue current of $Z$. 

1.32
Let us consider now the polynomial \( B(z) \) defined by:

\[
B(z) = \text{Res}_z \frac{1}{p_{n+1}(\zeta)} \begin{vmatrix}
g_1^1(z, \zeta) & \cdots & g_{n+1}^1(z, \zeta) \\
\vdots & \ddots & \vdots \\
g_n^1(z, \zeta) & \cdots & g_{n+1}^n(z, \zeta) \\
p_1(z) & \cdots & p_{n+1}(z)
\end{vmatrix} d\zeta_1^1 \cdots d\zeta_n^1.
\]

This polynomial is in fact of the form \( \sum_{j=1}^{n+1} A_j(z)p_j(z) \), with \( A_j \) polynomials of degree \( \leq n(D-1) \) whose coefficients are computed in terms of the values of derivatives of \( p_{n+1} \) and the coefficients of \( g_j^{\zeta} \) (as polynomials in \( z \)) evaluated over \( Z \).

The only problem is to show \( B(z) = 1 \). We fix values \( \sigma_j, 0 < \sigma_j < \varepsilon_j \).

Consider complex numbers \( \alpha_1, \ldots, \alpha_n \) sufficiently small and so chosen that:

1) All the common zeros of \( p_1-\alpha_1, \ldots, p_n-\alpha_n \) are simple and lie in \( \{|p_1| < \frac{1}{2}\sigma_1, \ldots, |p_n| < \frac{1}{2}\sigma_n\} \).

2) \( p_1-\alpha_1, \ldots, p_n-\alpha_n, p_{n+1} \) have no common zeros.

Note that \( p_1+\alpha_1, \ldots, p_n+\alpha_n \) still do not have any common zeros at \( \infty \). Let us denote \( Z_\alpha = \{z \in \mathbb{C}^n: p_1-\alpha_1 = \cdots = p_n-\alpha_n = 0\} \).

Then

\[
(73) \text{Res}_{z_\alpha} (v d\zeta_1^1 \cdots d\zeta_n^1) = \frac{1}{(2\pi i)^n} \int_{|p_1|=\sigma_1, |p_n|=\sigma_n} \frac{d\zeta_1^1 \cdots d\zeta_n^1}{v(\zeta) (p_1(\zeta)-\alpha_1) \cdots (p_n(\zeta)-\alpha_n)}
\]

where \( J(\zeta) = \frac{\delta(p_1-\alpha_1, \ldots, p_n-\alpha_n)}{\delta(\zeta_1, \ldots, \zeta_n)} = \frac{\delta(p_1, \ldots, p_n)}{\delta(\zeta_1, \ldots, \zeta_n)} \) and the last identity follows from Stokes' theorem. (Replace the contour by...)

1.33
little spheres about the distinct points of $Z_a$). This last identity is the effective computation of the residue current of $Z_a$, namely

$$\text{Res}_{Z_a} = \sum_{\xi \in Z_a} J(\xi)^{-1} \delta_{\xi}.$$ 

The first identity in (73) shows that $\text{Res}_{Z_a} \to \text{Res}_Z$ as $\sigma \to 0$, i.e. the residue currents at $\sigma = 0$ are continuous when acting on holomorphic $(n,0)$ forms. On the other hand, by Theorem 3 we have

$$\text{Res}_{Z_a} \left( \frac{1}{P_{n+1}(\zeta)} \right) \begin{vmatrix} g^1(z,\zeta) & \cdots & g^{n+1}(z,\zeta) \\ \vdots & \ddots & \vdots \\ p_1(z) - \alpha_1 & \cdots & p_{n+1}(z) \end{vmatrix} d\zeta_1 \cdots d\zeta_n = 1.$$ 

(72) (Note that the $g^j$ corresponding to $p_j$ and to $p_j - \alpha_j$ coincide). By continuity we obtain $B = 1$. This concludes the proof of Theorem 4.

Remarks 1. The can obviously obtain the same result without assuming the degrees of $p_1, \ldots, p_n$ coincide or that they are larger or equal than that of $P_{n+1}$.

2. The reasoning of Theorem 4 extends to a strongly coprime family of $n+1$ elements whose first $n$ members from a well-behaved family. Under the other conditions of Theorem 1 or Theorem 2, we obtain a series representation of the solutions of the Bezout equation which we computed in terms of the residue current associated to $Z$. This time the series converges after grouping of terms.
3. The interest of the theorems in this section lies in the search for explicit algorithms to obtain solutions $A_j$ for the algebraic Bezout equation $\sum_{1}^{n} A_j p_j = 1$ which satisfy Brownawell's estimate, $\deg A_j \leq 3 \mu n D^\mu$, $\mu = \min(n,m)$.

4. Conclusion

We have shown how explicit solutions to the analytic and algebraic Bezout equations can be obtained under natural restrictions on the original functions $f_1, \ldots, f_m$. This work has applications to the implementation of deconvolution for multidetector systems.

5. References.


5. C.A. Berenstein and D. Struppa, Small degree solutions for the polynomial Bezout equation, to appear in Linear Algebra and its Applications.


CHAPTER 2

EXACT DECONVOLUTION FOR MULTIPLE CONVOLUTION OPERATORS -
AN OVERVIEW PLUS PERFORMANCE CHARACTERIZATIONS FOR IMAGING SENSORS*

by

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1. INTRODUCTION

Throughout the last several years mathematical results have been present-
ed which form the foundations for the use of multiple (parallel) linear opera-
tors, each given by convolution with a distinct kernel (or impulse response),
in place of the use of a single such linear operator or, equivalently, in
place of the use of multiple (parallel) operators each with the identical
kernel [1] - [13]. See Fig. 1. The mathematical results cited above describe
the conditions under which compactly supported distributions $\mu_1, \mu_2, \ldots, \mu_m$
have associated to them compactly supported distributions $\nu_1, \nu_2, \ldots, \nu_m$
such that

$$\sum_{j=1}^{m} \mu_j \ast \nu_j = \delta,$$

where $\delta$ is the Dirac distribution on $\mathbb{R}^n$ and where $\ast$ denotes convolution.

We often refer to the $\mu_j$ as convolvers and to the $\nu_j$ as deconvolvers.

To introduce some explicit multiple operators and their role in deconvo-
lution, let us look at an example and the results of a computer simulation
[14]. Our example is for a case in which equation (1) holds for $m = 2$, and
it is outlined in Fig. 2.

In the center of Fig. 2 the block diagram of Fig. 1(a) is reproduced for
the case $m = 2$. (The operator $\phi$ in Fig. 1(a) will be discussed later.) We
shall be considering in this example the one-dimensional case, that is, func-
tions of one variable. To the left of the block diagram is the graph of the
input signal, that is, the function $f$, which here consists of the sum of
translated Gaussian pulses. (This function is one that is frequently used for
the evaluation of deconvolution algorithms; see for example [15].)

The input signal $f$ is acted on by two convolution operators, one with
kernel $\mu_1$ and one with kernel $\mu_2$. The resulting output functions are $g_1$
and \( g_2 \), respectively:

\[
g_1(x) = (f * \mu_1)(x) = \int_{-\infty}^{\infty} f(y) \mu_1(x-y) dy, \quad i = 1, 2, \quad x \in \mathbb{R}.
\]

In this example, the kernels \( \mu_1 \) and \( \mu_2 \) are essentially the simplest possible kernels with compact support: the kernel functions are each constant over some bounded intervals and zero elsewhere. (The support of the kernel function refers to (the closure of) the set on which the function is non-zero; the support is called compact if this set is bounded.) The kernel functions are shown in Fig. 2: \( \mu_1 \) is supported by the interval \([-1, 1]\), and \( \mu_2 \) by the interval \([-\sqrt{2}, \sqrt{2}]\).

The output functions \( g_1 \) and \( g_2 \) are also shown in Fig. 2. We often refer to these as the data functions, for in applications \( g_1 \) and \( g_2 \) are typically the results from two distinct measurement operations which are modeled by the "convolvers" \( \mu_1 \) and \( \mu_2 \). An application that is a physical realization of the situation presented in Fig. 2 is spectroscopy. In that case, the function \( f \) is an unknown density function with the wavelength of the optical radiation as the variable: \( f(x) \) is the energy per unit wavelength at wavelength \( x \). To make a measurement of nonzero energy from this density function it is necessary to integrate \( f \) over some range of wavelengths, for example, \( g_1(x) = \int_{x-\Delta}^{x+\Delta} f(y) dy \). We could also make the second measurement \( g_2(x) = \int_{x-\sqrt{2}\Delta}^{x+\sqrt{2}\Delta} f(y) dy \). A linear scaling of the wavelength variables converts these integrals into exactly the example in Fig. 2. In this interpretation, the peaks in the unknown spectrum \( f \) have a separation smaller than the lengths of wavelength intervals over which we integrate. Consequently, the data functions \( g_1 \) and \( g_2 \) retain no obvious indication of two peaks.
Of course, the classical deconvolution problem is to recover the unknown input function $f$ from only one data function, say $g_0 = \mu_0 * f$. The term multiple operator deconvolution is used to emphasize that in recovering $f$ we may use more than one data function. In fact, part of the problem is deciding how many are sufficient. More precisely, the issue is how many of which convolvers $\mu_1$ are sufficient for the recovery of $f$. As we shall see later, the two we have chosen for our example in Fig. 2 are in fact sufficient. They are in fact sufficient in a very strong manner. Not only is $f$ uniquely determined by $g_1$ and $g_2$, but in fact there exist linear operators given by convolution with distributions $\nu_1$ and $\nu_2$, both with support no greater than the larger of the supports of $\mu_1$ and $\mu_2$, such that

$$f = \nu_1 * g_1 + \nu_2 * g_2.$$  

We will give the explicit formulas for $\nu_1$ and $\nu_2$ later. Here, to finish our example and discussion of Fig. 2, we have shown to the right in Fig. 2 the result of recovering $f$ from $g_1$ and $g_2$ for a digital simulation [14]: discrete versions of "deconvolvers" $\nu_1$ and $\nu_2$ were constructed and, using discrete samples from the data functions $g_1$ and $g_2$, a discrete approximation of $f$ was constructed by the sum of discrete convolutions. A quite good reconstruction was obtained. (We attribute the asymmetrical ringing in the result to a combination of truncation error and the use of a loop in the simulation that recalculated a quantity that should have been calculated only once.)

We would like to emphasize one final point: the example is based on mathematical results regarding relationships between convolution operators and smooth functions on $\mathbb{R}^n$. Whenever we mention the use of discretization, it is always in the sense of a discrete approximation to a smooth function. When we
refer to discrete data, we mean samples of the underlying data functions. The reconstruction of the unknown input depends directly upon how well the discrete data can be used to approximate the data functions for a given choice of interpolating functions. In Fig. 2 it should be clear that if \( g_1 \) and \( g_2 \) were sampled at a rate of one sample per unit interval (\( \frac{1}{2} \) tick marks), one could hardly expect an accurate estimate of \( f \). The sampling rate for the results on Fig. 2 was 50 samples per unit interval (20 samples per tick mark interval).

Thus we always view the theory, the applications, and the algorithms in the following order. First the problems and results are stated in terms of continuous domains that model the applications. The accuracy of the algorithmic implementation of the results then is understood to depend on the sampling rate, with convergence as sampling rates increase. The specification of a possible minimum sampling rate for a given reconstruction problem is typically a problem-specific task in interpolation error estimates. In fact, the Nyquist sampling rate is sufficient for band limited functions \( f \).

The example in Fig. 2 illustrates why equation (1) is of interest for applications in which the convolver \( \mu_1 \) must correspond to a physical, analog device wherein the impulse response is dictated by a solid state or biological process. It is frequently possible to select such analog convolvers which satisfy approximately the multiple operator criteria such that equation (1) will hold. Then each associated deconvolver can be digitally implemented. The fact that the deconvolvers act linearly and have compact support means that their implementation is straightforward; their action as continuous linear operators implies stability. Most importantly, the evident high bandwidth of the overall operator is accomplished without any essential change in the response functions of the analog devices. The term overall operator
refers to the operator given by the kernel distribution \( \sum_{i=1}^{m} \mu_i * \nu_i = \delta \). Of course, because of practical constraints such as analog and digital approximations and computation time, the design objective for the overall operator would not be the identity operator with impulse response \( \delta \) but rather a high bandwidth approximation of the identity operator given by an impulse response \( \phi \). In terms of the distributions in equation (1), since convolutions commute,

\[
\sum_{i=1}^{m} (\mu_i * \phi) * \nu_i = \sum_{i=1}^{m} \mu_i * (\nu_i * \phi) = \phi.
\]  

In a sense \( \phi \) can be considered to be made up of "parts," each of which arises from one of the practical constraints just listed, along with a special part that is deliberately added to control the noise power spectrum of the output of the overall operator.

The publications on this subject have appeared primarily in the mathematical literature. The following issues regarding (1) have been addressed:

- sufficient conditions for the existence of solutions [1] - [4], [16];
- examples of sets of distributions that satisfy the sufficient conditions [5] - [7];
- construction of explicit solutions, that is, explicit formulas for the deconvolvers [7] - [9]; and construction and evaluation of approximate solutions [9], [11], [14].

Only recently have specific applications of (1) been mentioned. The work of Berenstein, Krishnaprasad, and Taylor [14] was one of the first times that (1) and contemporary mathematical methods for understanding the equation were applied to physical problems. This work also discussed the question of additive noise and the question of the continuity of the overall operator with respect to the distributions \( \mu_1, \mu_2, \ldots, \mu_m \). The noise question is in regard to noise added following the action of the operators defined by the \( \mu_i \).
while the continuity question is in regard to the dependence of the overall performance on either the actual analog approximations of the $\mu_1$ or the digital approximations of the $\nu_1$.

The approximation methods of [11] were motivated by this work of Berenstein et al. These methods exploit the approximation in (2). In conjunction with the analysis, a computer simulation for $\mathbb{R}^2$ was performed. This simulation dramatically illustrated (2) for imaging devices in which the analog convolvers were solid state photodetectors. With these results there was an increased interest in imaging applications. This led to the consideration of not just detectors but of linear systems consisting of sequences of operators with each operator of the multiple operator type. These activities led to the need to answer basic systems analysis questions.

In what follows three topics in multiple operator deconvolution are discussed. The first is that, in a sense that is relevant for applications, multiple operators are necessary for the deconvolution problem to be well-posed. We present examples to illustrate the ill-posedness of single operator deconvolution, and a theorem which shows that, except for an uninteresting case, $m$ must be greater than one in our problem statement.

The second topic is that of identifying convolvers $\mu_1, \ldots, \mu_m$ for which equation (1) holds and the construction of the deconvolvers.

The final topic is the major one here: measuring the utility of a multiple operator design. We describe the result of our application of standard methods of linear systems and random signals to the multiple operator type of system of equations (1) and (2). This analysis was necessary if one was to seriously consider multiple operator designs. While the extended bandwidth was well understood, analyzed, and even illustrated in simulations, the consequence of the introduction of noise and of design errors was not fully
understood. It was clear that since the operator was linear and continuous that there would be no instability due to noise (at least for smooth approximations), which is already an improvement over the case of single operator reconstruction methods [5], [14], [15]. However, the performance needed to be explicitly described so that standard tools such as resolution, equivalent bandwidth, and signal to noise ratio would be available for systems engineering design studies.

This investigation was motivated in large part by the potential application of these multiple operator methods to electo-optics, especially to imaging devices. We have in mind imaging devices that are for the detection, transformation, and display of electromagnetic radiation for a human observer as well as such devices for artificially intelligent "observers." The problems and the desired solutions have the flavor of this application. While the analysis and the results are in a sense general, much is framed and guided by the motivating problems.

This interest in electro-optics is made explicit in the last sections of this paper. There we examine convolvers and additive noises of the type encountered in imaging detector arrays. The method developed here for general performance comparisons is used to compare conventional detector arrays with arrays configured such that equation (1) holds (up to the resolution limitation imposed by a choice of optical sampling rate). In terms of the familiar parameters of resolution, modulation transfer function, and signal to noise, we show that the performance of the strongly coprime design exceeds that of the conventional design. Moreover, the comparison has a quantified, functional form suitable for system design trade-off studies.
2. THE NECESSITY OF MULTIPLE OPERATORS

Recall our mathematical problem statement from the Introduction: under what conditions do the compactly supported distributions $\mu_1, \mu_2, \ldots, \mu_m$ on $\mathbb{R}^n$ have associated to them compactly supported distributions $\nu_1, \nu_2, \ldots, \nu_m$ such that $\sum_{i=1}^{m} \mu_i \ast \nu_i = \delta$?

Here we offer some observations and a theorem which motivate this statement. A brief demystification of "compactly supported distributions" is given in the Appendix.

In Fig. 1(a) and in the problem statement, the integer $m$, the number of distributions $\mu_i$, is not specified beforehand. The number of distributions as well as the distributions $\nu_1, \ldots, \nu_m$ are parts of the problem. In the example above for $\mathbb{R}$ it was stated that $m = 2$ sufficed. When the $\mu_i$ are characteristic functions of cubes on $\mathbb{R}^n$, then there are choices for the $\mu_i$ such that $m = n + 1$ is sufficient.

Fig. 1(b) represents a different, more restricted problem: it represents a restriction of the problem statement above to $m = 1$. We show below that there is only one trivial case in which a single compactly supported distribution $\mu_0$ has an associated compactly supported distribution $\nu_0$ such that $\mu_0 \ast \nu_0 = \delta$. Before presenting this general result we briefly review three well known observations which help to motivate the problem statement.

For these remarks let $\mu_0$ be the characteristic function of the interval $(-1,1)$ in $\mathbb{R}$ ($\mu_0$ takes the value one on the interval and zero elsewhere).

Observation 1. There can be no such $\nu_0$ if we permit the input signals $f$ to belong to any space that contains $C(\mathbb{R})$, the continuous functions on $\mathbb{R}$.

If such a $\nu_0$ existed then, for any continuous $f$, $\nu_0 \ast (\mu_0 \ast f) = f$. But for $f$ continuous, $\mu_0 \ast f$ is (represented by) a continuous function and.
hence, for \( f_n(x) = \sin(n \pi x) \), \( n = 1, 2, \ldots \), it is easily checked that \( \mu_0 * f_n \) is zero everywhere.

The problematic functions \( f_n \) in Observation 1 can be eliminated by considering only functions in \( L^2(\mathbb{R}) \). For in this case we can consider Fourier transforms of \( f \), of \( \mu_0 \) (\( \mu_0 \) is in \( L^1(\mathbb{R}) \)), and of \( \mu_0 * f \) (in \( L^2(\mathbb{R}) \)). Let \( \hat{\mu}_0 \) denote the Fourier transform of \( \mu_0 \). Since \( \hat{\mu}_0(\omega) = 2 \frac{\sin \omega}{\omega} \), \( \omega \in \mathbb{R} \), then \( (\mu_0 * f) = \hat{\mu}_0 \hat{f} \) determines \( \hat{f} \) almost everywhere. Since the Fourier transform is an isometry from \( L^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \), \( f \) is uniquely determined by \( \mu_0 * f \). But still:

Observation 2. There can be no such \( \nu_0 \) even if we restrict the input signals \( f \) to belong in \( L^2(\mathbb{R}) \).

If such a \( \nu_0 \) existed, then let \([-M,M]\) be the support of \( \nu_0 \). Let \( \chi_{M+2} \) be the characteristic function \((-M+2,M+2)\) and let \( f_n(x) = \sin(n \pi x), n = 1, 2, \ldots \). Then \( \mu_0 * (f_n \chi_{M+2}) \) is zero on \((-M+1,M+1)\), hence \( \nu_0 * \mu_0 * (f_n \chi_{M+2}) \) is zero on \((-1,1)\).

The difficulty in Observation 2 is due to the fact that we seek \( \nu_0 \) with compact support. What if we dropped this requirement? This would be problematic for applications, for then to get better estimates of \( f \) from \( \mu_0 * f \) we would have to process larger and larger subsets of the domain of \( \mu_0 * f \). But even if we could tolerate noncompactness, there is still a problem of boundedness or continuity.

Observation 3. There can be no (non-compactly supported) continuous linear operator \( L \) from \( L^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \) such that \( L(\mu_0 * f) = f \) for all \( f \) in \( L^2(\mathbb{R}) \).

The difficulty is that convolution with \( \mu_0 \) does not carry \( L^2(\mathbb{R}) \) into \( L^2(\mathbb{R}) \). For example, let \( \psi_c \) be the approximate identity defined by \( \psi_c(x) = \frac{1}{2c} \mu_0(\frac{x}{c}), x \in \mathbb{R} \). Then the \( L^1 \) norm of \( \psi_c \) is \( \|\psi_c\|_1 = 1 \), while \( \|\psi_c\|_2 = \frac{1}{2c} \).
for the $L^2$ norm, and $\psi_\varepsilon \ast \mu_0$ converges to $\mu_0$ in $L^2$. If $L$ existed as described, then by continuity

$$L(\mu_0) = L(\lim_{\varepsilon \to 0} \psi_\varepsilon \ast \mu_0) = \lim_{\varepsilon \to 0} L(\psi_\varepsilon \ast \mu_0) = \lim_{\varepsilon \to 0} \psi_\varepsilon.$$

These observations illustrate ways in which the problem of inverting the convolution or solving the convolution equation $\mu_0 \ast f = g$ for a specific $\mu_0$ fails to be well-posed (that is, that the solution $f$ to $\mu_0 \ast f = g$ exists, is unique, and depends continuously on $g$) even with a variety of restrictions on $f$. Our problem statement in part seeks to find $m$ and $\mu_1, \ldots, \mu_m$ such that the inversion of the simultaneous distribution equations $\mu_1 \ast f = g_1, \ldots, \mu_m \ast f = g_m$ is well-posed, along with an extra condition on the form of the inverse. But possibly our observations were due entirely to an unfortunate choice for the distribution $\mu_0$. Possibly there is a compactly supported $\mu_1$ for which there exists a compactly supported distribution $\nu_1$ such that $\mu_1 \ast \nu_1 = \delta$. This issue is settled by the following theorem.

**Theorem.** Let $\mu_1$ and $\nu_1$ be compactly supported distributions on $\mathbb{R}^n$ such that $\mu_1 \ast \nu_1 = \delta$, the Dirac delta distribution or unit impulse (at the origin in $\mathbb{R}^n$). Then there exist $a \in \mathbb{R}^n$ and $C \in \mathbb{C}$, $C \neq 0$, such that $\mu_1 = C \delta_a$, the Dirac delta distribution translated to $a \in \mathbb{R}^n$.

**Proof.** See Appendix.

These observations and this theorem are primary examples of the difficulties that are avoided whenever we can use multiple operators $\mu_1, \ldots, \mu_m$ for which equation (1) holds. Of course, there is a vast literature and many approaches to address the difficulties of ill-posed problems such as the inversion of a single convolution equation [15], [17] and [18]. Our interest is in exploiting those cases in which using multiple operators we have a well-posed inverse problem. In the next section we identify some of those cases.
3. EXISTENCE AND CONSTRUCTION OF DECONVOLVERS

Our problem statement is typically viewed in two parts. The first part is existence, the identification of sets of convolvers $\mu_1, \ldots, \mu_m$ such that deconvolvers $\nu_1, \ldots, \nu_m$ exist. The second part is the explicit representation of the deconvolvers.

These problems are addressed using methods from the study of analytic functions of several complex variables. For if $\mu_1$ is the distribution on $\mathbb{R}^n$ with compact support, then its Fourier-Laplace transform $\hat{\mu}_1$ is an analytic function on $\mathbb{C}^n$. Moreover, $|\hat{\mu}_1(z)|$, $z \in \mathbb{C}^n$ increases for large $|z| = |(z_1, z_2, \ldots, z_n)| = \left(\sum |z_i|^2\right)^{1/2}$ in a manner which completely characterizes these analytic functions which are transforms of distribution of compact support (the Paley-Wiener-Schwartz theorem, see Appendix). Thus for a given set of convolvers $\mu_1, \ldots, \mu_m$, by taking transforms the existence of the deconvolvers is equivalent to the existence of solutions $\hat{\nu}_1, \ldots, \hat{\nu}_m$ of the analytic Bezout equation

$$\sum_{i=1}^m \hat{\mu}_i(z)\hat{\nu}_i(z) = 1, \quad z \in \mathbb{C}^n, \quad (3)$$

with $\hat{\nu}_1, \ldots, \hat{\nu}_m$ each in the Paley-Wiener class of functions.

In this form one quickly sees that a necessary condition on the convolvers $\mu_1$ is that they have no common zeros. Moreover, because the $\hat{\nu}_1$ are to satisfy certain growth conditions for large $|z|$, one has a stronger condition which turns out to be both necessary and sufficient.

**Theorem** [1], [16]. For the compactly supported distributions $\mu_1, \ldots, \mu_m$ on $\mathbb{R}^n$ there exists compactly supported distributions $\nu_1, \ldots, \nu_m$ such that

$$\sum_{i=1}^m \mu_i \ast \nu_i = \delta$$

if and only if there exist positive constants $c_1$ and $c_2$ and

2.11
a positive integer $N$ such that

$$\left(\sum_{1}^{\infty} |\mu_{i}(z)|^{2}\right)^{1/2} \leq c_{1} e^{-c_{2}|\Im z|(1+|z|)^{-N}}, \quad z \in \mathbb{C}^{n}. \quad (4)$$

A set of convolvers $\mu_{1}, \ldots, \mu_{n}$ that satisfies the inequality in the theorem is often referred to as strongly coprime.

The following sets of convolvers are known to be strongly coprime:

i) on $\mathbb{R}^{2}$, the characteristic functions of two disks wherein the ratio of the radii of the disks has any integer value from 2 to 200 [7];

ii) on $\mathbb{R}^{n}$, the characteristic functions of $n+1$ cubes with side lengths $s_{1}, \ldots, s_{n+1}$ such that $s_{1}^{2}, \ldots, s_{n+1}^{2}$ are integers that are pairwise relatively prime and at most one of these integers is a perfect square of integers [7], [11].

The example of the introduction falls within case ii. Another case which is known is the characteristic functions of certain rotations of three squares in the plane [7].

Explicit formulas for the deconvolvers have been found for the disks on $\mathbb{R}^{2}$ (case i) and for $n = 1$, and $n = 2$ in case ii [5] - [9] and [14]. For example, the explicit solutions for the deconvolvers $\nu_{1}, \nu_{2}$ for $n = 1$ in case ii is the following. For $i = 1,2$, $a_{i} > 0$, let $\chi_{i}$ be the characteristic function of the interval $(-a_{i}, a_{i})$, with $a_{2} = \sqrt{2}a_{1}$. Let $\phi$ be any smooth function. Then $\nu_{1}$ and $\nu_{2}$ such that $\chi_{1} \ast \nu_{1} + \chi_{2} \ast \nu_{2} = \delta$ are given by

$$\nu_{1} \ast \phi = \frac{1}{8a_{1}a_{2}} \chi_{2} \ast \phi - \frac{1}{24a_{1}^{2}a_{2}^{2}} \frac{d^{2}}{dx^{2}}(\chi_{2} \ast \phi) + \frac{d^{3}}{dx^{3}}((\chi_{2} K_{1}) \ast \phi)$$

where

2.12
and \( v_2 \) and \( K_2 \) are given by the permutation of indices. Note that the support of \( v_1 \) is contained in that of \( x_2 \), and the support of \( v_2 \) is contained in that of \( x_1 \).

Throughout the foregoing we have emphasized that we seek deconvolvers \( v_i \) which have compact support, and we have described necessary and sufficient conditions on the convolvers \( \mu_i \) for these to exist. The explicit determination of whether the \( \mu_i \) are strongly coprime and the explicit construction of the deconvolvers is typically too restrictive an approach for general engineering applications, and new numerical optimization approaches are being developed [10].

There is, however, a middle ground that we use when we have a set of strongly coprime convolvers but do not have explicit representations for compactly supported deconvolvers. These alternatives are due to the fact that the deconvolvers are not unique. To see this nonuniqueness, consider the one-dimensional example of the Introduction. Let \( \mu_1 \) and \( \mu_2 \) be compact supported and strongly coprime, and let \( v_1 \) and \( v_2 \) be compactly supported deconvolvers. Let \( \lambda \) be any compactly supported distribution. Then it is readily seen that both \( v'_1 = v_1 + \lambda \cdot \mu_2 \) and \( v'_2 = v_2 - \lambda \cdot \mu_1 \) are compactly supported and \( \sum_{i=1}^{2} \mu_1 \cdot v'_1 = \delta \). Going further, it was not necessary that \( \lambda \) be a compactly supported distribution. That is, we can construct deconvolvers that are not compactly supported.

We frequently use approximate deconvolvers even when they are not compactly supported. For example, let \( \varphi \) be any integrable, smooth function

\[
K_1(x) = \frac{a_1}{2\pi^2} \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k x}{a_2})}{k^2 \sin(\frac{\pi k}{a_2})}
\]
with integral over \( \mathbb{R} \) equal to 1. Then, for \( \varphi_t(x) = \frac{1}{t} \varphi \left( \frac{x}{t} \right) \), \( \nu_1 \ast \varphi_t \) is an approximate deconvolver and \( \nu_1 \ast \varphi_t \) converges to \( \nu_1 \) as \( t \to 0 \). We can choose, also, that the \( \nu_1 \ast \varphi_t \) be \( L^1 \) functions. Let \( \hat{\varphi} \) have compact support and be sufficiently differentiable. (The function \( \varphi \) is usually called a mollifier.)

Finally, we can construct approximate, non-compactly supported deconvolvers on \( \mathbb{R}^n \) solely from the knowledge that the convolvers are strongly coprime. Let \( \omega = (\omega_1, \ldots, \omega_n) = \text{Re} \ z = (\text{Re} \ z_1, \text{Re} \ z_2, \ldots, \text{Re} \ z_n) \) and for what follows we will need to consider the Fourier transform only on the real subspace \( \mathbb{R}^n \) of \( \mathbb{C}^n \). Since \( \hat{\mu}_1 \in C^\infty(\mathbb{R}^n) \), and with \( \varphi \in C^\infty(\mathbb{R}^n) \), with \( \hat{\varphi} \) compactly supported, let

\[
D_1(\omega) = \frac{\hat{\mu}_1(\omega)}{\sum_{j=1}^{m} \left| \hat{\mu}_j(\omega) \right|^2}, \quad \hat{h}_1 = D_1(\omega) \hat{\varphi}(\omega), \quad i = 1, \ldots, m.
\]

This defines \( h_1 \in L^1(\mathbb{R}^n) \) such that \( i \sum_{i=1}^{n} h_1 \ast \mu_i = \varphi \) (\( \bar{z} \) denotes the complex conjugate of \( z \)). This is the class of deconvolvers that are discussed in the last section, because from among these one can find optimal deconvolvers when there is additive noise.

4. PERFORMANCE OF OPTIMAL DECONVOLVERS.

While (5) is exhibited essentially by inspection, the result can be obtained in a more systematic fashion as well as in a more general form. We first recall some standard tools, apply these tools to a simple case, and then proceed to the more general form. The diagram in Fig. 3 represents an
operator $L$ acting on a function $f$. Let (temporarily) $f$ be bounded and in $C^w(R^n)$. Let $\mu_1, \mu_2, \ldots, \mu_m$ be an arbitrary set of $m$ distributions with compact support. For each linear operator defined by $\mu_1$ let $\eta_1$ be a sample function of a zero mean, wide-sense stationary random process that is added to the output of $\mu_1$, let $\eta_1 \in L^w(R^n)$, and let $N_1^2$ ($N_1 > 0$) be the noise power spectral density of the process (see, for example, [19] Ch. 4, Ch. 6). For each distinct $i$ and $j$ let $\eta_i$ be independent of $\eta_j$ and let each $\eta_j$ be independent of $f$. Let $\nu_1$ be defined by $(\nu_1 * \phi) = D_1 \hat{\phi}$, where $\hat{\phi}, D_1 \in C^C(R^n)$, $\hat{\phi}$ has compact support, and $r$ is sufficiently large so that $\left[D_1 \hat{\phi}\right] \in L^1(R^n)$. Let $g \in L^w(R^n)$ be defined by

$$g = Lf = \sum_{i=1}^{m} (\mu_i * f + \eta_i) * (\nu_1 * \phi).$$

(6)

In the usual manner, with $E$ denoting expectation,

$$E(g) = \sum_{i=1}^{m} \mu_i * f * (\nu_1 * \phi).$$

(7)

Let $T_y$ denote translation by $y$, $T'_y(x) = x + y$, let $\hat{\cdot}$ denote inverse Fourier transform, and let $\| \cdot \|_p$ denote the $L^p$ norm. Directly from the definition of wide-sense stationary and noise power spectral density it follows that

$$E\left\{ (g - E(g))(g - E(g)) * T'_y \right\} = \left[ \sum_{i=1}^{m} N_1^2 |D_1 \hat{\phi}|^2 \right]^2(y).$$

(8)

and, for $y = 0$, that

$$E\left\{ (g - E(g))^2 \right\} = \frac{1}{(2\pi)^n} \left[ \sum_{i=1}^{m} N_1^2 |D_1 \hat{\phi}|^2 \right]^2_1.$$

(9)
The simplest configuration for $L$ is all distributions equal, all deconvolvers trivial, and all random processes identically distributed:

$$\mu_1 = \mu_0, \quad \nu_1 = \delta, \quad N_1^2 = N_0^2 \quad \text{for } i = 1, 2, \ldots, m.$$  \hfill (10)

Then

$$E(g) = m\mu_0 \phi f, \quad E\{(g - E(g))^2\} = \frac{m}{(2\pi)^n} \|N_0^1 \phi \|_2^2.$$  \hfill (11)

The utility of (7) - (9) or of (11) is that if $L$ is followed by a linear operator $U$ with kernel $u$ (which could model a specific "end-use") then one can compare the function $E(U(g))^2$ with the constant function $E((U(g - E(g)))^2)$. In the case of the simplest configuration, (10) and (11), there are the following formulas and bounds.

$$(U(E(g)))^2 = E(Ug)^2 = (u \ast (m\mu_0 \phi f))^2 = m^2 \left((\hat{u} \hat{\mu}_0 \hat{\phi} f)^2\right)^2$$  \hfill (12)

$$\leq \left(\frac{m}{(2\pi)^n} \|\hat{u} \hat{\mu}_0 \hat{\phi} f\|_1\right)^2$$

$$\leq \left(\frac{m}{(2\pi)^n}\right)^2 \|\hat{u} \hat{\mu}_0 \hat{\phi} f\|_2^2 \|f\|_2^2 \quad \text{when } f \in L^2(\mathbb{R}^n);$$

and

$$E\{(U(g - E(g)))^2\} = \frac{m}{(2\pi)^n} \|\hat{u} \hat{N}_0 \hat{\phi}\|_2^2.$$  \hfill (13)

The function $E(Ug)$ is referred to as the signal, its square $E(Ug)^2$ is referred to as the signal power or energy, and $E((U(g-E(g)))^2)$ is referred to as the noise power. Typically the ratio of $E(Ug)^2$ to $E((U(g-E(g)))^2)$ is considered or, alternatively, the positive square root of the ratio. Here we shall consistently use the latter. If this ratio is evaluated at some distinguished point, the value defines a "signal to noise ratio."
point. Given \( L \) and for a given choice of \( \phi, f, U, \) and \( \Phi \) define the signal to noise ratio

\[
YNR(U_L) = \frac{\frac{\Phi U(E(g))}{E\left(U(g-E(g))^2\right)}}{1/2}.
\]  

(14)

For a fixed choice of \( \phi, f, U, \) and \( \Phi \), two operators \( L \) and \( L' \) can be compared and ordered by (14).

On the other hand, for a choice of \( \phi, f, U, \) and \( \Phi \), (14) is determined for the case of the trivial operator in (10) by the pair of functions

\[
\mu_0 \quad \text{and} \quad \sqrt{m} N_0.
\]

(15)

In general, let operators \( L \) and \( L' \) (for example, as in Fig. 3) have transfer functions and noise power spectral densities \( \hat{\mu}, \nu^2 \) and \( \hat{\mu}', \nu'^2 \), respectively. For a choice of \( U \) we shall say that \( UL \mid UL' \) (i.e., "\( UL \) divides \( UL' \)"") if there exists a "quotient" function \( \hat{q} \in L^\infty(\mathbb{R}^n) \) such that

\[
\hat{q} \hat{\mu} = \hat{\mu}'.
\]

If \( UL \mid UL' \) and \( |q|^2 \nu^2 \leq |\nu'|^2 \nu'^2 \), we say that \( UL \geq UL' \).

This definition is motivated by the following. As usual, let \( \phi \) be such that a linear operator \( \hat{\Phi} \) with kernel \( q \) can be associated with \( \hat{q} \) by considering \( \hat{\Phi} \). Let \( \hat{\Phi} \) be any continuous, translation invariant, linear operator. For fixed \( U \) if \( UL \geq UL' \), then \( \frac{YNR(U_L \Phi L)}{YNR(U_L \Phi L')} \geq 1 \). Consequently,

\[
\sup_{\hat{\Phi}} \frac{YNR(U_L \Phi L)}{YNR(U_L \Phi L')} \geq \sup_{\hat{\Phi}} YNR(U_L \Phi L').
\]

Next consider the operator \( L \) diagrammed in Fig. 3 for the case in which \( \mu_1, \mu_2, \ldots, \mu_m \) are distinct and strongly coprime (i.e., satisfy (4)). An obvious consequence is

\[
\sum_{i=1}^{m} |\hat{\mu}_i(\omega)|^2 > 0 \quad \text{and, equivalently,}
\]

\[
0 = (\hat{\mu}_1(\omega), \hat{\mu}_2(\omega), \ldots, \hat{\mu}_m(\omega)) \in \mathbb{C}^n, \ \omega \in \mathbb{R}^n.
\]

(16)

Consequently we can visualize (16) as is shown in Fig. 4a. A similar
Illustration can be used to visualize \( \hat{f}(\omega)(\hat{\mu}_1(\omega), \hat{\mu}_2(\omega), \ldots, \hat{\mu}_m(\omega)) = (\hat{f}(\omega)\hat{\mu}_1(\omega), \hat{f}(\omega)\hat{\mu}_2(\omega), \ldots, \hat{f}(\omega)\hat{\mu}_m(\omega)) \), except the "curve" passes through the origin if and only if \( \hat{f}(\omega) = 0 \). The power spectral densities are real and nonnegative (thus we write \( N_1^2 \) and choose \( N_1 \geq 0 \)). Assume

\[
N_1(\omega) > 0, \ \omega \in \mathbb{R}^n, \ i = 1, 2, \ldots, m. \tag{17}
\]

We can visualize (17) as is shown in Fig. 4b. The case of strongly coprime multiple operators has the useful feature that the consideration of (16) and (17) pointwise in conjunction with (7) - (9) uniquely determines an alternative choice for the \( D_1 \) of (5). This choice will be optimal in the sense it has the smallest \( E((g-E(g))^2) \) among all sets of deconvolvers.

**Proposition.** Let \( N_1 \in L^\infty(\mathbb{R}^n) \), \( N_1(\omega) > 0 \) for \( \omega \in \mathbb{R}^n \), \( i = 1, 2, \ldots, m \). Then there is a function \( D : \mathbb{R}^n \to \mathbb{C}^m \) uniquely determined (almost everywhere) by the condition that for each \( \omega \in \mathbb{R}^n \), \( D(\omega) = (D_1(\omega), D_2(\omega), \ldots, D_m(\omega)) \) solves:

\[
\text{Minimize } \sum_{i=1}^{m} |z_i|^2 N_i^2(\omega) \text{ on the set } \left\{ z \in \mathbb{C}^m : \sum_{i=1}^{m} z_i \hat{\mu}_i(\omega) = 1 \right\}. \tag{18}
\]

In fact

\[
D_i(\omega) = \frac{\frac{\hat{\mu}_i(\omega)}{N_i^2(\omega)}}{\sum_{j=1}^{m} \frac{|\hat{\mu}_j(\omega)|^2}{N_i^2(\omega)}}. \tag{19}
\]

**Proof.** Any \( z \) that satisfies (18) is clearly contained in the linear subspace of \( \mathbb{C}^m \) determined by the span of

\[
\{ (\hat{\mu}_1(\omega), 0, \ldots, 0), (0, \hat{\mu}_2(\omega), 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, \hat{\mu}_m(\omega)) \}. \tag{20}
\]

That is, \( z_i = 0 \) if \( \hat{\mu}_i(\omega) = 0 \). Equivalently, there exists \( \lambda = \ldots \)
\((\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{C}^m\) such that
\[
(z_1 N_1(\omega), z_2 N_2(\omega), \ldots, z_m N_m(\omega)) = (\lambda_1 \mu_1(\omega), \lambda_2 \mu_2(\omega), \ldots, \lambda_m \mu_m(\omega)).
\] (21)

Let \(\sum'\) denote \(\sum_{i=1}^{m} \mu_1(\omega) \neq 0\). Then (18) reduces to
\[
\text{Minimize } \sum' |\lambda_1|^2 |\mu_1(\omega)|^2 \text{ on } \left\{ \sum' \lambda_1 \left| \frac{\mu_1(\omega)}{N_1(\omega)} \right|^2 = 1 \right\}.
\] (22)

From this it follows that the \(\lambda_i\) are all real, so that (22) has the form
\[
\text{Minimize } \sum' \left| \lambda_i \hat{\mu}_i(\omega) \right|^2 \text{ on } \left\{ \sum' \lambda_i \left| \frac{\mu_i(\omega)}{N_i(\omega)} \right| = 1 \right\}.
\] (23)

With the new variable \(\lambda_i \left| \mu_i(\omega) \right|\), it is elementary to see that (23) has the unique solution
\[
\lambda_i \left| \hat{\mu}_i(\omega) \right| = \frac{\left| \mu_i(\omega) \right|}{N_i(\omega)} \left( \text{for } \hat{\mu}_i(\omega) \neq 0 \right).
\] (24)

Consequently, from (21), the unique \(z\) corresponding to the minimum is \(D(\omega)\) as in (19).

In addition to \(N_i > 0, i = 1, 2, \ldots, m\), we shall assume \(N_0 > 0\).

Further, we shall assume that the \(N_i\) are sufficiently differentiable and that \(\frac{1}{N_i} = O(|\omega|^p)\) for some integer \(p, 1 = 0, 1, 2, \ldots, m\) (that is, \(1/N_i\) does not grow faster than \(|\omega|^p\)). With this we can find \(\hat{\phi} = O(|\omega|^{-p'})\) so that \((D_1 \hat{\phi})^* \in L^2(\mathbb{R}^n)\) and for \(\hat{\phi}\) sufficiently smooth and with compact support then \((D_1 \hat{\phi})^* \in L^1(\mathbb{R}^n)\).

**Corollary.** For the choice of \(D_1\) from the Proposition,
\[ \sum_{i=1}^{m} \hat{\mu}_i D_1 = 1, \quad \left( \sum_{i=1}^{m} |D_1|^2 N_1^2 \right)^{1/2} = \left( \frac{m}{\sum_{j=1}^{N_1^2} \frac{1}{N_2^2}} \right)^{1/2} \quad (25) \]

Let \( L_0 \) identify the trivial configuration of \( L \) in (10) and let \( L_s \) identify the strongly coprime configuration. Unless explicitly indicated to the contrary, \( L_s \) indicates that the deconvolvers \( D_1 \) of (19) are used. The first of the functions in (25) is the transfer function for \( L_s \) and the second is the square root of the noise power spectral density. The corresponding functions for \( L_0 \) are (15). The mollifier \( \hat{\Phi} \) is suppressed but understood. From (25) obviously \( L_s | L \) for any operator \( L \). (With an abuse of notation we use \( L, L_0 \) and \( L_s \) to denote configurations consisting of linear operators and additive noises.) From (15) and (25) the quotient \( \hat{q} \) for \( L = L_0 \) is \( \hat{m}_0 \). Let \( N_2^2 \) denote the noise power spectral density of \( L_s \). In the sense discussed earlier let \( D_0 \) denote the linear operator associated with \( \hat{m}_0 \). That \( L_s | L_0 \) with quotient \( \hat{m}_0 \) means \( L_0 = D_0 L_s \). Then \( D_0 L_s \) has functions corresponding to (25) (transfer function, square root of noise power spectral density) given by

\[ \hat{\mu}_0 \sum_{i=1}^{m} \hat{\mu}_1 D_1 = \hat{\mu}_0, \quad |\hat{\mu}_0| N_s = m |\hat{\mu}_0| \left( \sum_{i=1}^{m} |D_1|^2 N_1^2 \right)^{1/2} = \left( \frac{m}{\sum_{j=1}^{N_1^2} \frac{1}{N_2^2}} \right)^{1/2} \sqrt{m N_0}. \quad (26) \]

By definition \( L_s \geq L_0 \) if \( |\hat{\mu}_0| N_s(\omega) \leq \sqrt{m N_0}(\omega) \) on the support of \( \hat{u} \), and \( L_0 \geq L_s \) if \( \|L_0\|_{L_s} \) and \( |\hat{\mu}_0| N_s(\omega) \geq \sqrt{m N_0}(\omega) \) on the support of \( \hat{u} \). Thus, whether \( \|L_0\|_{L_s} \geq \|L_s\|_{L_s} \) or \( \|L_s\|_{L_s} \geq \|L_0\| \) holds depends, in part, on whether one of the following inequalities holds on the support of \( \hat{u} \) : from (15) and (26)
In (27) the notation means that the upper inequality symbol on the left is to be paired with the upper inequality symbol on the right and lower left with lower right.

The comparison in (27) can in special cases be viewed from a slightly different perspective. First, view the left side of the second inequality in (27) as the Fourier transform of a kernel. Define

\[
\hat{\epsilon}(\omega) = \left[ \sum_{i=1}^{m} \frac{N_0^2(\omega)}{N_1^2(\omega)} |\hat{\mu}_1(\omega)|^2 \right]^{1/2} \leq \sqrt{m} |\hat{\mu}_0(\omega)|.
\]  

(28)

We refer to \( \hat{\epsilon} \) as the envelope transfer function corresponding to the envelope operator \( \mathcal{E} \) for a given strongly coprime \( L_s \) in comparison with a given \( L_0 \). If \( \sqrt{m} \hat{\epsilon} \) acts on \( L_s \), then the pair of functions associated with \( \sqrt{m} \hat{\epsilon} L_s \) is

\[
\sqrt{m} \hat{\epsilon}, \sqrt{m} N_0.
\]  

(29)

Recall that the pair for \( L_0 \) is given by (15) (rewritten for convenience)

\[
\hat{\mu}_0, \sqrt{m} N_0.
\]  

(15)

That is, the composition of \( \sqrt{m} \hat{\epsilon} \) with \( L_s \) has a noise power spectral density equal to that of \( L_0 \). If, for example, \( \hat{\mu}_0 \) is real and positive, then it makes sense to compare (29) with (15). It is easy to check that the condition \( \sqrt{m} \hat{\epsilon} \geq \hat{\mu}_0 \) (on the support of \( \hat{\mu} \)) coincides with our definition \( \mathbb{W}_s \geq \mathbb{W}_0 \), and \( \sqrt{m} \hat{\mu}_0 \geq \sqrt{m} \hat{\epsilon} \) coincides with what we mean by \( \mathbb{W}_0 \geq \mathbb{W}_s \). These two inequalities are precisely the content of the comparison of the right side of (27). One could say that \( \sqrt{m} \hat{\epsilon} \) is the normalization of \( L_s \) to the noise.
power spectral density of \( L_0 \).

For either point of view we consider \( \mathcal{W}_> = \{ \omega \in \mathbb{R}^n : \hat{\epsilon}(\omega) \geq \sqrt{n} |\hat{\mu}_0(\omega)| \} \)
and \( \mathcal{W}_< = \{ \omega \in \mathbb{R}^n : \hat{\epsilon}(\omega) \leq \sqrt{n} |\hat{\mu}_0(\omega)| \} \). For all \( \hat{u} \) such that \( \hat{u} \) has support in \( \mathcal{W}_> \) it follows from (27) and the definitions that \( \mathcal{W}_s \geq \mathcal{W}_0 \).
Consequently,

\[
\frac{\mathcal{YNR}(\mathcal{W}_0 L_s)}{\mathcal{YNR}(\mathcal{W}_0)} = \frac{|\hat{u} \hat{\phi} N_0|_2}{|u \frac{\sqrt{n} \hat{\mu}_0}{\hat{\epsilon}} \phi N_0|_2} \geq 1. \tag{30}
\]

where \( D_0 \) is used to denote the linear operator corresponding to the transfer function \( \hat{\mu}_0 \) of \( L_0 \).

Assume \( \hat{\mu}_0(0) \neq 0 \) and define

\[
\Omega_0 = \{ \omega \in \mathbb{R}^n : \forall t \in [0,1) |\hat{\mu}_0(t\omega)| > 0 \}. \tag{29}
\]

Note that for \( \mathbb{R}^1 \) the usual definition of limiting resolution is \( \sup \Omega_0 \). If \( \text{supp}(\hat{u}) \) is compact and \( \text{supp}(\hat{u}) \subset \Omega_0 \), then \( \mathcal{W}_0^{-1} \) makes sense, consequently \( \mathcal{W}_0 \mathcal{W}_s \). Hence, if \( \text{supp}(\hat{u}) \) is compact and \( \text{supp}(\hat{u}) \subset \mathcal{W}_c \cap \Omega_0 \), then \( \mathcal{W}_0 \geq \mathcal{W}_s \). Consequently,

\[
\frac{\mathcal{YNR}(\mathcal{W}_s)}{\mathcal{YNR}(\mathcal{W}_0^{-1} L_0)} = \frac{|\hat{u} \frac{1}{\sqrt{n} \hat{\mu}_0} \phi N_0|_2}{|\hat{u} \frac{1}{\hat{\epsilon}} \phi N_0|_2} \leq 1. \tag{31}
\]

In general the inequality cannot be extended to all of \( \mathcal{W}_c \cap \Omega_0 \) because of the behavior of \( 1/\hat{\mu}_0 \) on the boundary.

There is no information regarding \( \frac{\mathcal{YNR}(\mathcal{W}_s)}{\mathcal{YNR}(\mathcal{W}_0)} \) implied by either \( \mathcal{W}_s \geq \mathcal{W}_0 \)
or \( \mathcal{W}_0 \geq \mathcal{W}_s \). Additional information is needed. For example, it may be sufficient to know the effect of the so-called "boost" \( \mathcal{W}_0 \rightarrow \mathcal{W}_0^{-1} L_0 \). In
particular, if supp(\(\hat{u}\)) is compact and supp(\(\hat{u}\)) \(\subset\mathbb{O}_0\) then

\[
\text{supp}(\hat{u}) \subset \mathcal{W}_> \quad \text{and} \quad \frac{\mathcal{N}_R(\mathfrak{W}^{-1}_0L_0)}{\mathcal{N}_R(\mathfrak{W}_0)} \geq 1 \quad \Rightarrow \quad \frac{\mathcal{N}_R(\mathfrak{W}_S)}{\mathcal{N}_R(\mathfrak{W}_0)} \geq 1, \tag{32a}
\]

and

\[
\text{supp}(\hat{u}) \subset \mathcal{W}_< \quad \text{and} \quad \frac{\mathcal{N}_R(\mathfrak{W}^{-1}_0L_0)}{\mathcal{N}_R(\mathfrak{W}_0)} \leq 1 \quad \Rightarrow \quad \frac{\mathcal{N}_R(\mathfrak{W}_S)}{\mathcal{N}_R(\mathfrak{W}_0)} \leq 1. \tag{32b}
\]

For supp(\(\hat{u}\)) \(\subset\mathbb{R}^n-\mathbb{O}_0\), or even for supp(\(\hat{u}\)) \(\cap\)(\(\mathbb{R}^n-\mathbb{O}_0\)) \(\neq\) \(\emptyset\), it is often the case in applications that \(\mathfrak{W}_0\) is not defined. Since \(\mathfrak{W}_s\) is defined for all \(\mathfrak{W}\) it makes sense in such cases to consider \(\mathfrak{W}_s \geq \mathfrak{W}_0\).

5. EXAMPLES: OPTIMAL DECONVOLVERS FOR CHARACTERISTIC FUNCTIONS OF SETS IN \(\mathbb{R}^n\)

Collections of sets in \(\mathbb{R}^n\) such that the characteristic functions of the sets in the collection are strongly coprime have been described in Section 3.

Let \(\chi_S\) denote the characteristic function of a set \(S\). Consider the case in which the noise power spectral density has the form \(\|\chi_S\|_{L^1}^2\), where \(\|\chi_S\|_{L^1}\) is the \(L^1\) norm of \(\chi_S\) (equivalently, the Lebesgue measure of the set). For such a case, let sets \(S_1,S_2,...,S_m\) be chosen so that, for \(\mu_i = \chi_{S_i}\), the \(\mu_1,\mu_2,...,\mu_m\) are strongly coprime. Then, from the Proposition in the previous section,

\[
D_1(\omega) = \frac{\hat{\mu}_1(\omega)}{|\mu_1|_1} \quad \text{and} \quad \text{(25) becomes}
\]

\[
D_1(\omega) = \sum_{j=1}^{m} \frac{|\hat{\mu}_j(\omega)|^2}{|\mu_j|_1^2}
\]

and (25) becomes

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Let $S_0$ be any set, let $\mu_0 = \chi_{S_0}$ be its characteristic function, and consider this to be the convolver in $L_0$ defined by (10) (i.e., $m$ parallel, identical convolvers). Let the noise power spectral density have the same form as above, $N_0^2 = \|\mu_0\|_1^2 N_0^2$. From (27) and (28) one obtains an envelope transfer function $\hat{e}_d$ and the associated comparison for these two: a convenient renormalization by the constant $\|\mu_0\|_1^{1/2}$ is made in

$$\hat{e}_d(\omega) = \frac{\hat{e}(\omega)}{\|\mu_0\|_1^{1/2}} = \left( \sum_{i=1}^{m} \frac{|\hat{\mu}_i(\omega)|^2}{\|\mu_i\|_1} \right)^{1/2} \geq \sqrt{m} \frac{|\hat{\mu}_0(\omega)|}{\|\mu_0\|_1^{1/2}}.$$  

For an explicit example let $S_1 \subset \mathbb{R}^2$ be the region in a focal plane of an imaging device which corresponds to a single light sensitive detector. The exposure time interval is assumed fixed and the image is assumed constant. Then $\mu_1 = \chi_{S_1}$ is the idealized response function of the detector. (The actual shape of the response function can be incorporated into the mollifier $\phi$ of Fig. 3. That is, if $\alpha_1$ is the actual detector response function, and if the deviation of $\alpha_1$ from $\mu_1$ is due to, say, a material diffusion process that is common to all detectors, then $\alpha_1 = \mu_1 \ast \psi$, where $\psi$ models the diffusion. Such a common $\psi$ would not be deconvolved; one would use $\sum_{i=1}^{m} \mu_i \ast \psi \nu_1 = \psi$.) Then $\mu_1$ is what is referred to as the "detector MTF." The form of the noise power spectral density above is that of many noise processes in electro-optical detectors, that is, that the noise power spectral density is proportional to the area of the detector. Such is the case for infrared detectors and this proportionality is contained in the definition of the fami-
liar "D." The above form is also valid for the so-called background limited case. It also has this form for $\mathbb{R}^3$ when the time interval is included as the third dimension. Further, a background limited slit detector corresponds to the above forms for $\mathbb{R}^1$ with the slit width as the coordinate. (The background limited case consists of a relatively small signal of interest superimposed on a relatively large constant signal so that the noise in the signal of interest is due to the "shot" noise of the constant signal.) (For detector characteristics discussed above see, for example, [20, Ch.2].)

In Fig. 5, Fig. 6, and Fig. 7 the transfer functions for such cases are shown. In Fig. 5, a comparison is shown for the example in $\mathbb{R}^1$ described in the Introduction. The characteristic functions $\mu_1$ and $\mu_2$ for the two intervals $(-1,1)$ and $(-\sqrt{2},\sqrt{2})$, respectively, are strongly coprime. The envelope transfer function $\hat{e}_d$ is shown and is compared with the transfer function for the two identical, parallel convolvers as in (35) where $\mu_0 = \mu_1$. The choice $\mu_0 = \mu_1$ is used rather than $\mu_0 = \mu_2$ in this comparison because $\mu_1$ is "better" than $\mu_2$ in the sense that the first zero of $\hat{\mu}_1$ (i.e., its bandwidth) is greater than the first zero of $\hat{\mu}_2$. Recall from the scaling property for Fourier transforms on $\mathbb{R}^1$ that $\mu_1(x) = \mu_2(\sqrt{2}x)$ for all $x \in \mathbb{R}^1$ if and only if $\sqrt{2}\hat{\mu}_1(\sqrt{2}x) = \hat{\mu}_2(x)$ for all $x \in \mathbb{R}^1$. In making the comparison we use two (i.e., $m = 2$) identical parallel convolvers rather than only one so as not to bias the comparison in favor of the strongly coprime case. In terms of the applications to electro-optics, the $m$ strongly coprime detectors must make $m$ sequential measurements of the optical signal, and the noise for each measurement is assumed independent. Consequently, there is an inherent noise averaging due to these $m$ measurements, and we wish to have the number of independent measurements the same for both cases. The case of sequential measurement with identical detectors for noise averaging is exactly
what is done in so-called "time delay integration" for scanned infrared
detector arrays [21]. Fig. 5 illustrates the consequence of the strongly
coprime condition: the envelope response is approximately an envelope for the
modulus of the other two responses and, correspondingly, is without zeroes.
Also, it can be observed that the envelope response decreases approximately as
$1/|\omega|$.

In Fig. 6 and Fig. 7 the envelope transfer function is shown for an
example in $\mathbb{R}^2$, the case of three squares $Q_1, Q_2, Q_3$ of side length $1, \sqrt{2}, \sqrt{3}$,
respectively. The characteristic functions of these three squares are strongly
coprime. The comparison (35) is illustrated by graphing the modulus of the
corresponding transfer functions for two subsets of $\mathbb{R}^2$: the $\omega_1$-axis
$\{\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_2 = 0\}$ (see Fig. 6) and the diagonal $\{\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 :
\omega_1 = \omega_2\}$ (see Fig. 7). All graphs use the Euclidean distance as abscissa,
$|\omega| = (\omega_1^2 + \omega_2^2)^{1/2}$. The comparison illustrated in Fig. 6 and Fig. 7 is for
$\hat{\mu}_0 = \chi_{Q_1}$. (As before, $\chi_{Q_1}$ has the greatest bandwidth and the scaling prop-
erty for $\mathbb{R}^n$ has the form $\mu_1(x) = \mu_2(kx)$ for $k > 0$ and for all $x \in \mathbb{R}^n$
if and only if $k^n \hat{\mu}_1(k \omega) = \hat{\mu}_2(\omega)$ for all $\omega \in \mathbb{R}^n$.) The comparison is
essentially the same as that for the two intervals in $\mathbb{R}^1$. The difference
between the $\omega_1$-axis and the diagonal illustrates that approximately the
envelope response decreases as $|\omega|^{-1}$ along the $\omega_1$-axis and as $|\omega|^{-2}$ along
the diagonal.

From (35) (and as illustrated by the figures) the following statements
can be made. These are stated as "observations" because the results can not
be given in terms of explicit inequalities. Some notation is helpful. Define
\begin{equation}
\Omega_1 = \{\omega \in \mathbb{R} : \forall t \in [0, 1] \ |\mu_1(t \omega)| > 0 \} \text{ and } \Omega = \bigcap_{i=1}^m \Omega_1. \tag{36}
\end{equation}

**Observations.** Let $\mu_1, \mu_2, \ldots, \mu_m \in L^1(\mathbb{R}^n)$ be strongly coprime characteristic
functions of sets in \( \mathbb{R}^n \) as considered above. With each \( \mu_1 \) let there be associated as in Fig. 3 an additive wide-sense stationary noise with noise power spectral density of the form \( \| \mu_1 \| N_2^2 \). Let \( L_s \) be the configuration in Fig. 3 with deconvolvers determined by the Proposition. Let \( L_0 \) be the trivial configuration as in (10) with \( \mu_0 = \mu_1, N_0 = N_1 \).

Observation 1: For \( u \) with \( \text{supp}(u) \subset \Omega, \). \( u_s = u_0 \).

Observation 2: For \( u \) with \( \text{supp}(u) \subset \mathbb{R}^n - \bigcup_{i=2}^m \Omega_i, \), \( u_s \geq u_0 \).

Observation 3: For \( u \) with \( \text{supp}(\hat{u}) \subset \bigcup_{i=2}^m \Omega_i - \Omega, \), \( u_s \leq u_0 \).

Observation 4: For \( u \) with \( \text{supp}(\hat{u}) \) compact, \( \text{supp}(\hat{u}) \subset \Omega_1, \) let \( \tilde{\mu}_0 \) be the boost on \( u, \). \( u_0 \rightarrow u_0^{-1} \) (see (31) and (32)). If Observation 3 can be neglected then

\[
\frac{\gamma NR(u_0^{-1}L_0)}{\gamma NR(u_0)} \geq 1 \implies \frac{\gamma NR(u_s)}{\gamma NR(u_0)} \geq 1.
\]

As discussed at (31) it is not possible to extend this to all of \( \Omega_1 \), for \( \mu_1 = 0 \) on the boundary of \( \Omega_1 \). However, it still is desirable to have a means to compare \( L_s \) with the more well known, more thoroughly studied trivial configurations. In the next section this is accomplished by pushing the troublesome set \( \{ \tilde{\mu}_1 = 0 \} \) out toward infinity.

6. MORE COMPARISONS: STRONGLY COPRIME VERSUS CHANGE OF SCALE

Let \( L_s \) be the same as above. In the above \( L_s \) was compared with \( L_0 \), where \( L_0 \) was chosen to be \( \mu_1 \) and \( N_1^2 = \| \mu_1 \| N_2^2 \). In these cases \( \mu_1 \) was the "best" in the sense \( \Omega_1 \subset \Omega_i, i = 1, 2, \ldots, m \). Here \( L_s \) will be compared with a one parameter family of such \( L \). Define \( L_0 \) by the trivial configura-

2.27
tion of a parallel, identical \( \mu_{<\omega>} \) as in (10), where \( N_\omega^2 = |\mu_{<\omega>}|_1 N_\omega^2 \) and \( \mu_{<\omega>}(x) = \mu_1 \left( \frac{x}{\omega} \right), \omega > 0. \)

The primary result of this section is

**Observation for Fixed Number Of Channels.** Fix the number of parallel convolvers in both \( L_s \) and \( L_\omega \) to be \( m. \) Let the convolvers be characteristic functions of cubes on \( \mathbb{R}^n \) and let the additive noise be as above. Assume that \( U \) is such that \( \text{supp}(u) \subseteq \bigcup_{j=1}^m \{ \omega \in \mathbb{R}^n : \omega_j = 0, j \neq j \}. \) Then for \( n \geq 2 \)

\[
\forall L_s \geq \forall L_\omega \quad \text{for all} \quad 0 < \alpha \leq 1. \tag{40}
\]

**Corollary to Observation.** For the conditions in the Observation above, it is advantageous to construct \( L_s \) using sets that are as large as possible.

**Application of the Corollary.** In parallel scanned imaging systems with square detectors wherein the systems are ranked using some \( U \) meeting the conditions of the Observation (e.g., horizontal or vertical bars), the detector size should be sufficiently large so that the array of detectors fills the image, and the detector sizes in the array should constitute a strongly coprime collection. (This application depends on sufficiently high sampling rates. See Section 1.)

The Observation is illustrated in Fig. 8 for \( n = 2. \) For Fig. 8 \( L_s \) is as in Fig. 6: in the notation just above \( L_s \) is configured from the parallel convolvers \( \mu_{<1>}, \mu_{<\sqrt{2}>}, \mu_{<\sqrt{3}>}, \) and \( \mu_1 \) is the characteristic function of the unit square. For this \( L_s \) the envelope transfer function \( \hat{e}_d \) is compared with \( \sqrt{3} \frac{|\mu_{<\omega>}|}{\|\mu_{<\omega>}\|_1}^{1/2} \), as in (35), for \( \omega = 1, 0.5, 0.2, \) and \( 0.1. \) For \( \omega = 1 \)

see Fig. 6; for \( \omega = 0.5, 0.2, \) and \( 0.1, \) see Fig. 8. The observation in (40) is clearly evident. (Here we neglect Observation 3 of the last section by
means of a broad interpretation of \( \omega \) in Observation 1.)

The Observation (40) depends on the following properties. The first, which is again an approximation, is that for \( A_j = \{ \omega \in \mathbb{R}^n : \omega_1 = 0, 1 \neq j \} \), the \( \omega_j \)-axis,

\[
\hat{\varepsilon}_d |_{A_j}(\omega) \leq C|\omega|^{-1}. \tag{41}
\]

The second is that for \( n \geq 2 \), for \( \omega \in A_j \),

\[
\frac{|\hat{\mu}_{\omega}(\omega)|}{\|\mu_{\omega}\|_1^{1/2}} \leq C|\omega|^{-1} \Rightarrow \frac{\sqrt{m}}{\|\mu_{\omega}\|_1^{1/2}} \leq C|\omega|^{-1}.
\]

Fig. 9 and Fig. 10 show two counterexamples for cases not addressed in the Observation. Fig. 9 is for the case of the diagonal in \( \mathbb{R}^2 \), and Fig. 10 is for \( n = 1 \). The Observation fails on the diagonal \( D \) = \( \{ \omega = (\omega_1, \omega_2) : \omega_1 = \omega_2 \} \) because

\[
\hat{\varepsilon}_d |_{D}(\omega) \leq C|\omega|^{-2}. \tag{42}
\]

It fails for \( \mathbb{R} \) because (41) holds.

If in place of characteristic functions of cubes one uses characteristic functions of disks on \( \mathbb{R}^2 \), then the relationship between \( \hat{\varepsilon}_d \) and \( \frac{\sqrt{m}|\hat{\mu}_{\omega}|}{\|\mu_{\omega}\|_1^{1/2}} \) is intermediate between that of the \( \omega_j \)-axis and that of the diagonal for

\[
\hat{\varepsilon}_d (\omega) \leq C|\omega|^{-3/2}. \tag{43}
\]

The significance of Observation (40) is that it provides a qualitative lower bound for the performance of the strongly coprime configuration. To the extent performance is characterized for the \( M_\alpha \), the "envelope" consisting of the collection over all \( \alpha \) is a lower bound for the performance of \( M_\mathbf{s} \).

All of the above has focused on performance away from the origin. If the
I figures are rescaled so that the $\mu_{\alpha}$ appear fixed with a sequence of $L_\alpha$ constructed from convolvers of increasing support, the observation indicates that nothing is sacrificed away from zero while the envelope transfer function near zero is substantially increased. That is, $\mu_{L_\alpha} \geq \mu_{\alpha}$ represents a substantial enhancement near $\omega = 0$, not merely approximately identical performance. On the other hand, this uniform improvement is for the case of $U$ supported by the axes. For the cases off the axes for cubes and for the case of disks there is a trade-off between some loss away from zero and the gain near zero.

APPENDIX: REVIEW OF DISTRIBUTIONS OF COMPACT SUPPORT

We present here a short review of some properties of distributions of compact support in $\mathbb{R}^n$ (sometimes called generalized functions). The implicit reference throughout this Appendix is the very clear monograph [22].

The simplest example of a distribution of compact support is obtained from a continuous function $F$ of compact support. (That is, $F$ is zero outside a bounded set in $\mathbb{R}^n$.) One associates to $F$ a scalar valued linear operator $T_F$ acting on infinitely differentiable functions $f$ by the rule

$$\langle T_F, f \rangle := \int_{\mathbb{R}^n} F(x)f(x)dx.$$ 

The fact that $\text{supp } F$ is bounded is what makes the above integral finite and well defined for an arbitrary $f$. This linear operator $T_F$ is called a distribution of compact support.

If the function $F$ were not only continuous but also continuously differentiable, we can similarly associate a distribution to each of its partial
derivatives. Their relationship to $T_F$ is given by integration by parts:

$$\langle T_{\frac{\partial F}{\partial x_j}} f \rangle = \int_{\mathbb{R}^n} \frac{\partial F}{\partial x_j} f \, dx = -\int_{\mathbb{R}^n} F \frac{\partial f}{\partial x_j} \, dx = -\langle TF, \frac{\partial f}{\partial x_j} \rangle.$$ 

This motivates defining the derivative $\frac{\partial}{\partial x_j} T_F$ of the distribution $T_F$ by the formula

$$\langle \frac{\partial}{\partial x_j} T_F, f \rangle := -\langle T_F, \frac{\partial f}{\partial x_j} \rangle,$$

which does not require that $F$ be differentiable. One defines in the same way more distributions $D^\alpha T_F = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} T_F$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, by

$$\langle D^\alpha T_F, f \rangle := (-1)^{|\alpha|} \langle T_F, D^\alpha f \rangle.$$

Further, for any pair of continuous functions $F, G$ of compact support, and any multiindices $\alpha, \beta$, the operator $D^\alpha T_F + D^\beta T_G$

$$\langle D^\alpha T_F + D^\beta T_G, f \rangle := \langle D^\alpha T_F, f \rangle + \langle D^\beta T_G, f \rangle$$

is a linear operator on the space of infinitely differentiable functions.

The collection of all finite sums of this kind is precisely the space of distributions of compact support in $\mathbb{R}^n$. That is, for a distribution of compact support $\mu$ we mean a scalar valued linear operator which can be represented in the form

$$\mu = \sum_{|\alpha| \leq N} D^\alpha T_{F_\alpha},$$

for some integer $N$ and continuous functions $F_\alpha$ of compact support ([22], p. 74)).
Let us indicate how the unit impulse $\delta_a$ at a point $a \in \mathbb{R}$ can be written in the above form. Let

$$L(x) = \begin{cases} 0 & \text{if } x < a \\ x-a & \text{if } x \geq a \end{cases}$$

and let $\phi$ be any twice continuously differentiable function of compact support such that for some $\epsilon > 0$ $\phi(x) = 1$ for $a - \epsilon \leq x \leq a + \epsilon$. Then

$$(\phi L)'' = \phi''L' + \phi L''$$

and the function $\phi'L'$ is continuous in spite of the discontinuity of $L'$ at $x = a$ since $\phi' \equiv 0$ near $a$. Letting $F = \phi L$ and $G = -(\phi''L + 2\phi'L')$ we get

$$\delta_a = \frac{d^2}{dx^2}TF + TG.$$

The space of infinitely differentiable functions in $\mathbb{R}^n$ is usually denoted by $\mathcal{E}$ (or $\mathcal{E}(\mathbb{R}^n)$ or $C^\infty(\mathbb{R}^n)$) and the space of distributions of compact support by $\mathcal{E}'$ (or $\mathcal{E}'(\mathbb{R}^n)$).

We can essentially work with distributions of compact support as if they were ordinary continuous functions of compact support. In particular, for a distribution $T \in \mathcal{E}'$ it makes sense to compute its Fourier-Laplace transform $\hat{T}(\zeta)$ for $\zeta \in \mathbb{C}^n$. Namely, let $\zeta \cdot x = \zeta_1 x_1 + \ldots + \zeta_n x_n$ ($x \in \mathbb{R}^n$, $\zeta \in \mathbb{C}^n$), then

$$\hat{T}(\zeta) := \langle T, e^{-i\zeta \cdot x} \rangle,$$

where $i = \sqrt{-1}$. For instance, for the unit impulse $\delta_a$ at a point $a \in \mathbb{R}^n$ we have $\hat{\delta}_a(\zeta) = e^{-i\zeta \cdot a}$.

From the above representation of the distributions of compact support one can see that the function $\zeta \mapsto \hat{T}(\zeta)$ has two properties. First, it is an analytic function in the whole of $\mathbb{C}^n$ (i.e., an entire function). Second, there are positive constants $A, B, N$ such that $\hat{T}$ satisfies everywhere the estimate

$$|\hat{T}(\zeta)| \leq A e^{-B|\zeta|^N}.$$
\[ |\hat{T}(\zeta)| \leq A(1+|\zeta|)^N e^{B|\Im \zeta|}. \]

\( \Im \zeta = (\Im \zeta_1, \ldots, \Im \zeta_n). \) It is usual to call the space of functions satisfying these two properties the Paley-Wiener class, \( \PW(\mathbb{C}^n). \)

The classical Paley-Wiener theorem for functions of compact support in \( L^2 \) can be extended to the case of distributions of compact support as follows.

**Theorem (Paley-Wiener-Schwartz).** The Fourier-Laplace transform is a one-to-one correspondence between the spaces \( \mathcal{E}'(\mathbb{R}^n) \) and \( \PW(\mathbb{C}^n). \)

Recall now that if \( F \) and \( G \) are continuous functions of compact support then their convolution \( F \ast G \) is given by

\[
F \ast G(x) = \int_{\mathbb{R}^n} F(x-y)G(y)dy = \int_{\mathbb{R}^n} G(x-y)F(y)dy
\]

and it is again a continuous function of compact support. Moreover, if one of them is continuously differentiable, say \( F \), then \( \frac{\partial}{\partial x_j} (F \ast G) = \left( \frac{\partial F}{\partial x_j} \right) \ast G. \) This observation allows for the definition of convolution of distributions of compact support in such a way that if \( T, S \in \mathcal{E}' \) then \( T \ast S = S \ast T \in \mathcal{E}'. \) The unit impulse at the origin \( \delta \) acts as the unit of this product \( T \ast \delta = T. \) Furthermore, \( (T \ast S)(\zeta) = \hat{T}(\zeta)\hat{S}(\zeta), \) so that the convolution becomes ordinary product of the analytic functions \( \hat{T} \) and \( \hat{S}. \)

In this paper we were interested in solving the equation

\[
\mu_1 \ast \nu_1 + \ldots + \mu_m \ast \nu_m = \delta,
\]

where \( \mu_1, \ldots, \mu_m \) are given in \( \mathcal{E}' \) and we need \( \nu_1, \ldots, \nu_m \in \mathcal{E}'. \) Using the Fourier-Laplace transform this equation is equivalent to the analytic Bezout equation.
\[ \hat{\mu}_1(\zeta)f_1(\zeta) + \ldots + \hat{\mu}_m(\zeta)f_m(\zeta) = 1, \]

with unknowns \( f_1, \ldots, f_m \in \text{PW}(\mathbb{C}^n) \). In particular, to solve this equation is necessary that the functions \( \hat{\mu}_1, \ldots, \hat{\mu}_m \) have no common zeros in \( \mathbb{C}^n \).

In the text we raised the question of whether one could find a deconvol-\( \ldots \)
er \( \nu \in \mathcal{E}' \) for a single convolutor \( \mu \in \mathcal{E}' \), i.e.,

\[ \mu * \nu = \delta \text{ or equivalently } \hat{\mu}(\zeta)\hat{\nu}(\zeta) = 1. \]

We need that \( \mu(\zeta) \neq 0 \) for every \( \zeta \in \mathbb{C}^n \). It follows that there is an analytic function \( h \) in \( \mathbb{C}^n \) such that

\[ \hat{\mu}(\zeta) = e^{h(\zeta)}. \]

Since \( \hat{\mu} \) is in the Paley-Wiener class one can show without difficulty that

\[ |h(\zeta)| \leq A + B|\text{Im } \zeta| \]

for some positive constants \( A, B \). Therefore, by the Liouville theorem, \( h \) must be of the form

\[ h(\zeta) = -i(a_1 \zeta_1 + \ldots + a_n \zeta_n) + c \]

for some \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n, \ c \in \mathbb{C} \). This proves, by the the Paley-Wiener-Schwartz theorem, that \( \mu = C\delta_a \), for some constant \( C \in \mathbb{C} \).

Having no common zeros is not enough to solve the analytic Bezout equation, the necessary and sufficient condition is the following.

**Theorem** [1], [16]. Given \( \mu_1, \ldots, \mu_m \in \mathcal{E}'(\mathbb{R}^n) \), the necessary and sufficient condition for the existence of deconvolvers \( \nu_1, \ldots, \nu_m \in \mathcal{E}'(\mathbb{R}^n) \) is that there are constants \( \varepsilon > 0, M > 0, C > 0 \) such that

\[ \left( \sum_{j=1}^{m} |\hat{\mu}_j(\zeta)|^2 \right)^{1/2} \geq \varepsilon e^{-C|\text{Im } \zeta|} \frac{1}{(1 + |\zeta|)^M} \text{ for all } \zeta \in \mathbb{C}^n. \]

2.34
This is a purely existential theorem and the work in [5] - [10] consists in obtaining explicit choices of deconvolvers \( \nu_1, \ldots, \nu_m \). This is not trivial. A portion of the work [7] and [11] also consists in finding simple and practical examples of \( \mu_1, \ldots, \mu_m \) that satisfy the above necessary and sufficient condition for the existence of deconvolvers.
REFERENCES


Figure 1. (a) Multiple parallel linear operators with distinct distributions $\mu_1$. Single operator (b) and the multiple parallel operators with identical distributions $\mu_0$ (c).
Fig. 3. Multiple operator configuration consisting of $m$ parallel convolvers $\mu_1, \mu_2, \ldots, \mu_m$, $m$ noise signals $\eta_1, \eta_2, \ldots, \eta_m$, and $m$ deconvolvers $\nu_1, \nu_2, \ldots, \nu_m$. 
Figure 6  $R^2$; abscissa = $\omega_1$-axis  

solid = $\hat{\epsilon}_d$; broken: $\sqrt{3} \frac{|\hat{\mu}_0|}{|\mu_0|^{1/2}}$, $\hat{\mu}_0 = \hat{\chi}_0$.
Figure 7  \( R^2 \); abscissa = diagonal  solid = \( \hat{c}_d \); broken: \( \sqrt{3} \frac{|\hat{\mu}_0|}{|\mu_0|^{1/2}} \), \( \hat{\mu}_0 = \hat{\chi}_0 \).
Figure 8  \( r^2 \); abscissa = \( \omega \)-axis  

solid: \( \hat{\varepsilon}_d \); broken: \( \sqrt{3 \frac{|\hat{\mu}_{<\alpha>}|}{\|\mu_{<\alpha}>\|_{1}^{1/2}}} \), \( \alpha = 0.5, 0.2, 0.1 \)
CHAPTER 3

A local version of the two-circles theorem

by

C.A. Berenstein and R. Gay
1. Introduction.

One of the oldest questions in integral geometry has been that of recovering a function $f$ in $\mathbb{R}^n$ from the knowledge of its average over balls. It is easy to see that unless $f$ decays sufficiently fast at infinity the average over all balls of a fixed radius could vanish without $f$ being identically zero. It is not always possible to assume such decay but a very elegant result of Zalcman [20] and, independently, Brown-Schreiber-Taylor [10], describes explicitly a countable set $E_n$ such that averages over all balls of radii $r_1$, $r_2$ suffice as long as $r_1/r_2 \notin E_n$. This "two circles" theorem can be described as saying that the map

$$C(\mathbb{R}^n) \to C(\mathbb{R}^n) \oplus C(\mathbb{R}^n)$$

$$f \to (\int_{B(x,r_1)} f(y) dy, \int_{B(x,r_2)} f(y) dy)$$

is injective if and only if $r_1/r_2 \notin E_n$.

$$(B(x,r) = \{y \in \mathbb{R}^n : |x-y|<r\})$$. Under slightly stronger conditions on the quotient $r_1/r_2$ this map has also a continuous and explicit inverse [8]. This result and other variants of the so-called Pompeiu problem have been generalized to symmetric spaces 'see the surveys [21], [1] for positive results and their limitations).

In practical situations of a tomographic nature one is limited to balls that fit into a fixed region $\Omega$. One could take smaller and smaller balls when approaching the boundary $\partial \Omega$ of $\Omega$, this is roughly the situation when we consider the case...
Ω = unit ball of \( \mathbb{R}^n \) as the hyperbolic space, but it is clear that it might be hard to accomplish if we are dealing with physical devices whose size cannot be made infinitesimally small or cannot even be changed at will. It is this kind of problem that we call a local version of the two-circles theorem. The main difference with the above mentioned results is that we do not have any longer the whole group of Euclidean motions at our disposal which was the crucial ingredient lying behind the two circles theorem and its generalizations. The inversion formula of [8] would allow us to reconstruct \( f \) away from \( \partial \Omega \) but gives no indication of whether we could change the values of \( f \) in a collar-like region near \( \partial \Omega \) without affecting its average.

There is some recent work on systems of convolution equations in convex domains which deals with this type of question [4] but the hypotheses required are far too restrictive to be satisfied by our simple looking problems. Nevertheless, using a combination of ideas from classical harmonic analysis and results of Cormack-Quinto on the Radon transform on spheres [12] we are able to prove the following.

**Theorem.** - Let \( r_1, r_2 \) be positive numbers, \( r_1/r_2 \notin \mathbb{E}_n \), \( \Omega \) an open subset of \( \mathbb{R}^n \) such that every point lies in an open ball contained in \( \Omega \) of radius strictly larger than \( r_1 + r_2 \). If \( f \in C(\Omega) \) satisfies

\[
\int_{B(x,r_j)} f(y)dy = 0 \quad \text{for every} \quad B(x,r_j) \subseteq \Omega, \ j = 1,2
\]
then $f \equiv 0$. Furthermore, this statement does not hold if $\Omega$ fails the above geometrical restriction.

The method of proof allows us to generalize this theorem greatly, providing in particular new local mean-value theorems for harmonic functions.

We will like to express our appreciation to Professor L. Zalcman who called our attention to these problems.

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2. Preliminaries.

We will follow the standard notation for distributions found in [14]. We denote $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r \} (r > 0)$, $\overline{B}(x,r)$ its closure and $\chi_r$ the characteristic function of $B(0,r)$. Let $\Omega$ be an open set in $\mathbb{R}^n$, $\Omega_r = \{x \in \Omega : d(x, \Omega^c) < r\}$. For a locally-integrable function $f$ in an open set $\Omega$ the average

\[ A_r(f,x) = \frac{1}{\omega_r^m} \int_{B(x,r)} f(y) dy \]

is defined for $x \in \Omega_r$. Here $\omega_n$ is the volume of $B(0,1)$. If we let $\nu_r = \chi_r / \omega_r^m$, we can interpret this average as a convolution and hence it makes sense to define it for $f \in \mathcal{D}'(\Omega)$ giving a distribution $A_r(f)$ in $\mathcal{D}'(\Omega_r)$, namely

$A_r(f) = f * \nu_r$. Therefore, for uniqueness questions, if the averages of $f$ are zero, by restriction ourselves to $\Omega_\varepsilon, \varepsilon > 0$ small, we can assume $f \in C^\varepsilon$. Henceforth, all distributions with vanishing averages will be assumed to be $C^\varepsilon$ functions in $\Omega$. 


For $r > 0$, we denote by $\sigma_r$ the distributions defining the spherical average 

$$\lambda_r(f,x) = \int_{S^{n-1}} f(x+ry) d\sigma(y) = (\sigma_r * f)(x)$$

do is the normalized Lebesgue measure on $S^{n-1}$.

For $T \in E'$ the Fourier transform

$$\hat{T}(\zeta) = \langle T_x, e^{-i(x|\zeta)} \rangle, \quad (x|\zeta) = \sum_j x_j \zeta_j,$$

is an entire function in $\mathbb{C}^n$ which satisfies, for some $A, N > 0$, the estimates

$$|\hat{T}(\zeta)| < A(1+|\zeta|)^N \exp(H(\text{Im}\zeta)).$$

where $\zeta = \xi + in$, $\xi, n \in \mathbb{R}^n$, $\text{Im}\zeta = n$ and $H$ is the supporting function of the support of $T$, i.e.: 

$$H(n) = \text{Max}\{(x|n): x \in \text{supp}T\}$$

Note that $H$ is also the supporting function of $\text{cv}(\text{supp}T)$, the convex hull of $\text{supp}T$. The Fourier transform is an isomorphism between the convolution algebra $E'(\mathbb{R}^n)$ and $\hat{E}'(\mathbb{R}^n)$, the algebra of entire functions of exponential type and polynomial growth on the real axis.

A distribution $T$ will be called invertible (or $T$ is slowing decreasing) if whenever $S \in E'(\mathbb{R}^n)$ and $S/T$ is an
entire function, then there is a distribution \( U \in \mathcal{E}'(\mathbb{R}^n) \) such that \( \hat{U} = \frac{\hat{S}}{\hat{T}} \), that is

\[
S = T * U
\]

and we have the identity

\[
H_S = H_T + H_U
\]

or, what amounts to the same thing

\[
cv(\text{supp}U*T) = cv(\text{supp}U) + cv(\text{supp}T),
\]

where, for two sets \( A, B \subseteq \mathbb{R}^n \) we have \( A \pm B = \{ x \pm y; x \in A, y \in B \} \).

We will need to use that \( \hat{\mu_T} \) is an invertible distribution. This will follow from the explicit formula for \( \hat{\mu_T} \) given below and the characterization of invertible distributions: \( T \) is invertible if and only if there is a positive constant \( a \) such that for all \( \xi \in \mathbb{R}^n \)

\[
\text{Max}\{|T(\xi+n)|; n \in \mathbb{R}^n, |n| < a \log(2+|\xi|)} > (a + |\xi|)^{-a}
\]

The Fourier transform of a radial distribution \( T \) is a radial function, i.e.: if:

\[
\langle T, foA^{-1} \rangle = \langle T, f \rangle
\]
for every $A \in O(n)$ then

$$\hat{T}(\xi) = \hat{T}(A \cdot \xi)$$

for every $A \in O(n)$, $\xi \in \mathbb{C}^n$, and depends, for $\xi \in \mathbb{R}^n$, only on $|\xi|$. Hence we consider the associated even entire function $\tilde{T}$ of one variable by

$$(8) \quad \tilde{T}(|\xi|) = \hat{T}(\xi) \quad \text{and} \quad \tilde{T}(\xi) = \tilde{T}(\sqrt{\xi_1^2 + \ldots + \xi_n^2})$$

Let us call $E'_o(\mathbb{R}^n)$, the space of radial distributions. This correspondence establishes an isomorphism between the algebras $\hat{E}'_o(\mathbb{R}^n)$ and $\hat{E}'_o(\mathbb{R})$. Using this notation we have

$$(9) \quad \tilde{\sigma}_r(t) = n^{-\frac{n-2}{2}} \frac{\Gamma(n)}{\frac{1}{2}} \frac{J_n (rt)}{(rt)^{n/2}}$$

$$(10) \quad \tilde{\sigma}_r(t) = 2^{-\frac{n-2}{2}} \frac{\Gamma(n)}{\frac{1}{2}} \frac{J_n (rt)}{(rt)^2}$$

and, more generally, if $f(x) = \phi(|x|)$ is a radial function of compact support

$$(11) \quad \hat{f}(\xi) = \tilde{f}(t) = \frac{(2\pi)^{n/2}}{t^{\frac{n-2}{2}}} \int_0^\infty \phi(p) \frac{J_{n-2}(pt)p^{n/2}}{2} dp \quad (|\xi|=t)$$

To show that $\tilde{\sigma}_r$ is invertible it is now sufficient to recall the asymptotic development of the Bessel functions [19] on the positive real axis.
(12) \[ J_\nu(t) = \sqrt{\frac{2}{\pi}} t^{-\nu/2} \cos(t-\frac{\pi\nu}{4}) + O(t^{-3/2}) \]

It follows, for \( |\xi| > 1 \) and some \( C > 0 \)

\[
\max \{|\hat{u}_\nu(\xi+n)| : n \in \mathbb{Z} \cap \mathbb{R}, |n| < \pi \} \leq C|\xi|^{-\frac{n+1}{2}}
\]

which is the condition of invertibility.

From (12) we also obtain MacMahon's asymptotic development of the positive zeros \( a_k, \nu \) of \( J_\nu \)

\[ 0 < a_1, \nu < a_2, \nu < \ldots \]

(13) \[ a_k, \nu = (2k+1)\frac{\pi}{2} + (2\nu+1)\frac{\pi}{4} + O(1/k) \]

which will be used further on.


Let \( \Omega \) be an open convex set in \( \mathbb{R}^n \) and \( K = \text{cv}(\text{supp}\mu) \), \( \mu \in E'(\mathbb{R}^n) \). We say that a function \( f \in C^\infty(\Omega-K) \) is mean-periodic with respect to \( \mu \) if

(14) \[ \mu*f(x) = \langle \mu, f(x-y) \rangle = 0 \quad \text{for all} \quad x \in \Omega \]

If an exponential-polynomial, that is a finite linear combination of terms of the form \( x^j e^{i(x|\xi)} \)
\( (x^j = x_1^j \ldots x_n^j, j_k \in \mathbb{N}, 1 < k < n) \), is mean-periodic with respect to \( \mu \) then the frequencies \( \zeta \) must satisfy
\[
\hat{\mu}(\zeta) = 0 \quad \text{since}
\]
\[
(15) \quad (\mu \ast e^{i(\cdot|\zeta)}) (x) = \hat{\mu}(\zeta) e^{i(x|\zeta)}
\]

When the zeros of \( \hat{\mu} \) are simple no non constant monomials can appear. More generally if a monomial \( x^j \) appears with non zero coefficient then
\[
\left( \frac{\partial |j|}{\partial \zeta^j} \right) \hat{\mu}(\zeta) = 0
\]
for the corresponding frequency \( \zeta \).

For \( n = 1 \) there is a well-known series development for such functions in terms of the exponential polynomial solution of the same convolution equation (14) due to L. Schwartz [18], [15], [13]. The case of interest for us is \( n > 2, \mu \) invertible. In this case, a development in terms of integrals over the zero set of \( \hat{\mu} \) has been proved when \( \Omega = \mathbb{R}^n \) [6]. For \( \Omega \) arbitrary convex set, a similar development has been proved in [4] but only for a very restrictive class of invertible distributions. In all these cases one obtains also some knowledge of the behavior of the terms involved in this development. Unfortunately, the distributions \( \mu_r \), though invertible, do not satisfy the conditions required in [4], as was shown (for a different reason) in [3]; moreover we are interested in \( \Omega = B(0, R) \). Therefore we cannot depend on any of the previously known
results. We obtain here a series development without additional information on the coefficients that appear in it; nevertheless the existence of this development is all we need later.

**Proposition 1** - Let \( \Omega \) be an open convex subset in \( \mathbb{R}^n \) \((n>2)\), \( \mu \in \mathcal{E}'(\mathbb{R}^n) \) an invertible distribution, 
\( K = cv(\text{supp}\mu) \). Any function \( f \in C^\infty(\Omega-K) \), mean periodic with respect to \( \mu \) can be written as

\[
\sum_{j>1} p_j(x) \quad (x(\Omega-K))
\]

with \( p_j \) exponential-polynomials also mean-periodic with respect to \( \mu \), and the series is convergent in the \( C^\infty \)-topology of \( \Omega - K \). Furthermore, given a sequence \( (s_j)_{j>1} \) of positive numbers, letting \( p_0 = 0 \), we can chose the \( p_j \) so that the absolute value of all frequencies in \( p_{j+1} \) exceeds the largest absolute value of the frequencies in \( p_j \) by at least \( s_{j+1} \).

**Proof.** Let us show first that, for any \( s > 0 \), the exponential polynomials which are mean-periodic with respect to \( \mu \) and whose frequencies lie outside the ball of center 0 and radius \( s \) in \( \mathbb{C}^n \) are dense in the space \( N = \{ f \in C^\infty(\Omega-K) : \mu * f = 0 \text{ in } \Omega \} \). \( N \) is a closed subspace of a Frechet space and we only need to show that if \( \nabla \in \mathcal{E}'(\Omega-K) \) is orthogonal to the above exponential-polynomials then \( \nabla \) is orthogonal to \( N \). Hence \( (\nabla) \) is divisible by \( \hat{\mu} \) at every point of \( \mathbb{C}^n \setminus \overline{B}(0,s) \). Since \( n > 2 \), by Hartogs' theorem, \( (\nabla) / \hat{\mu} \) is an entire function. Since \( \mu \) is invertible there is a distribution \( T \in \mathcal{E}'(\mathbb{R}^n) \) such that
\[ v = \mu \ast T \]

We need to know where is the support of \( T \). By (6)

\[ \text{cv}(\text{supp} v) = \text{cv}(\text{supp} \mu) + \text{cv}(\text{supp} T) \]

or

\[ \text{cv}(\text{supp} T) - K = \text{cv}(\text{supp} v) \subseteq \Omega - K \]

By the Hahn - Banach theorem one concludes that

\[ \text{cv}(\text{supp} T) \subseteq \Omega \]

Hence \( \langle v, f \rangle = (v \ast f)(0) = (T \ast \mu \ast f)(0) = \langle T, \mu \ast f \rangle = 0 \) for \( f \in \mathbb{N} \).

To end the proof of the proposition, we pick an exhaustion of \( \Omega - K \) by convex compacts sets \( K_j \), hence we can find \( P_1 \), exponential-polynomial with frequencies lying in

\[ \{ \xi \in \mathbb{R}^n : \hat{\mu}(\xi) = 0, |\xi| > s_1 \} \]

such that

\[ \sup_{K_1}|f-P_1| < 1. \]

Let \( \sigma_1 \) = maximum of the absolute values of frequencies in \( P_1 \). We can find \( P_2 \) with frequencies in

\[ \{ \xi \in \mathbb{R}^n : \hat{\mu}(\xi) = 0, |\xi| > s_2 + \sigma_1 \} \]

such that

\[ \max_{|\alpha| \leq 1} \sup_{K_2}|D^\alpha (f-P_1-P_2)| < 1/2 \]
Continuing in this fashion we obtain the desired expansion.

Remark. One can eliminate the requirement of $\nu$ being invertible by using [14, 16.4.1].

From (9) we know that the zero variety of $\tilde{\nu}_r$ is the union of the hypersurfaces

\begin{equation}
\zeta^2 = \zeta_1^2 + \ldots + \zeta_n^2 = \lambda_k^2
\end{equation}

\[ k = 1, 2, \ldots \]

where $\lambda_k = a_k, n/2/r$. We disregard temporarily the dependence on $r$ though it will play a role later on. Furthermore the function $\tilde{\nu}_r(t)$ vanishes at $t = \lambda_k$ with multiplicity one, in fact

\begin{equation}
\frac{d}{dt} \tilde{\nu}_r(t) = -n \frac{\Gamma(n/2) J_{(n/2) + 1}(rt)/(rt)^{n/2}}{r^{n-2}}
\end{equation}

and well known properties of Bessel functions show that this expression does not vanish for $t = \lambda_k$. Using the asymptotic expressions (12) and (13) we obtain

\begin{equation}
0 \neq \frac{d}{dt} (\lambda_k) = r n \frac{2^{(n-1)/2} \Gamma(n/2)(-1)^{k+1}/(\lambda_k r)^2}{(n+1)/(n+3)}
\end{equation}

We introduce some auxiliary radial distributions $T_{r,k}$ by the formula

\begin{equation}
\tilde{T}_{r,k}(t) = \frac{\tilde{\nu}_r(t)}{t^2 - \lambda_k^2}
\end{equation}
They are even and entire since \( \tilde{\mu}_r(\pm \lambda_k) = 0 \). Hence they correspond to radial distributions (in fact \( C^1 \) functions) whose supports are contained in the support of \( \mu_r \), i.e. \( \overline{B}(0,r) \).

Furthermore they satisfy

\[
(\Delta + \frac{\lambda^2}{k^2}) \tilde{T}_{r,k} = -\mu_r \quad \text{and} \quad (21)
\]

\[
\tilde{T}_{r,k}(\lambda_k) = \frac{\tilde{\mu}_r(\lambda_k)}{2\lambda_k} = \text{const.}(-1) \lambda_k^{k+1} \frac{n+3}{2} + O(k^{n+5}). \quad (22)
\]

We remark that these distributions have conspicuously appeared in previous work on the Pompeiu problem [2], [7].

**Proposition 2** - Let \( r > 0 \) be fixed. For any \( \rho, 0 < \rho < \infty \), we can decompose \( \sigma_\rho \) in the following form

\[
(23) \quad \sigma_\rho = \nu_\rho + \mu_r \ast S_\rho,
\]

where \( S_\rho \) is a radial distribution, whose support satisfies

\[
(24) \quad \text{supp } S_\rho \subseteq \overline{B}(0, \max(r, \rho) - r)
\]

and \( \nu_\rho \) is given explicitly by:

\[
(25) \quad \nu_\rho = - \sum_{k > 1} \frac{\tilde{\sigma}_\rho(\lambda_k)}{\lambda_k^2 \tilde{T}_{r,k}(\lambda_k)} \Delta \tilde{T}_{r,k},
\]

hence \( \text{supp } \nu_\rho \subseteq \overline{B}(0,r) \).
Proof. We consider the series

\[ g(t) = \sum_{k>1} \frac{\tilde{\sigma}_p(\lambda_k)}{\lambda_k^2 T_{r,k}(\lambda_k)} t^2 \tilde{T}_{r,k}(t) \]

The coefficients \( \lambda_k^{-2} \tilde{\sigma}_p(\lambda_k)/\tilde{T}_{r,k}(\lambda_k) \) are uniformly bounded by a constant depending only on \( p \) as it can be seen from (10), (13) and (22), since \( \lambda_k \sim \text{const.} k \). Therefore, if \( |t| < R \), \( \eta_k > 2R \) we have \( |t^2 \tilde{T}_{r,k}(t)| < \text{const.} k^{-2} \) which guarantees the convergence of the series, and shows \( g \) is an even entire function. We can obtain more precise estimates by picking a sequence of circles of center \( 0 \) and radii

\[ R_j = (4j+n+5)r/4r, \ j = 1, 2, \ldots \]

Decomposing the sum into those terms where \( \lambda_k < 2R_j \) and \( \lambda_k > 2R_j \) one can estimate the second sum over \( |t| = R_j \) by

\[ \max_{|t| = R_j} |t^2 \tilde{\nu}_r(t)| \cdot C_0(p) \]

The first (finite) sum can be estimated by

\[ C_1(p)(\max_{|t| = R_j} |t^2 \tilde{\nu}_r(t)|) \left( \max_{0 < \lambda_k < 2R_j} \frac{1}{|\lambda_k^2 - t^2|} \right) \]

where \( \Omega_j, \varepsilon \) is the region obtained from \( |t| < R_j \) by removing disks of radius \( \varepsilon \), \( 0 < \varepsilon \) very small, about \( \pm \lambda_k \). One can then see, without difficulty, that the last sum is estimated by \( \text{const.} \varepsilon^{-1} \) In any case we obtain as a final estimate
\[ \max \left| g(t) \right| < C(\rho) \max \left| t^{2-}\tilde{\nu}(t) \right|, \]
\[ |t| < R_j \quad |t| < R_j \]

Thus \( g \) defines a radial distribution of order 2, \( \nu_{\rho} \), by \( \tilde{\nu}_{\rho} = g \), one can see \( \nu_{\rho} \) is given explicitly by (25).

We also have

\[ \tilde{\sigma}_{\rho} - \tilde{\nu}_{\rho} = \tilde{\mu}_r h, \]

with \( h \) even entire function since \( g(t\lambda_k) = \nu_{\rho}(t\lambda_k) = \tilde{\sigma}_{\rho}(t\lambda_k) \) by (26). Since \( \mu_r \) is an invertible distribution it follows

\[ h = \tilde{S}_{\rho} \quad \text{for some} \quad S_{\rho} \in \mathcal{E}_r^\prime(\mathbb{R}^n). \]

The identity (6) gives

\[ \text{cv}(\text{supp}(\sigma_{\rho} - \nu_{\rho})) = \text{cv}(\text{supp}S_{\rho}) + \text{cv}(\text{supp} \mu_r). \] (27)

There are two cases to consider. If \( \rho < r \), then the support on the left hand side of (27) is contained in \( B(o, r) \) and

\[ \text{cv}(\text{supp} S_{\rho}) = \{0\}, \]

which says \( S_{\rho} \) is a polynomial in the Laplace operator; if \( \rho > r \) then the left hand side of (27) is contained in \( B(o, \rho) \), which says \( \text{cv}(\text{supp} S_{\rho}) \subseteq B(o, \rho - r) \).

Remark The decomposition we have just given in proposition 2 works also if we replace \( \sigma_{\rho} \) by any radial distribution. We need only to change \( (t/\lambda_k)^2 \) by \( (t/\lambda_k)^{2q} \) with \( q \) convenient non-negative integer. In particular there is such a decomposition with \( \sigma_0 = \delta \), the Dirac mass at the origin (take \( q > \frac{n+1}{4} \)).

3.14
Corollary 3 - Let \( f \) be a \( \mu_r \) -mean-periodic function in \( C^\infty(B(0,R)) \) \((R>r)\). Let \( |x_0| < R - r \). Then, for any \( \rho \), \( 0 < \rho < R - |x_0| \) we have

\[
(28) \quad \lambda_\rho(f,x_0) = (\nu_\rho * f)(0) = - \sum_{k \geq 1} A(T_{r,k} * f)(x_0) \frac{\tilde{\sigma}_\rho(\lambda_k)}{\lambda_k^2 T_{r,k}(\lambda_k)}
\]

**Proof.** It suffices to use \( (2) \) and \( (23) \). \( \square \)

4. **Local two-circles theorem.**

Let \( r_1, r_2 \) be two positive numbers and consider the distributions \( \mu_{r_1}, \mu_{r_2} \). They will have no common, mean-periodic, exponential-polynomials if and only if \( \hat{\mu}_{r_1} \) and \( \hat{\mu}_{r_2} \) have no common zeros. By \( (17) \) this occurs if and only if

\[
r_1/r_2 \neq \text{quotient of two zeros of } J_{n/2}
\]

The set

\[
E_n = \{a_{k,n/2}/a_{j,n/2}:1 \leq j,k \leq n\}
\]

is the exceptional set described in the two-circles theorem.

**Proposition 4** Let \( R > r_1 + r_2, r_1/r_2 \notin E_n \). The only function in \( C^\infty(B(0,R)) \) which is mean-periodic with respect to both \( \mu_{r_1} \) and \( \mu_{r_2} \) is the zero function.

**Proof.** We assume \( r_1 < r_2 \). Let \( f \in C^\infty(B(0,R)) \) be \( \mu_{r_1} \) -mean periodic. By proposition 1 we have
\[ f(x) = \sum_{j \geq 1} P_j(x) (|x|<R) \]

where the frequencies appearing in the exponential sums \( P_j \) lie in

\[ \{ \zeta \in \mathbb{C}^n : \mu_{r_1}(\zeta) = 0 \} = \bigcup_{k \geq 1} \{ \zeta \in \mathbb{C}^n : \zeta^2 = (a_k, n/2/r_1)^2 \} = \bigcup_{k \geq 1} V_k. \]

We fix now \( k > 1 \), and consider \( T_{r_1} * f \) which is in \( C^\infty(B(0,R-r_1)) \), furthermore

\[ (29) \quad T_{r_1} * f = \sum_{j \geq 1} T_{r_1} * P_j \]

If \( P_j(x) = \sum \alpha_{j,l} e^{i(x \cdot \xi_j,l)} \) then

\[ T_{r_1} * P_j = \sum \alpha_{j,l} T_{r_1} \theta_{\xi_j,l} e^{i(x \cdot \xi_j,l)} \]

but \( T_{r_1} \theta_{\xi_j,l} \neq 0 \) only if \( \xi_j,l \in V_k \) in which case we obtain the value \( T_{r_1} \theta_{\lambda_k} = 0 \) (where \( \lambda_k \) is computed with respect to \( r_1 \)). Therefore

\[ (30) \quad T_{r_1} * f = \sum \alpha_{j,l} T_{r_1} \theta_{\lambda_k} P_j,k \]

where \( P_j,k \) is the sum of the terms in \( P_j \) whose frequencies lie in \( V_k \). This series is convergent in \( C^\infty(B(0,R-r_1)) \). We convolve now with \( \mu_{r_2} \). We obtain

\[ (31) \quad \mu_{r_2} * (T_{r_1} * f) = \sum \alpha_{j,l} \mu_{r_2} \theta_{\lambda_k} T_{r_1} \theta_{\lambda_k} P_j,k \]
since $\mu_{r_2}$ is also a radial distribution. The expansion (31) is valid in $C^0(B(0,R-r_1-r_2))$. Since $f$ is also $\mu_{r_2}$-mean-periodic we have

$$0 = (\mathcal{T}_{r_1,k}^{*}\mu_{r_2}^{*}f)(x) = \tilde{\mu}_{r_2}(\lambda_k)(\mathcal{T}_{r_1,k}^{*}f)(x)$$

for $|x| < R - r_1 - r_2$. The hypothesis $r_1/r_2 \notin \mathbb{E}_n$ now implies that $\tilde{\mu}_{r_2}(\lambda_k) \neq 0$. Hence

$$\mathcal{T}_{r_1,k}^{*}f)(x) = 0 \text{ for } |x| < R - r_1 - r_2$$

On the other hand we have (by (22))

$$(\Delta + \lambda_k^2)(\mathcal{T}_{r_1,k}^{*}f) = -(f^{*}\mu_{r_1}) = 0 \text{ in } |x| < R - r_1$$

hence $\mathcal{T}_{r_1,k}^{*}f$ is a real analytic function in $|x| < R - r_1$. We conclude that

$$\mathcal{T}_{r_1,k}^{*}f)(x) = 0 \text{ for } |x| < R - r_1$$

Applying now corollary 3, formula (28), we have

$$\lambda_{\rho}(f,x) = 0 \text{ whenever } |x| < R - r_1, 0 < \rho < R - |x|$$

(We are allowed to take $\rho = 0$ by continuity). In particular
\[ f(x) = 0 \text{ for } |x| < R - r_1 \]

To finish the proof of the proposition we need to show \( f \) is zero in the remaining annulus, we do that using (34). It is at this point that we use Cormack-Quinto [12]. For any \( y \in B(0,R) \), consider \( R(f)(y) = \lambda |y|/2(f,y/2) \). This is the Radon transform on spheres through the origin discussed in [12]. We want to show \( Rf(y) = 0 \). We only need to verify that the conditions stated in (34) are valid. Here \( \rho = |y|/2, x = y/2, \) hence

\[ R - |x| = R - \frac{|y|}{2} = R - \rho > R/2 > \rho \]

The only condition left to see is that \( |x| < R - r_1 \). We have \( 2r_1 < r_1 + r_2 < R \) hence \( r_1 < R/2 \) and \( R - r_1 > R/2 \), therefore \( |x| < R - r_1 \) holds.

By [12, corollary 2] \( f(y) = 0 \). (We note that in [12], they require that \( f \in C^\infty(\mathbb{R}^n) \) while we only have \( f \in C^\infty(B(0,R)) \) but the proof of corollary 2 depends on an explicit inversion formula for the Radon transform on spheres which uses, for each \( y \), values of \( f \) in a neighborhood of \( B(0,|y|) \).)

Remark. The crucial point of the proof above is (32). One does not really need the whole strength of Proposition 1 to obtain it. One can get by using the density of the exponential polynomial solutions in the sub-space \( N \) introduced in Proposition 1. Nevertheless, we feel that the proof is clearer using the expansion (16) as we have done.
We want to show that the condition \( R > r_1 + r_2 \) is sharp. It is easier to show this under the slight restriction that \( r_2/r_1 \) is not too well approximated by elements in \( E_n \).

**Definition.** - For \( N > 0 \), we say that a positive number is \( N \)-well approximated by points in \( E_n \) if, for every \( l > 1 \), there are indices \( j,k \) such that

\[
|a_k/a_j| < \frac{1}{lj^N}
\]

where \( a_k = a_{k,n/2} \)

**Proposition 5**  For any \( N > 2 \), the set of numbers \( N \)-well approximates by \( E_n \) has zero measure in \( (0,\infty) \)

*Proof.* Given \( p,q, \) \( 0 < p < r < q \) and \( v > 0 \), from (13) we have

\[ a_k,v = a_k = (2k+1)\pi/2 + (2v+1)\pi/4 + O(1/k) \]

Therefore, if \( r \) satisfies (35), for \( l > 1 \), we have

\[
|x.j - k + Ar + B| < C
\]

for some constants \( A,B,C \). Hence

\[
pj - C_1 < k < qj + C_2
\]

for some constant \( C_1, C_2 > 0 \). Hence the cardinal of the set of \( k \) satisfying (36) is bounded by \( (q-p)j + L, \) \( L \) constant \( > 0 \).
Now, the set of $N$-well approximated numbers in $[p,q]$ is

\[(37) \quad \bigcup_{l \geq 1} \bigcup_{j,k \geq 1} \{ r : p < r < q, \left| r - \alpha_k / a_j \right| < 1 / l j^N \} \]

For $l$ fixed, the Lebesgue measure

\[\left| \bigcup_{j,k} \{ r : p < r < q, \left| r - \alpha_k / a_j \right| < 1 / l j^N \} \right| < \frac{2}{l} \sum_{j=1}^{C_3} \frac{(q-p)j+1}{l^{N}} < \frac{C_3}{l} \]

$(C_3 > 0)$ since $N > 2$. Therefore the set (37) has zero measure and by letting $q = p + 1$, $p \in \mathbb{N}$ we obtain the proposition.

It is interesting to compare proposition 5 with [8, Lemma 2.1] where examples of numbers which are not 2-well approximated by $E_n(n=2)$ are discussed. It might be that these include all rationals $\not\perp 1$ or all quadratic irrationals $\not\perp 1$, but no such theorem seems to be known. Also, it is easy to see that, for $N < 1$, every positive number is $N$-well approximated by $E_n$.

**Proposition 6.** Let $r_1, r_2$ be two positive numbers such that $r_2/r_1$ is not $N$-well approximated by $E_n$. Denote by $\lambda_k$ the positive zeros of $\tilde{\nu}_{r_1}$. There is a positive constant $C$ such that

\[(38) \quad \left| \tilde{\nu}_{r_2}(\lambda_k) \right| > \frac{C}{N^{n-1} \kappa} \]

**Proof.** Let us denote $a_k = a_{k,n/2}$. Recall that $\lambda_k = \alpha_k / r_1$ and that

\[\tilde{\nu}_{r_2}(t) = \text{const.} \frac{J_{n/2}(r_2 t)}{(r_2 t)^{n/2}}\]
From the asymptotic development (13) we have

\[ a_{k+1} - a_k = \pi + O(1/k) \]

Hence, if \( k \) is fixed and \( j_k \) is chosen such that \(|r_{2\lambda_k} - a_j|\) is minimal we have

\[ \varepsilon_k = |r_{2\lambda_k} - a_j| < \frac{\pi}{2} + O(1/k). \]

Let us distinguish two cases: \( \varepsilon_k < \frac{\pi}{4} \) or not. In the second case we have

\[
|\cos(r_{2\lambda_k} - \frac{(n+1)\pi}{4})| = |\cos(\pm\varepsilon_k + (2j_k + 1) \frac{\pi}{2} + O(1/k))| \\
= |\sin(\varepsilon_k + O(1/k))| > \frac{\sqrt{2}}{2} + O(1/k) > c_0 > 0
\]

for large \( k \). In this case the asymptotic development (12) gives the estimate

\[ |\tilde{\mu}_{r_2}(\lambda_k)| > C_1 k^{-\frac{n+1}{2}} \]

for some \( C_1 > 0 \) and all large \( k \).

By hypothesis we have that for all \( j, k \)

\[ \left| \frac{r_2}{r_1} - \frac{a_1}{a_k} \right| > \frac{C_2}{k^N} (C_2 > 0) \]
Therefore \( \varepsilon_k > C_3/k^{N-1} \) (\( C_3 > 0 \)). Suppose also \( \varepsilon_k < \pi/4 \). By the mean-value theorem there is a \( \xi \) between \( \alpha_{j_k} \) and \( r_2k \) such that

\[
\frac{J_{n/2}(r_2k)}{(r_2k)^{n/2}} = -r_2 \frac{J_{n/2+1}(\xi)}{\xi^{n/2}} \cdot (r_2k - \alpha_{j_k})
\]

(Recall \( J_{n/2}(\alpha_{j_k}) = 0 \).) Note that \( \delta_k = |\xi - \alpha_{j_k}| < \varepsilon_k < \pi/4 \).

Again by (12) we have to estimate

\[
\cos(\xi - \frac{\pi}{4} - \frac{(n+2)\pi}{4}) = \cos(\pm \delta_k + (2\delta_k + 1)\frac{\pi}{2} + (n+1)\frac{\pi}{4} - \frac{(n+2)\pi}{4} + O(1/k))
\]

\[
= \cos(\pm \delta_k + j\cdot\pi + O(1/k))
\]

\[
= \pm \cos \delta_k + O(1/k).
\]

Then

\[
|\tilde{u}_{r_2}(\lambda_k)| > \frac{C_4}{k^{N-1+n+1/2}} \quad (C_4 > 0).
\]

Since \( N > 1 \) the estimate (38) holds in both cases. 

**Proposition 7** Let \( f \) be a function in \( L^1_{locc}(B(0,R)) \), \( g \in L^1_{locc}(B(0,R)) \), \( \text{supp} g \subset B(0,r) \), \( g \) radial. For \( |x_0| < R - r \) and \( \rho < R - r - |x_0| \) we have

\[
\lambda_{\rho}(f*g,x_0) = (\lambda_{|\cdot|}(f,x_0)*g(\cdot))(y) \quad (|y| = \rho)
\]

3.22
(The notation indicates that we are convolving in the variable denoted by a dot).

Proof. Recall that the average \( \lambda_\rho(f,x_0) \) can also be computed by

\[
\lambda_\rho(f,x_0) = \int_{O(n)} f(x_0 + Ay) dA
\]

where \( y \) is any point with \(|y| = \rho\), \( O(n) \) is the orthogonal group and \( dA \) is the normalized Haar measure. Let

\[
\phi(y) = (\lambda_\rho(f,x_0) * g(\cdot))(y)
\]

we have

\[
\phi(y) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(x_0 + Ay - x) dA \right) g(x) dx
\]

\[
= \int_{O(n)} \left( \int_{\mathbb{R}^n} f(x_0 + Ay - x) g(x) dx \right) dA
\]

Set \( u = Ax \) then \( g(x) = g(u) \) and \( dx = du \) Hence

\[
\phi(y) = \int_{O(n)} \left( \int_{\mathbb{R}^n} f(x_0 + Ay - u) g(u) du \right) dA
\]

\[
= \lambda_\rho(f * g,x_0). \quad \Box
\]

Corollary 8. Let \( g \) be radial integrable function of compact support and \( \alpha \) a positive number. Then

3.23
Proof. Let \( \xi \in \mathbb{R}^n \) be any vector with \( |\xi| = a \), then

\[
\left( g(x) * \frac{J_{n-2}(a|x|)}{2^{n-2}} \right)(y) = \tilde{g}(a) \frac{J_{n-2}(a|y|)}{2^{n-2}}
\]

On the other hand

\[
\chi_{\rho}(e^{i(\xi \cdot \cdot)}, 0) = \tilde{\sigma}_{\rho}(a)
\]

\[
= \frac{n-2}{2} \Gamma(n/2) \frac{J_{n-2}(a \rho)}{(a \rho)^{n-2}}
\]

Applying now to (42) proposition 7 we obtain the desired formula (41).

Proposition 9. Let \( r_1, r_2 \) be two positive numbers such that \( r_2/r_1 \) is not N-well approximated by \( E_n \). Let \( R \) be any number, \( \max(r_1, r_2) < R < r_1 + r_2 \). Then there is a non zero radial function \( f \in C^\infty(B(0, R)) \) which is mean periodic with respect to \( \mu_{r_1} \) and \( \mu_{r_2} \).

Proof. Let \( \phi \in \mathcal{D}(0, r_1) \), \( \xi = 0 \) such that \( \text{supp} \phi \subseteq [R-r_2, r_1] \)

It follows from [16, theorem 2.1 page 247] that \( \phi \) admits a series development of the form
\( \lambda_k = a_k, n/2/r_1 \). This is the Sturm-Liouville expansion for a boundary value problem singular at \( t = 0 \) and derivative equal to zero at \( t = r_1 \). It can be seen by successive integrations of parts that

\[
|a_k| = O(k^{-p}) \text{ for every } p > 0.
\]

Since \( r_2/r_1 \) is not \( N \)-well approximated by \( E_n \) we see that

\[
b_k = a_k/\tilde{u}_{r_2}(\lambda_k)
\]

satisfies the same estimates as \( a_k \) (Proposition 6). Hence the function

\[
f(x) = \sum_{k \geq 1} b_k \frac{J_{n-2}(\lambda_k |x|)}{\left(\lambda_k |x|\right)^2}
\]

is a \( C^\infty \) radial function in \( \mathbb{R}^n \), \( f \not\equiv 0 \). And, from corollary 8, it follows that \( f \) is \( \mu_{r_1} \) mean-periodic. Furthermore

\[
(\mu_{r_2} * f)(x) = \sum_{k \geq 1} b_k \tilde{u}_{r_2}(\lambda_k) \frac{J_{n-2}(\lambda_k |x|)}{(\lambda_k |x|)^2}
\]

\[
= \phi(|x|)
\]
which is zero in $B(0, R-r_2)$ and therefore the function $f$
restricted to $B(0, R)$ is $\mu_{r_1}$ and $\mu_{r_2}$ mean-periodic. $\Box$

The above propositions can be summarized by the following:

**Theorem 10.** Let $r_1 > 0$ and $r_2 > 0$ be such that

$$r_2/r_1 \in \mathbb{E}_n.$$ The necessary and sufficient condition on a open
set $\Omega$ of $\mathbb{R}^n$ so that the only distribution $T \in D'(\Omega)$ which
can be mean-periodic with respect to both $\mu_{r_1}$ and $\mu_{r_2}$ is
$T = 0,$ is that $\Omega$ is the reunion of balls of radii strictly
larger than $r_1 + r_2.$

An amusing corollary of theorem 10 is the following:

**Corollary 11** If $r_2/r_1 \in \mathbb{E}_2,$ $r_1 + r_2 < R$ and $f \in C(B(0,R))$
then the conditions

$$\int_{\partial B(z,r_j)} f(\zeta) d\zeta = 0 \text{ for every } z, |z| < R - r_j (j=1,2),$$

imply that $f$ is holomorphic in the disk $B(0,R)$.

5. **Generalizations**

After the paper was written, we became aware of the work of
J.D. Smith [17], in which local versions of certain two-circle
theorems are also proved. Smith's results, which require
$R > 2r_1 + r_2$ are less sharp than Proposition 4; nor does the
method of proof seem to generalize to the other problems discussed
in [20], e.g. the converse of the mean value property for harmonic
functions. The aim of this section is to show that the methods
used above do generalize.

3.26
Definition. We say that a radial distribution $\nu$ of compact support is hyperbolic if:

(i) $\nu$ is invertible, and

(ii) there is a constant $C$ such that every zero $\lambda$ of $\tilde{\nu}$ satisfies

$$|\text{Im}\lambda| \leq C \log(2+|\lambda|).$$

Theorem 12 Let $\nu_1, \nu_2, \ldots$ be a (possibly infinite) family of radial distributions of compact support, $\text{cv}(\text{supp}\nu_j) = \overline{B}(0, r_j)$. Suppose $\{z \in \mathbb{C}^n : \nu_j(z) = 0 \; \forall j\} = \emptyset$, $\nu_1$ is hyperbolic, and $R - r_1 > \sup r_j$. Then $\{f \in D(B(0, R) : \nu_j \ast f = 0 \; \forall j\} = \{0\}$.

Proof: Due to the condition on $R$ we can assume $f \in C^\infty(B(0, R))$ as done before. The proof that leads to (32) can be repeated almost verbatim just using for each $\lambda_k$, zero of $\tilde{\nu}_1$, a convenient $\nu_j(j > 2)$ with $\tilde{\nu}_j(\lambda_k) \neq 0$. We obtain

$$T_{k,s} \ast f(x) = 0 \text{ for } |x| > R - r_1 - \sup r_j,$$

where $T_{k,s}(t) = \tilde{\nu}_1(t)(t^2 - \lambda_k^2)^{-s}$, $1 < s < m_k$, $m_k =$ multiplicity of $\lambda_k$ as a root of $\tilde{\nu}_1$.

On the other hand

$$(-1)^s(\Delta + \lambda_k^2)^s(T_{k,s} \ast f) = \nu_1 \ast f = 0 \text{ in } B(0, R-r_1),$$

therefore, $T_{k,s} \ast f$ is real analytic, and hence
as before. It is at this point we have to be more careful to prove the correct version of Proposition 2. It will be replaced by the following:

**Lemma 13** Let \( \Lambda = \{ \lambda_k \} \) = set of distinct zeros of \( \tilde{\mu}_1 \), then
\[
\Lambda = \bigcup_{j=0}^{\infty} \Lambda_j,
\]
where the \( \Lambda_j \) are finite and mutually disjoint sets. There is also a positive integer \( q \) such that for any \( \rho, 0 < \rho < \infty \) we can write

\[
(50) \quad \sigma_\rho = \nu_\rho + \mu_1^* S_\rho,
\]

where \( \nu_\rho, S_\rho \) are radial distributions satisfying

\[
(51) \quad \text{supp } \nu_\rho \subseteq \overline{B}(0, r_1)
\]
and

\[
(52) \quad \text{supp } S_\rho \subseteq \overline{B}(0, \max(r_1, \rho) - r_1).
\]

Furthermore,

\[
(53) \quad \nu_\rho = \sum_{j=0}^{\infty} \Delta^q \nu_{\rho, j},
\]

a convergent series in \( E'_\omega(\mathbb{R}^n) \), each \( \nu_{\rho, j} \) a finite linear combination of the distributions \( T_{k, s'} \), \( \lambda_k \in \Lambda_j \), \( 1 < s < m_k \) (if \( m_0 > 1 \) then one denotes by \( \Delta^q \nu_{\rho, 0} \) not only a finite linear combination of \( \Delta^q T_{k, s'} \), \( \lambda_k \in \Lambda_0 \) but also of

\( T_{0, m_0}, \Delta T_{0, m_0}, \ldots, \Delta^{q-1} T_{0, m_0} \)).
Once this lemma has been proved, the proof of Theorem 12 is achieved the same way as it was done in Proposition 4 and we note that the hypotheses imply $2r_1 < R$.

**Proof of Lemma 13** The proof of this lemma proceeds as in Proposition 2 by interpolating the values of $\tilde{\sigma}_p$ on the variety of zeros of $\tilde{u}_1$ (counted with multiplicities). We have to repeat with due care the procedure used in [13], [15], [18] since we need the precise statement (51), (52), and (53).

First we note that as in [5, lemma 4] (cf. also [11, p. 50]), the condition of hyperbolicity and the minimum modulus theorem allow us to construct a family of a Jordan quadrilaterals $\Gamma_k$, $k \in \mathbb{Z}$ symmetric with respect to the real axis and enjoying the following properties:

1. **(54)** for some $d > 0$ the horizontal sides lie on the curves

   $$\text{Im } z = \pm \log(d + |Rez|),$$

   and the vertical sides are arcs of circles.

2. **(55)** $0 \notin \text{int } \Gamma_0$ which is symmetric with respect to the origin (i.e. if $z \in \Gamma_0$ then $-z \notin \Gamma_0$ also).

3. **(56)** for $k \neq 0$, $\Gamma_{-k}$ is the symmetric of $\Gamma_k$ with respect to the origin.
(57) for \( j \neq k \), \( \text{int} \Gamma_j \cap \text{int} \Gamma_k = \emptyset \), furthermore, for some positive number \( a \) we have that if \( z \in \Gamma_j \),
\[ \text{dist}(z, \Gamma_k) > (a + |z|)^{-a} \]
for any \( k \neq j \).

(58) for some positive constant \( b \) we have:
\[ \text{diam} \Gamma_j < b(1 + |z|)^b \]
and
\[ \text{length} \Gamma_j < b(1 + |z|)^b, \]
for any \( z \in \text{int} \Gamma_j \), any \( j \).

(59) there is a constant \( c > 0 \) such that for any \( j \), and any \( z \in \Gamma_j \) we have
\[ |\tilde{\mu}_j(z)| > (c + |z|)^{-c}, \]
and this inequality is valid even for those \( z \) such that
\[ \text{dist}(z, \Gamma_j) < 1/2 (a + |z|)^{-a} \]
(the same \( a \) as in (57)).

(60) \[ A \subseteq \bigcup_{j=}^{\infty} \text{int} \Gamma_j. \]

(61) for some \( d > 0 \): if \( j > 1 \), \( z \in \Gamma_j \), then \( |z| > j/d \).

(62) \[ A_0 = A \text{ int} \Gamma_0, A_j = A \text{ int } \Gamma_j \cap \text{int} \Gamma_{-j}, j > 1. \]
For the sake of definiteness we will index the points in $\Lambda$ so that $\lambda_0 = 0$, and, for $k > 1$, either $\Re \lambda_k > 0$ or $\Re \lambda_k = 0$ and $\Im \lambda_k > 0$, and, finally, $\lambda_{-k} = -\lambda_k$.

Now consider the even entire function

$$f(t) = t^{2q} \tilde{\nu}_1(t),$$

for $q$ a positive integer to be chosen conveniently later on.

We note that if $t \notin \overline{\text{int } T_j} \cup \overline{\text{int } T_j}$ then

$$\phi_j(t) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{\tilde{\sigma}(s)}{f(s)} \frac{ds}{s-t} + \frac{1}{2\pi i} \int_{\Gamma_{-j}} \frac{\tilde{\sigma}(s)}{f(s)} \frac{ds}{s-t}$$

(where we disregard the second term if $j = 0$) is an even function which is a linear combination of terms of the form $(t^2 - \lambda_k^2)^{-s}$, for $\lambda_k \in \Lambda_j$ and $1 < s < m_k$ if $k > 1$, $1 < s < m_0 + 2q$ if $k = 0$. Hence $\phi_j$ can be defined as a rational function throughout $\mathbb{C}$ and the function $f(t) \phi_j(t)$ is an even entire function. We want to show now that $q$ can be chosen so that

$$g(t) = \sum_{j=0}^{m} f(t) \phi_j(t)$$

is in $\tilde{E}'(\mathbb{R}^n)$ and the series converges in the topology of $\tilde{E}'(\mathbb{R}^n)$.

In fact, we have that for $|\Im t| < \log(\delta + |\Re t|)$ there is some $N > 0$ such that
(66) \[ |\tilde{\sigma}_p(t)| < C(p) (1+|t|)^N \]

and also

(67) \[ |\tilde{\nu}_1(t)| < C_0 (1+|t|)^N. \]

Therefore, for some \( N_1 > 0 \) sufficiently large, if \( \text{dist}(t, \text{int} T_j \cup \text{int} T_{-j}) > 1 \) we have by (66), (59) and (58), that with respect to an arbitrary point \( z \in \text{int} T_j \), which we can take it to be the point in the positive real axis closest to the origin,

\[ |\phi_j(t)| < (N_1 + |z|)^N |z|^{-2q} < \text{const.} \ j^{-2} \]

by (61) (just take \( 2q > N_1 + 2 \)). Therefore, under the same condition on \( t \) we have

(68) \[ |f(t) \phi_j(t)| < C_1 \ j^{-2} (1+|t|)^M e^{-r_1 |\text{Im} t|} \]

Using the condition (58) on the diameter of \( T_j \) and (67), this estimate remains valid throughout \( \mathcal{C} \), after possibly increasing \( C_1, M \). This shows that the \( \nu_p \in \mathcal{E}'(\mathbb{R}^n) \) defined by

\[ \tilde{\nu}_p(t) = g(t) \]

satisfies (51). It is also clear that the distributions \( \nu_{p,j} \) such that \( t^{2q} \nu_{p,j} = f(t) \phi_j(t) \) have the properties required
by (53) (with special care taken if $\omega_0 \neq 0$). To end the proof of the lemma we only have to show that $g(t) - \tilde{\sigma}^p(t)$ is divisibly by $\tilde{\mu}_1(t)$, the rest is the same as in Proposition 2. Note that if $t \in \Omega_j \cup \Omega_{j'}$ then we have

$$\psi_j(t) = \frac{1}{2\pi i} \int_{\Gamma_j + \Gamma_{-j}} \frac{\sigma(s)}{f(s) s-t} ds$$

is holomorphic and, we pick a new residue at $s = t$ yielding

$$f(t) \psi_j(t) = f(t) \phi_j(t) + \sigma(t).$$

This concludes the proof of Lemma 13 and Theorem 12. □

We give here the local version of Delsarte's theorem for harmonic functions.

**Corollary 14** Let $H_n = \{\xi/n: \xi \in \Omega(0, \infty), \tilde{\sigma}_1(\xi) = \tilde{\sigma}_1(n) = 1\}$. If $R > r_1 + r_2$, $r_1/r_2 \notin H_n$, and $u$ is a continuous function in $B(0, R)$ satisfying

$$u(x) = \lambda_{r_1}^x (u, x), \quad \text{for all } x, |x| < R - r_1,$$

and

$$u(x) = \lambda_{r_2}^x (u, x), \quad \text{for all } x, |x| < R - r_2,$$

then $u$ is harmonic in $B(0, R)$.

**Proof:** From the asymptotic development of the Bessel functions and the formula (10), it follows that the radial distributions $\mu_j$
defined by

\[ \tilde{\mu}_j(t) = t^{-2}(\sigma - \delta)(t) = (\sigma_{r_j}(t) - 1)/t^2 \]

are hyperbolic. The hypothesis on \( r_1/r_2 \) guarantees these two entire functions have no common zeros. Theorem 13 shows now that the distribution \( \Delta u \) is zero in \( B(0,R) \).

Remark As mentioned in [20], Delsarte proved this theorem in \( \mathbb{R}^n \). He also showed that \( H_n \) is finite and \( H_3 = \{1\} \). Hence, at least for dimension 3, any pair of distinct positive value \( r_1, r_2 \) would work in the above corollary.

The several other results in [20] can now be carried over to the local case without difficulty. It remains as an open question for the moment the elimination of the invertibility condition on \( \mu_1 \), which could probably be done following the Euclidean summation method of [6]. More interesting, in our view, is to try to extend this theorem to non-compact symmetric spaces of rank 1 or even to the Euclidean group thus obtaining a local version of the Pompeiu problem considered in [9].

As an example of this let us mention the following corollary of Theorem 13.

**Corollary 15** Let \( R > \sqrt{n} a \), if \( f \in L^1_{\text{loc}}(B(0,R)) \) has zero integral over any \( n \)-cube of side \( a \) contained in \( B(0,R) \), then \( f = 0 \) a.e.

**Proof** Following the ideas from [9] we see we can consider all radial distributions \( \mu \) whose Fourier transforms are of the form
where $Q$ is the cube $[-a/2,a/2]^n$ and $T$ is a distribution of compact support in the ball $B(0,\varepsilon)$, $\varepsilon + \sqrt{n} a < R$. Then, for any such $\mu$, \( \text{cv}(\text{Supp}\mu) \subset B(0,r) \), and $f$ will satisfy the equations:

$$\mu * f = 0 \text{ in } B(0,R-r).$$

Since this set of distributions generates the same closed ideal in $E'(\mathbb{R}^n)$ as those are considered in [9, p. 602], then their Fourier transforms have no common zeros [9, section 9]. It only remains to find a distribution that plays the role of $\mu_1$ in Theorem 13. The easiest one is obtained when

$$T = \frac{\partial^{2n} \chi_Q}{\partial x_1^2 \cdots \partial x_n^2}.$$

An easy computation shows that in this case, for

$$\mu_1 = \text{average over } O(\mu) \text{ of } \frac{\partial^{2n}}{\partial x_1^2 \cdots \partial x_n^2} \chi_Q,$$

we have

$$\tilde{\mu}_1(t) = \text{const. } t^{(n/2)+1} J_{3n-2}(\sqrt{n} at/2)$$

which is clearly hyperbolic. (For $n = 2$, this can be obtained from Sonine second finite integral [19, p. 376].)
6. References


CHAPTER 4

INVERSION OF THE LOCAL POMPEIU TRANSFORM

by

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1. Introduction.

The prototype of the Pompeiu transform and of the problems we will consider here is the Two-Disks problem we discuss presently. Let $E$ be the collection of positive quotients of zeros of the Bessel function $J_1$. This countable set is precisely the exceptional set for the following theorem (proved independently in [12] and [21]).

**Theorem.** Let $r_1, r_2 > 0, r_1/r_2 \in E$. Then a continuous function in the plane is identically zero if and only if all its averages over any disk of radius either $r_1$ or $r_2$ vanish.

If we let $\chi_1, \chi_2$ be the characteristic functions of the disks $B(0, r_1), B(0, r_2)$ we are saying that the map

$$P : f \mapsto (\chi_1 \ast f, \chi_2 \ast f)$$

is injective. In [7], [8] it was shown that a stronger arithmetical condition was necessary and sufficient for the existence (and the explicit construction) of two distributions $\nu_1, \nu_2$ of compact support such that

$$\nu_1 \ast (\chi_1 \ast f) + \nu_2 \ast (\chi_2 \ast f) = f.$$  

Because $\nu_1, \nu_2$ might be distributions (the map $P$ is smoothing), it is more natural to consider $P$ in the Fréchet space $\mathcal{E} = \mathcal{E}(\mathbb{R}^2)$ of $C^\infty$ functions. Then we have that $r_1/r_2 \in E$ if and only if

$$P : \mathcal{E} \rightarrow \mathcal{E}^2$$

is injective. The existence of $\nu_1, \nu_2$ solving (1.2) is equivalent to $\text{Im } P$ being closed, hence $P$ is right invertible. One can in this case also show that

$$\text{Im } P = \{(g, h) \in (\mathcal{E}(\mathbb{R}^2))^2 : \mu_2 \ast g = \mu_1 \ast h\}$$
and

\[ P^{-1} : \text{Im} \rightarrow \mathcal{S} \]

\[(g, h) \mapsto \nu_1 g + \nu_2 h. \]

The exact arithmetical condition is just the existence of a constant \( A > 0 \) such that

\[ |r_1/r_2 - \xi/\eta| \geq \frac{1}{A} \eta^{-A}. \]

for any pair \( \xi, \eta > 0, \ J_1(\xi) = J_1(\eta) = 0. \)

Since (1.2) is an explicit deconvolution formula, L. Zalcman asked in [20] whether some sort of explicit reconstruction formula would also exist under the sole condition \( r_1/r_2 \in \mathcal{E} \). The theorem below answers this question in the affirmative. Naturally coupled to this question is the following local problem. Suppose we know the averages of \( f \) on disks of radii \( r_1 \) and \( r_2 \), only when those disks lie in some fixed disk \( B(0, R) \), could these data determine \( f \) in \( B(0, R) \)? A priori, formula (1.2), if valid, would only determine \( f \) in \( B(0, R') \) for some convenient \( R' < R \). This is just a consequence of the fact that (1.2) is given by convolutions, and \( \nu_1, \nu_2 \) do not have support at the origin. This is the nature of what was called elsewhere the local Pompeiu problem. It is shown in [2] that if \( r_1 + r_2 < R, \ r_1/r_2 \in \mathcal{E} \), then \( f \) is determined in the whole disk \( B(0, R) \) by those local averages. We will show here that one has even a local reconstruction formula.

These local theorems are also valid for other sets instead of disks, when one takes all rigid motions. The injectivity was shown in [3]. We give here a method to construct a local inverse in this more general situation.

Finally, we would like to mention that even for the injectivity question in \( \mathbb{R}^2 \), it is very hard to see when one can work just with translations of a
finite family of sets. The antecedent of several such results is the Three-
Squares Theorem [5] where it is shown that if the averages of a continuous
function \( f \) in the plane vanish on every square of sides either \( r_1, r_2 \) or
\( r_3 \) (with sides parallel to the axes), then \( f \) is identically zero if and
only if \( r_1, r_2, r_3 \) are \( \mathbb{Q} \)-linearly independent. In [8], [9], explicit decon-
volution formulas were found for this case under extra arithmetical assump-
tions on the triple \( r_1, r_2, r_3 \). The local version of the Three-Squares Theorem
requires that we place ourselves in a square of side \( R \). In [4] it is shown
that if \( R > r_1 + r_2 + r_3 \), then the uniqueness still holds. Its proof is
akin to the methods used in this paper.

The motivation for this work lies in trying to find an algorithmic decon-
volution approach (with due care for error bounds and noise behavior) in a
situation where part of the scene is obstructed from our view but the object
we are looking for lies very close to this obstruction. In the case of the
Radon transform this is sometimes called the Hole problem or the Bagel problem
[15]. It is interesting to note that these two problems are related, since
one way to find an inversion formula is to use Cormack-Quinto’s Spherical
Radon Transform and its explicit inverse.

2. Preliminaries.

We recall some notation and basic properties of the Pompeiu transform.
Let \( E_1, \ldots, E_N \) be a collection of compact sets in \( \mathbb{R}^n \) of positive measure
and let \( M(n) \) be the group of Euclidean motions in \( \mathbb{R}^n \), then the (global)
Pompeiu transform (associated to \( E_1, \ldots, E_N \)) is the map [10]

\[
P : C(\mathbb{R}^n) \to C(M(n))^N
\]
Given by

\[(2.2) \quad (Pf)(g) = \left[ \int_{gE_1} f dx, \ldots, \int_{gE_N} f dx \right].\]

Given an open set \( U \subseteq \mathbb{R}^n \) we can define open sets \( G_j \subseteq M(n) \) by

\[(2.3) \quad G_j = \{ g \in M(n) : gE_j \subseteq U \} \]

(these sets could be empty). We define the local Pompeiu transform

\[(2.4) \quad P : C(U) \rightarrow \bigoplus_{j=1}^N C(G_j) \]

defined exactly by the same formula (2.2). The family \( E_1, \ldots, E_N \) is said to have the Pompeiu property (respectively, the local Pompeiu property with respect to \( U \)) whenever the map (2.1) (respectively (2.4)) is injective. The most interesting case is the case of a single set with relatively nice boundary. One can prove the following theorem.

**Theorem 1** [1], [6]. Let \( \Omega \) be an open bounded set in \( \mathbb{R}^n \), \( E = \overline{\Omega}, E^c \)
connected and \( \partial E \) Lipschitz. Then \( E \) (or \( \Omega \)) has the Pompeiu property if and only if there is no positive eigenvalue for the overdetermined Neumann problem

\[(N) \begin{cases}
\Delta u + \omega u = 0 & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0, \quad u = 1 & \text{in } \partial \Omega.
\end{cases}\]

As a corollary, one obtains two classes of examples:

(1) A single ball never has the Pompeiu property;

(2) A set \( \Omega \) satisfying the hypotheses of Theorem 1 whose boundary is not a real analytic hypersurface has the Pompeiu property.

Examples of sets having the Pompeiu property and also having real analytic boundary exist, for instance, any ellipsoid which is not a ball. Since a
single ball does not have the Pompeiu property, the following theorem gives the best possible answer.

**Theorem 2** [12], [21]. Let $Z_n = \{\xi/\eta : \xi > 0, \eta > 0, J_{n/2}(\xi) = J_{n/2}(\eta) = 0\}$. A pair of balls $B(0,r_1), B(0,r_2)$ has the Pompeiu property if and only if $r_1/r_2 \notin Z_n$.

To consider the local Pompeiu property it is clear we have to start with a family that satisfies the global Pompeiu property. We restrict ourselves to open sets $U$ that can be covered by balls of a fixed radius $R$. In this case we might as well suppose that $U$ is the ball $B(O,R)$. In this setting it is easy to state the local version of Theorem 2.

**Theorem 3** [2]. Let $r_1, r_2 > 0, r_1/r_2 \notin Z_n$ and $R > r_1 + r_2$. Then the pair $B(0,r_1), B(0,r_2)$ has the local Pompeiu property with respect to the ball $B(O,R)$. Under the additional restriction that $r_1/r_2$ is badly approximated by elements of $Z_n$, the condition $R > r_1 + r_2$ is also necessary.

It is quite possible that $R > r_1 + r_2$ is always a necessary condition in Theorem 3. When considering a single set $E$ having the global Pompeiu property, in order to decide whether $E$ has the local Pompeiu property with respect to $B(O,R)$, it is natural to try to measure $R$ against a value $r$ such that $E \subseteq B(x_0,r)$ for some $x_0$. The condition that replaces $r_1 + r_2 < R$ is $2r < R$. We are compelled to introduce an extra technical condition due to the fact that we know nothing about $\partial E$. This condition, which we will explain below, is called hyperbolicity. There are a number of simple conditions that imply it (see [3]), for instance.

1. Near an extreme point $x_1$, $x_1 \in E \cap \partial B(x_0,r)$, $E$ coincides with a polyhedral angle with vertex $x_1$. The fact that the walls or edges are straight plays no role. For instance, $E$ is a cube in $\mathbb{R}^n$.

2. There is an extreme point $x_1$, $x_1 \in E \cap \partial B(x_0,r)$, such that $x_1$ is
a point of strict convexity for \( \partial E \) and \( \partial E \) is of class \( C^3 \) near \( x_1 \).

(3) In \( \mathbb{R}^2 \), near a point \( x_1, x_1 \in E \cap \partial B(x_0, r) \), \( \partial E \) is a Jordan curve, sufficiently smooth.

**Theorem 4 [3].** Assume \( E \subseteq \mathbb{B}(x_0, r) \) has a hyperbolic point \( x_1 \in E \cap \partial B(x_0, r) \). Assume further that \( E \) has the global Pompeiu property and \( 2r < R \). Then \( E \) has the local Pompeiu property with respect to \( B(0, R) \).

We remark that the conditions \( r_1 + r_2 < R \) in Theorem 3 and the corresponding \( 2r < R \) in Theorem 4 mean that the open sets \( G \) (respectively \( g \) ) of \( M(n) \) are not to small. On the other hand, they impose no restriction, when \( U \) is the complement of a closed convex set \( K \). This is akin to say that the Pompeiu transform possesses the Hole property, as is the case for the Radon transform (in that case \( K \) must be compact).

The proofs of Theorems 1 and 4 depend on the introduction of a countable family of radial distributions associated to the set \( E \). For that purpose we need to introduce the concept of the radialized version of a distribution (or circular symmetrization) with respect to a point \( x_0 \). Namely, let \( T \in \mathcal{D}'(\mathbb{R}^n) \) then \( \mathcal{R}_{x_0} T = \mathcal{R} T \) is defined by

\[
\langle \mathcal{R}_{x_0} T, \varphi \rangle = \langle T(x), \int_{SO(n)} \psi(k(x-x_0)+x_0)dk \rangle,
\]

where \( \psi \in \mathcal{D}(\mathbb{R}^n) \), \( dk \) is the Haar measure on \( SO(n) \). In case \( T = \chi_E \), \( E \) a compact set in \( \mathbb{R}^n \), then \( \mathcal{R}_{x_0} T \) is a function of compact support with values in \( [0, 1] \) which is a radial function with respect to \( x_0 \). \( \mathcal{R}_{x_0} T(x) = \varphi(|x-x_0|) \).

\[
\varphi(t) = \sigma(E \cap \partial B(x_0, t)),
\]

where \( \sigma \) is the normalized surface area. We consider \( \varphi \) as a radial Borel function in \( \mathbb{R}^n \), of compact support, and therefore its Fourier transform \( \mathcal{F} \varphi \)
is an entire function in \( \mathbb{C}^n \) in the Paley-Wiener class, which is radial, i.e., if we let

\[
\phi(\zeta) = \mathcal{F}_\varphi(\zeta,0,\ldots,0)
\]

then \( \phi \) is an entire function in \( \mathbb{C} \), even, satisfying the inequality

\[
|\phi(\zeta)| \leq A \exp(r|\text{Im} \, \zeta|),
\]

whenever \( E \subseteq \overline{B}(x_0, r) \). In fact, \( \phi \) is the Fourier-Bessel transform of the radial function \( \varphi \)

\[
\phi(\zeta) = (2\pi)^{n/2} \int_0^\infty \varphi(t) J_{(n-2)/2}(\zeta t)t^{n-1} \, dt,
\]

where \( J_\ell(z) = J_\ell(z)/z^\ell \), \( J_\ell \) being the Bessel function of order \( \ell \). We also denote \( \phi = \tilde{\phi} \). Conversely, \( \phi \) determines \( \mathcal{F}_\varphi \) by

\[
\mathcal{F}_\varphi(\zeta_1, \ldots, \zeta_n) = \phi(\sqrt{\zeta_1^2 + \ldots + \zeta_n^2}).
\]

The smallest \( r \) which appears in (2.8) determines the smallest disk of center \( x_0 \) such that \( E \subseteq \overline{B}(x_0, r) \). We call exterior radius of \( E \), \( r^* = r^*(E) \), the minimum value of \( r \), with respect to all \( x_0 \), such that this inclusion holds.

For fixed \( x_0 \) and any \( \alpha \in \mathbb{N}^n \), we can construct a corresponding entire function of one complex variable \( \phi_\alpha \) by

\[
\phi_\alpha(\zeta) := (\mathcal{R}_{x_0} (D_x^\alpha \chi_E))^{-}(\zeta),
\]

where \( D_\alpha = \frac{\partial^{|\alpha|}}{\partial x_\alpha} \). It is convenient to introduce the distributions \( \mu_\alpha = \mathcal{R}_{x_0} D_\alpha \chi_E \), which are radial with respect to \( x_0 \). These definitions are justified by the following.

**Lemma 4** [12]. \( E \) has the global Pompeiu property if and only if the functions
\[ \phi, \quad \alpha \in \mathbb{N}, \text{ have no common zeros.} \]

An even entire function \( \phi \) of a complex variable is said to be hyperbolic if the following three conditions are satisfied.

(i) For some constant \( A > 0 \), it satisfies the estimate

\[ |\phi(\zeta)| \leq A(1 + |\zeta|)A\exp(A|\text{Im} \ \zeta|). \]

(ii) For some constant \( a > 0 \) and any real value \( \xi \)

\[ \max\{|\phi(\xi + \eta)| : |\eta| \leq a \log(2 + |\xi|)| \geq (a + |\xi|)^{-a}. \]

(iii) For some constant \( c > 0 \) we have that

\[ \phi(\zeta) = 0 \Rightarrow |\text{Im} \ \zeta| \leq c \log(2 + |\zeta|). \]

The first condition is simply that \( \phi \in \mathcal{F}(\mathcal{C}'(\mathbb{R})) \). The second is usually called invertibility, it means that the principal ideal \( \mathcal{F}(\mathcal{C}'(\mathbb{R})) \) is closed. It coincides with the property that \( \mathcal{F}(\mathcal{G}) \cap \mathcal{F}(\mathcal{C}') = \mathcal{F}(\mathcal{C}')(\mathbb{R}) \). The third one says that all the zeros of \( \phi \) are almost real.

In [3], it is shown that a number of natural conditions on \( \mathcal{G} \) imply that \( \phi_0 \), the radialized function of \( \chi_\mathcal{G} \), is hyperbolic.

The idea of Lemma 4, and a fortiori Theorems 1 and 4, is that everything is reduced to the study of the radial case, as follows. Let us fix \( x_0 \), which we take to be the origin for simplicity. Then the condition

\[ \int_{g \mathcal{G}} f dx = 0 \quad \forall \ g \in \mathcal{M}(n) \ ( \text{or} \ \forall \ g \in C) \]

can be rewritten as an infinite system of convolution equations

\[ (2.12) \quad \forall \rho \in \text{SO}(n): \ \chi_{\rho \mathcal{G}} * f(x) = 0 \quad \forall \ x \in \mathbb{R}^n \ ( \text{or} \ |x| < R - r). \]

The proof of the injectivity of the (local) Pompeiu transform is based on the principle that if there is a non-zero solution of (2.12), we can assume it
to be \( C^\infty \) and show there must be a non-zero radial \( C^\infty \) function \( g \) in \( \mathbb{R}^n \) (respectively in \( B(0,R) \)) solution of the denumerable system of convolution equations

\[
\forall \alpha \in \mathbb{N}^n \quad \mu_\alpha \ast g(x) = 0 \quad \forall x \in \mathbb{R}^n \quad \text{or} \quad |x| < R - r.
\]

where the \( \mu_\alpha \) are the distributions defined in (2.11). In particular, they involve derivatives (see [12], [3] for the details).

In order to find an inversion formula for the local Pompeiu transform we must make this principle a bit more precise. This depends on the following lemma.

**Lemma 5.** Let \( E \subseteq \overline{B}(0,r) \) and \( U = B(0,R), R > 2r \). Then for every \( x \) such that \( |x| < R - r, \alpha \in \mathbb{N}^n \), and any \( f \in \mathcal{E}(B(0,R)) \) we have

\[
\mu_\alpha \ast f(x) = \int_{SO(n)} \delta^{(\alpha)}(y), P(f)[\left[\begin{array}{cc}
-k^{-1} & x-ky \\
0 & 1
\end{array}\right]] dk,
\]

where \( M(n) \) is considered as the group of \((n+1)\times(n+1)\) matrices of the form

\[
\begin{bmatrix}
k x
\end{bmatrix}, \quad k \in SO(n), \quad x \in \mathbb{R}^n,
\]

and \( \mathbb{R}^n \) is identified to the affine subspace \( \{x_{n+1} = 1\} \) of \( \mathbb{R}^{n+1} \).

**Proof.** The identity (2.14) is an easy computation and the only thing to verify is that for \( |x| < R - r \), the image of \( E \) by \( \begin{bmatrix}
-k^{-1} & x-ky \\
0 & 1
\end{bmatrix} \) is contained in \( B(0,R) \) (recall that \( \text{supp} \delta^{(\alpha)}(y) = \{0\} \)). This is evident since \((-k^{-1})E \subseteq \overline{B}(0,r)\). Hence, translation by \( x \) keeps it inside \( B(0,R) \).

Given a compact set \( E \), let \( E_\tau = \{x + \tau : x \in E\} \) the translate of \( E \) by \( \tau \in \mathbb{R}^n \). Consider the functions

\[
\psi_\tau(t) := (R_{x_0} E_\tau)^\sim(t).
\]

It is easy to see they are real analytic with respect to the parameter \( \tau \). In
analogy with Lemma 4 we have

**Lemma 4'**. \( E \) has the global Pompeiu property if and only if there is an open set \( \mathcal{F} \subseteq \mathbb{R}^n \) such that the family of functions \( \{ \psi_t \}_{t \in \mathcal{F}} \) has no common zero.

Another minor technical lemma will be useful to us.

**Lemma 6.** Let \( P \) be a homogeneous harmonic polynomial of degree \( k \) and \( \sigma_P \) the normalized measure on the sphere \( \partial B(0, \rho) \) \( (0 < \rho) \), considered as a radial distribution in \( \mathbb{R}^n \). Then, with \( |x| = r \), we have

\[
(2.15) \quad P(x) \sigma_P(r) = \frac{(-1)^k}{2^{k-1}(k-1)!} \frac{\rho^{2-n}}{\Omega_n} P\left( \frac{\partial}{\partial x} \right) \left( \rho^2 - |x|^2 \right)^{k-1} \chi_{B(0,\rho)}(x),
\]

where \( \Omega_n = 2\pi^{n/2}/\Gamma(n/2) \) is the surface of the unit sphere in \( \mathbb{R}^n \).

**Proof.** If we consider the Fourier transform of \( P(x) \sigma_P(r) \), we have

\[
\mathcal{F}(P(x) \sigma_P(r))(\zeta) = P\left( \frac{i}{\partial \zeta_1}, \ldots, \frac{i}{\partial \zeta_n} \right) \sigma_P(\zeta)
\]

\[
= i^k \frac{(n-2)/2 \pi^{n/2}}{\zeta} P\left( \frac{\partial}{\partial \zeta} \right) J_{(n-2)/2}(\rho \sqrt{\zeta_1^2 + \ldots + \zeta_n^2}).
\]

where \( J_{\nu}(z) = J_{\nu}(z)z^{-\nu} \). We can now apply an identity from [16, p. 126] to obtain

\[
P\left( \frac{\partial}{\partial \zeta} \right) J_{(n-2)/2}(\rho \sqrt{\zeta_1^2 + \ldots + \zeta_n^2}) = P(\zeta) \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^k J_{(n-2)/2}(\rho \sqrt{\zeta_1^2 + \ldots + \zeta_n^2})
\]

\[
= (-1)^k \rho^{2n} P(\zeta) J_{(n/2)+k-1}(\rho \sqrt{\zeta_1^2 + \ldots + \zeta_n^2}).
\]

On the other hand, the function \( J_{(n/2)+k-1}(\rho \sqrt{\zeta_1^2 + \ldots + \zeta_n^2}) \) is the Fourier transform of a rather simple function. Namely, if \( t = |\zeta| \),

\[
\int_{\mathbb{R}^n} (\rho^2 - |x|^2)^{k-1} \chi_{B(0,\rho)}(x) e^{-ix \cdot \zeta} dx
\]
\[
\psi(x) = \psi_{p, k}(x) = (\rho^2 - |x|^2)^{k-1} \chi_B(0, \rho)(x),
\]
we have
\[
\mathcal{F}(P(x) \sigma_p(x))(\zeta) = (-1)^{k-1} \frac{\Gamma(n/2)}{\pi^{n/2} \rho^{2-n}} P(\zeta) \hat{T}(\zeta)
\]
\[
= (-1)^{k} \frac{\Gamma(n/2) \rho^{2-n}}{\pi^{n/2} \rho^{2-n} (k-1)!} \mathcal{F}\left(\frac{\partial}{\partial x} T(r)\right)(\zeta).
\]
Therefore
\[
P(x) \sigma_p(x) = \frac{\rho^{2-n}}{\rho^n} \frac{(-1)^k}{(k-1)!} \mathcal{F}\left(\frac{\partial}{\partial x} T(r)\right)_{p, k}(r).
\]
which is the formula (2.15). \qed

In what follows we will need to determine the radial distribution in \( \mathbb{R}^n \),
\( \mu = \mu_{n, k, \alpha, \rho'} \) that has as Fourier-Bessel transform
\[
(2.16) \quad \tilde{\mu}(t) = j_{(n/2)+k-1}(\rho t)/(t^2 - \alpha^2),
\]
where \( \alpha \) is a zero of the numerator. It is easy to find the even function \( \varphi \)
in \( \mathbb{R} \) whose Fourier transform coincides with \( \tilde{\mu} \). Therefore, what we need is
to have a systematic way of obtaining \( \mu \) from \( \varphi \). In fact, this type of
relation underlies all the proofs about the local properties of the Pompeiu
transform.

We have the following setup. Let $\mathcal{S}_0'(\mathbb{R}^n)$ be the space of radial distributions in $\mathbb{R}^n$ and $(\mathcal{E}'(\mathbb{R})^\circ)^\vee$ the space of entire functions in the Paley-Wiener class which are radial when restricted to $\mathbb{R}^n$. Then, if $\mathcal{Y}_{n}$ denotes the Fourier transform in $\mathbb{R}^n$, $\mathcal{S}B_n$ the Fourier-Bessel transform in $\mathbb{R}^n$, we have

\begin{equation}
\begin{array}{ccc}
\mathcal{E}_0'(\mathbb{R}^n) & \xrightarrow{\mathcal{Y}_{n}} & (\mathcal{E}'(\mathbb{R}))^\circ \\
\gamma_n & \circ & (\mathcal{E}'(\mathbb{R}))^\circ \\
B_n & & S \\
\mathcal{E}_0'(\mathbb{R}) & \xrightarrow{\gamma_1} & (\mathcal{E}'(\mathbb{R}))^\circ \\
\gamma_1 & \circ & (\mathcal{E}'(\mathbb{R}))^\circ \\
\end{array}
\end{equation}

where $SF(t) = F(t,0,...,0) (= F(\zeta))$ for any $\zeta \in \mathbb{C}^n$ such that $\zeta_1^2 + ... + \zeta_n^2 = t$. $S^{-1}f(\zeta) = f(\sqrt{\zeta_1^2 + ... + \zeta_n^2})$. The isomorphism $B = B_n$ is defined by the commutative diagram and it is called a transmutation operator.

We remark that $B$ is an isomorphism of convolution algebras, as such it has been originally studied by Bochner, Leray, Levitan [11], [17].

It is easier to find first $B^{-1}$ explicitly. Namely, let $\mu$ be a smooth radial function of compact support in $\mathbb{R}^n$, then if $r = \sqrt{x_1^2 + ... + x_n^2}$

$$S\mathcal{G}_n(t) = \int_{-\infty}^{\infty} e^{-itx_1}dx_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(r)dx_2...dx_n.$$  

Comparing with the above diagram we see immediately that

$$\varphi(x_1) = B^{-1}\mu(x_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(r)dx_2...dx_n$$

$$= \frac{2\pi(n-1)/2}{\Gamma(n-1/2)} \int_{0}^{\infty} \mu(\sqrt{x_1^2+y^2})y^{n-2}dy$$

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We introduce a change of variable, so that

\[
\varphi(x_1) = \frac{n-1/2}{\Gamma(n-1/2)} \int_{x_1}^{\infty} \mu(\sqrt{s})(s^2-x_1^2)^{(n-3)/2} ds.
\]

Let \( \varphi(x_1) = \Phi(x_1^2), \ v = x_1^2 \), then

\[
\Phi(v) = \frac{n-1/2}{\Gamma(n-1/2)} \int_{v}^{\infty} \mu(\sqrt{s})(s-v)^{(n-3)/2} ds.
\]

To invert this expression, we distinguish two cases. First, the case \( n \) is odd, \( n = 2\ell + 1, \ \ell \geq 1 \). Then

\[
\mu(\sqrt{s}) = \frac{1}{\pi} \left[ \frac{d}{dv} \Phi \right]_{v=s}.
\]

Hence

(2.18) \[ \mu(s) = (B_{2\ell+1}\varphi)(s) = \frac{1}{\pi} \left[ \left( -\frac{1}{2x} \frac{d}{dx} \right)^\ell \varphi \right]_{x=s}^2. \]

The case of \( n = 2\ell, \ \ell \geq 1 \), is a bit harder. First we find, using Laplace transforms, that

\[
\mu(\sqrt{s}) = \frac{1}{\pi} \left[ \frac{d}{dv} \left( \int_{v}^{\infty} (t-v)^{-1/2} \Phi(t) dt \right) \right]_{v=s}.
\]

Returning to the original function \( \varphi \) we get

(2.19) \[ \mu(s) = (B_{2\ell}\varphi)(s) = \frac{1}{\pi} \left[ \left( -\frac{1}{2x} \frac{d}{dx} \right)^\ell \left( \int_{x}^{\infty} (t^2-x^2)^{-1/2} \varphi(t) dt \right) \right]_{x=s}. \]

We can now apply these formulas to find the distributions \( \mu = \mu_{n,k,\alpha,\rho} \).
defined by (2.16).

Let \( \nu = \frac{n-2}{2} + k \), \( 1/c_\nu := 2^\nu \sqrt{\pi} \Gamma(\nu+\frac{1}{2}) \), then

\[
J_\nu(\rho t) = c_\nu \rho^{-2\nu} \int_{-\rho}^{\rho} e^{-its} (\rho^2 - s^2)^{\nu-(1/2)} ds.
\]

Hence, if \( J_\nu(\rho a) = 0 \), we rewrite

\[
\frac{J_\nu(\rho t)}{t^2 - a^2} = \frac{1}{2a} \left[ \frac{J_\nu(\rho t)}{t-a} - \frac{J_\nu(\rho t)}{t+a} \right].
\]

\[
J_\nu(\rho t) = c_\nu \rho^{-2\nu} \int_{-\rho}^{\rho} e^{-its} e^{-1as} (\rho^2 - s^2)^{\nu-(1/2)} ds,
\]

and

\[
J_\nu(\rho t) = c_\nu \rho^{-2\nu} \int_{-\rho}^{\rho} e^{-its} e^{1as} (\rho^2 - s^2)^{\nu-(1/2)} ds.
\]

Therefore

\[
\frac{J_\nu(\rho t)}{t^2 - a^2} = \frac{1}{2a} \int_{-\rho}^{\rho} e^{-its} \psi_{\alpha,\nu}(s) ds,
\]

where \( \psi_{\alpha,\nu}(s) \) is the even function which, for \( s \geq 0 \), is given by

\[
(2.20) \quad \psi_{\alpha,\nu}(s) = 2 \int_{s}^{\rho} \sin \alpha(s-u)(\rho^2 - u^2)^{\nu-(1/2)} du.
\]

To find \( \mu \) we apply the transmutation operator \( B_n \) to the function

\[
\varphi_{\alpha,\nu} = \frac{c_\nu \rho^{-2\nu}}{2a} \psi_{\alpha,\nu}.
\]

For instance, for \( n = 2 \),

\[
(2.21) \quad \mu_{2,k,\alpha,\rho}(r) = -\frac{1}{\pi^2} e^{-k/\rho} \chi_{[0,\rho]}(r) \int_{r}^{\rho} (\tau^2 - r^2)^{-1/2} \left( \int_{\tau/\rho}^{1} (1-\zeta^2)^{k-(1/2)} \cos \alpha(\tau-\rho \xi) d\zeta \right) d\tau.
\]
Finally, we recall a simple property of the Bessel functions.

Lemma 7. Let $a_1, a_2, a_3 > 0$ and $m \in \mathbb{Z}^+$. Let

$$\theta(z) = J_{n/2}(a_1 z) J_{n/2}(a_2 z) J_{(n/2)+m-1}(a_3 z).$$

Then there are constants $L_m, A_m > 0$, such that for any integer $k \geq L_m$ there exists $\rho_k \in \{k, k+1\}$ so chosen that if either $|z| = \rho_k$ or $|\Im z| \geq 1$ then

$$|\theta(z)| \geq A_m |z|^{-(1/2)(3n+1+2m)} e^{(a_1^2+a_2^2+a_3^2)|\Im z|}.\tag{2.22}$$

Here $n \geq 2$ is considered fixed throughout. If we are given $\delta > 0$ in such a way that $a_j \in [\delta, \delta^{-1}]$ then the constants $A_m, L_m$ can be made to depend only on $\delta$ and $m$. They actually become explicit in the proof.

Proof. The function $\theta$ is an even entire function, therefore to prove (2.22) we can assume we have $\Re z \geq 0$. In this case we can use the following asymptotic development of the Bessel function $J_{\nu}$ $(\nu \geq 1)$ (see [13])

$$|J_{\nu}(z) - \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\pi}{4}(2\nu+1))| \leq 3 \sqrt{\frac{\pi}{2}} \frac{|\Im z|}{|z|^{3/2}} \left\{ \frac{14\nu^2-1}{8} \exp\left(\frac{\pi |4\nu^2-1|}{8|z|^3}\right) \right\}.\tag{2.23}$$

Clearly, when $|z| \geq \frac{\pi}{8} |4\nu^2-1|$ we obtain

$$|J_{\nu}(z) - \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\pi}{4}(2\nu+1))| \leq 3e \sqrt{\frac{\pi}{2}} |4\nu^2-1| |\Im z| |z|^{3/2}.$$  

On the other hand, if $V = \{(2\ell+1)\pi/2 : \ell \in \mathbb{Z}\}$, $d(z, V) = \min\{1, \text{dist}(z, V)\}$, then the cosine satisfies the Lojasiewicz inequality

$$|\cos z| \geq \frac{1}{\pi e} d(z, V) e^{|\Im z|}.$$  

Considering the expression (2.23) we see that we need to know how many zeros does the product $\cos(a_1 z - \frac{\pi}{4}(n+1))\cos(a_2 z - \frac{\pi}{4}(n+1))\cos(a_3 z - \frac{\pi}{4}(n+2m-1))$ have.
in an interval \([l, l+1]\); as we have at most \(\frac{a_1 a_2 a_3}{n}\) such zeros, we can find a value \(\rho \in [l, l+1]\) such that the distance from \(\rho\) to any of these zeros is at least \(\frac{1}{2} \frac{\pi}{\pi + a_1 a_2 a_3}\). Hence, on the circle \(|z| = \rho\), we have

\[
|\cos(a_1 z - \frac{\pi}{4}(n+1))| \geq \min\{1, a_1\} e^{a_1|\text{Im} z|}.
\]

On the other hand, if \(|\text{Im} z| \geq 1\), then if \(a_j = \min\{1, a_j\}\)

\[
|\cos(a_1 z - \frac{\pi}{4}(n+1))| \geq \frac{a_1}{n} e^{a_1|\text{Im} z|},
\]

that is, the previous inequality is valid both for \(|z| = \rho\) and for \(|\text{Im} z| \geq 1\). Therefore, on every such circle as well as on \(|\text{Im} z| \geq 1\) we have that

\[
|I_n(z)| \geq \frac{a_1 e^{a_1|\text{Im} z|}}{2e^{\sqrt{2\pi}(\pi + a_2 a_3)|a_1 z|^{1/2}}}
\]

if we assume \(|z| \geq \frac{3}{4} e^{2\pi(n^2 + a_2 a_3 + a_3)^2/\min^2\{a_1, a_2, a_3\}}\). The same kind of estimates hold for the other two factors in \(\theta\). Let

\[
L_m = \frac{3}{4} e^{2\pi(n + a_2 a_3)(n+2m)^2/\min^2\{a_1, a_2, a_3\}}
\]

then for \(l \geq L_m, |z| = \rho\) or for \(|\text{Im} z| \geq 1, |z| \geq L_m\) we have

\[
|\theta(z)| \geq \frac{a_1^{(n+1)/2} a_2^{(n+1)/2} a_3^{((n+1)/2) - m} a_1 a_2 a_3 e^{(a_1 + a_2 + a_3)|\text{Im} z|}}{[2e^{\sqrt{2\pi}(\pi + a_2 a_3)^2}]^3 |z|^{1/2}(3n+2m+1)}
\]

Remarks. 1) The sequence \(\rho\) depends both on the \(a_j\) and on \(m\).

2) We do not really use the full strength of the estimate (2.23). It would be enough to use Weber's estimates [19, § 7.33].

We conclude this preliminary section with upper bounds on the Bessel functions, which depend on the fact that the order \(\nu\) is either an integer or half an integer. From [19, §2.2 and §3.32] we have for \(\Re z > 0\) and \(q\) an
Integer \( t \geq 0 \)

\[
J_q(z) = \frac{1}{\pi} \int_0^\pi \cos(q\theta - z \sin \theta) d\theta
\]

\[
J_{q+1/2}(z) = (-1)^q (\frac{z}{2n})^q \int_0^\pi iz \cos \theta P_q(\cos \theta) \sin \theta d\theta,
\]

where \( P_q \) is the Legendre polynomial of degree \( q \). From [14, 8.917] we know that

\[
|P_q(\cos \theta)| \leq 1.
\]

Therefore

\[
(2.26) \quad |J_q(z)| \leq |\text{Im } z|, \quad |J_{q+1/2}(z)| \leq \sqrt{\frac{2}{\pi}} |z|^{1/2} |\text{Im } z|,
\]

which has the advantage of being a global inequality, but loses \( |z|^{1/2} \) and \( |z| \) respectively, from the asymptotic expansions. This is a minor loss for us, and we will use this estimate below.

3. Reconstruction of a function in the case of two disks.

Let \( Z_n = \{\xi/\eta : \xi > 0, \eta > 0, J_{n/2}(\xi) = J_{n/2}(\eta) = 0\} \). It is known that if \( r_1 > 0, r_2 > 0 \) and \( r_1/r_2 \notin Z_n \), then the Pompieu transform

\[
P : f \mapsto (x_B(0,r_1)^*f, x_B(0,r_2)^*f)
\]

is injective. Moreover, this condition on \( r_1/r_2 \) is also necessary for the injectivity [12]. In case \( r_1/r_2 \) is badly approximated by elements of \( Z_n \), that is, in case there are constants \( c > 0, N > 0 \) such that
one can construct explicitly distributions $\nu_1, \nu_2$ with $\text{supp} \, \nu_1 \subseteq \tilde{B}(0, r_2), \text{supp} \, \nu_2 \subseteq \tilde{B}(0, r_1)$, such that (8)

\[
(3.2) \quad \chi_{B(0, r_1)} * \nu_1 + \chi_{B(0, r_2)} * \nu_2 = \delta.
\]

Therefore, under the arithmetic assumption (3.1) any function (or distribution) $f$ in $\mathbb{R}^n$ can be reconstructed from the knowledge of its Pompeiu transform $Pf = (\chi_{B(0, r_1)} * f, \chi_{B(0, r_2)} * f)$. This process (3.2) is known as deconvolution. In case we only have a function $f$ defined in a ball $B(0, R)$, with $R > r_1 + r_2$, even if we use the deconvolvers $\nu_1, \nu_2$, we will only be able to reconstruct $f$ in $B(0, R-(r_1+r_2))$. For this reason, we will give a different kind of explicit inversion process for this local problem. It turns out that the arithmetic condition (3.1) will not be necessary, only $r_1/r_2 \in \mathbb{Z}$.

In particular, this answers on the affirmative a question posed by Zalcman [21] for the case $R = \infty$. In fact, our procedure is basically a method to approximate $f$ with an estimate of the error. In this sense, we are dealing with several possible procedures that depend on the way we want to represent a given unknown signal $f$. For the sake of concreteness we will describe this in detail when we represent $f$ in terms of the Fourier expansion in $B(0, R)$ by spherical harmonics. In a forthcoming paper we will explain the modifications necessary if one wants to use wavelets or the Radon transform on spheres.

To begin, let us point out that if $f$ is not $C^\infty$ in $B(0, R)$, then we can first consider $f * \varphi_{\epsilon}$, $\text{supp} \, \varphi_{\epsilon} \subseteq B(0, \epsilon)$, $0 < \epsilon < 1$, and since $Pf$ is given by convolutions we will have $P(f * \varphi_{\epsilon}) = Pf * \varphi_{\epsilon}$. The first component of $Pf$, $\chi_{B(0, r_1)} * f$, is known in $B(0, R-r_1)$, hence that of $P(f * \varphi_{\epsilon})$. 

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\( \chi_{B(0,r)} \), will be known in \( B(0,R-c-r) \). Similarly for the second component. Hence, replacing \( R \) by \( R-c \), and then taking \( c \) so small that \( R-c > r_1 + r_2 \), we can assume \( f \) is \( C^\infty \).

Given a function \( f \in C^\infty(B(0,R)) \), we have a Fourier expansion of the form

\[
(3.3) \quad f(x) = \sum_{m=0}^{\infty} \left( \sum_{k} a_{m,k}(r)S_{m,k}(\omega) \right), \quad x = rw, \quad \omega \in S^{n-1},
\]

where \( \{S_{m,k}\} \) is an orthonormal basis (with respect to the normalized measure of the unit sphere) of the spherical harmonics of degree \( m \). That is, 
\( r^m S_{m,k}(\omega) = H_{m,k}(x) \) is a harmonic homogeneous polynomial of degree \( m \). The dimension of the vector space \( \text{Span}(\{S_{m,k}\}) \) is \( O(m^{n-2}) \) [18]. The coefficients

\[
a_{m,k}(r) = \int_{S^{n-1}} f(r\omega)S_{m,k}(\omega)d\sigma_1(\omega)
\]

satisfy an inequality of the form

\[
|a_{m,k}(r)| \leq C \frac{r^N}{N^m}
\]

for any \( N > 0 \). Another way to write down this formula is as the action of the distribution \( r^{-m}\sigma_r(x) \) on a smooth function

\[
(3.4) \quad a_{m,k}(r) = \langle r^{-m}\sigma_r(x), H_{m,k}(x)f(x) \rangle = \langle r^{-m}H_{m,k}(x)\sigma_r(x), f(x) \rangle
\]

\[
= (-1)^m r^{2-n-m} \frac{1}{2^{m-1}(m-1)!} \Omega_n \int_{r,m} (|x|, f(x))
\]

where, as given by Lemma 6,

\[
T_{r,m}(|x|) = (r^2 - |x|^2)^{m-1} \chi_{B(0,r)}(x).
\]

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the idea now will be to write \( T_{r,m} \) as

\[
T_{r,m} = g_1 \ast \chi_{B(0,r_1)} + g_2 \ast \chi_{B(0,r_2)}
\]

with control on the supports of \( g_1 \) and \( g_2 \). In this way we reduce the problem to a decomposition of radial functions. We fix some notation first.

Choose a strictly increasing sequence of positive numbers \( \epsilon_k \) with limit

\[
\lim_{k \to \infty} \frac{R}{r_1 + r_2} = 1,
\]

and a corresponding increasing sequence of radii

\[
R_k = (r_1 + r_2)(1 + \epsilon_k), \quad k \geq 1, \quad R_0 = 0.
\]

We want to decompose the radial distribution \( T_{r,m} \), and the decomposition will depend on the value \( k \) such that \( r \in [R_{k-1}, R_k] \). We recall that the Fourier-Bessel transform of \( T_{r,m} \) is given by

\[
\mathcal{T}_{T_{r,m}}(t) = (2\pi^n)^{n/2} (2\pi)^{m-1} j_{(n/2)+m-1}(r t).
\]

**Proposition 8.** Let \( r_1, r_2 > 0, r_1/r_2 \notin \mathbb{Z}_n, r_1 + r_2 < R \), \( \{\epsilon_k\} \) fixed as above. For every \( r, m \) with \( R_{k-1} \leq r < R_k \) there are two sequences of radial distributions \( \mu_t, \nu_t \) whose Fourier-Bessel transforms satisfy

\[
|j_{(n/2)+m-1}(r z) - (j_{n/2}(r_1 z) \mu_t(z) + j_{n/2}(r_2 z) \nu_t(z))| \leq c' \left( \frac{1}{t} \right) r^{-m+3/2} \|z\|^{-m+3/2} e^{r \epsilon_k ||m z||}
\]

for \( r \geq c m^2 \)

\[
|\mu_t(z)| \leq c r^{-m+3/2} \|z\|^{-m+2} e^{r \epsilon_k ||m z||}
\]

\[
|\nu_t(z)| \leq c r^{-m+3/2} \|z\|^{-m+2} e^{r \epsilon_k ||m z||}
\]

In this statement, we use the notation \( \|z\| = \max(1,|z|) \), \( c, c' \) being strictly positive constants depending only on \( r_1, r_2, R, \epsilon_1 \) and \( n \), while \( c'' \)

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depends also on $\ell$ and $m$. The quantity $\beta_k$ denotes $\epsilon_k(r_1+r_2)$.

**Proof.** We are going to apply Lemma 7 with $a_1 = r_1$, $a_2 = r_2$, $a_3 = \beta_k$. The corresponding value $L_1 \leq 3 e^{2\pi(n+2m)}(n+2m)^2 \leq c m^2 \epsilon_k^2$. We let $q$ be an even integer to be fixed later on. To simplify the notation we denote $f(z) = J_{(n/2)+m-1}(rz)$.

The idea of the proof is to profit from the simple fact

\[(3.8)\quad \frac{1}{2\pi i} \int_{|\zeta|=\rho_1} \frac{f(\zeta)}{\zeta-z} \, d\zeta = \begin{cases} f(z) & \text{if } |z| < \rho_1 \\ 0 & \text{if } |z| > \rho_1. \end{cases} \]

The usual method of interpolation theory consists in rewriting \((3.8)\) as follows

\[(3.9)\quad f(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho_1} \frac{f(\zeta)}{\zeta^q(\zeta)} \frac{(\zeta^q(\zeta) - z^q(\zeta))}{\zeta-z} \, d\zeta + \frac{1}{2\pi i} z^q(\zeta) \int_{|\zeta|=\rho_1} \frac{f(\zeta)}{\zeta^q(\zeta)} \frac{d\zeta}{\zeta-z} \quad \text{for } |z| < \rho_1\]

and

\[(3.10)\quad \frac{1}{2\pi i} \int_{|\zeta|=\rho_1} \frac{f(\zeta)}{\zeta^q(\zeta)} \frac{(\zeta^q(\zeta) - z^q(\zeta))}{\zeta-z} \, d\zeta = -\frac{z^q(\zeta)}{2\pi i} \int_{|\zeta|=\rho_1} \frac{f(\zeta)}{\zeta^q(\zeta)} \frac{d\zeta}{\zeta-z} \quad \text{for } |z| > \rho_1.\]

We define an even entire function $H_\ell$ by

\[(3.11)\quad H_\ell(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho_1} \frac{f(\zeta)}{\zeta^q(\zeta)} \frac{(\zeta^q(\zeta) - z^q(\zeta))}{\zeta-z} \, d\zeta,\]

that is, the interpolation of $f$ on the zeros of $z^q(z)$. Note that $H_\ell$ is
even independently of the parity of the integer \( q \); we took \( q \) even to simplify the computations below, but this is not necessary.

We are going to show the family \( \{ H_{\ell} \}_{\ell \geq 1} \) is a bounded family in \( \hat{\mathcal{G}}'(\mathbb{R}) \). For that purpose we recall the upper bounds for \( f \) and \( \theta \) that we obtain from (2.26),

\[
|f(z)| \leq e^{r_{n}-(n-3)/2-m_{n} \| z \|^{-(n-3)/2} - n^{m}} e^{r_{1}} \| z \|^{m} z |
\]

(3.12)

\[
|\theta(z)| \leq e^{R_{k}(r_{1}r_{2})^{-1/2} - ((n-3)/2) - m_{n} \| z \|^{-1/2} - m} e^{R_{k} \| z \|^{m} z} \]

(3.13)

We note that for \( n \) even these two estimates are too big by a power \( \| z \|^{1/2} \) and \( \| z \|^{3/2} \) respectively. Also the constants \( e^{r} \) and \( e^{R_{k}} \) can be replaced by \( e^{R} \). Choose \( \ell \geq cm^{2}/r_{k}^{2} \geq 1_{m} \geq 2 \), then for \( |\zeta| = \rho_{\ell} \) we have:

\[
|\theta(\zeta)| \geq c_{1} \beta^{-((n-1)/2) - m - (1/2)(3m+2m+1)} e^{R_{k} \| z \|^{m} z},
\]

where \( c_{1} = c_{1}(r_{1}, r_{2}, R) \) is given by

\[
c_{1} = \frac{c_{1}(r_{1}, r_{2}, R)}{(2eV\pi(n+R))^{3}}
\]

(4.22)

For \( |z| > \rho_{\ell} \) we can use (3.10) to estimate \( H_{\ell} \). We let \( q = n + 3 \) if \( n \) is odd, \( q = n + 4 \) if \( n \) is even, then

\[
|H_{\ell}(z)| \leq e^{r_{n}((n-1)/2) + m_{n} \| z \|^{-(n-3)/2} - m} e^{q \| z \|^{m} \| z \|^{m-3}}
\]

\[
\leq c_{2} e^{2R_{k} r_{1} r_{2} \beta_{k} (2eV\pi(n+R))^{3}} \leq c_{2}'(r_{1}, r_{2}, R) = c_{2}'
\]

where

\[
c_{2} \leq \frac{c_{2}}{\min(1, r_{1}, r_{2}, \beta_{1})^{3}} \leq c_{2}'(r_{1}, r_{2}, R) = c_{2}'
\]
(we have taken into account here the fact that the upper estimate for \( \Phi \) can be improved by \( |z|^{3/2} \) when \( n \) is even). For \( |z| < \rho_\ell \) we use (3.9), so that

\[
|H_\ell(z)| \leq |f(z)| + c_2 \rho_\ell^{-(13/2)-m} \frac{||z||^{-(n/2)-m+(11/2)} e^{R_k |\text{Im} z|}}{\rho_\ell^{|z|}}.
\]

These estimates are fine when \( |z| \leq \ell - 1 \) or \( |z| \geq \ell + 1 \), since then \( |z| - \rho_\ell \geq 1 \). In this case we have

\[
|H_\ell(z)| \leq 2 c_2 R^{-(13/2)-m} ||z||^{-(n/2)-m+(11/2)} e^{R_k |\text{Im} z|}.
\]

In the remaining annulus, \( \ell - 1 \leq |z| \leq \ell + 1 \), we can use the maximum principle to obtain

\[
|H_\ell(z)| \leq c^n \rho_\ell^{-(13/2)-m} ||z||^{-(n/2)-m+(11/2)} e^{R_k |\text{Im} z|}.
\]

The new constant \( c^n \) is not substantially different from \( c_2 \), and it is independent of \( m \) since if \( m + \frac{n}{2} \geq 11 \), \( \frac{||z||^{(n/2)+m-(11/2)}}{(\ell-1)^{(n/2)+m-(11/2)}} \leq c_3 \), independent of \( m \) since \( \ell \geq c \ell^2 / \ell^2 \). The other values of \( m \) are handled by the same reasoning. We conclude that \( \{H_\ell' \} \) is bounded in \( \text{Exp}(C) = \{\text{entire functions of exponential type}\} \). To conclude that \( \{H_\ell\} \) is bounded in \( \mathcal{E}'(\mathbb{R}) \) we need estimates on the real axis. From (3.14) we know that if \( x \in \mathbb{R} \setminus \ell - 1, \ell + 2 \) then

\[
|H_\ell(x)| \leq 2 c_2 R^{-(13/2)-m} ||x||^{-(n/2)-m+(11/2)}.
\]

As it was done in [2] we write

\[
H_\ell(x) = H_\ell(x) - H_{\ell+3}(x) + H_{\ell+3}(x),
\]

hence, by (3.17), the last term satisfies the correct inequality when \( x \in [\ell - 1, \ell + 2] \), and we only need to estimate the difference \( (H_\ell - H_{\ell+3})(x) \). Let \( \Gamma_\ell \) be the Jordan curve which is the boundary of the region \( \{\text{Re} z \geq 0, \text{Re} z \leq \ell \} \).
\[\rho \leq |z| \leq \rho_{e_3}, \ |\text{Im} \ z| \leq 1 \] and \( \Gamma' \) its symmetric with respect to the imaginary axis. They are chosen with a convenient choice of orientation so that
\[
H_\ell(x) - H_{\ell+3}(x) = \frac{1}{2\pi i} \left[ \int_{\Gamma} + \int_{\Gamma'} \right] \frac{r(\xi)(\xi q(\xi) - xq(x))}{\xi q(\xi)(\xi - x)} \, d\xi.
\]

For \( \xi \in \Gamma_\ell \) and \( x \in [\ell-1, \ell+2] \), we have
\[
\left| \frac{\xi q(\xi) - xq(x)}{\xi - x} \right| \leq \max \left| \frac{d}{dw} q(w) \right|_{|w-x| \leq 5} \leq c_3' e^{6R (r_1 r_2)^{-\frac{n}{2}} - \frac{(n-3)}{2} - \frac{m}{2} - \frac{n}{2} - m + \frac{11}{2}};
\]
by the same reasoning that led to (3.16) the constant \( c_3' \) is independent of \( m \). For \( \xi \in \Gamma'_\ell \) and \( x \in [\ell-1, \ell+2] \), we can use
\[
\left| \frac{\xi q(\xi) - xq(x)}{\xi - x} \right| \leq \max \left| \frac{d}{dw} q(w) \right|_{|w-x| \leq 5} \leq c_3' e^{6R (r_1 r_2)^{-\frac{n}{2}} - \frac{(n-3)}{2} - \frac{m}{2} - \frac{n}{2} - m + \frac{11}{2}}.
\]
We conclude that for \( x \in [\ell-1, \ell+2] \), we have
\[
|H_\ell(x)| \leq c_3' r^{-\frac{n-3}{2} - m} x^{-\frac{n}{2} - m + \frac{11}{2}},
\]
which is therefore valid everywhere. By applying the proof of the Phragmén-Lindelöf theorem to the function \( (z + \frac{1}{m})(n/2) + m - (11/2) \rho_k z H_\ell(z) \) on the quadrant \( \text{Re} z \geq 0, \text{Im} z \geq 0 \) we see that
\[
|H_\ell(z)| \leq c_4' r^{-(n-3)/2 - m} z^{-(n/2) - m + (11/2)} e^{R_k \text{Im} z},
\]
where \( c_4' \) is independent of \( m \). This argument works in the other three quadrants. This shows that this sequence \( \{H_\ell\}_\ell \) is bounded in \( \mathcal{B}'(\mathbb{R}) \). Moreover, we can show its limit is \( f \) and one can estimate the error by the previous
procedure.

If $|z| < \rho$, formula (3.10) yields (see (3.14)):

$$(3.20) \quad |H_{\ell}(z) - f(z)| \leq c_2 r^{-((n-3)/2)-m} \frac{\|z\|-(n/2)-m+(11/2)}{\rho^2 - |z|^2} e_{R_k} |\operatorname{Im} z|$$

$$\leq 2 c_2 r^{-((n-3)/2)-m} |z|^{-(n/2)-m+(13/2)} e_{R_k} |\operatorname{Im} z|$$

if $|z| \leq \frac{\nu}{2}$. For $|z| \geq \frac{\nu}{2}$ we have from (3.19) and the estimates for $f$ that

$$(3.21) \quad |H_{\ell}(z) - f(z)| \leq |H_{\ell}(z)| + |f(z)|$$

$$\leq \frac{c_4'}{\ell} r^{-((n-3)/2)-m} |z|^{-(n/2)-m+(13/2)} e_{R_k} |\operatorname{Im} z|,$$

$c_4'$ is independent of $m$ and bigger than $2c_2$, hence the last estimate is valid everywhere as stated in (3.5).

We can describe $\tilde{u}_{\ell}$ and $\tilde{v}_{\ell}$ explicitly using the Residue Theorem. Let us do it for the case the sequence $(c_k)$ has been chosen with the additional requirement that the pairs $j_{n/2}(r_j z)$ and $j_{(n/2)+m-1}(\beta_k z)$ $(j = 1, 2)$ have no common zeros. We note that if there were a common zero the multiplicity of this zero for the product of the two functions would be exactly two and the formula below can be modified easily. From the definition of $H_{\ell}$ we can write for example when $n$ is odd

$$H_{\ell}(z) = z^{n+3} j_{n/2}(r_1 z) j_{n/2}(r_2 z) g_3, \ell(z) + z^{n+3} j_{n/2}(r_1 z) j_{(n/2)+m-1}(\beta_k z) g_2, \ell(z)$$

$$+ z^{n+3} j_{n/2}(r_2 z) j_{(n/2)+m-1}(\beta_k z) g_1, \ell(z) + o(z) P(z),$$

where

$$g_1, \ell(z) = \sum_{|a| < \rho} \frac{f(a)}{a^{n+3} j_{n/2}(r_2 a) j_{(n/2)+m-1}(\beta k a)} j_{n/2}(r_1 a) j_{(n/2)+m-1}(\beta k a),$$

$$j_{n/2}(r_1 a) = 0.$$
and analogous formulas for $g_{2,\ell}$ and $g_{3,\ell}$. $P$ is the polynomial of degree $n + 2$ (in fact, $P$ must be an even function) given by

$$P(z) = \text{Res}_{\zeta=0} \left[ f(\zeta) \frac{z^{n+2} + \zeta^{n+3} + \ldots + \zeta^{n+2}}{\zeta^{n+3} \theta(\zeta)} \right].$$

We define

$$\tilde{\mu}_\ell(z) = z^{n+3} j_{n/2}(r_1 z) g_{3,\ell}(z) + z^{n+3} j_{n/2}(r_1 z) g_{2,\ell}(z),$$

$$\tilde{\nu}_\ell(z) = z^{n+3} j_{n/2}(r_1 z) g_{1,\ell}(z) + P(z) j_{n/2}(r_1 z) j_{n/2}(r_2 z).$$

Hence

$$H_\ell(z) = \tilde{\mu}_\ell(z) j_{n/2}(r_1 z) + \tilde{\nu}_\ell(z) j_{n/2}(r_2 z),$$

and the estimates (3.6) and (3.7) follow from the asymptotic estimates of the Bessel functions. Even though we need to replace $n + 3$ by $n + 4$ in case $n$ is even, the estimates remain the same by the remark made after (3.13).

The previous proposition leads to the following

**Theorem 9.** Let $r_1, r_2 > 0$, $r_1 / r_2 \in \mathbb{Z}$, $r_1 + r_2 < R$, $\{c_k\}_{k \geq 1}$ a strictly increasing sequence of positive numbers with limit $\frac{R}{r_1 + r_2} - 1$, $R_k = (r_1 + r_2)(1 + c_k)$ $(k \geq 1)$, $R_0 = 0$. For any $r > 0$, $r \in [R_{k-1}, R_k]$, and $S$ a normalized spherical harmonic of degree $m$ there are two sequences of distributions $\psi_\ell, \phi_\ell$ of order $\leq n + 3$ with compact support in $B(0, R - r_1)$ (respectively $B(0, R - r_2)$) such that for $\ell \geq m^2$ and any function $f$ in $C^\infty(B(0, R))$

$$\int_{S^{n-1}} f(rw) S(\omega) d\sigma(\omega) - \langle \psi_\ell, \chi_{B(0, r_1)} f \rangle - \langle \phi_\ell, \chi_{B(0, r_2)} f \rangle$$

$$\leq \frac{\gamma}{\ell} (R-r)^{-(n-3)/2} \max_{|x| \leq R_k} |\partial^\alpha f(x)|,$$

$$\gamma = \max_{|x| \leq R'} |\partial^\alpha f(x)|.$$
where \( N = \left( \frac{1}{2}(n+13) \right) + 1 \), \( R'_k = \frac{2}{3} R + \frac{1}{3} R_k \) and \( \gamma > 0 \) is a constant depending on \( r_1, r_2, R, n, \varepsilon_1 \).

**Proof.** Let \( m \geq 1 \) and let \( H \) be the harmonic polynomial homogeneous of degree \( m \) such that \( H(\rho \omega) = \rho^m S(\omega), \ \omega \in S^{n-1}, \ \rho > 0 \). From (3.4) we have

\[
\int_{S^{n-1}} f(rw) S(w) d\sigma(w) = \frac{r^{2-n-m}}{2^{m-1}(m-1)! \Omega_n} <T(\|x\|), \frac{\partial}{\partial x} f(x)>,
\]

where \( T(\|x\|) = (r^2 - |x|^2)^{m-1} \chi_{B(0,r)}(x) \). We have seen that the Fourier-Bessel transform of \( \frac{r^{2-n-m}}{2^{m-1}(m-1)! \Omega_n} T(\|x\|) \) is given by

\[
t \mapsto 2^{(n-2)/2} \Gamma(\frac{n}{2}) r^m j_{(n/2) + m-1}(rt).
\]

We can apply Proposition 8 and find explicit radial distributions \( \mu_t, \nu_t \) with \( \text{supp } \mu_t \subseteq B(0,R_k-r_1), \text{supp } \nu_t \subseteq B(0,R_k-r_2) \) such that the radial distribution \( \tau_t \), whose Fourier Bessel transform is given by

\[
\tilde{\tau}_t(z) = 2^{(n-2)/2} \Gamma(\frac{n}{2}) r^m \left[ j_{(n/2) + m-1}(rz) - j_{n/2}(r_1 z) \tilde{\mu}_t(z) - j_{n/2}(r_2 z) \tilde{\nu}_t(z) \right]
\]

satisfies

\[
|\tilde{\tau}_t(z)| \leq c \frac{r^{-(n-3)/2}}{z^\|z\|-(n/2)-m+(13/2)} R_k \text{Im } z \quad (t \geq \text{const. } m^2).
\]

If we define the distributions

\[
\psi_t(x) = \frac{(-1)^m r^m}{r^n \Omega_n} \frac{\partial}{\partial x} H(\frac{\partial}{\partial x}) \mu_t(|x|)
\]

\[
\varsigma_t(x) = \frac{(-1)^m r^m}{r^n \Omega_n} \frac{\partial}{\partial x} H(\frac{\partial}{\partial x}) \nu_t(|x|)
\]

then

4.27
\[
\int_{S^{n-1}} f(r\omega)S(\omega)d\sigma(\omega) - \langle \nabla_t \chi_{B(0, r_1)} * f \rangle - \langle \nabla_t \chi_{B(0, r_2)} * f \rangle = \langle \nabla_t H_{\frac{\partial}{\partial x}} f \rangle.
\]

We will now estimate \( \langle \nabla_t H_{\frac{\partial}{\partial x}} f \rangle \) using Plancherel's formula. Let \( \psi_k \) be a smooth radial function which is identically one on the ball \( B(0, \frac{1}{3}R + \frac{2}{3}R_k) \), with support in \( B(0, \frac{2}{3}R + \frac{1}{3}R_k) \). One can find them in such a way that

\[
\max_{x, |\alpha|=q} \frac{\delta^\alpha}{\delta x^\alpha} \psi_k(x) \leq \tau_q (R-R_k)^{-q} \leq \tau_q (R-R)^{-q}.
\]

The constants \( \tau_q \) are independent of \( R \) and \( k \). Then

\[
|\langle \nabla_t H_{\frac{\partial}{\partial x}} f \rangle| = |\langle \nabla_t H_{\frac{\partial}{\partial x}} (\psi_k f) \rangle| \leq \int_{\mathbb{R}^n} \tau_k (|\xi|)(\xi)(\psi_k f)(\xi) d\xi 
\]

\[
\leq c\tau^{-\frac{n-3}{2}} \int_{\mathbb{R}^n} \|\| \|\|^{(n/2)+(11/2)} \int_{S^{n-1}} |S(\omega)|^2 d\sigma(\omega) dp 
\]

\[
\leq c'\tau^{-\frac{n-3}{2}} \left( \int_{S^{n-1}} |S(\omega)|^2 d\sigma(\omega) \right)^{1/2} \int_0^\infty \|\|^{(n/2)+(11/2)} \max_{\omega \in S^{n-1}} |(\psi_k f)(\omega)| dp.
\]

On the other hand

\[
|(\psi_k f)(\xi)| \leq \tau'_N \|\xi\|^{-N}(R-r)^{-N} \max_{|x| \leq R'} \frac{\delta^\alpha}{\delta x^\alpha} f(x),
\]

where \( N \geq \frac{n+13}{2} \) and \( R' = \frac{2}{3}R + \frac{1}{3}R_k \).

This concludes the proof of Theorem 9 for \( m \geq 0 \). For \( m = 0 \) everything works the same way, except that \( T \) is now the normalized measure of the sphere of radius \( r \), with Fourier-Bessel transform equal to \( \int_{S(n-2)/2} j_{(n-2)/2}(rt) \) up to a constant.

The estimate (3.23) leaves us with a little problem for small values of
r. In the case \( 0 \leq r < R - (r_1 + r_2) \), we can use an explicit formula to approximate \( \delta \) in the form

\[
\delta \approx \chi_B(0,r_1)^{a_1} \chi_B(0,r_2)^{a_2},
\]

\( a_j \) radial distributions of fixed order, \( \text{supp} \ a_1 \subseteq \tilde{B}(0,r_2) \), \( \text{supp} \ a_2 \subseteq B(0,r_1) \). Hence we compute directly

\[
f(x) \approx [a_1 \chi_B(0,r_1)]^B + [a_2 \chi_B(0,r_2)]^B
\]

for \( |x| < R - (r_1 + r_2) \), the precise estimate of the error being of the form as in (3.21) (without the dependence on \( r \) and \( R - r \)); we use as a function \( \theta \) just the product \( j_{n/2}(r_1 z) j_{n/2}(r_2 z) \).

If we choose the sequence \( \epsilon_k \) in Theorem 9 so that every zero of \( \Theta(z) = j_{n/2}(r_1 z) j_{n/2}(r_2 z) j_{(n/2)+m-1}(\beta_k z) \) is simple, the distributions \( \mu_{\ell}, \nu_{\ell} \), and a fortiori \( \mu_{\ell} \) and \( \nu_{\ell} \) can be made completely explicit. We only need to know the explicit form of a radial distribution whose Fourier-Bessel transform has the form

\[
\frac{j_{(n/2)+m-1}(\rho z)}{z^2 - \alpha^2}, \quad (m \geq 1)
\]

when \( \alpha \) is a zero of \( j_{(n/2)+m-1}(\rho z) \).

These distributions have been computed in section 2; recall for example that when \( n = 2 \), \( \mu_{m,2,0}(t) = \frac{j_{(n/2)+m-1}(\rho t)}{t^2 - \alpha^2} \), with

\[
\mu_{m,2,0}(r) = -\frac{1}{2\pi} \chi_{[0,\rho]}(r) \int_0^\rho (\tau^2 - r^2)^{-1/2} \left[ \int_1^{(1 - \xi^2)\alpha^2(1/2)} \frac{d\xi}{\tau / \rho} \right] \cos \alpha(\tau - \rho \xi) d\xi.
\]

The sequence of approximations for the Fourier coefficients

\[
\frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{i\theta m} d\theta, \quad m \geq 1, \quad r \in [R_{k-1}, R_k]
\]

can be written as

4.29
\[
\left\{ \left< u_{r,m,\ell} \chi_{B(0, r_1)} \right> + \left< \bar{u}_{r,m,\ell} \chi_{B(0, r_2)} \right> \right\}_{\ell \in \mathbb{N}},
\]

where

\[
u_{r,m,\ell} = \frac{(-1)^m}{2\pi r_1^2} \left( \frac{\partial}{\partial x} \pm \frac{\partial}{\partial y} \right)^m \nu_{\ell}
\]

\[
u_{r,m,\ell} = \frac{(-1)^m}{2\pi r_2^2} \left( \frac{\partial}{\partial x} \pm \frac{\partial}{\partial y} \right)^m \nu_{\ell}
\]

and

\[
\mu_{\ell} = -\Delta^3 \left[ x_{B(0, r_2)} \ast \sum_{0 < \alpha < \rho_{\ell}} \frac{j_m (r_2 \alpha)}{\alpha^5 r_2 j_1 (r_2 \alpha) j_1 (r_2 \alpha)^2} \frac{J_m (r_1 \alpha)}{\beta_k j_1 (r_1 \alpha)} \frac{\mu_{\ell_1, \alpha, j_1}}{\beta_k^2} \right]^{m-1} \left( \beta_k^2 - 1 \right)^{m-1}
\]

\[
\nu_{\ell} = -\Delta^3 \left[ \frac{1}{2^{m-1}(m-1)! n} \frac{(\beta_k^2 - 1)^{m-1}}{\beta_k^{2m}} x_{B(0, r_k)} \ast \sum_{0 < \alpha < \rho_{\ell}} \frac{j_m (r_1 \alpha)}{\alpha^5 r_1 j_1 (r_1 \alpha) j_1 (r_1 \alpha)^2} \frac{J_m (r_2 \alpha)}{\beta_k j_1 (r_2 \alpha)} \frac{\mu_{\ell_1, \alpha, j_1}}{\beta_k^2} \right]^{m-1} \left( \beta_k^2 - 1 \right)^{m-1}
\]

The polynomial \( S_k + T_k z^2 + U_k z^4 \) is defined as

4.30
4. Local inversion of the Pompeiu transform for the case of a square.

In this section we will show how to reconstruct a function $f$ defined in the disk $B = B(0, R)$ of $\mathbb{R}^2$ in terms of its averages over squares of side $2a$, with $2\sqrt{2}a < R$. We will also see that this inversion procedure generalizes to practically any set $E$ with the Pompeiu property.

The idea is to reduce ourselves to the situation of the previous section. Let $Q$ be the square $[-a,a] \times [-a,a]$, $\chi_Q$ its characteristic function. Then

$$\left(4.1\right) \quad T = \frac{\partial^2 \chi_Q}{\partial x \partial y} = \delta(a,a) - \delta(-a,a) + \delta(-a,-a) - \delta(a,-a).$$

It is immediate that $\mathcal{R}_0 T = 0$. On the other hand, this is not true for some derivatives of $T$. We compute directly the Fourier-Bessel transform of $\mathcal{R}_0 \left( \frac{\partial^{k}}{\partial z^k} T \right)$. Let $t$ be a positive number, choose $\zeta \in \mathbb{C}^2$, $|\zeta| = t$, and identify $z = x + iy$ with $(x,y) \in \mathbb{R}^2$, then

$$\left[ \mathcal{R}_0 \left( \frac{\partial^{k}}{\partial z^k} T \right) \right]^{-1}(t) = \langle \mathcal{R}_0 \left( \frac{\partial^{k}}{\partial z^k} T \right)(z), e^{-iz \cdot \zeta} \rangle$$

$$= \langle \frac{\partial^{k}}{\partial z^k} T(z), \mathcal{R}_0(e^{-iz \cdot \zeta}) \rangle$$

$$= \langle \frac{\partial^{k}}{\partial z^k} T(z), \mathcal{R}_0(e^{-iz |z|}) \rangle$$

$$= \langle \frac{\partial^{k}}{\partial z^k} T(z), \mathcal{J}_0(t |z|) \rangle$$

$$= (-1)^k \langle T(z), \frac{\partial^{k}}{\partial z^k} \mathcal{J}_0(t |z|) \rangle$$

4.31
(as shown in Lemma 6) $t^{2k}<l(z)\left(\frac{z}{2}\right)^kJ_k(t|z|)$.

That is,

$$
(4.2) \quad \left[ \mathcal{R}_0 \frac{\partial^k}{\partial z^k} \right]^{-1} (t) = \begin{cases} 
0 & \text{if } k \equiv 2 \pmod{4} \\
4 \left(\frac{a}{2}\right)^k 2^k e^{-i(n/4)} k_j \left(\sqrt{2} at\right) & \text{if } k \equiv 2 \pmod{4}.
\end{cases}
$$

In particular,

$$
(4.3) \quad \left[ \mathcal{R}_0 \frac{\partial^2}{\partial z^2} \right]^{-1} (t) = -ia^2 t^4 J_2 (\sqrt{2} at) \\
(4.4) \quad \left[ \mathcal{R}_0 \left( \frac{\partial^6}{\partial z^6} \right) \right]^{-1} (t) = \frac{ia^6}{16} t^4 J_6 (\sqrt{2} at).
$$

The two functions above have only the origin as a common zero of order 4, by a well known property of the Bessel functions (see [19, p. 485]). On the other hand, the function $(\mathcal{R}_0 \chi_0)^{-1}(0) \neq 0$. If we use these three functions with a reasoning similar to that in the last section we would need $R > 3\sqrt{2} a$ to obtain an inversion formula which will be even a bit more complicated. (We will return to this point below.) We sidestep this question by working only with the radial distributions $\mathcal{R}_0 \left( \frac{\partial^2}{\partial z^2} \right)$ and $\mathcal{R}_0 \left( \frac{\partial^6}{\partial z^6} \right)$ to first reconstruct $\Delta^2 f$. Namely, consider the two radial distributions $\mu_1, \mu_2$ such that

$$
\tilde{\mu}_1(t) = -ia^2 J_2 (\sqrt{2} at) \\
\tilde{\mu}_2(t) = \frac{ia^6}{16} t^4 J_6 (\sqrt{2} at),
$$

whose Fourier Bessel transforms have no common zeros and which are supported in $B(0, \sqrt{2} a)$. One can modify Lemma 7 by considering the function

$$
\theta(t) = \tilde{\mu}_1(t) \tilde{\mu}_2(t) J_m (c2\sqrt{at}),
$$

with $0 < \varepsilon < R - 2\sqrt{a}$. The inequality (2.22) is replaced by
Whenever \( |z| = \rho'_l \) or \( |\text{Im } z| \geq 1, \ l \geq L'_m \), for some \( \rho'_m, L'_m > 0 \).

The proof of Theorem 9 furnishes distributions \( \psi^r_l, \phi^r_l \) such that for any function \( f \in C^\infty(B(0,R)) \)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta})e^{-im\theta}d\theta - <\psi^r_l, \mu_1*f> - <\phi^r_l, \mu_2*f> \leq \frac{2^r}{l} \max_{|\alpha| \leq N} \frac{\partial^{2|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} f(z) \\text{for } |z| \leq R'_k
\]

where, as before, the distributions \( \psi^r_l, \phi^r_l \) depend also on \( r, m \) have support in \( B(R - \sqrt{2a}) \), their order and the constant \( N \) do not depend on \( r \) or \( m \);
\( R'_k \) has the same meaning as in Theorem 9, that is \( R'_k = \frac{2}{3}R + \frac{1}{3} \sqrt{2a(1 + \frac{1}{k})} \).

Applying (4.6) to \( \Delta^2 f \), we can compute its Fourier coefficients in \( B(0,R) \) in terms of the functions \( \varphi_1 = \left[ R_0 \left[ \frac{\partial^2}{\partial z^2} \right] \right]*f \) and \( \varphi_2 = \left[ R_0 \left[ \frac{\partial^6}{\partial z^6} \right] \right]*f \) in \( B(0,R - \sqrt{2a}) \), since \( \Delta^2 \mu_1 = R_0 \left[ \frac{\partial^2}{\partial z^2} \right] \) and \( \Delta^2 \mu_2 = R_0 \left[ \frac{\partial^6}{\partial z^6} \right] \). By Lemma 5, we can compute the functions \( \varphi_1, \varphi_2 \) in terms of the Pompeiu transform of \( f \). In other words, we can explicitly recover the Fourier coefficients of \( \Delta^2 f \) in terms of \( P(f) \).

We need now to show how to compute the Fourier coefficients of \( f \) in terms of \( P(f) \) and \( \Delta^2 f \). This can be accomplished by observing that

\[
(R_0 \chi_Q)^*(0) = \text{area}(Q) = 4a^2
\]

and that \( (R_0 \chi_Q)^*(z) \) has an asymptotic behavior of the same type as that of the Bessel functions, as it was shown in [3, Section 6]. Hence we can apply the same procedure as above to the two radial distributions \( R_0 \chi_Q \) and \( \Delta^2 \delta_0 \).
and in this way, we can approximate the Fourier coefficients of \( f \) in \( B(0,R) \) as closely as we want in terms of the "data" \( P(f) \).

**Remark 1.** The procedure for the local inversion of the Pompeiu transform associated to a square is valid for a very general class of domains having the Pompeiu property. In fact, let \( E \) be a compact subset of \( \mathbb{R}^n \) having the global Pompeiu property. Let us assume that for some base point \( x_0 \), the boundary of \( E \) contains a hyperbolic point \( p \) in the sense of [3, section 2]. One can assume that \( \partial B(x_0,r) \cap E = \{p\} \), where \( r = \text{dist}(x_0,p) \) and \( E \subseteq \overline{B}(x_0,r) \). This is just a minimal regularity condition on the extremal point \( p \) of \( E \). By Proposition 5 in [3], we have that near the real axis we have the asymptotic development

\[
(4.7) \quad (\mathcal{R}_{x_0} \chi_E)(t) \sim \frac{\text{const}}{t \nu_0} \cos(t \nu_1),
\]

where \( \nu_0, \nu_1 \) are some positive constants related to the geometry of \( \partial E \) near \( p \). Outside a horizontal strip there is a corresponding good lower bound. This behavior is the same as that of the Bessel functions.

Let \( \mu_1 \) be \( \mathcal{R}_{x_0} \chi_E \). If we assume that \( R > 3r \), let \( \epsilon_0 > 0 \) be such that \( 3r + 3\epsilon_0 < R \). For any \( \epsilon > 0 \) we can consider the set \( E_\epsilon \) which is the translate of \( E \) along the unit direction \( \overrightarrow{x_0p} \) by \( \epsilon \), then

\[
(4.8) \quad (\mathcal{R}_{x_0} \chi_{E_\epsilon})(t) \sim \frac{\text{const}}{t \nu_0} \cos(t(r+\epsilon) \nu_1),
\]

since the only extremal point of \( E_\epsilon \) of interest is just \( p + \epsilon \overrightarrow{x_0p} \). It is then immediate that we can choose \( \epsilon \leq \epsilon_0 \) so that \( \tilde{\mu}_1 \) and \( (\mathcal{R}_{x_0} \chi_{E_\epsilon})(t) \) have only finitely many common zeros. Let us choose \( \epsilon'_0 \) with such a property, \( \nu_2 = \mathcal{R}_{x_0} \chi_{E_{\epsilon'_0}} \), and \( z_1, \ldots, z_N \) the common zeros of \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \). The argument in [3, Proposition 5] shows that one cannot only translate along the ray \( \overrightarrow{x_0p} \).
but even along a small open cone $\tau$ with vertex $0$ in such a way that the translate of $p$ remains the only extremal point in $E_{\tau} \cap \delta B(x_0,|p+\tau-x_0|)$ for $\tau \in \mathbb{I}$. By Lemma 4' the corresponding functions $\psi_\tau$ have no common zero if $|\tau| \leq \epsilon_0$.

For each $z_j$ we can choose distinct $\tau_j$ such that $(\mathcal{R}_{x_0} \chi_{E_{\tau_j}})(z_j) \neq 0$, hence there is a linear combination of these finitely many functions which will not vanish at any of the points $z_1, \ldots, z_N$. This linear combination $\mu_3$ will automatically have a good asymptotic development. The radial distributions (with respect to $x_0$) $\mu_1, \mu_2, \mu_3$ are supported by the balls $B(x_0, r_1)$, $B(x_0, r_2)$, $B(x_0, r_3)$ respectively, $r_1 + r_2 + r_3 < R$ and the method of the previous section applies with small modifications. Lemma 7 allows us to compute $\mu_j \cdot f$ in terms of $P(f)$, in this way one obtains an inversion formula in any ball of radius $R > 3r$.

**Remark 2.** In comparing the previous remark with the construction for the square one notes that the construction for the square could be further simplified if we knew that $(\mathcal{R}_{x_0} x_Q)^{\sim}$ and $\mathcal{R}_{x_0} \left( \frac{\partial^2}{\partial z^2} - I \right)^{\sim}$ have no common zeros. Though we believe this is true we have not been able to prove it.
References


CONCLUSIONS

We have demonstrated the feasibility of building super-resolution systems using multiple detectors. The deconvolution method provides a real time linear implementation of the reconstruction problem which is robust with respect to noise and perturbations of the overall system.

The next task that is starting to be carried out by doctoral students of Dr. Berenstein's in Maryland, Drs. Gay and Yger in Bordeaux, and Dr. Taylor in Michigan, is to incorporate postprocessing features adequate for particular tasks. For instance, in many problems we need not only to reconstruct the pixel by pixel values of signal to a desired degree of resolution but also we need to automatically identify it, e.g. using segmentations, Voronov diagrams, etc. How to incorporate this image (post) processing into the deconvolutions is the challenge.

Another thing that still remains to be implemented, since it requires considerable manpower, is to write a user friendly menu to design and simulate multidetector systems. It is for this task we feel the newer completely numerical algorithm of finding deconvolvers might be most useful.

The local Pompeiu problem that "peeled-off" the deconvolution problem presents new challenges. The nature of the inversion formulas for Chapter 4, indicates that the deconvolution and local deconvolution formulas must be well
adapted to implementation in terms of wavelets. If this were the case, it will also help to produce a significant data reduction and reduction of processing time in ATR and similar problems.

Finally, the purely algebraic problems of the size and degree of solutions to the Bezout equation have applications to robotics, control of distributed parameter systems, computational geometry, etc., and they show the unexpected payoffs of the use of the powerful methods of several complex variables in applied mathematics.