A Diffusion on a Fractal State Space

by

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FSU Technical Report Number M-812
US Army Research Office Technical Report Number D-108

September, 1989

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Research partially supported by the U.S. Army Research Office Grant number DAAL03-86-K-0094.

AMS 1980 subject classifications: 60J60

Key words: Diffusions, fractals

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is defined

Abstract: We define a fractal in the plane known as the Vicsek Snowflake by constructing a skeletal lattice graph and then rescaling spatial dimensions to give a sequence of lattices that converges to a fractal. By defining a simple random walk on the skeletal lattice and then rescaling both time and space, we define a sequence of random walks on the approximating lattices that converge weakly to a limiting process on the snowflake. ~~We show that~~ this limit has continuous sample paths and the strong Markov property, and that it is the unique diffusion limit of random walk on the snowflake in a natural sense. ~~We show that~~ this diffusion has a scaling property reminiscent of Brownian motion, and we introduce a coupling argument to show that the diffusion has transition densities with respect to Hausdorff measure on the snowflake.

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1. Introduction:

Construct a figure in the unit square by the following recursive procedure. Let $\mathcal{G}_0$ denote the unit square. Construct $\mathcal{G}_1$ by deleting from $\mathcal{G}_0$ four squares, each with edge length $\frac{1}{3}$, centered along the four edges of $\mathcal{G}_0$. $\mathcal{G}_1$ will consist of five squares with edge $\frac{1}{3}$ whose corners overlap. At stage n, $\mathcal{G}_n$ will consist of 5^n squares, each with edge length 3^{-n}. To construct $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$, take each square $\mathcal{S}$ composing $\mathcal{G}_n$, and delete the four square centered along the edges of $\mathcal{S}$ with edges of length 3^{-n-1}. $\mathcal{G}_{n+1}$ then consists of 5^{n+1} squares with edges of length 3^{-n-1}.

Take $\mathcal{G} = \cap_{n=1}^{\infty} \mathcal{G}_n$. Some easy topology shows that $\mathcal{G}$ is a closed connected set, with Lebesgue measure 0. In fact, it is not hard to show that $\mathcal{G}$ has finite Hausdorff $\log_3 5$-dimensional measure. In the spirit of Mandelbrot, $\mathcal{G}$ is a fractal with starter polygon $\mathcal{G}_1$. Extensive treatments of such fractal sets have been given by various authors. (See, for example Hutchinson[12] or Barnsley and Demko[4]).

A number of authors have treated the problem of constructing diffusions on nested fractals. Particular attention has been paid to diffusions on the Sierpinski gasket, a fractal constructed from a unit equilateral triangle by successsively deleting "middle" triangles. Goldstein[10] and Kusuoka[13] constructed a Brownian motion on the Sierpinski gasket, using a decimation-invariance property. Barlow and Perkins[3] have studied this Brownian motion comprehensively. Browniand motion on the Sierpinski gasket is broadly similar to the natural diffusion on the Vicsek snowflake, and the results of these authors are generally similar to those in the present work. I fully acknowledge the priority of their results. More recently, Lindstrom[14] has constructed a Brownian motion on any fractal set satisfying a general set of nesting axioms from a sequence of random walks, provided that the distribution of the random walk satisfies a non-degeneracy condition.

The first objective of this paper is to construct a diffuion on the Vicsek snowflake, starting from an non-degenerate random walk model. In an important respect, the problem of defining diffusions on the snowflake is more complicated than defining diffusions on the Sierpinski gasket. On the snowflake, one can define a variety of random walk models that are symmetric under the natural symmetries of the square. A natural question is whether one can construct a diffusion for any such model. Another is whether the diffusion on the fractal is unique such diffusion are unique. The snowflake seems to be the simplest nested fractal where such questions arise. For the snowflake, the answer is that if the random walk is not degenerate then the a unique diffusion limit exists independent of the underlying random walk model. The corresponding problem for general nested fractals remains unsolved at the time of this writing.

2. Constructing the Diffusion:

Consider the following system of transformations:

$$M_1 : x \to 3^{-1}x \qquad M_4 : x \to (3^{-1} \cdot x) + (2,0)$$

$$M_2 : x \to 3^{-1} \cdot x + (1,1) \qquad M_5 : x \to (3^{-1} \cdot x) + (0,2) \tag{2.1}$$

$$M_3 : x \to 3^{-1} \cdot x + (2,2)$$

By inspection, M_1, .., M_5 are strict contractions, with fixed points $(0,0)$, $(1\frac{1}{2}, 1\frac{1}{2})$, $(3,3)$, $(3,0)$ and $(0,3)$, respectively. For bounded subsets A of $\mathbf{R}^2$, define $M(A) = \cup_1^5 M_i(A)$. It is well-known to geometers that the transformation M has a unique compact invariant set, whose Hausdorff dimension may easily be computed as $\log_3 5$.(See, for example, Barnsley and Demko [4], Dubins and Freedman [6] or Hutchinson [12]). We will call this set the *bounded Vicsek snowflake*, and denote it by Γ_b.

For future reference, we establish the following definition. Let $M_{i_1}, ..., M_{i_n}$ be some sequence of the transformations $M_1, ..., M_5$. Let $\mathcal{S} = M_{i_1} \circ ... \circ M_{i_n}(\Gamma_b)$. We will call $\mathcal{S}$ a *square* of Γ_b. We will also need an unbounded version of our state space. Let $\Gamma = \cup_{n=0}^{\infty} 3^n \Gamma_b$. Γ also has Hausdorff dimension $\log_3 5$, and has the property that $\frac{1}{3}\Gamma = \Gamma$. We will call Γ the *unbounded Vicsek snowflake.*

We wish to construct a diffusion process on Γ and study its basic properties. We shall do this by defining random walks on a sequence of lattices that approximate Γ, which we now construct.

In the unit square, let U denote the complete graph on the corners of $[0,1]^2$. Let $\mathcal{V}(U)$ and $\mathcal{E}(U)$ denote the vertices and edges of U, respectively.

Define a new graph U_0 with vertex set $\mathcal{V}(U_0)$ and edge set $\mathcal{E}(U_0)$ by taking

$$\begin{aligned} \mathcal{V}(U_0) &= \mathcal{V}(U) \cup [\mathcal{V}(U) + (1,1)] \cup [\mathcal{V}(U) + (0,2)] \cup [\mathcal{V}(U) + (2,0)] \cup [\mathcal{V}(U) + (2,2)] \\ \mathcal{E}(U_0) &= \mathcal{E}(U) \cup [\mathcal{E}(U) + (1,1)] \cup [\mathcal{E}(U) + (0,2)] \cup [\mathcal{E}(U) + (2,0)] \cup [\mathcal{E}(U) + (2,2)] \end{aligned} \tag{2.2}$$

where the arithmetic is done componentwise. We call U_0 the *unit snowflake lattice*. Note that U_0 lies in the square $[0,3]^2$; we will call the points $(0,0), (0,3), (3,3)$, and $(0,3)$ the *outer corners* of U_0.

Inductively, we construct a sequence of graphs, using same procedure that yielded U_0 from U. That is, if $n > 0$, let

$$\begin{aligned}
\mathcal{V}(U_n) &= \mathcal{V}(U_{n-1}) \cup [\mathcal{V}(U_{n-1}) + (1,1)] \cup [\mathcal{V}(U_{n-1}) + (0,2)] \cup [\mathcal{V}(U_{n-1}) + (2,0)] \\
&\quad \cup [\mathcal{V}(U_{n-1}) + (2,2)] \\
\mathcal{E}(U_n) &= \mathcal{E}(U_{n-1}) \cup [\mathcal{E}(U_{n-1}) + (1,1)] \cup [\mathcal{E}(U_{n-1}) + (0,2)] \cup [\mathcal{E}(U_{n-1}) + (2,0)] \\
&\quad \cup [\mathcal{E}(U_{n-1}) + (2,2)]
\end{aligned} \tag{2.3}$$

As with U_0, say that $(0,0), (0, 3^{n+1}), (3^{n+1}, 3^{n+1})$, and $(0, 3^{n+1})$ are the outer corners of U_n.

Let $G = G_1 = \cup_{n=0}^{\infty} U_n$. Then G is an infinite graph, with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G)$, which we shall call the *unbounded snowflake lattice*, or, simply, the *snowflake lattice*. Let $\mathbf{0}$ denote the point $(0,0)$, and let $\mathbf{1}$ denote $(3,3)$.

We mention two key properties of G_1. First, if we let $\Gamma_\infty = \cup_{n=1}^{\infty} 3^{-n} \cdot \mathcal{V}(G)$, then Γ_∞ is dense in Γ. Thus, we may have some hope that a suitably scaled sequence of random walks will converge to some process on Γ_∞. Second, , $3 \quad \mathcal{V}(G) \subset \mathcal{V}(G)$, and $G \setminus 3 \cdot G$ contains no infinite connected component. We call this the "branching", "nesting" or *self-similarity* property of G. property of random walks on the lattices G_n.

Let x be a vertex of G, and suppose our particle is at x at time n. Let N_x be the number of points adjacent to x in G. Suppose our random walk is at x at time n. Then at time $n+1$, we choose a vertex adjacent to y according to the following distribution:

If $N_x = 3$:

$P[X_{n+1} = y] = p$	if x and y are diagonally adjacent
$P[X_{n+1} = y] = (1-p)/2$	if x and y are vertically or horizontally adjacent

If $N_x = 6$:

$$P[X_{n+1} = y] = p/2 \qquad \text{if } x \text{ and } y \text{ are diagonally adjacent}$$

$$P[X_{n+1} = y] = (1-p)/4 \qquad \text{if } x \text{ and } y \text{ are vertically or horizontally adjacent}$$

Here, $0 \leq p < 1$ is a fixed but arbitrary parameter. This defines a random walk on G, which we denote by X^p. Thus, if $X_0^p = (0,0)$ then $P[X_1^p = (1,1)] = p$ while $P[X_1^p = (1,0)] = (1-p)/2$. If $X_1^p = (1,1)$, then $P[X_2^p = (0,0)] = p/2$ and $P[X_2^p = (1,0)] = (1-p)/4 = P[X_2^p = (2,1)]$

We begin by studying the special case where $p = (1-p)/2 = \frac{1}{3}$, and $X_0^p = 0$. This will define a simple random walk on the graph G, starting from 0. We call this discrete-time Markov chain. which we will call *random walk on the snowflake lattice,*. For convenience, we will write $X^{1/3} = X$. o

Let T_n^1 and T_n^2 be the sequences of times between visits by X_n to distinct points of $3 \cdot G$ and $3^2 \cdot G$ respectively. Since $3^2 \cdot G \subset 3 \cdot G$, $T_j^2 = \sum_{a(j)}^{b(j)} T_k^1$. By the nesting property of the lattice, the distribution of $b(j)-a(j)$ is the same as the distribution of T_k^1, and the Markov property of X_n shows that $b(j)-a(j)$ and $T_k^1, k = a(j), ..., b(j)$ are independent. are independent random variables, equal in distribution to T. Thus, for each j T_j^2 has the distribution of the second generation of a branching process with offspring distribution equal to that of T.

Similarly, let T_i^n be the times between visits to distinct points of $3^n \cdot G$. A similar argument shows that for each i, T_i^n has t' · distribution of the n^{th} generation of a branching process, again with offspring distribution equal to that of T.

Let $f(u)$ be the generating function of T. By direct calculation, we can show that

$$f(u) = \frac{u^3}{(3-2u)(12-12u+u^2)}. \tag{2.4}$$

(See Section 4 for the details of the computation.) Differentiating f shows that $ET = 15$ and $\text{Var}(T) = 114$, so the branching process $\{T^n\}$ is obviously supercritical. Since T is always strictly greater than 0, $\{T^n\}$ has extinction probability 0. Since this branching property of the hitting times of X_n plays a key role in the remainder of this section, we will review some standard theory of branching processes.

Theorem 2.1. *Let Z_n be a branching process, with $Z_0 = 1$ and let f be the generating function of the offspring distribution. Suppose that $1 < f'(1-) = m < \infty$ and $f''(1-) < \infty$. Let $W_n = Z_n m^{-n}$. Then there exists a random variable W with $EW = 1$ such that $W_n \to W$ a. s. and in L^2 $P[W = 0] = P[Z_n = 0$ for some $n]$. If $\phi(u) = e^{-uW}$, then satisfies Abel's functional equation $\phi(u) = f(\phi(u/m))$.*

Proof: This is Theorems 1 and 2, and equation (5) in Athreya and Ney [2].

In particular, $W_n \to W$ in distribution.

The theorem implies that $15^{-n}T^n$ converges in distribution to some random variable W, with $EW = 1$. As T^n can never be 0, W is strictly positive almost surely, and $\phi(\lambda) = Ee^{-\lambda W}$ satisfies $\varphi(\lambda) = f(\varphi(15^{-1}\lambda)$.

For $m = 0, 1, \ldots$, define a stochastic process on Γ by $Y_m(t) = 3^{-m}X([15^m t])$. ($[x]$ denotes the greatest integer less than or equal to x). Observe that for each n, $Y_n(t)$ is a random walk on $3^{-n} \cdot G$. Let $D[0,\infty]$ be the set of functions $\omega : \mathbf{R}^+ \to \Gamma$ that are right continuous and have left hand limits for all t.

Theorem 2.2. *The sequence of processes $\{Y_n(t)\}_{t\geq 0}$ is tight in $D[0,\infty]$. If $Y_{n'}(t)$ is a subsequence of $\{Y_n(t)\}$ converging weakly to a process Y_t, then Y_t has continuous sample paths.*

The theorem follows from an estimate of the moments of the displacements $||Y_n(t) - Y_n(s)||$, which we state as a lemma.

Lemma 2.3. *For $n = 1, 2, \ldots, 0 \leq s < t < \infty, \gamma > 0$,*

$$E||Y_n(t) - Y_n(s)||^\gamma \leq 2\sqrt{2}^\gamma \cdot \left[3^{-n\gamma} + C \cdot (t-s)^{\rho\gamma}\right] \tag{2.5}$$

where $\rho = \log_{15} 3$ and C is a constant independent of n.

Proof: Let $\mathcal{Q} = \{q \in Q : q = p \cdot 15^{-n}, p, q \in \mathbf{Z}\}$. Since $Y_n(t)$ jumps only at points in $\mathcal{Q}$, it will suffice to estimate $E\|Y_n(q) - Y_n(r)\|^\gamma$, for arbitrary $q, r \in \mathcal{Q}$. We make the following displacement estimate for random walk $\{X_n\}$ on the snowflake lattice:

$$P[\|X_n - X_m\| > 2\sqrt{2} \cdot 3^k] \leq P[T_k < n - m] \tag{2.6}$$

To establish this estimate, we observe that $2\sqrt{2} \cdot 3^k$ is the diameter of two diagonally adjoining squares in the $3^k \cdot G$. If $\|X_n - X_m\| > 2\sqrt{2} \cdot 3^k$, then between epochs m and n, X_k must visit two distinct points of $3^k \cdot G$.

To use estimate (6), we write

$$\begin{aligned}
E\|Y_n(q) - Y_n(r)\|^\gamma & \\
&\leq (2\sqrt{2})^\gamma \cdot 3^{-\gamma n} P\left[\|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{-n}\right] \\
&\quad + (2\sqrt{2})^\gamma \cdot \sum_{i=-n}^{\infty} 3^{i\gamma} P\left[2\sqrt{2} \cdot 3^i \leq \|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{i+1}\right] && (7) \\
&\leq (2\sqrt{2})^\gamma \cdot 3^{-\gamma n} P\left[\|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{-n}\right] \\
&\quad + (2\sqrt{2})^\gamma \cdot (1 - 3^{-\gamma}) \cdot \sum_{i=-n}^{\infty} 3^{i\gamma} P\left[\|Y_n(q) - Y_n(r)\| > 2\sqrt{2} \cdot 3^i\right] && (8) \\
&\leq (2\sqrt{2})^\gamma \cdot 3^{-\gamma n} P\left[\|X(15^n \cdot q) - X(15^n \cdot r)\| \leq 2\sqrt{2}\right] \\
&\quad + (2\sqrt{2})^\gamma \cdot (1 - 3^{-\gamma}) \cdot \sum_{k=0}^{\infty} 3^{(k-n)\gamma} P\left[\|X(15^n \cdot q) - X(15^n \cdot r)\| > 2\sqrt{2} \cdot 3^k\right] && (9) \\
&\leq (2\sqrt{2})^\gamma \cdot \left[3^{-\gamma n} + (1 - 3^{-\gamma}) \cdot \sum_{k=0}^{\infty} 3^{(k-n)\gamma} P\left[15^k \cdot T_k \leq 15^{-k+n} \cdot (q - r)\right]\right] && (10)
\end{aligned}$$

Inequality (8) comes from substituting

$$\begin{aligned}
&P\left[2\sqrt{2} \cdot 3^i < \|Y_n(q) - Y_n(r)\| \leq 2\sqrt{2} \cdot 3^{i+1}\right] = \\
&\qquad P\left[2\sqrt{2} \cdot 3^i < \|Y_n(q) - Y_n(r)\|\right] - P\left[2\sqrt{2} \cdot 3^{i+1} < \|Y_n(q) - Y_n(r)\|\right]
\end{aligned}$$

into the estimate and rearranging the terms. (9) follows from the definition of $Y_n(q)$, and (10) is from estimate (6).

To estimate $P\left[15^k \cdot T_k \leq 15^{-k} \cdot 15^n \cdot (q-r)\right]$ we first estimate the Laplace transforms of $\{T_k\}$. Let ϕ_k and ϕ be the Laplace transforms of T_k and W, respectively. As T_k has the distributions of a branching process and $15^{-k}T_k \rightarrow W$, it is not hard to show that $\phi_k \uparrow \phi$. So, it will suffice to estimate ϕ.

To estimate $\phi(15^k \cdot (q-r)^{-1})$, let $h(u) = -\log(\phi(u))$ be the cumulant generating function of W. h satisfies the functional equation $h(u) = \log\left(f\left(\exp\left(h(15^{-1} \cdot u)\right)\right)\right)$, where f is the generating function of $T = T^1$. Let $1 < s < 15$. Since h is non-decreasing, we have,

$$\begin{aligned}\frac{h(su)}{h(u)} \leq \frac{h(15u)}{h(u)} &= h(u)^{-1}\log\left(f\left(\exp(h(15^{-1} \cdot u)\right)\right) \\ &= \left(\frac{1}{\log(\phi(u))}\right) \cdot \log\left(\frac{\phi(u)^3}{(3-2\phi(u))(12-12\phi(u)+\phi(u)^2)}\right) \\ &= 3 - \frac{\log((3-2\phi(u))(12-12\phi(u)+\phi(u)^2))}{\log(\phi(u))}\end{aligned} \tag{2.11}$$

The second term goes to 0 as $u \rightarrow \infty$. Since h is monotone increasing,

$$\limsup_{u\rightarrow\infty} h(su)/h(u) \leq 3, \quad 1 \leq s \leq 15. \tag{2.12}$$

This shows that h is a function of dominated variation, which implies the existence of constants C_1 and C_2, such that $C_1 u^\rho \leq h(u) \leq C_2 u^\rho$, where $\rho = \log_{15} 3$. (See Feller[8], de Haan and Stadtmüller[11]). Restating this in terms of ϕ shows that $\phi(t) \leq \exp(-C_1 t^\rho)$.

We now apply a standard technical result on Laplace transforms.

Lemma 2.4. *Let U be the distribution function of a random variable, let $\psi(u)$ be its Laplace transform, and let $a > 0$. Then, for any $t > 0$, $U(t) \leq e^{-a} \cdot \psi(t^{-1})$.*

Proof: This is proved in the Corollary to Theorem XIII.5.1 in Feller [9].

Applying these two lemmas to our series gives

$$\sum_{k=0}^{\infty} 3^{(k-n)\gamma} P\left[15^k \cdot T_k \le 15^{-k} \cdot 15^n \cdot (q-r)\right]$$

$$\le e^{1/15} \cdot \sum_{k=0}^{\infty} 3^{(k-n)\gamma} \phi(15^{k-n} \cdot (q-r)^{-1}) \tag{2.13}$$

$$\le e^{1/15} \cdot \sum_{k=-\infty}^{\infty} 3^{(k-n)\gamma} \phi(15^{k-n} \cdot (q-r)^{-1}) \tag{2.14}$$

$$\le e^{1/15} \cdot \left[\sum_{m=1}^{\infty} 3^{-m\gamma} + \sum_{m=0}^{\infty} 3^{m\gamma} \exp\left(-15^{m\rho} \cdot C_1 (q-r)^{-\rho}\right)\right] \tag{2.15}$$

$$< \infty$$

To complete the proof of the estimate, let $j(t) = [\log_{15} t] + 1$ for $t > 0$ and let $F(t) = e^{1/15} \cdot 3^{\gamma j(t)} \cdot \sum_{i=-\infty}^{\infty} 3^{i\gamma} \phi(15^i)$. Substitute inequality (2.14) into the left-hand term in (2.10) and let $m = k - n$, to get

$$E\|Y_n(q) - Y_n(r)\|^\gamma \le (2\sqrt{2})^\gamma \cdot \left[3^{-\gamma n} + (1 - 3^{-\gamma}) \cdot e^{1/15} \cdot \sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m \cdot (q-r)^{-1})\right] \tag{2.16}$$

$$\le (2\sqrt{2})^\gamma \cdot \left[3^{-\gamma n} + 3^{\gamma j(q-r)} (1 - 3^{-\gamma}) \cdot e^{1/15} \cdot \sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m)\right] \tag{2.17}$$

From (15), $\sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m) = C < \infty$. Since $j(q-r) \le \log_{15}(q-r) + 1$, $3^{\gamma j(q-r)} \cdot e^{1/15} \le (q-r)^{\rho\gamma}$. Subsitituting these expressions into (2.17) gives

$$E\|Y_n(q) - Y_n(r)\|^\gamma \le \left[3^{-\gamma n} + (q-r)^{\rho\gamma} \cdot (1 - 3^{-\gamma}) \cdot \sum_{m=-\infty}^{\infty} 3^{m\gamma} \phi(15^m)\right] \tag{2.18}$$

which proves the lemma.

Using our estimate, we can deduce both weak convergence of $Y_{n'}(t)$ and sample continuity of Y_t. Weak convergence of $\{Y_n(t)\}$ follows from the standard result on convergence of stochastic processes.

Theorem 2.5. *Suppose that*

$$(X_n(t_1), \ldots, X_n(t_k)) \to_D (X(t_1), \ldots, X(t_k))$$

holds whenever $t_1 < \ldots < t_k$ are points where $X(t)$ is almost surely continuous, and that

$$E\left[\|X_n(t) - X_n(t_1)\|^\gamma \|X_n(t_2) - X_n(t)\|^\gamma\right] \le (F(t_2) - F(t_1))^{2\alpha}$$

for $t_1 \le t \le t_2$, $n \ge 1$, $\gamma > 0$, $\alpha > \frac{1}{2}$, and F a nondecreasing continuous function. Then $X_n \rightarrow_D X$.

Proof: This is Theorem 15.6 in Billingsley [5].

Proof of the main theorem: To show that there exists a weakly convergent sequence $\{Y_{n'}(t)\}$, it suffices to show that $\{Y_n\}$ is tight. Let $s \le u \le t$ and consider

$$E\left[\|Y_n(u) - Y_n(s)\|^\gamma \|Y_n(t) - Y_n(u)\|^\gamma\right] \tag{2.19}$$

If $|s - t| < 15^{-n}$ then

$$E\left[\|Y_n(u) - Y_n(s)\|^\gamma \|Y_n(t) - Y_n(u)\|^\gamma\right] = 0 \tag{2.20}$$

for any $s \le u \le t$, because $Y_n(t)$ jumps only at integral multiples of 15^{-n}, and if $|s - t| < 15^{-n}$ then there can be at most one such multiple in $[s, t]$. So, suppose that $|s - t| \ge 15^{-n}$. Apply Hölder's inequality and the monotonicity of $E\|Y_n(u) - Y_n(s)\|^\gamma$ to see that

$$\begin{aligned} & E\Big[\|Y_n(u) - Y_n(s)\|^\gamma \|Y_n(t) - Y_n(u)\|^\gamma\Big] \\ & \quad \le E\|Y_n(t) - Y_n(s)\|^{2\gamma} \\ & \quad \le (2\sqrt{2})^{2\gamma}\left[3^{-n2\gamma} + (1 - 3^{-2\gamma}) \cdot (t - s)^{2\gamma\rho}\right] \end{aligned} \tag{2.21}$$

But, since $|s - t| \ge 15^{-n}$, $3^{-n2\gamma} \le (s - t)^{2\gamma\rho}$. Substitute this into (2.21) to get

$$E\left[\|Y_n(u) - Y_n(s)\|^\gamma \|Y_n(t) - Y_n(u)\|^\gamma\right] \le 2 \cdot (2\sqrt{2})^{2\gamma} \cdot (t - s)^{2\gamma\rho} \tag{2.22}$$

This suffices to prove tightness.

Choose a subsequence n' so that j $Y_{n'}(t) \rightarrow_D Y_t$. To show that the sample paths of Y_t are continuous with probability 1, let $n \rightarrow \infty$ in estimate (2.6), giving

$$E\|Y_t - Y_s\|^\gamma \leq\leq C \cdot (t-s)^{\rho\gamma} \tag{2.23}$$

Kolmogorov's criterion for sample path continuity applies, which completes the proof of our main theorem.

Although we have written our proofs for the special case where $Y_0 = \underline{0}$, the proof will also work for $Y_0 = x$ for an arbitrary $x \in \Gamma_\infty$, with a minor modification of the sequences $Y_n(t)$.

For $n = 1, 2, \ldots$ and $m < n$ let $S^m_{n,k}$ be the time of the k^{th} visit of the random walk $Y_n(t)$ to G_m and let $T^m_{n,k} = S^m_{n,m,k} - S^m_{n,k-1}$ be the k^{th} interarrival time for $S^m_{n,k}$. Previously, we observed that $Y_n(S^m_{n,k})k = 1, 2, \ldots$ is a random walk on G_m, for all $m < n$. We can extend this property to Y_t.

Proposition 2.6. *Let $Y_{n'} \rightarrow Y$. $T_{n',m,k}$ converges in distribution to a random variable $T_{m,k}$ for every $k > 0$; furthermore, the sequence $\{T_{m,k}\}_{k=1}^\infty$ is independent and identically distributed for all m. $Y_{n'}(S_{n',m,k})$ converges weakly to $Y(S_{m,k})$ where $\{Y(S_{m,k})\}_{k=1}^\infty$ is a random walk on the lattice G_m.*

Proof: We have already shown that $T^m_{n',k}$ is distributed as the $n'-m$ generation of a branching process with offspring generating function $f(u)$. Applying the theorem about branching processes cited at the beginning of this section shows that $15^{-n'-m}T_{n',m,k} \rightarrow_D 15^{-m}W$ also. For each n', and m $\{T^m_{n',k}\}_{k=1}^\infty$ is a sequence of independent random variables. Thus $\{T^m_k\}_{k=1}^\infty$ is also an independent sequence.

For any n', the sequence $Y_{n'}(S^m_{n',k})$ is a random walk on G_m. As for $\{Y(S_{m,k})\}_{k=1}^\infty$, $Y(S^m_k) \in G_m, k = 1, 2, \ldots$ with probability 1. It is straightforward, although tedious, to show that for sites $x_1, \ldots, x_\nu \in G_m$, $\{\omega : Y(S^m_i, \omega) = x_i, i = 1, \ldots, \nu\}$ is a continuity set for the distribution of Y_t. The second statement of the proposition follows.

A consequence of this proposition is

Corollary 2.7. *The sample paths of Y_t are uniformly continuous in probability.*

Proof: The corollary follows from Skorokhod's lemma, upon observing that for $u > v$

$$P\left[\|Y_u - Y_v\| \leq 2\sqrt{2}\cdot 3^{-n}\Big|Y_s, 0 \leq s < v\right] \geq P\left[T_{n,k(v)} < u - v\Big|Y_s, 0 \leq s < v\right] \tag{2.24}$$

$$= P\left[T_{n,k(v)} < u - v\right] \tag{2.25}$$

$$\geq 1 - 15^{-n}/(u - v) \tag{2.26}$$

where $k(v)$ is the smallest k such that $S_{n,k} \geq v$. Inequality (24) follows from the fundamental estimate, (25) uses the fact that $\{T_{n,k}\}$ is an i. i. d. sequence of random variables, and (26) is Markov's inequality.

To show that Y_t is a diffusion, we must show that it has the strong Markov property. We consider those stopping times of Y_t that are also stopping times of the embedded random walks $Y(S_k^n)$, in an appropriate sense. Let C be the class of stopping times T of Y_t such that

i. For some n, $Y_T \in G_n$ with probability 1.

ii. If $m \geq n$ and $T = \sum_{k=1}^{N_m} T_{m,k}$, then N_m is a stopping time for the random walk $Y(S_{m,k})$.

Clearly $S_{n,k} \in C$ for any n and k, so the class C is not vacuous.

Proposition 2.8. *Let $T \in C$. Then Y_t satisfies the strong Markov property with respect to T. That is for any bounded, measurable functions $f : \mathrm{R}^d \to \mathrm{R}$, $g : \mathrm{R}^k \to \mathrm{R}$, and $s_1...s_d$, $t_1...t_k$,*

$$\begin{aligned} &E\Big[g(Y(s_1 \wedge T), ..., Y(s_d \wedge T))f(Y(t_1 + T), ..., Y(t_k + T))\Big] \\ &\quad = E\Big[g(Y(s_1 \wedge T), ..., Y(s_d \wedge T))E_T^Y[f(Y(t_1), ..., Y(t_k))]\Big] \end{aligned} \tag{2.27}$$

Proof: By the monotone class theorem, it suffices to prove the results for $g(x_1, ..., x_d)$ and $f(y_1, ..., y_k)$

bounded and continuous. For $n = 1, 2, \ldots$ let

$$Y_{n,t} = \sum_{l} Y(S_{n,l}) 1[S_{n,l} \le t < S_{n,l+1}] \tag{2.28}$$

Obviously, $\sup_t |Y_t - Y_{n,t}| \le \sqrt{2} \cdot 3^{-n}$. Thus, $Y_{n,t} \to Y_t$ uniformly with probability 1, as $n \to \infty$. By the strong Markov property of random walks on G_n and the independence of $T_{n,k}$,

$$\begin{aligned} &E\big[g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) f(Y_n(T + t_1), \ldots, Y_n(T + t_k))\big] \\ &= E\Big[g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) E^{Y}_{n,T}\left[f(Y_n(T + t_1), \ldots, Y_n(T + t_k))\right]\Big] \end{aligned} \tag{2.28}$$

Since $T \in G_1$ almost surely,

$$E[f(Y_n(T + t_1), \ldots, Y_n(T + t_k)) | Y_{n,T}] = \sum_{p} E^{x}_{p}[f(Y_n(t_1), \ldots, Y_n(t_k))] \cdot P[Y_{n,T} = x_p]. \tag{2.29}$$

The distribution of $Y_{n,T}$ is fixed, so, x_p and $P[Y_{n,T} = x_p]$ are fixed as n varies. Since f and g are bounded and continuous,

$$g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) \to g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) \quad \text{a.s.} \tag{2.29}$$

$$f(Y_n(T + t_1), \ldots, Y_n(T + t_k)) \to f(Y(T + t_1), \ldots, Y(T + t_k)) \quad \text{a.s.} \tag{2.30}$$

$$f(Y_n(t_1), \ldots, Y_n(t_k)) \to f(Y(t_1), \ldots, Y(t_k)) \quad \text{a.s.} \tag{2.31}$$

As $n \to \infty$, the dominated convergence theorem gives

$$\begin{aligned} &E[g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) f(Y_n(t_1 + T), \ldots, Y_n(t_k + T))] \\ &\qquad \to E[g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) f(Y(t_1 + T), \ldots, Y(t_k + T))] \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} &E\Big[g(Y_n(s_1 \wedge T), \ldots, Y_n(s_d \wedge T)) E^{Y_T}[f(Y_n(t_1), \ldots, Y_n(t_k))]\Big] \\ &\qquad \to E\Big[g(Y(s_1 \wedge T), \ldots, Y(s_d \wedge T)) E^{Y}_{T}[f(Y(t_1), \ldots, Y(t_k))]\Big], \end{aligned} \tag{2.33}$$

which proves the proposition.

We next use this limited form of the strong Markov property to show that the laws P^x, $x \in \Gamma_\infty$ are uniformly weakly continuous in x. We do this by studying escape times, the time required for the process to reach G_k, starting from G_n for $n > k$. Clearly, escape times are stopping times in the class C, so the limited Markov property in the preceding proposition applies to them. We begin by proving a lemma for Markov chains

Lemma 2.9. *Let P be an $N \times N$ stochastic matrix. Let Q be the space of measures on $\{1, ..., N\}$, with the total variation norm. Then, the transformation $\mu \to \mu P$ is a strict contraction on Q if and only if no rows of P are mutually singular.*

Proof: Let λ and μ be two distinct measures in Q. Without loss of generality, suppose that $\lambda_j - \mu_j > 0, j = 1, ..., M$, $\lambda_j - \mu_j \leq 0, j = M+1, ..., N$. Let $\nu = \sum_1^M (\lambda_j - \mu_j) = \sum_{M+1}^N (\mu_j - \lambda_j)$. As μ and λ are distinct, $\nu > 0$. Let $\nu\alpha_j = (\lambda_j - \mu_j), j = 1, ..., M$ and $\nu\beta_j = \nu^{-1}(\mu_j - \lambda_j), j = M+1, ..., N$. Then,

$$d(\lambda P, \mu P) = \frac{1}{2}\sum_i \left|\sum_j (\lambda_j - \mu_j) P_{j,i}\right| \tag{2.34}$$

$$= \frac{1}{2}\sum_i \left|\sum_{j=1}^M (\lambda_j - \mu_j) P_{j,i} - \sum_{k=M+1}^N (\mu_k - \lambda_k) P_{k,i}\right| \tag{2.35}$$

$$= \nu\frac{1}{2}\sum_i \left|\sum_{j=1}^M \alpha_j P_{j,i} - \sum_{k=M+1}^N \beta_k P_{k,i}\right| \tag{2.36}$$

$$\leq \nu \sum_{j=1}^M \sum_{k=M+1}^N \alpha_j \beta_k \sum_i \frac{1}{2}|P_{j,i} - P_{k,i}| \tag{2.37}$$

$$\leq \nu \max_{j,k} d(P_j, P_k) \tag{2.38}$$

Inequality (2 37) above follows because $f(x, y) = |x - y|$ is convex in both x and y; the other relationships are straightforward. If $\max_{j,k} d(P_j, P_k) < 1$, $d(\mu P, \lambda P) < \nu = d(\mu, \lambda)$. If $d(P_i, P_j) = 1$ for some i and j, then take $\mu = \delta_i$, $\lambda = \delta_j$ to get $d(\mu, \lambda) = 1 = d(\mu P, \lambda P)$.

Let $x \in G_j$. Let Y_t^x denote the continuous process constructed at the beginning of this section, starting from x. For $k < j$, let $T_k = \inf\{t : Y_t^x \in G_k\}$. Then T_k is a stopping time such that $Y(x, T_k) \in G_k$ almost surely. By the strong Markov property of the embedded Markov chains $Y(T_{k,n})$ $\{Y(x, T_k), k < j\}$ form a discrete time Markov chain.

With probability 1, $Y(x, T_k)$ will be one of the four corners of the square $\mathcal{S}_k$ on level k enclosing x. Number these four corners 1, 2, 3 and 4, starting with 1 at the Northeast corner and proceeding clockwise around the square. If $Y(x, T_k) = i, i = 1, 2, 3, 4$, then the transition probabilities $P[Y(x, T_{k-1}) = j | Y(x, T_k) = i]$ will depend on which of the five level-k squares of $\mathcal{S}_{k-1}$ x belongs to. Number the four outside squares I, II, III, and IV, starting in the Northeast corner and numbering clockwise around the outer squares. Let V denote the center square of $\mathcal{S}_{k-1}$. By using the generating functions given in *display number*, it is not hard to calculate:

$$P^{(I)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/4 & 1/12 & 1/12 & 1/12 \\ 1/2 & 1/6 & 1/6 & 1/6 \\ 3/4 & 1/12 & 1/12 & 1/12 \end{bmatrix}; \qquad P^{(II)} = \begin{bmatrix} 1/12 & 3/4 & 1/12 & 1/12 \\ 0 & 1 & 0 & 0 \\ 1/12 & 3/4 & 1/12 & 1/12 \\ 1/6 & 1/2 & 1/6 & 1/6 \end{bmatrix}$$

$$P^{(III)} = \begin{bmatrix} 1/6 & 1/6 & 1/2 & 1/6 \\ 1/12 & 1/12 & 3/4 & 1/12 \\ 0 & 0 & 1 & 0 \\ 1/12 & 1/12 & 3/4 & 1/12 \end{bmatrix}; \qquad P^{(IV)} = \begin{bmatrix} 1/12 & 1/12 & 1/12 & 3/4 \\ 1/6 & 1/6 & 1/6 & 1/2 \\ 1/12 & 1/12 & 1/12 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{(V)} = \begin{bmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/2 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/2 \end{bmatrix}$$

Direct calculation shows that

$$\sup_{i,j} d(P^{\alpha}(i), P^{\alpha}(j)) = \tfrac{1}{2} \alpha = I, II, III, IV \text{ and } \sup_{i,j} d(P^{(V)}(i), P^{(V)}(j)) = \tfrac{1}{6} \qquad (2.39)$$

Let $BC(\Gamma)$ denote the bounded, real-valued continuous functions on Γ. For $g \in BC(\Gamma)$, $x \in \Gamma$, $t > 0$, let $P_t g(x) = E^x g(Y_t)$.

Theorem 2.10. *For all $t > 0$, $P_t : BC(\Gamma) \to BC(\Gamma)$.*

Proof: Fix $t > 0$. For $g \in BC(\Gamma)$, we must show that $P_t g(\cdot) \in BC(\Gamma)$. To do this, it will suffice to show that $P_t g(x), x \in G_\infty$ is uniformly continuous and observe that a uniformly continuous function on G_∞ has a unique uniformly continuous extension to Γ.

Let $x, y \in G_\infty$, choose M sufficiently large that $x, y \in G_M$, and suppose x and y lie within the same square on level m of G_∞, say the square $\mathcal{S}_m$. *A fortiori,* x and y lie in the same square $\mathcal{S}_n$ on level n of G_∞. for all $n < m$.

Let $\epsilon > 0$, and let g be a continuous function on Γ with $\|g\| \le 1$. Then

$$\begin{aligned} &|E^x g(Y_t) - E^y g(Y_t)| \\ &\qquad \le |E^x g(Y_t) - E^x g(Y_{t+T})| + |E^x g(Y_{t+T}) - E^y g(Y_{t+T})| \\ &\qquad\quad + |E^y g(Y_t) - E^y g(Y_{t+T}))|. \end{aligned} \tag{2.40}$$

We estimate the terms separately.

Note that

$$E^x g(Y(t+T_n)) = \sum_{i=1}^{4} \lambda_i^x E^i g(Y_n(t)); \qquad E^y g(Y(t+T_n)) = \sum_{i=1}^{4} \lambda_i^y E^i g(Y_n(t)) \tag{2.41}$$

where λ^x and λ^y are the escape distributions of $Y_n(t)$ on the corners of $\mathcal{S}_n$ starting from x and y, respectively. Then,

$$|E^x g(Y_n(t+\tau_k)) - E^y g(Y_n(t+\tau_k))| \le \sum_{i=1}^{4} |\lambda_i^x - \lambda_i^y| E^z g(Y_n(t)) \le \sum_{i=1}^{4} |\lambda_i^x - \lambda_i^y| \tag{2.42}$$

Choose $\delta > 0$ such that $|u - v| < \delta$ implies $|g(u) - g(v)| < \frac{\epsilon}{2}$, and let $A_h = \{\omega : \sup_{t \le u < v \le t+h} \|Y_v - Y_u\| > \delta\}$. We have shown that the paths of Y_t are uniformly continuous in probability, thus, choose $h > 0$ so that $P[A_h] \le \frac{\epsilon}{4}$. Then,

$$|E^x g(Y_t) - E^x g(Y_{t+T})| \le E^x |g(Y_t) - g(Y_{t+T})| \tag{2.43}$$

$$\leq E^x|g(Y_t) - g(Y_{t+T})|1([T \geq h] \cup [A_h^c]) \tag{2.44}$$

$$+ E^x|g(Y_t) - g(Y_{t+T})|1[T < \eta, A_h] \tag{2.45}$$

$$\leq 2 \cdot \frac{\epsilon}{2} + 2 \cdot \frac{\epsilon}{2} + \epsilon \tag{2.46}$$

$$= 3\epsilon \tag{2.47}$$

Similarly, we can estimate $|E^x g(Y(t)) - E^x g(Y(t + T_n))| \leq 3\epsilon$ for suitable n.

To complete the estimate, choose n sufficiently large so that $P^x[T_n \geq \eta] < \frac{\epsilon}{2}$ Then, if we apply this to the preceding inequalities, we get

$$|E^x g(Y_n(t)) - E^y g(Y_n(t))| < 3\epsilon + \epsilon + 3\epsilon = 7\epsilon \tag{2.48}$$

If we choose k and m as in the preceding paragraphs, and let $|x - y| < 3^{-m}$ then there must exist some $z \in \Gamma_\infty$ such that x and z and y and z each lie within a common square of level m of Γ. So,

$$\begin{aligned} |E^x g(Y_n(t)) - E^y g(Y_n(t)| &\leq |E^x g(Y_n(t)) - E^z g(Y_n(t)| + |E^z g(Y_n(t)) - E^y g(Y_n(t)| \\ &< 14\epsilon \end{aligned} \tag{2.49}$$

Let $n \to \infty$. Then if $|x - y| < 3^{-m}$, $|E^x g(Y_t) - E^y g(Y_s)| < 14\epsilon$.

This shows that $E^x g(Y_t)$ is a uniformly continuous function of x, for all $x \in \Gamma_\infty$, all t, and for g an arbitrary uniformly continuous function.

Proposition 2.11. *Y_t is a Markov process.*

Proof: Fix $t > 0$ and consider $S_{n,[15^n t]}$ where $[x]$ denotes the greatest integer less than x. Then, using our earlier notation,

$$S_{n,[15^n t]} = \sum_{j=1}^{[15^n t]} T_{n,j} = \frac{[15^n t]}{15^n} \cdot \left(15^n \cdot \frac{1}{[15^n t]} \sum_{j=1}^{[15^n t]} T_{n,j} \right) \tag{2.50}$$

As $n \to \infty$,

$$15^n \cdot \frac{1}{[15^n t]} \sum_{j=1}^{[15^n t]} T_{n,j} \to 1 \text{ a.s.}; \qquad \frac{[15^n t]}{15^n} \to t \tag{2.51}$$

The first limit follows by applying the law of large numbers to the i. i. d. sequence $T_{n,j}$. The second is simple analysis. Thus, $S_{n,[15^n t]} \to t$ almost surely, as $n \to \infty$.

If $g(x_1, ..., x_d)$ and $f(y_1, ..., y_k)$ are bounded continuous functions and $s_1 < ... < s_d < t < t_1 < ... < t_k$ then, by Proposition 1.8, for any n,

$$\begin{aligned} &E[g(Y(s_1 \wedge S_{n,[15^n t]}), \ldots, Y(s_d \wedge S_{n,[15^n t]}))f(Y(t_1 + S_{n,[15^n t]}), \ldots, Y(t_k + S_{n,[15^n t]}))] \\ &= E[g(Y(s_1 \wedge S_{n,[15^n t]}), \ldots, Y(s_d \wedge S_{n,[15^n t]}))E[f(Y(t_1), \ldots, Y(t_k))]|Y(S_{n,[15^n t]})] \end{aligned} \tag{2.52}$$

Let $n \to \infty$, apply the continuity of f and g, Theorem 1.10, and the sample continuity of Y_t. We get

$$E[g(Y_{s_1}, ..., Y_{s_d})f(Y_{t_1+t}, ..., Y_{t_k+t})] = E[g(Y_{s_1}, ..., Y_{s_d})E^{Y_t}[f(Y_{t_1}, ..., Y_{t_k})]] \tag{2.53}$$

This proves the proposition.

To show that Y_t is a strong Markov process, we will apply the following theorem:

Theorem 2.12. *Let X_t be a process satisfying*

i. For all $f \in BC(\Gamma)$, and all $t > 0$, $0 \le f \le 1$ implies $0 \le Ef(X_t) \le 1$.

ii. For all $s,t > 0$, $Ef(X_{s+t}) = E_s^X(Ef(X_t))$.

iii. For all $f \in BC(\Gamma)$, $E^x f(X_t) - f(x) \to 0$ uniformly in x as $t \to 0$. Let $\mathcal{F}_t^\circ = \sigma(X_s : s \le t)$ and $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s^\circ$. Then if T is an $\mathcal{F}_{t+}$ stopping time, $E^\mu[\theta_T \eta | \mathcal{F}_{T+}] = E^{X(T)}[\eta]$, for any measure μ and any bounded measurable η.

Proof: This theorem is given in Williams [17], as Theorem III.15.3.

To apply the theorem, it remains to show that $E^x f(Y_t) - f(x) \to 0$ uniformly in x as $t \to 0$ for any bounded continuous f.

Consider the operator $\tilde{P}_t : BC(\Gamma) \rightarrow BC(\Gamma_\infty)$ defined by $\tilde{P}_t g(x) = E^x g(Y_t), x \in \Gamma_\infty$. Since $E^x g(Y_t)$ is uniformly continuous for $x \in \Gamma_\infty$, we can extend $E^x g(Y_t)$ to a uniformly continuous function defined for all $x \in \Gamma_\infty$ and all $t > 0$. Let $P_t g$ denote this extension of $\tilde{P}_t g$. To apply the theorem from Williams [17] stated earlier, we need only prove the following

Proposition 2.13. *For all $f \in BC(\Gamma)$, and all $t > 0$, $0 \leq f \leq 1$ implies $0 \leq P_t f \leq 1$, and $P_t f(x) - f(x) \rightarrow 0$ uniformly in x as $t \rightarrow 0$.*

Proof: Let $f \in BC(\Gamma)$. If $0 \leq f \leq 1$, then $0 \leq E^x f(Y_t) \leq 1$ (a. s.), since conditional expectation is a positive operator. Since Γ, BC(Γ), and $[0, \infty)$ are all separable, we can modify P on a single null set N to get $0 \leq E^x f(Y_t) \leq 1$.

We apply the following lemma.

Lemma 2.14. *Let $P_t : BC(\Gamma) \rightarrow BC(\Gamma)$ be a substochastic Markov semigroup. If $P_t f(x) \rightarrow f(x)$ as $t \downarrow 0$ for all $x \in \Gamma$ and $f \in BC(\Gamma)$, then $P_t f(x) - f(x) \rightarrow 0$ uniformly in x as $t \rightarrow 0$, for all $f \in BC(\Gamma)$.*

Proof: This paraphrases formula III.8.2.iv. in Williams [17].

Let $\epsilon > 0$. Let $x \in \Gamma_\infty$, and let $g \in$ BC(Γ). Given $\epsilon > 0$, choose δ such that $|y - x| < \delta$ implies that $|g(x) - g(y)| < \frac{\epsilon}{2}$. $P^x[|Y_t - x| > \delta] \downarrow 0$ as $t \downarrow 0$. Choose t sufficiently small that $P^x[|Y_t - x| > \delta] \leq \frac{\epsilon}{4}$. Then

$$
\begin{aligned}
|P_t g(x) - g(x)| & \\
&\leq |E^x(g(Y_t) - g(x))1[|Y_t - x| \leq \delta]| + |E^x(g(Y_t) - g(x))1[|Y_t - x| > \delta]| \\
&\leq \tfrac{\epsilon}{2} P^x[|Y_t - x| \leq \delta] + 2 \cdot P^x[|Y_t - x| > \delta] \\
&\leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon
\end{aligned}
\tag{2.54}
$$

This completes the proof of the proposition.

In terms of Williams [17], this makes P_t an FD semigroup and Y_t an FD process. Applying the theorem that we stated earlier in this section to Y_t, we conclude

Corollary 2.15. *Y_t has the strong Markov property.*

Henceforth, we will refer to Y_t as the *snowflake diffusion*.

In fact we can show that the snowflake diffusion is the unique diffusion limit for any of the random walks X^p, within a constant change of time scale

By Proposition 2.6, random walks on all the lattices G_n are embedded in the snowflake diffusion. We also note in passing that for $Y_0 = 0$, $3Y_t =_D Y_{15t}$; we will return to this point in the next section at greater length. These two facts form the basis for the proof of the following proposition.

Proposition 2.16. *Within a non-random change in time scale, Y_t is the unique limit in distribution of the processes $Y_n(t)$.*

Proof: For each n, define the sequence $T_1^n, T_2^n, \ldots$ as in proposition 2.6. for each k, T_k^n is the sum of k independent random variables equal to $15^{-n}W$ in distribution. If $\mathrm{Var}(W) = \sigma^2$, then $ET_k^n = k \cdot 15^{-n}$ and $\mathrm{Var}(T_k^n) = k\sigma^2$.

For arbitrary n and t,

$$Y_n(t) = 3^{-n}X([15^n t]) =_D 3^{-n}Y(T^n([15^n t])) = Y\left(15^{-n}T^n([15^n t])\right). \tag{2.55}$$

Note that $t - 15^{-n} \leq 15^{-n} \cdot [15^n t] \leq t + 15^{-n}$. Now, T_k^n is the sum of k independent random variables, with distribution $15^{-m} \cdot W$. Thus,

$$E(15^{-m} \cdot T([15^m t])) = 15^{-m} \cdot E(T([15^m t])) = 15^{-m} \cdot [15^{-m} \cdot t] \to t \tag{2.56}$$

$$\sigma^2(15^{-m} \cdot T([15^m t])) = 15^{-2m} \cdot \sigma^2(T([15^m t])) = 15^{-2m} \cdot 15^{-m} \cdot \sigma^2 = 15^{-m} \cdot \sigma^2 \to 0 \tag{2.57}$$

as $n \to \infty$, where $\sigma^2 = \sigma^2(W)$. Thus, as $n \to \infty$, $Y_n(t) \to_P Y_t$.

By the same argument, for any $t_1 < \ldots < t_k$ and any convergent subsequence $Y_{n''}(t)$ converges in distribution. Then by the same analysis as in the preceding paragraph,

$$(Y_{n''}(t_1), \ldots, Y_{n''}(t_k)) \to_D (Y(t_1), ..., Y(t_m)) \tag{2.58}$$

Thus all limits of $\{Y_n(t)\}_{n=1}^{\infty}$ have the same finite dimensional distributions. Since the finite dimensional distributions of a stochastic process determine the process, it follows that Y_t is the unique limit of the processes $\{Y_n(t)\}_{n=1}^{\infty}$.

Now we consider again the general model for random walk proposed at the beginning of this section. Let X^p be random walk on the snowflake lattice with parameter p. Let $p_0 = p$ and for $n = 1, 2, \ldots$, let p_n be the probability that X^p starts from 0 and reaches $(3^n, 3^n)$ before either $(3^n, 0)$ or $(0, 3^n)$. It is not hard to see that in general $p_n \neq p_{n-1}$. We can find a recurrence relationship by calculating computing p_1 as a function of p_0 Observe that we can identify vertices of U_1 that are reflections of one another across diagonals of the square. Thus, we can identify the following sets of vertices

$$\begin{array}{lll} a = \{(0,0)\} & b = \{(0,1),(1,0)\} & c = \{(1,1)\} \\ d = \{(1,2),(2,1)\} & e = \{(0,2),(1,3),(2,0),(3,1)\} & f = \{(2,2)\} \\ g = \{(2,3),(3,2)\} & & \end{array} \tag{2.59}$$

For each $i \in \{a, \ldots, g\}$, let q_i be the probability that X_n starts from vertex i and reaches $(3,3)$ before either

(0,3) or (3,0). Using the Markov property of X^p, we get the following system of equations for q_i

$$\begin{aligned}
q_a &= (1-p)q_b + pq_c & q_b &= \tfrac{1}{2}(1-p)q_a + pq_b + \tfrac{1}{2}(1-p)q_c \\
q_c &= \tfrac{1}{2}pq_a + \tfrac{1}{2}(1-p)q_b + \tfrac{1}{2}(1-p)q_d + \tfrac{1}{2}pq_f & q_d &= \tfrac{1}{4}(1-p)q_c + \tfrac{1}{2}pq_d + \tfrac{1}{2}(1-p)q_e + \tfrac{1}{4}(1-p)q_f \\
q_e &= \tfrac{1}{2}(1-p)q_d + \tfrac{1}{2}pq_e & q_f &= \tfrac{1}{2}pq_c + \tfrac{1}{2}(1-p)q_d + \tfrac{1}{2}(1-p)q_g + \tfrac{1}{2}p \\
q_g &= \tfrac{1}{2}(1-p)q_f + pq_g + \tfrac{1}{2}(1-p)
\end{aligned} \tag{2.60}$$

Applying Cramer's rule, we get

$$q_a = \frac{(\frac{1}{2})^5 \cdot (1-p)^3 \cdot (1+p)^3}{(\frac{1}{2})^5 \cdot (1-p)^3 \cdot (1+p)^3 \cdot (4-3p)} = \frac{1}{4-3p} \tag{2.61}$$

Formula (2.61) gives us a recurrrence for p_1 in terms of p_0. Furthermore, we see that $p = \frac{1}{3}$ is a fixed point for this recurrence formula, and that for any $p_0 < 1$, $p_n \to \frac{1}{3}$ as $n \to \infty$.

For $n = 1, 2, \ldots$, let $T^{n,p} = T(n,p)$ be the number of steps X^p requires to pass from 0 to some other vertex in $3^n \cdot G$. In analogy with the branching property of $X^{1/3}$, it is not hard to see that $T^{n,p} = \sum_1^B T_k^{n-1,p}$, where $T_k^{n-1,p} k = 1, 2, \ldots$ is a sequence of random variables distributed as $T^{n-1,p}$, B is distributed as $T(1, p_{n-1})$ and B and $T_k^{n-1,p} k = 1, 2, \ldots$ are mutually independent.

Let $m_n = ET(1, p_n)$, and let $M_n = \prod_{i=1}^n m_i$. Then, $ET^{n,p} = M_n$, and an argument analogous to that for branching processes shows that $T^{n,p} \cdot M_n^{-1}$ is a positive martingale converging a. s. and in distribution to some random variable $\tilde{W}$.

Lemma 2.17. *Within a constant scaling factor,* $\tilde{W} =_D W$.

Proof: Let

$$f_n(u) = Eu^{T(1,p_n)}, \qquad g_n(\lambda) = E\exp\left(-\lambda T(n,p) \cdot M_n^{-1}\right) \tag{2.62}$$

Then from the branching property of $T^{n,p}$ outlined in the preceding two paragraphs

$$g_n(\lambda) = f_n\left(g_{n-1}(\lambda \cdot m_n^{-1})\right) \tag{2.63}$$

Since $T^{n,p} \cdot M_n^{-1} \to_D \tilde{W}$, $g_n(\lambda) \to g(\lambda) = E\exp(-\lambda\tilde{W})$ as $n \to \infty$. $p_n \to 1/3$, so $f_n(u) \to f(u)$, and $m_n \to 15$. Substituting these limits in (2.63) gives

$$g(\lambda) = f(g(\lambda \cdot 15^{-1})), \tag{2.64}$$

the same functional equation satisfied by W. It is well known that the solution to Abel's functional equation is unique, except for a constant multiplier of the argument. (See, for example, Seneta [16], Theorem 3.1.) This implies that $\tilde{W} =_D W$ within a constant change of scale.

Let $Y_n^p(t) = 3^{-n}X^p([M_n t])$. Arguments analogous to those in Theorem 2.2 , Lemma 2.3, Propositions 2.6, 2.8, 2.11, and 2.13 show that we can find a diffusion $\tilde{Y}_t$ on Γ and a subsequence n' such that $Y_{n'}^p \to \tilde{Y}$ weakly. Let $\{\tilde{T}_n\}$ be the sequence of times when $\tilde{Y}_t$ visits distinct vertices of G_1. Note that $\tilde{T} =_D \tilde{W}$. As $p_n \to 1/3$, $\{\tilde{Y}(\tilde{T}_n)\} =_D \{X^{1/3}\}$

Theorem 2.18. *$\tilde{Y}_t =_D Y_t$, within a constant change of time scale.*

Proof: As a consequence of Lemma 2.17, we can choose a time scale for $\tilde{Y}_t$ so that $3\tilde{Y}_t =_D \tilde{Y}_{15t}$. The proof now proceeds as in Proposition 2.16. Let

$$Y_m^*(t) = 3^{-m}\tilde{Y}(\tilde{T}([15^m t])) = \tilde{Y}(15^{-m} \cdot \tilde{T}([15^m t])). \tag{2.65}$$

As $\tilde{Y}_t$ has continuous sample paths, $Y_m^*(t) \to \tilde{Y}_t$ almost surely. On the other hand, $Y_m^*(t) =_D Y_m(t)$, and $Y_m(t) \to_D Y_t$. Thus, $\tilde{Y}_t =_D Y_t$. This proves the theorem.

3. Scaling Properties:

In this chapter, we will study some of the detailed sample path properties of the snowflake diffusion. Let $n \geq 0$, let x and y be points on $G_\infty \cap [0,3]$ such that $x, y \in \mathcal{S}$ for some square $\mathcal{S}$ on level n. Let $T_y = \inf\{t : Y_t = y\}$ be the first visit of the diffusion on Γ_b to y; set $T_y = \infty$ if this set is empty.

We begin by proving two technical lemmas.

Lemma 3.1. $E^x T_y < 369 \cdot 3^{-n}$.

Proof: Suppose $x, y \in \mathcal{S}$. There exists $m \geq n$ such that $x, y \in G_m$. We will establish our estimate by induction on m.

Suppose that $m = n$. Then x and y are necessarily corners of $\mathcal{S}$. Let $T_1, T_2, \ldots$ be the times between successive visits to G_n Then, $T_y = \sum^{N(y)} T_j$ where

$$N_y = \sum_{i=1}^{M} \left(\sum_{j=1}^{N(i)} R_{i,j} + 1 \right) \tag{3.1}$$

M is the number of corners of $\mathcal{S}$ that Y_t visits before T_y, N_i is the number of excursions to $G_n \setminus \mathcal{S}$ between the time Y_t hits the $i - 1^{st}$ and i^{th} distinct corners of S, and $R_{i,j}$ the number of points in $G_n \setminus \mathcal{S}$ that Y_t visits on the j^{th} such excursion.

The strong Markov property shows that M has a geometric distribution with parameter $\frac{1}{3}$. For $i = 1, 2, \ldots$ N_i either is identically 0 or else has a geometric distribution with parameter $\frac{1}{2}$. To estimate $ER_{i,j}$, note that each excursion outside of $\mathcal{S}$ is a random walk on a finite graph, and $R_{i,j}$ is the number of steps the walk takes to return to its starting point. It is well-known that this expected time is equal to twice the total number of edges in the graph divided by the degree of the starting vertex; see Gobel and Jagers [9] for this and other general results about random walks on finite graphs. Thus, $ER_{i,j}$ is proportional to the number of the edges in the graph cut out of G_n by $\mathcal{S}$. This number, in turn, is less than the number of edges in G_m. It is not hard to compute that G_n has $6 \cdot 5^{n+1}$ edges, so, $ER_{i,j} \leq 20 \cdot 5^n$

Recall that $ET_i = 15^{-n}$, and apply Wald's identity.

$$\begin{aligned} E^x T_y &= EM \cdot (EN(i) \cdot ER_{i,j} + 1) \cdot ET \\ &\leq 3 \cdot (2 \cdot 20 \cdot 5^n + 1) \cdot 15^{-n} \\ &\leq 41 \cdot 3^{-(n-1)} \end{aligned} \tag{2.2}$$

W now proceed by induction. Suppose that if $x, y \in G_m$ with $|x - y| \leq 3^{-n}$, then $E^x T_y \leq 82 \cdot \sum_n^m 3^{-i+1}$. Let $x, y \in G_{m+1}, |x - y| \leq 3^{-n}$. There exist points v and w in $\mathcal{S} \cap \mathcal{G}_{\updownarrow}$ such that x and v and y and w are adjoining corners of squares on G_{m+1}. (x and v may be identical, and so forth.) The strong Markov property gives

$$\begin{aligned} E^x T_y &\leq E^x T_v + E^v T_w + E^w T_y \\ &\leq 41 \cdot 3^{-m} + 82 \cdot \sum_{n-1}^{m-1} 3^{-i} + 41 \cdot 3^{-m} \\ &= 82 \cdot \sum_{n}^{m+1} 3^{-i+1} \end{aligned} \tag{2.3}$$

completing the inductive step.

Finally, if $x, y \in \mathcal{S} \cap -\infty$, then

$$E^x T_y \leq 82 \cdot \sum_n^{\infty} 3^{-i+1} = 369 \cdot 3^{-n} \tag{2.4}$$

This completes the proof of the lemma.

Corollary 2.2. *If $x, y \in G_\infty \cap [0,3]$, such that $|x - y| < 3^{-n}$, then $E^x T_y \leq 738 \cdot 3^- n$.*

Proof: If $|x - y| < 3^{-n}$, then either they lie in a single square $\mathcal{S}$ or level n, or else they lie in two such adjoining squares. In either case, the preceding lemma gives the desired bound.

Recall that 3 $G_n = G_{n-1}$ This fact and the uniqueness of the snowflake diffusion limit imply the following

Lemma 3.3. *Let $x \in \Gamma$. Then $3Y_t^x =_D Y_{15t}^{3x}$.*

Proof: Let $x \in \Gamma_\infty$. Using the notation of section 2. consider the sequence of processes $3 \cdot Y_n(x.t)_{-}n = 1^\infty$. We have shown that $Y_n(x,t) \to Y_t^x$ as $n \to \infty$, so, trivially, $3 \cdot Y_n(x,t) \to 3 \cdot Y_t^x$

On the other hand,

$$3 \cdot Y_n(x,t) = 3 \cdot 3^{-n} X(3^n x, [15^n t]) = 3^{-n+1} X(3^{n-1} 3x, [15^{n-1}(15t)]) = Y_{n-1}(3x, 15t) \qquad (3.4)$$

Thus, $3 \cdot Y_n(x,t) \to Y_{15t}^{3x}$. Since both of these limits are unique, it follows that $3Y_t^x =_D Y_{15t}^{3x}$, for $x \in \Gamma_\infty$.

For general x in Γ choose a sequence x_n in Γ_∞ converging to x. Then, for each n, $3Y_t^{x_n} =_D Y_{15t}^{3x_n}$. As Y has the Feller property,

$$3Y_t^{x_n} \to_D 3Y_t^x \text{ and } Y_t^{3x_n} \to_D Y_t^{3x} \qquad n \to \infty \qquad (3.5)$$

Therefore, $3Y_t^x =_D Y_{15t}^{3x}$.

Corollary 3.4. $3Y_t^0 =_D Y_{15t}^0$.

For starting points in G_∞ other than 0, a result similar to the preceding corollary also holds. Let $x \in \Gamma_\infty$. and define $M_x : y \to 3^{-1} \cdot (y - x) + x$. Then $M_x x = x$.

Corollary 3.5. *Let $x \in G_\infty$. Then there exists a stopping time $T_x > 0$ such that $M_x Y_t^x =_D Y_{15t}^x, 0 \le t \le T_x$*

Proof: Suppose that $x \in G_n$. Then there exists either one or two squares S on level n. such that x is a corner of S However, if x is a corner of S, then there is some square S' on level $n+1$ contained in S such that x is also a corner of S'. Clearly, M_x : S $\to$ $S\prime$. Let U' be the union of any squares S' on level $n+1$ adjoining x Then, the proof of the fractal scaling law will continue to work if we set T_x to be the first time the process started from x exits U'' and T_x satisfies the requirements given in the statement of the corollary.

We shall call the scaling property stated in either corollary the *unbounded fractal scaling law.*

If we restrict our attention to Γ_b, then our scaling properties are not quite so tidy. However, a closely related property does apply, which arises from the invariance of Γ_b under a family of contractions. Define

$$
\begin{aligned}
&N_1 : x \to 3 \cdot x \\
&N_2 : x \to -3 \cdot x + (6,6) \\
&N_3 : x \to 3 \cdot x - (6,6) \\
&N_4 : x \to 3 \cdot x - (1\tfrac{1}{2}, 4\tfrac{1}{2}) \\
&N_5 : x \to 3 \cdot x - (4\tfrac{1}{2}, 1\tfrac{1}{2})
\end{aligned}
\quad \text{and} \quad
N(x) = \begin{cases} N_1(x), & x \in [0,1]^2 \\ N_2(x), & x \in [1,2]^2 \\ N_3(x), & x \in [2,3]^2 \\ N_4(x), & x \in [0,1] \times [2,3] \\ N_5(x), & x \in [2,3] \times [0,1] \end{cases}
\tag{3.6}
$$

Note that N is a continuous mapping of Γ_b onto Γ_b

Lemma. *Let $x \in \Gamma$. Then, $N(Y_t^x) =_D Y_{15t}^{N(x)}$.*

Proof: Inspection will verify that if X_n is a random walk on G_k starting from x, then $N(X_n^k)$ is a random walk on G_{k-1} starting from $N(x)$, for any $k1$. We have shown in Section 2 that $X^k_{[15^k t]} \to Y_t$ weakly. Then

$$N(X^k_{[15^k t]}) = N(X^k_{[15^{k-1}(15t)]}) \to Y_{15t} \tag{3.7}$$

and since N is continuous, $N(X^k_{[15^k t]}) \to N(Y_t)$.

We shall call this scaling property the *bounded fractal scaling law.* As an important consequence, note that for any $x \in \Gamma_b$ and any measurable set A, $P^{N(x)}[Y_{15t} \in A] = P^x[Y_t \in N^{-1}(A)]$.

Our next step is to show that two independent copies of the snowflake diffusion restricted to Γ_b meet in finite time with probability 1. First we prove two technical lemmas. For $x, y \in \Gamma$, let $d(x, y)$ denote the Euclidean distance between x and y. Let $D[0,\infty]$ denote the space of functions $\omega : \mathbb{R}^+ \to \Gamma$ which are right continuous and have left limits for all $t > 0$.

Lemma 3.7. *For any $t, \epsilon > 0$ and any compact set K, let*

$$H(t,\epsilon) = \{\omega : \inf d(\omega(u), K) > \epsilon, 0 < u < t + \epsilon\}.$$

Then $H(t, \epsilon)$ is an open subset in the topology $\mathcal{T}$ on $D[0,\infty)$ defined by convergence in the Skorokhod metric on compact intervals $[0,p]$.

Proof: Let $\omega, \psi \in D[0,\infty]$. For any $p > 0$, let $\rho_p(\mu, \nu)$ equal the infimum of those $\epsilon > 0$ for which there exists a continuous, increasing function $\lambda : [0,p] \to [0,p]$ such that

i. $\sup\{0 < t < p : |\lambda(t) - t|\} \leq \epsilon$

ii. $\sup\{0 < t < p : d(\mu(\lambda(t)), \nu(t))\} \leq \epsilon$

It is easy to see that ρ_p is a pseudo-metric and that the topology $\mathcal{T}$ is induc by the family of pseudo-metrics $\{\rho_p\}_{p>0}$.

Suppose $\omega \in H(t,\epsilon)$. Let $a = \inf\{d(\omega(u), K), 0 < u < t + \epsilon\}$ By hypothesis. $a - \epsilon = \delta > 0$. Choose $\nu \in D[0,\infty]$ with $\rho_t(\omega, \nu) < \frac{\delta}{2}$. There exists a strictly increasing continuous function $\lambda : [0,t] \to [0,t]$ satistfying *i.* and *ii.* By the triangle inequality, for any $u \in [0,t]$,

$$d(\nu(u), K) \geq d(\omega(\lambda(u)), K) - d(\omega(\lambda(u)), \nu(u)) > \epsilon + \delta - \frac{\delta}{2} > \epsilon; \tag{3.8}$$

since $\lambda : [0,t] \to [0,t]$, $d(\omega(\lambda(u)), K) > \epsilon + \delta$, for $0 \leq u \leq t$. Thus, $\nu \in H(t, \epsilon)$, so H_ϵ is open.

Let $T = \inf\{t : \omega(t) \in K\}$. For every $\epsilon > 0$, $H(t,\epsilon) \subset \{\omega : T(\omega) > t\}$. Let $C[0,\infty]$ denote the continuous functions from $\mathbb{R}^+$ to Γ_b.

Lemma 3.8. $\{\omega \in C[0,\infty] : T(\omega) > t\} = \cup_{k=1}^{\infty} H(t, k^{-1})$

Proof: Suppose ω is continuous, and that for every $k > 0$, there exists $s_k \in [0,t]$ such that $d(\omega(s_k), K) \leq k^{-1}$. As $[0,t]$ is compact, there exists a convergent subsequence $s_{k'}$ with $s_{k'} \to s \in [0,t]$. As ω is continuous.

$\omega(x_{k'}) \to \omega(x)$. However, $d(\omega(x_{k'}), K) \to 0$, so $\omega(s) \in K$, and $T(\omega) \leq s < t$. Thus, if $T(\omega) > t$ then $\inf\{d(\omega(s), K), 0 \leq s \leq t\} > k^{-1}$, for some k.

Using the continuity of ω again, there must be some m such that $d(\omega(s), K) > k^{-1}, t \leq s < t + m^{-1}$. If not, there would exist a sequence $x_\nu \to t$ such that $d(\omega(x_\nu), K) \leq k^{-1}$, forcing $d(\omega(t), K) \leq k^{-1}$. Choose $N = \min\{k, m\}$ to give $\omega \in H_{N^{-1}}$.

To restate the previous result

Corollary 3.9. *$T : C[0,\infty) \to [0,\infty)$ is a lower semicontinuous function.*

Proof: Since $\cup_{k=1}^{\infty} H_{k^{-1}}$ is an open set, this follows by definition.

We now proceed to prove the main result of this section.

Theorem 3.10. *Let Y_t and Y_t' be two independent copies of the diffusion on Γ_b. Let $T_M = \inf\{u : Y_u = Y_u'\}$. Then $T_M < \infty$ a. s.*

Proof: In Aldous [1] it is shown that if X_t and X_t' are two independent copies of a continuous time random walk on a finite graph H, then there exists some constant D such that $ET_M \leq D \max_{i,j} E^i T_j$, where the maximum is taken over all pairs of states i, j. By Markov's inequality, it follows that $P[T_M > t] \leq t^{-1} \cdot D\max_{i,j} E^i T_j$.

Let $Y_n(t)$ be a random walk in continuous time on G_n. We have shown in the preceding section that for any pair of vertices x and y, such that $||x - y|| < 3^{-i}$ then $E^i T_j \leq 738 \cdot 3^{-i}$. If we regard $Y_n(t)$ as a sequence of processes on Γ_b, then $P[T_M^{(n)} > t] \leq t^{-1} \cdot 738D$ for all n.

Consider the set $\Delta = \{(x, x) : x \in \Gamma_b\}$. Clearly, T_M is the first hitting time on Δ for the process (Y_t, Y_t').

As before, let $H_\epsilon = \{\omega : \inf d(\omega(u), \Delta) > \epsilon, 0 < u < t + \epsilon$. Since $H_\epsilon \subset \{T_M < t\}$, it follows that

$$P[Y_n(t) \in H_\epsilon] \leq P[T_M > t] \leq t^{-1} \cdot 738D \tag{3.9}$$

for all n. As H_ϵ is open, Prokhorov's theorem and the Lemma 3.1 show that

$$P[Y_t \in H_\epsilon] \leq \liminf_{n \to \infty} P[Y_n(t) \in H_\epsilon] \leq t^{-1} \cdot 738D \tag{3.9}$$

As Y_t has a. s. continuous sample paths, the preceding equation and Lemma 3.8 show that

$$P[T_M > t] = \liminf_{n \to \infty} P[Y_t \in H_{n^{-1}}] \leq t^{-1} \cdot 738D \tag{3.10}$$

Letting $t \to \infty$ gives $P[T_M = \infty] = 0$, which is what we proposed to prove.

We next consider invariant measures for Y_t. For the moment, restrict the random walks $Y_n(t)$ to Γ_b. (Or, equivalently, restrict them to $[0,3]^2$). Since each Y_n is now a random walk on a finite graph, it has a unique reversible stationary measure, which we shall call μ_n. Regard Y_n as a Γ_b-valued stochastic process, and μ_n as a measure on Γ_b. Since Γ_b is compact, $\{\mu_n\}$ has a weakly convergent subsequence $\{\mu_{n'}\}$.

Proposition. *Y_t has a stationary distribution μ, and $\mu_{n'} \to_D \mu$.*

Proof: To show the proposition, we apply weak convergence. We have previously shown that

$$E\left[\|Y_n(u) - Y_n(s)\|^\gamma \, \|Y_n(t) - Y_n(u)\|^\gamma\right] \leq D(t-s)^{\rho\gamma} \tag{3.11}$$

for any $\rho, \gamma > 0$ and any $s \leq u \leq t$. Since Γ_b is compact, the sequence μ_n is weakly precompact. Let n' be a sequence of integers such that $\mu_{n'}$ converges, say to μ. If we let $Y_{n'}^{\mu}$ be the stationary version of $Y_{n'}$ then standard results on weak convergence (See, again, Theorem 15.6 in Billingsley) show that $Y_{n'}^{\mu}$ converges weakly to a process Y^μ, where Y^μ is a version of Y with stationary distribution μ.

Theorem 3.11. *The distribution of Y_t converges μ in total variation norm*

Proof: Let $\mu_t(y,A) = P^y[Y_t \in A]$. Let Y_t' be an independent stationary diffusion on Γ_b Couple Y_t to Y_t' by letting Y and Y' move independently prior to T_M but specifying that they move identically afterwards. Then for any measurable A,

$$|\mu_t(A) - \mu(A)| \leq P[Y_t \neq Y_t'] \leq t^{-1} \cdot 738D \tag{3.12}$$

and the inequality is uniform over all measurable A. The theorem follows.

Corollary 3.12. *μ is the unique stationary distribution for Y_t.*

Theorem 3.13. *μ is normalized Hausdorff $\log_5 3$-dimensional measure, restricted to Γ_b.*

Proof: Since μ is a stationary measure, we apply the preceding theorem and the bounded fractal scaling law to get

$$P^{\underline{0}}[Y_t \in N^{-1}(A)] - \mu(N^{-1}(A)) \tag{3.13}$$

$$P^{\underline{0}}[Y_t \in N^{-1}(A)] = P^{\underline{0}}[Y_{15t} \in A] - \mu(A) \tag{3.14}$$

as $t - \infty$. So, μ satisfies the equation $\mu(A) = \mu(N^{-1}(A))$. Theorem 4.4.1 in Hutchinson [12] shows that there is a unique measure on Γ_b that satisfies this equation. Since ρ-dimensional Hausdorff measure restricted to Γ_b also satisfies this equation, it follows that μ is ρ-dimensional Hausdorff measure on Γ_b.

Now consider Y_t on Γ. If we let μ be Hausdorff $\log_3 5$-dimensional measure restricted to Γ, we have the following

Corollary 3.14. *μ is an invariant measure for Y_t.*

Proof: Normalize μ so that $\mu(\Gamma_b) = 1$, and let f be a continuous function on Γ with compact support.

Choose N sufficiently large so that $\Gamma \cap [0,3^N]$ contains the support of f. Then for $n > N$, μ is an invariant measure for $Y_t^{(n)}$, the diffusion process restricted to $\Gamma \cap [0,3^N]$. Thus, as $n \to \infty$,

$$\int f d\mu = \int E^x[f(Y_t^{(n)})]\mu(dx) \to \int E^x[f(Y_t)]\mu(dx) \qquad (3.15)$$

As the continuous functions with compact support are dense in L^1, the result follows.

We have shown in Theorem 3.11 that μ_t converges to Hausdorff $\log_3 5$-dimensional measure in total variation norm for Y_t restricted to Γ_b. This, together with the scaling laws, implies that Y_t has a transition density with respect to Hausdorff measure.

Recall the transformations $N_1, ..., N_5$ and N defined in (3.6), Let $\mu_t(\cdot) = P^0[Y_t \in \cdot]$ and let $\mu(\cdot)$ denote Hausdorff $\log_3 5$-dimensional measure, restricted to Γ.

Theorem 3.15. *$\mu_t \ll \mu$ for all $t > 0$.*

Proof: We begin by proving the theorem for Y_t restricted to Γ_b. Let $B \subset \Gamma_b$ be a set with $\mu(B) = 0$. and suppose that $\mu_t(B) = q > 0$. We observe in passing that $\mu(B) = 0$ iff $\mu(N(B)) = 0$, and that $N^{-1}(N(B)) \supset B$. Let $B_\infty = \bigcup_1^\infty N^k(B)$. Since $\mu(B) = 0$, $\mu(N^k(B)) = 0$ for all k, so $\mu(B_\infty) = 0$. On the other hand, $B \subset N^{-k}(B_\infty)$ for all k. By the bounded fractal scaling law,

$$\mu_{15^k t}(B_\infty) = \mu_t(N^{-k}(B_\infty)) \geq \mu_t(B) = q \qquad (3.16)$$

Thus, $\liminf_{k \to \infty} \mu_{15^k t}(B_\infty) \geq q$. But this contradicts the fact that $\mu_t \to \mu$ in total variation norm as $t \to \infty$. By contradiction, $\mu_t(B) = 0$. Thus, $\mu_t \ll \mu$.

To show absolute continuity for the unbounded process, let $Y_t^{(n)}$ denote the diffusion process restricted to $\Gamma \cup [0,3^n]$. As $n \to \infty$, $Y_t^{(n)} \to Y_t$ in total variation norm. The proof for Y_t restricted to Γ_b shows that

$P[Y_t^{(n)} \in \cdot]$ is absolutely continuous with respect to Hausdorff measure for all n. Therefore, it follows that $P[Y_t \in \cdot]$ is also absolutely continuous with respect to Hausdorff $\log_5 3$-dimensional measure on Γ.

4. Computing Generating Functions

We can determine the generating function of the distribution of T^1 by elementary calculations. Let $k(u) = E^0 u^\tau$ where τ is the time required to cross from 0to $(1,1)$. τ has a geometric distribution with $p = 1/3$, so $k(u) = u/(3-2u)$, trivially.

Suppose $X_0 = 0$, and let σ denote the first hitting time on an outer corner, other than $\mathbf{0}$. Let $f(u) = E^0 u^\sigma$. We calculate the generating functions of some hitting times for corner $\mathbf{1}$, as a preliminary to calculating the generating function of σ.

Let μ denote the first hitting time on $\mathbf{1}$, on the on the set where $\mathbf{1}$is the first outer corner of U_0 that the walk visits. As the distribution of the random walk is not affected by symmetries of U_0, the g depends only on the graph distance between the random walk's starting point and $\mathbf{1}$. Thus we can identify vertices of Γ_b by

$$\begin{aligned} a &= \{(0,1),(1,0),(0,2),(1,3),(2,0),(3,1)\} \\ b &= \{(1,1),(1,2),(2.1)\} \\ c &= \{(2,2)\} \\ d &= \{(2,3),(3,2)\} \end{aligned} \tag{4.1}$$

For i a vertex in U_0, let $g_i(u) = E^i u^\mu$. g then satisfies the following system of equations

$$\begin{aligned} g_a &= \tfrac{1}{3}ug_a + \tfrac{1}{3}ug_b & \qquad g_b &= \tfrac{1}{3}ug_a + \tfrac{1}{3}ug_b + \tfrac{1}{6}ug_c \\ g_c &= \tfrac{1}{2}ug_b + \tfrac{1}{3}ug_d + \tfrac{1}{6}u & \qquad g_d &= \tfrac{1}{3}ug_c + \tfrac{1}{3}ug_d + \tfrac{1}{3}u \end{aligned} \tag{4.2}$$

Direct calculation shows that this system of equations has the solutions:

$$\begin{aligned} g_a &= \frac{u^3}{3(2-u)(18-15u+u^2)} & \qquad g_b &= \frac{u^2(3-u)}{3(2-u)(18-15u+u^2)} \\ g_c &= \frac{2u(3-2u)}{(2-u)(18-15u+u^2)} & \qquad g_d &= \frac{u(12-10u+u^2)}{(3-u)(18-15u+u^2)} \end{aligned} \tag{4.3}$$

Now, let λ be the first hitting time for any corner other than 0. Again we can use symmetry to identify vertices, in the groups

$$\begin{aligned} w &= \{(0,1),(1,0)\} \\ x &= \{(1,1),(1,2),(2,1)\} \\ y &= \{(1.2),(2,1),(2,2)\} \\ z &= \{(0,2),(1,3),(2,0),(3,1),(2,3),(3,2)\} \end{aligned} \tag{4.4}$$

Then $h_i(u) = E^i u^\lambda$ satisfies

$$\begin{aligned} h_w &= \tfrac{1}{3}uh_w + \tfrac{1}{3}uh_x & \qquad h_x &= \tfrac{1}{3}uh_x + \tfrac{1}{3}uh_y \\ h_y &= \tfrac{1}{6}uh_w + \tfrac{1}{3}uh_y + \tfrac{1}{3}uh_z + \tfrac{1}{6}u & \qquad h_z &= \tfrac{1}{3}uh_y + \tfrac{1}{3}uh_z + \tfrac{1}{3}u \end{aligned} \tag{4.5}$$

with solutions

$$\begin{aligned} h_w &= \frac{u^3}{(2-u)(18-15u+u^2)} & \qquad h_x &= \frac{u^2(3-u)}{(2-u)(18-15u+u^2)} \\ h_y &= \frac{2u(9-3u-u^2)}{3(2-u)(18-15u+u^2)} & \qquad h_z &= \frac{u(36-30u+5u^2)}{3(2-u)(18-15u+u^2)} \end{aligned} \tag{4.6}$$

Using the generating functions g, h, and k, we can now calculate $f(u)$. Start at 0and condition on N, the number of returns to 0before hitting a different outer corner. We use the symmetry of U_0 and the generating functions we have just calculated, to get

$$E^0u^\sigma = k(u)\sum_0^\infty h_x(u)[g_c(u)k(u)]^n = k(u)\frac{h_x(u)}{[1-g_c(u)k(u)]} \tag{4.7}$$

If we substitute the generating functions we have already calculated, we get

$$E^0u^\sigma = \left(\frac{u}{3-2u}\right)\left[\frac{\dfrac{u^3}{(2-u)(18-15u+u^2)}}{1-\dfrac{2u(3-2u)}{(2-u)(18-15u+u^2)}\dfrac{u}{(3-2u)}}\right] \tag{4.8}$$

$$= \frac{u^3}{(3-2u)(12-12u+u^2)} \tag{4.6}$$

$$= f(u) \tag{4.9}$$

The symmetry of U_0 shows that $f(u)$ is the generating function for the distribution of the time to cross between any two distinct corners of G.

5. Acknowledgments:

I wish to acknowledge my advisor David Aldous for his guidance and support in completing this work. I also wish to thank Ben Hambly for pointing out several errors in an earlier version of this paper.

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SECURITY CLASSIFICATION OF THIS PAGE (When Date Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ARO 23699.24-MA	2. GOVT ACCESSION NO. N/A	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) A Diffusion on a Fractal State Space		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) William Bernard Krebs		8. CONTRACT OR GRANT NUMBER(s) DAAL03-86-K-0094
9. PERFORMING ORGANIZATION NAME AND ADDRESS Florida State University Department of Statistics Tallahassee, FL 32306-3033		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE September, 1989 13. NUMBER OF PAGES 39
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

N/A

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Diffusions, fractals

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We define a fractal in the plane known as the Vicsek Snowflake by constructing a skeletal lattice graph and then rescaling spatial dimensions to give a sequence of lattices that converges to a fractal. By defining a simple random walk on the skeletal lattice and then rescaling both time and space, we define a sequence of random walks on the approximating lattices that converge weakly to a limiting process on the snowflake. We show that this limit has continuous sample paths and the strong Markov property, and that it is the unique diffusion limit of random walk on the snowflake in a natural sense. We show that this

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20. ABSTRACT (continued. . .)

diffusion limit of random walk on the snowflake in a natural sense. We show that this diffusion has a scaling property reminiscent of Browniar motion, and we introduce a coupling argument to show that the diffusion has transition densities with respect to Hausdorff measure on the snowflake.