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APPLICATION OF A GENERALIZED LEIBNIZ RULE FOR CALCULATING ELECTROMAGNETIC FIELDS WITHIN CONTINUOUS SOURCE REGIONS

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... here holds for any function of \vec{F} and \vec{r}' , and shows that, analogously, one correction term is required per functional limit of the triple integral in each of the derivatives in the curl expression. This generalized three-dimensional Leibniz rule is then tailored to solve the specific problem of determining electromagnetic fields in the source region by direct differentiation of the vector potential. $\int_V \dots$

Finally, in applying the generalized expression to calculate the fields, we simply insert the vector potential to evaluate the magnetic field, and then insert the curl of the vector potential to evaluate the electric field. Since the shape of the principal volume can be chosen arbitrarily we choose a convenient pillbox/slab to simplify extracting and evaluating the relevant correction terms in the generalized Leibniz expression. Reducing the correction terms yields identically the expressions for the depolarizing dyadic previously calculated for an arbitrary as well as for a pillbox/slab volume. This process also reveals that the Leibniz correction terms are directly related to the depolarizing dyadic which represents the difference between Maxwellian and cavity-defined fields.

Preface

I thank Dr. Arthur Yaghjian for guidance and discussions of the source region problem, and J. Ducharme for typing the manuscript.



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Application of a Generalized Leibniz Rule for Calculating Electromagnetic Fields Within Continuous Source Regions

1. INTRODUCTION

In the following treatment of electromagnetic fields in source regions we calculate E and H due to a continuous current source by direct differentiation of the vector potential, using a principal volume approach similar to that in Yaghjian's article on Maxwellian and cavity fields¹. This paper, however, presents an alternative differentiation method and interpretation of the mathematics governing transposition of differential and integral operators when the integration limits depend on the differentiation variable. What most calculus texts term the "generalized Leibniz rule" pertains specifically to this subtle point when differentiating 1-D integrals. Therefore, it seemed appropriate to employ a three-dimensional version to rigorously perform the curl of a volume integral.

After searching the literature for such a 3-D version of Leibniz' rule and finding no previous examples we derived the 3-D generalized Leibniz rule from first principles. When subsequently applied to the vector potential in a current region, the 3-D generalized rule produced the rigorous expressions for the "cavity" as well as "Maxwellian" electric and magnetic fields. One merit in using this direct approach is that it relates the additional terms arising from the 3-D generalized Leibniz rule to the difference between the Maxwellian and

(Received for Publication 11 Jan. 1989.)

¹Yaghjian, A.D. (1985) Maxwellian and Cavity Electromagnetic Fields Within Continuous Sources, *Am. J. Phys.* **53**:859.

cavity fields, that is, to the source dyadic term. This study also encourages further applications of the 3-D Leibniz rule derived here, wherever differentiation is required of integrals with limits that depend on the differentiation variables. Such possibilities lie in the analogous electrostatics problem², and perhaps in the theory of fluid dynamics.

2. DIRECT DIFFERENTIATION OF THE VECTOR POTENTIAL IN A CURRENT REGION

From the expressions for the fields in terms of the vector potential \mathbf{A} of a volume current density \mathbf{J} , with the $e^{-i\omega t}$ time dependence suppressed, the \mathbf{H} field requires one curl operation,

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} (\nabla \times \mathbf{A}(\mathbf{r})), \quad (1)$$

and the \mathbf{E} field two in succession¹,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{i\omega\epsilon_0\mu_0} [\mu_0\mathbf{J}(\mathbf{r}) - \nabla \times \nabla \times \mathbf{A}(\mathbf{r})], \quad (2)$$

where

$$\mathbf{A}(\mathbf{r}) = \mu_0 \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \frac{\mathbf{J}(\mathbf{r}') e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (3)$$

Though we will only treat current sources initially, we will generalize the results later to include polarization and magnetization sources as well. To unambiguously define the "Maxwellian" fields inside the source region we use a limiting principal volume, V_δ , in Eq. (3) to eliminate the singularity of the Green's function right at the observation point. We may relate these mathematically defined Maxwellian fields to operationally defined "cavity" fields by cutting a hole, V_c , in the source, inserting a test charge, q , measuring the force exerted on the test charge by the now external sources, and calculating the fields, from

² Portis, Alan M. (1978) *Electromagnetic Fields, Sources, and Media*, John Wiley & Sons, New York, p. 40-41.

$\mathbf{E} = \lim_{q \rightarrow 0} \frac{\mathbf{F}_{\text{ext}}}{q}$. The difference between the Maxwellian electric field and the operationally defined cavity electric field is contained in a source dyadic term, which emerges as the ultimate product of this analysis. It is important to realize that V_c is fixed during the differentiation with respect to \mathbf{r} whereas in Eq. (3), V_δ shrinks around the observation point, moving with the differentiation variable, \mathbf{r} . Thus, $V - V_\delta$ varies with \mathbf{r} . The mathematical repercussion of this is apparent whenever we take the curl of \mathbf{A} (that is, differentiate with respect to \mathbf{r}). Specifically, the limits of the integral in Eq. (3) depend on the differentiation variable, which prevents a casual interchange of the ∇ and \int operations.

To overcome the restriction on interchanging this differentiation and integration, Yaghjian changes the integration variable from \mathbf{r}' to $\mathbf{r}'' = \mathbf{r}' - \mathbf{r}$, to remove the dependence of the limits of integration of the differentiation variable, \mathbf{r} , and proceeds to evaluate the fields using, initially, a spherical principal volume.¹ Since Leibniz' rule already governs differentiation of integrals, an obvious alternative to Yaghjian's change of variable is to extend and apply the generalized Leibniz' rule to the 3-D differentiation of the vector potential volume integral. This straightforward method retains both the singularity and the dependence of the limits of integration on \mathbf{r} , and still yields identical results for the fields.

References to multi-dimensional forms of Leibniz' rule are rare and apparently address only the case of fixed integration limits rather than the "generalized Leibniz rule" which allows variable limits of integration. For example, Osgood states, "In the case of multiple integrals, we assume that the region of integration is fixed³. Cases arise in hydromechanics in which the region varies with the parameters, but the treatment does not belong to the elements of the calculus." A 3-D rule is required for the problem at hand, so at this point we embark on a derivation of the 3-D generalized Leibniz rule.

3. FORMULATION OF 3-D GENERALIZED LEIBNIZ RULE

In Cartesian coordinates, the curl operator can be written as a sum of partial derivatives,

$$\nabla \times \mathbf{K} = \hat{x} \left(\frac{\partial K_z}{\partial y} - \frac{\partial K_y}{\partial z} \right) + \hat{y} \left(\frac{\partial K_x}{\partial z} - \frac{\partial K_z}{\partial x} \right) + \hat{z} \left(\frac{\partial K_y}{\partial x} - \frac{\partial K_x}{\partial y} \right) \quad (4)$$

where \mathbf{K} , in our problem, is the vector potential volume integral. We wish to apply the familiar 1-D Leibniz rule to each of the six partial derivatives above. Consider first the generalized 1-D Leibniz rule:

³ Osgood, (1925) *Advanced Calculus*, MacMillan, New York, p. 463.

$$\frac{\partial}{\partial x} \int_{a_1(x)}^{a_2(x)} f(x, x') dx' = \int_{a_1(x)}^{a_2(x)} \frac{\partial}{\partial x} f(x, x') dx' + \sum_{n=1}^2 (-1)^n f(x, a_n(x)) \frac{da_n}{dx} \quad (5)$$

Eq. (5) shows that exchanging the differential and integral operations adds (subtracts) an additional term for each variable upper (lower) limit. We will find that the curl of a volume integral will, in its most general form, require six additional terms (one for each limit in the volume integral); these will generate a total of 36 additional terms. Fortunately, a proper choice of principal volume geometry will later eliminate most of these terms.

We derive the 3-D rule in similar fashion to the 1-D rule, applying first the chain rule and then the fundamental theorem of integral calculus. As an example, we write explicitly one term from Eq. (4):

$$\frac{\partial K_x}{\partial z} = \frac{\partial}{\partial z} \int_{a_5}^{a_6} \int_{a_3}^{a_4} \int_{a_1}^{a_2} F_x(r, r') dV',$$

where K depends on the variables $a_1, a_2 (r, y', z')$, $a_3, a_4 (r, z')$, $a_5, a_6 (r)$ and r . The derivative of each component of K can be expanded, using the chain rule, as a sum of the derivatives of each parameter, holding the other six constant.

$$\frac{\partial K_x}{\partial z} = \sum_{n=1}^6 \frac{\partial K_x}{\partial a_n} \frac{\partial a_n}{\partial z} + \frac{\partial K_x}{\partial r} \frac{\partial r}{\partial z} \left| \begin{array}{l} a_1, a_2, a_3, \\ a_4, a_5, a_6 \text{ constant} \end{array} \right. \quad (6)$$

The fundamental theorem of integral calculus is now applied to the volume integrals as follows:

$$\frac{\partial K_x}{\partial a_1} = \frac{\partial}{\partial a_1} \int_{a_5}^{a_6} \int_{a_3}^{a_4} \int_{a_1}^{a_2} F_x(r, x', y', z') dx' dy' dz' = - \int_{a_5}^{a_6} \int_{a_3}^{a_4} F_x(r, x', y', a_1) dy' dz'. \quad (7)$$

Similar expressions hold for $a_2 - a_6$ as well. If we apply the chain rule to each partial derivative in Eq. (4), and in turn the fundamental theorem to each term in Eq. (6), after collecting terms we can write the generalized 3-D Leibniz rule in a form analogous to the 1-D rule⁴, with the help of the ϵ_{ijk} Levi-Civita symbol. Specifically,

$$\begin{aligned}
 (\nabla \times \mathbf{K})_i &= \epsilon_{ijk} \partial_j K_k \\
 &= \epsilon_{ijk} \int_{a_5}^{a_6} \int_{a_3}^{a_4} \int_{a_1}^{a_2} \partial_j F_k dV \\
 &+ \epsilon_{ijk} \left[\sum_{n=1}^2 (-1)^n \left(\frac{\partial a_n}{\partial j} \right) \int_{a_5}^{a_6} \int_{a_3}^{a_4} F_k(\mathbf{r}, a_n, x'_2, x'_3) dx'_2 dx'_3 \right. \\
 &+ \sum_{n=3}^4 (-1)^n \left(\frac{\partial a_n}{\partial j} \right) \int_{a_5}^{a_6} \int_{a_1}^{a_2} F_k(\mathbf{r}, x'_1, a_n, x'_3) dx'_1 dx'_3 \\
 &\left. + \sum_{n=5}^6 (-1)^n \left(\frac{\partial a_n}{\partial j} \right) \int_{a_3}^{a_4} \int_{a_1}^{a_2} F_k(\mathbf{r}, x'_1, x'_2, a_n) dx'_1 dx'_2 \right]
 \end{aligned} \tag{8}$$

Note that summation over repeated indices is implied and thus each surface integral is a summation of 12 additional terms. This is the central result which we will use to calculate the electric and magnetic fields in the source region given the vector potential. We emphasize that the field calculation is done to verify that the square-bracketed terms in Eq. (8) give rise to the source dyadic term.

4. TAILORING THE 3-D GENERALIZED LEIBNIZ RULE

Returning to our specific problem of evaluating the vector potential integral, Eq. (3), let us choose initially a disc-shaped (pillbox) principal volume to evaluate the surface integrals. Choosing a disc principal volume, rather than a sphere or cube, in this case is advantageous

⁴ Kaplan, (1952), *Advanced Calculus*, Addison Wesley, Reading, MA, p. 221.

because it greatly simplifies the 3-D Leibniz expression by eliminating all but 2 of the possible 36 additional terms.

We wish to evaluate the vector potential \mathbf{A} by summing the contributions from the surface of the disc principal volume to the edges of the source region, as the height of the disc, 2ϵ , shrinks around the observation point \mathbf{r} (Figure 1). We will, accordingly, substitute V_ϵ and $\lim_{\epsilon \rightarrow 0} V_\delta$ for V_δ and $\lim_{\delta \rightarrow 0} V_\delta$ on the surface integrals. Obviously, the outer limits of integration are fixed, since the source volume is finite. Furthermore, in Appendix A we prove that the truncated disc can be replaced by the thin slab in Figure 2 where the disc radius extends to the edges of the source region.

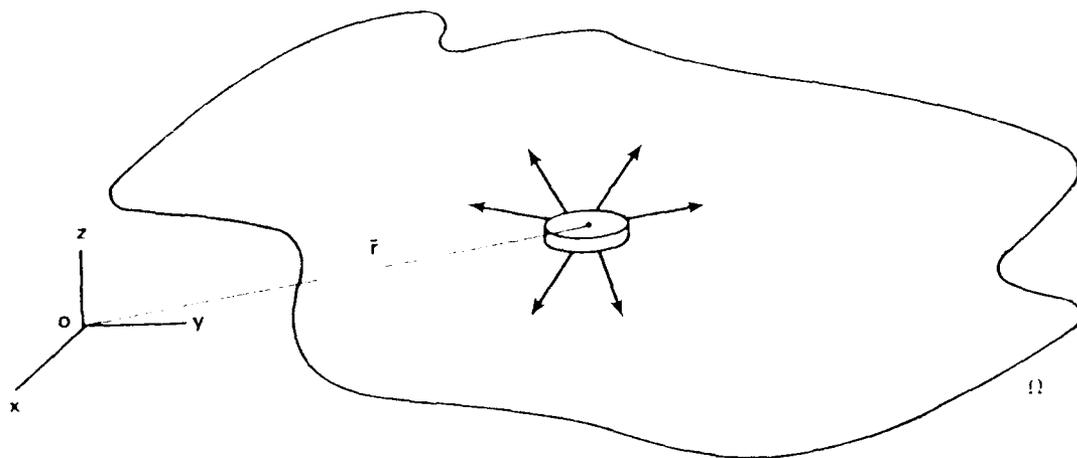


Figure 1. 'Pillbox', or 'Disc' Principal Volume.

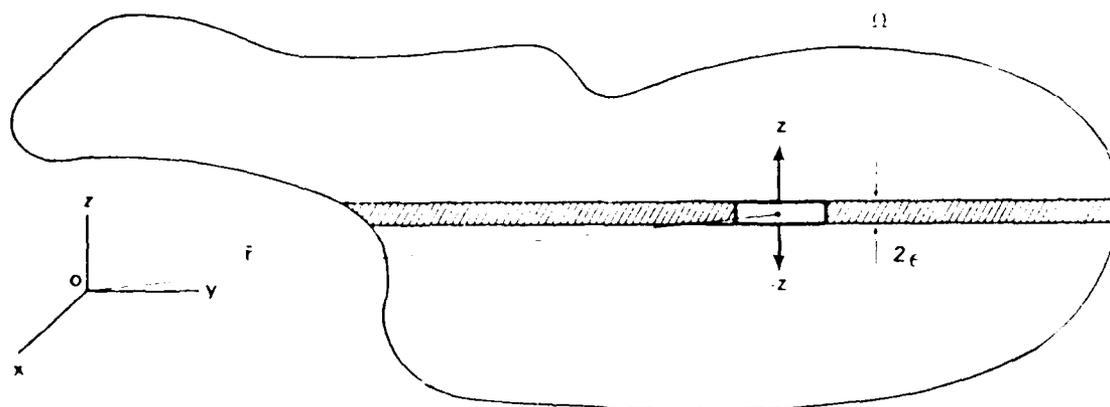


Figure 2. 'Slab' Principal Volume.

Next, divide the volume of integration into the volumes above and below the thin slab. Each of these two volumes are singly connected and thus we can apply our generalized 3-D Leibniz rule [Eq. (8)] to each region. The limits in the x- and y- directions are constant, and the only variable limits remaining when Eq. (8) is applied to each region are the z' integrations. We now integrate over the source volume from z' = z + ε to Ω_{top}, and z' = z - ε to Ω_{bot}, and from x', y' = Ω_{left} to Ω_{right} where Ω represents the distant boundary of the volume current, J. The only additional Leibniz terms demanding consideration are those corresponding to the z' limits of integration. In particular, our bounded current region (excluding a slab principal volume) in Eq. (8) yields:

$$\begin{aligned} \nabla \times \lim_{\delta \rightarrow 0} \iiint_{V - V_\delta} \mathbf{F}(\mathbf{r}, \mathbf{r}') dV' &= \lim_{\delta \rightarrow 0} \iiint_{V - V_\delta} \nabla \times \mathbf{F}(\mathbf{r}, \mathbf{r}') dV' \\ &+ \lim_{\epsilon \rightarrow 0} \left\{ \hat{\mathbf{x}} \left[\iint_{S_\epsilon} \mathbf{F}_y(\mathbf{r}, x', y', z' = z + \epsilon) dx'dy' - \iint_{S_\epsilon} \mathbf{F}_y(\mathbf{r}, x', y', z' = z - \epsilon) dx'dy' \right] \right. \\ &\left. + \hat{\mathbf{y}} \left[- \iint_{S_\epsilon} \mathbf{F}_x(\mathbf{r}, x', y', z' = z + \epsilon) dx'dy' + \iint_{S_\epsilon} \mathbf{F}_x(\mathbf{r}, x', y', z' = z - \epsilon) dx'dy' \right] \right\} \end{aligned} \quad (9)$$

where the limit representing the surface of the principal volume splits into two terms, one for the upper slab surface and one for the lower slab surface. This expression is particular to our geometry but holds for any sufficiently well-behaved vector function F.

5. CALCULATION OF MAXWELLIAN FIELDS

To calculate the Maxwellian magnetic field $\mathbf{H}(\mathbf{r})$ from Eq. (1) at points within the source region, we simply differentiate the vector potential \mathbf{A} in Eq. (3) using the 3-D generalized Leibniz rule [Eq. (9)]

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} \left[\nabla \times \lim_{\delta \rightarrow 0} \iiint_{V - V_\delta} \mathbf{F}(\mathbf{r}, \mathbf{r}') dV' \right] \quad (10)$$

Comparing this with Eq. (9), we set $\mathbf{F}(\mathbf{r}, \mathbf{r}') = \mathbf{J}(\mathbf{r}') \psi(|\mathbf{r} - \mathbf{r}'|)$,

where $\psi(\mathbf{r}, \mathbf{r}') = \frac{e^{ik(|\mathbf{r} - \mathbf{r}'|)}}{(|\mathbf{r} - \mathbf{r}'|)}$ to give:

$$\begin{aligned} \mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} & \left[\lim_{\delta \rightarrow 0} \iiint_{V - V_\delta} \nabla \times \mathbf{J}(\mathbf{r}') \psi(|\mathbf{r} - \mathbf{r}'|) dV' \right. \\ & + \lim_{\epsilon \rightarrow 0} \left\{ \hat{\mathbf{x}} \iint_{S_\epsilon} \left(J_y \psi \Big|_{z'=z+\epsilon} - J_y \psi \Big|_{z'=z-\epsilon} \right) dS' \right. \\ & \left. \left. + \hat{\mathbf{y}} \iint_{S_\epsilon} \left(-J_x \psi \Big|_{z'=z+\epsilon} + J_x \psi \Big|_{z'=z-\epsilon} \right) dS' \right\} \right] \quad (11) \end{aligned}$$

Up to this point we have dealt with the problem of rigorously interchanging the curl and integral operators in Eq. (10), but have simply assumed that the curl and limit could be interchanged. This assumption can be proven valid by dividing the volume of integration ($V - V_\delta$) into two regions, an exterior region ($V - V_d$) and an annular region ($V_d - V_\delta$), where V_d is a small but finite fixed volume enclosing V_δ . Since $V - V_d$ does not depend on δ , the curl and limit for this part of the volume integration can be interchanged immediately. If V_d is made small enough so that \mathbf{J} can be expanded in a power series over V_d , the annular volume integration over $V_d - V_\delta$ can be performed and the curl taken before or after the limit is taken. This procedure shows directly that the result remains independent of whether the curl or limit is taken first, thus proving that the curl and limit can be interchanged in Eq. (10).

To evaluate the four surface integrals, we note that if $J_y \psi \Big|_{z'=z+\epsilon}$ (upper) = $J_y \psi \Big|_{z'=z-\epsilon}$

(lower) and if $J_y \psi|_{z=z+\epsilon}$ (upper) = $J_y \psi|_{z=z-\epsilon}$ (lower), all the surface integrals cancel. Since J approaches the same value on the top and bottom surfaces of the slab, as the slab becomes very thin, we need only consider $\psi = \frac{e^{ik\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$. Obviously, for

$z' = z + \epsilon$ or $z' = z - \epsilon$, $(z - z')^2 = \epsilon^2$, so $\psi_{z'=z+\epsilon} = \psi_{z'=z-\epsilon}$. That is, the Green's function has the same value on either surface of the slab, and thus it follows (see Appendix A) that both pairs of terms vanish in Eq. (11)*, leaving the rather anti-climactic result for the magnetic field,

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} \left(\lim_{\delta \rightarrow 0} \int_{V - V_\delta} \nabla \times \mathbf{J}(\mathbf{r}') \psi(|\mathbf{r} - \mathbf{r}'|) dV' \right) \quad (12)$$

As Yaghjian states, "The merit of the rigorous procedure does not become obvious until it is applied a second time to find the Maxwellian electric field."¹ The electric field calculation proceeds from Eq. (2) in the same fashion as that for the magnetic field, except we differentiate the vector potential twice: $\mathbf{E}(\mathbf{r}) \propto \nabla \times \nabla \times \mathbf{A}(\mathbf{r})$. The first differentiation, we know from the magnetic field calculation, yields no additional Leibniz terms, so we may exchange the differential and integral signs,

$$\nabla \times \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J} \psi dV' = \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \nabla \times \mathbf{J} \psi dV'. \quad (13)$$

Differentiating again we write,

$$\nabla \times \left(\nabla \times \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J} \psi dV' \right) = \nabla \times \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \nabla \times \mathbf{J} \psi dV'. \quad (14)$$

where, as explained above, we may interchange the curl and limit. To evaluate the RHS of

* for the case where $x = x'$ and $y = y'$, $\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} \frac{e^{ik\epsilon}}{\epsilon} dS'$ becomes ik on both surfaces

Eq. (14), we apply the same 3-D Leibniz rule, this time setting $\mathbf{F} = \nabla \times \mathbf{J}\Psi$ in Eq. (9). Thus,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) = & \frac{1}{i\omega\epsilon_0\mu_0} \left(\mu_0 \mathbf{J}(\mathbf{r}) - \mu_0 \lim_{\substack{\delta \rightarrow 0 \\ \epsilon \rightarrow 0}} \left\{ \iiint_{\mathbf{v} - \mathbf{v}_s} \nabla \times \nabla \times \mathbf{J}(\mathbf{r}') \psi(|\mathbf{r} - \mathbf{r}'|) dV \right. \right. \\ & + \hat{\mathbf{x}} \left[\iint_{S_\epsilon} (\nabla \times \mathbf{J}\Psi)_{y|z'=z+\epsilon} dx'dy' - \iint_{S_\epsilon} (\nabla \times \mathbf{J}\Psi)_{y|z'=z-\epsilon} dx'dy' \right] \\ & \left. \left. + \hat{\mathbf{y}} \left[-\iint_{S_\epsilon} (\nabla \times \mathbf{J}\Psi)_{x|z'=z+\epsilon} dx'dy' + \iint_{S_\epsilon} (\nabla \times \mathbf{J}\Psi)_{x|z'=z-\epsilon} dx'dy' \right] \right\} \right). \end{aligned} \quad (15)$$

To evaluate the two pairs of Leibniz surface integrals in Eq. (15), we first use the following simple vector identity to transform the integrand $\mathbf{F} = \nabla \times \mathbf{J}\Psi$:

$$\nabla \times \mathbf{J}\Psi = \nabla \times \Psi \mathbf{J} = \nabla \Psi \times \mathbf{J} + \Psi (\nabla \times \mathbf{J}). \quad (16)$$

Noting that $\nabla \times \mathbf{J}(\mathbf{r}') = 0$, since ∇ operates only on \mathbf{r} , we can rewrite the surface integrals in Eq. (15) as:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left\{ \hat{\mathbf{x}} \left[\iint_{S_\epsilon} (\nabla \Psi \times \mathbf{J})_{y|z'=z+\epsilon} dx'dy' - \iint_{S_\epsilon} (\nabla \Psi \times \mathbf{J})_{y|z'=z-\epsilon} dx'dy' \right] \right. \\ \left. + \hat{\mathbf{y}} \left[\iint_{S_\epsilon} -(\nabla \Psi \times \mathbf{J})_{x|z'=z+\epsilon} dx'dy' + \iint_{S_\epsilon} (\nabla \Psi \times \mathbf{J})_{x|z'=z-\epsilon} dx'dy' \right] \right\}. \end{aligned} \quad (17)$$

Next, we examine the x- and y- components of the vector product $\nabla \Psi \times \mathbf{J}$:

$$\left(\nabla\psi \times \mathbf{J}\right)_x = \left(\nabla\psi\right)_y J_z - \left(\nabla\psi\right)_z J_y \quad (18)$$

$$\left(\nabla\psi \times \mathbf{J}\right)_y = \left(-\nabla\psi\right)_x J_z + \left(\nabla\psi\right)_z J_x$$

where the components of the gradient of the scalar Green's function can be written

$$\nabla\psi = \sum_{i=1}^3 \frac{e^{ik|\mathbf{r}-\mathbf{r}'|} [ik|\mathbf{r}-\mathbf{r}'| - 1]}{4\pi|\mathbf{r}-\mathbf{r}'|^3} \left(x_i - x'_i\right) \quad (19)$$

$x_1 = x$
 $x_2 = y$
 $x_3 = z$

Recall that in the magnetic field expression, the scalar Green's function ψ had equal magnitude on the upper and lower surfaces of the slab principal volume. This was due to the z dependence being quadratic (that is, $(z - z')^2 = \epsilon^2$ regardless of z). In the case of the electric field, though, the factor of $(z - z')$ corresponding to $i = 3$ in $\nabla\psi$ is linear. The result, upon substituting the components from Eq. (19) into Eq. (18), and subsequently, Eq. (18) into Eq. (17), is

$$\begin{aligned} & - \lim_{\epsilon \rightarrow 0} \left\{ \hat{x} \iint_{S_\epsilon} \left[\gamma(x - x') J_z(x', y', z + \epsilon) + \gamma\epsilon J_x(x', y', z + \epsilon) \right. \right. \\ & \left. \left. - \gamma(x - x') J_z(x', y', z - \epsilon) + \gamma\epsilon J_x(x', y', z - \epsilon) \right] dS' \right. \\ & + \hat{y} \iint_{S_\epsilon} \left[\gamma(y - y') J_z(x', y', z + \epsilon) + \gamma\epsilon J_y(x', y', z + \epsilon) \right. \\ & \left. \left. - \gamma(y - y') J_z(x', y', z - \epsilon) + \gamma\epsilon J_y(x', y', z - \epsilon) \right] dS' \right\}, \end{aligned} \quad (20)$$

where the shorthand notation $\gamma = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|} [ik|\mathbf{r}-\mathbf{r}'| - 1]}{4\pi|\mathbf{r}-\mathbf{r}'|^3}$ has been introduced.

At first glance, it appears that the pairs of terms containing $\gamma(x - x')$ and $\gamma(y - y')$ will cancel immediately while the terms containing ϵ will add. Indeed, the following evaluation of

these integrals proves this to be true. The analysis used to evaluate the terms in Eq. (20) is similar to the H field case, of Appendix B:

(1) Within a continuous current distribution, opposing currents on the upper and lower surfaces of a slab of infinitesimal thickness ϵ are equal. To be certain, however, we expand $J(z + \epsilon)$ and $J(z - \epsilon)$ in a Taylor series about z :

$$J_1(z + \epsilon) = J_1(z) + \epsilon J_1'(z) + \frac{\epsilon^2}{2!} J_1''(z) + \dots \quad (21)$$

$$J_1(z - \epsilon) = J_1(z) - \epsilon J_1'(z) + \frac{\epsilon^2}{2!} J_1''(z) + \dots$$

Subtracting the two gives $2\epsilon J_1'(z)$; adding gives $2J_1(z)$. Thus, in the limit as $\epsilon \rightarrow 0$, all the current terms become zero, because they are each multiplied by a factor ϵ at some point. Therefore, whenever $\int \nabla \Psi$ is shown to be finite, we can discard the product $\epsilon J \nabla \Psi$.

(2) The gradient of the scalar Green's function $\nabla \Psi$ yields slightly different integrands for the x- and y- components than for the z- component. Translating into polar coordinates, according to Figure 3 we have:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \epsilon \int_{S_z} 2 \gamma (x - x') dS' &= \lim_{\epsilon \rightarrow 0} \epsilon \int_0^{2\pi} \int_0^\infty \frac{e^{ik\sqrt{\rho'^2 + \epsilon^2}} ik\rho'^2 \cos \theta' d\rho' d\theta'}{2\pi (\rho'^2 + \epsilon^2)} \\ &- \lim_{\epsilon \rightarrow 0} \epsilon \int_0^{2\pi} \int_0^\infty \frac{e^{ik\sqrt{\rho'^2 + \epsilon^2}} \rho'^2 \cos \theta' d\rho' d\theta'}{2\pi (\rho'^2 + \epsilon^2)^{3/2}} \end{aligned} \quad (22)$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{S_z} 2 \gamma \epsilon dS' &= \lim_{\epsilon \rightarrow 0} \epsilon \int_0^{2\pi} \int_0^\infty \frac{e^{ik\sqrt{\rho'^2 + \epsilon^2}} ik\rho' d\rho' d\theta'}{2\pi (\rho'^2 + \epsilon^2)} \\ &- \lim_{\epsilon \rightarrow 0} \epsilon \int_0^{2\pi} \int_0^\infty \frac{e^{ik\sqrt{\rho'^2 + \epsilon^2}} \rho' d\rho' d\theta'}{2\pi (\rho'^2 + \epsilon^2)^{3/2}} \end{aligned}$$

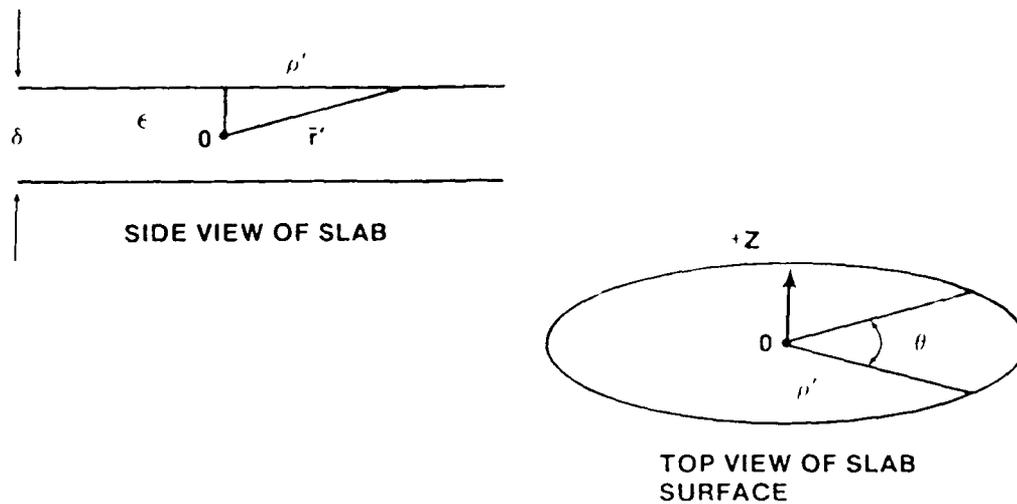


Figure 3. Polar Coordinates for Evaluating Surface Integrals from Leibniz Rule Application

Because we will be taking the limit as $\epsilon \rightarrow 0$ and each of these integrals is multiplied by a zero current term, we can discard all portions that are finite as $\epsilon \rightarrow 0$. First, note the graphs of each integrand in Figure 4, revealing the validity of the following approximations:

- a. Since the singularities only occur as $\rho' \rightarrow 0$ and $\epsilon \rightarrow 0$, we discard the finite portion of the curves above $\rho' = d$, thereby establishing d as the upper limit on the surface integration.
- b. For very small ρ' and ϵ , $e^{ik\sqrt{\rho'^2 + \epsilon^2}} \approx 1$.

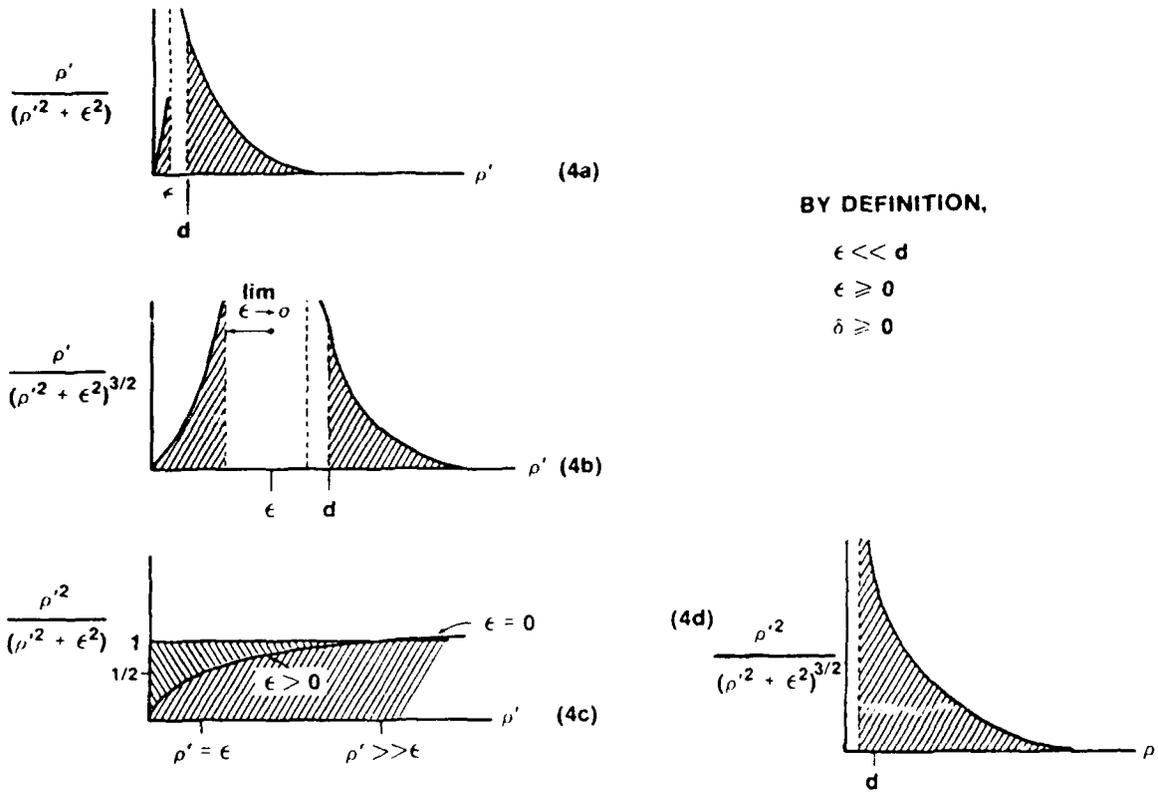


Figure 4. Graphs Depicting Infinite and Finite Portions of Integrands Involving $\nabla\psi$.

With these assumptions, the first of the four integrals, depicted in Figure 4c, is finite and vanishes in the limit as $\epsilon \rightarrow 0$.

Because the singularity in the remaining three integrals is higher order, we evaluate these integrals directly. First, we note that the $d\theta'$ integral conveniently yields a factor of 2π . All ρ' integrals are tabulated, and are evaluated as follows:

$$\begin{aligned}
 \text{(a)} \quad & - \lim_{\epsilon \rightarrow 0} \epsilon \int_0^d \frac{\rho'^2 d\rho'}{(\rho'^2 + \epsilon^2)^{3/2}} = - \lim_{\epsilon \rightarrow 0} \epsilon \left(- \frac{d}{\sqrt{d^2 + \epsilon^2}} + \ln \left(d + \sqrt{d^2 + \epsilon^2} \right) - \ln \epsilon \right) \\
 & = - \lim_{\epsilon \rightarrow 0} \epsilon \ln(2d) - \lim_{\epsilon \rightarrow 0} \epsilon \ln \epsilon = 0.
 \end{aligned}$$

(The last term vanishes by L'Hospital's rule.)

$$(b) \lim_{\epsilon \rightarrow 0} \epsilon ik \int_0^d \frac{\rho' d\rho'}{(\rho'^2 + \epsilon^2)} = \lim_{\epsilon \rightarrow 0} ik \frac{\epsilon}{2} \ln \left(\frac{d^2}{\epsilon^2} + 1 \right).$$

Using the approximation that as $\epsilon \rightarrow 0$, $\frac{d^2}{\epsilon^2} \gg 1$, we have $\lim_{\epsilon \rightarrow 0} ik \frac{\epsilon}{2} (\ln d^2 - \ln \epsilon^2)$. Since $\ln d^2$ is finite, we discard it, and $\lim_{\epsilon \rightarrow 0} -ik \epsilon \ln \epsilon = 0$. [This result also vanishes by L'Hospital's rule, as in (a) above.]

$$(c) \lim_{\epsilon \rightarrow 0} \epsilon \int_0^d \frac{\rho' d\rho'}{(\rho'^2 + \epsilon^2)^{3/2}} = \lim_{\epsilon \rightarrow 0} \epsilon \left[\frac{-1}{\sqrt{\rho'^2 + \epsilon^2}} \right]_0^d = - \lim_{\epsilon \rightarrow 0} \epsilon \left[\frac{1}{\sqrt{d^2 + \epsilon^2}} - \frac{1}{\epsilon} \right] = 1.$$

Thus, only the last integral in Eq. (22) survives.

This non-zero result for the fourth integral is, in fact, the crucial discrepancy arising from the double curl operation in Eq. (2). It is particular to the slab/disc principal volume. We will quickly see that it generates the correct source dyadic term \mathbf{L}_δ , for a slab/disc. Inserting the results of Eq. (21) and (22) into Eq. (20), we find

$$- \lim_{\epsilon \rightarrow 0} \left\{ \hat{x} \iint \left[\gamma (\mathbf{x} - \mathbf{x}') \left(2\epsilon J'_z \right) - \gamma \epsilon 2J'_x \right] dS' + \hat{y} \iint \left[\gamma (\mathbf{y} - \mathbf{y}') \left(2\epsilon J'_z \right) - \gamma \epsilon 2J'_y \right] dS' \right\} \quad (23)$$

$$= \hat{x} J_x(\mathbf{r}) + \hat{y} J_y(\mathbf{r}).$$

Finally, we insert Eq. (23) into the electric field expression,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{i\omega\epsilon_0} \left[\mathbf{J}(\mathbf{r}) - \lim_{\delta \rightarrow 0} \iiint_{\mathbf{v} = \mathbf{v}_\delta} \nabla \times \left[\nabla \psi (|\mathbf{r} - \mathbf{r}'|) \times \mathbf{J}(\mathbf{r}') \right] dV' - \hat{x} J_x(\mathbf{r}) - \hat{y} J_y(\mathbf{r}) \right].$$

and combine terms to get the final electric field result:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{i\omega\epsilon_0} \left[- \lim_{\delta \rightarrow 0} \iiint_{V - V_\delta} \nabla \times \left[\nabla \psi (|\mathbf{r} - \mathbf{r}'|) \times \mathbf{J}(\mathbf{r}') \right] dV' + \hat{\mathbf{z}} J_z(\mathbf{r}) \right], \quad (24)$$

where $\hat{\mathbf{z}} J_z(\mathbf{r})$ is the scalar product of the current, J_z , and the pillbox/slab source dyadic, $\mathbf{L}_\delta = \hat{\mathbf{z}}\hat{\mathbf{z}}^5$.

It is desirable, at this point, to express the \mathbf{E} field in terms of the electric dyadic Green's function, $\mathbf{G}_E(\mathbf{r}, \mathbf{r}') = \left(\frac{\nabla\nabla}{k^2} + \mathbf{I} \right) \psi (|\mathbf{r} - \mathbf{r}'|)$. To do this we employ the identity

$$\nabla \times (\nabla \times \mathbf{J}\psi) = \mathbf{J} \cdot [\nabla\nabla - \nabla^2] \psi \quad \text{and invoke the homogeneous scalar wave equation,}$$

$$\nabla^2 \psi = -k^2 \psi, \quad (\mathbf{r} \neq \mathbf{r}').$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{i\omega\epsilon_0} \left\{ - \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J}(\mathbf{r}') \cdot \left[k^2 \mathbf{I} + \nabla\nabla \right] \psi (|\mathbf{r} - \mathbf{r}'|) dV' + \hat{\mathbf{z}} J_z(\mathbf{r}) \right\} \quad (25)$$

$$= i\omega\mu_0 \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J}(\mathbf{r}') \cdot \left[\mathbf{I} + \frac{\nabla\nabla}{k^2} \right] \psi (|\mathbf{r} - \mathbf{r}'|) dV' + \hat{\mathbf{z}} J_z(\mathbf{r}) / i\omega\epsilon_0$$

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E dV' + \frac{\hat{\mathbf{z}} J_z(\mathbf{r})}{i\omega\epsilon_0}.$$

Analogously, we can express $\mathbf{H}(\mathbf{r})$ [Eq. (12)] in terms of the magnetic dyadic Green's function,

⁵ Yaghjian, A.D. (1980), Electric Dyadic Green's Functions in the Source Region, *Proc. IEEE*, **68**: 258.

$$\mathbf{H}(\mathbf{r}) = \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J}_E(\mathbf{r}') \cdot \mathbf{G}_H dV \quad (26)$$

where

$$\mathbf{G}_H(\mathbf{r}, \mathbf{r}') = \nabla' \times \psi l.$$

6. CALCULATION OF GENERIC SOURCE DYADIC GIVEN A PARTICULAR SOURCE DYADIC

Having calculated the source dyadic corresponding to a designated principal volume, V_δ^D , one can derive a general expression valid for an arbitrary principal volume, V_δ . This process improves our appreciation of how the Leibniz rule surface integrals relate to the general form of the source dyadic, L_δ , also a surface integral. Restated mathematically, we can show that our result for the electric field, excluding a disc principal volume, is,

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \lim_{\delta \rightarrow 0} \int_{V - V_\delta^D} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E(\mathbf{r}, \mathbf{r}') dV + \frac{\hat{z} \mathbf{J}_z(\mathbf{r})}{i\omega\epsilon_0} \quad (27)$$

can be rewritten and simplified to yield the general expression,

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E(\mathbf{r}, \mathbf{r}') dV + \frac{\mathbf{L}_\delta \cdot \mathbf{J}(\mathbf{r})}{i\omega\epsilon_0}$$

where

$$\mathbf{L}_\delta = \frac{1}{4\pi} \int_{S_\delta} \frac{\hat{n} \hat{e}_r}{r^2} dS$$

and

$$\hat{e}_r = \frac{(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|}.$$

We accomplish this by circumscribing the disc principal volume with the arbitrary principal volume⁶ (or vice versa), as shown in Figure 5.

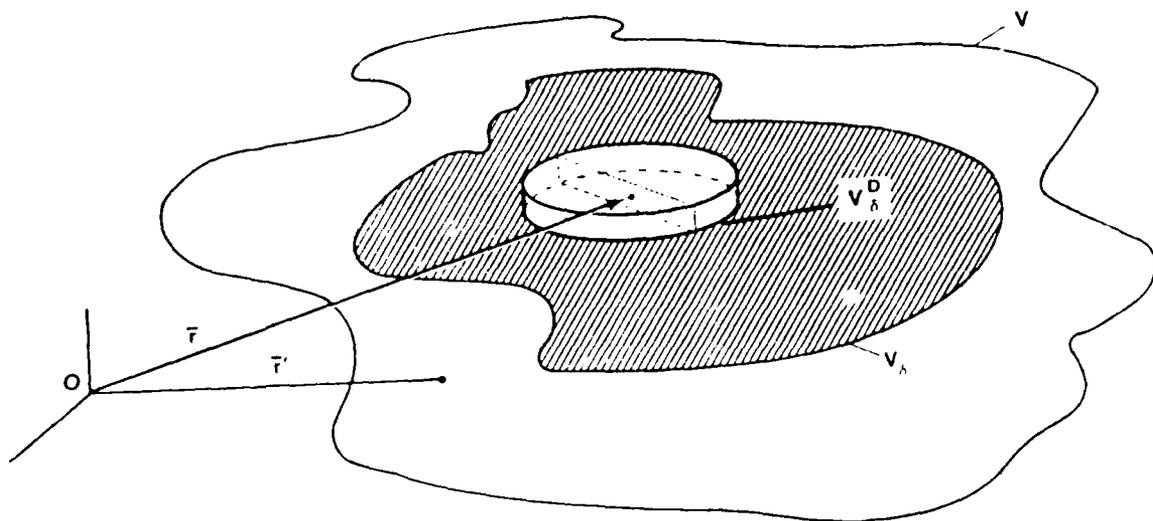


Figure 5. Arbitrary Principal Volume Circumscribing Disc-shaped Principal Volume.

We can rewrite the electric field Eq. (27) in terms of V_δ .

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \lim_{\delta \rightarrow 0} \left\{ \int_{V - V_\delta} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E(\mathbf{r}, \mathbf{r}') dV' + \int_{V_\delta - V_\delta^D} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E(\mathbf{r}, \mathbf{r}') dV' \right\} + \frac{\hat{z}J_z(\mathbf{r})}{i\omega\epsilon_0}.$$

⁶ Yaghjian, A.D. (1982), A Delta-Distribution Derivation of the Electric Field in the Source Region, *Electromagnetics*, 2:161-167.

The task is to solve the second integral.

$$\lim_{\delta \rightarrow 0} \int_{V_\delta - V_\delta^D} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E(\mathbf{r}, \mathbf{r}') dV'$$

by considering the integral over just the Green's function,

$$\lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{G}_E dV = \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \left(\frac{\nabla \nabla \psi}{k^2} + \psi \right) dV.$$

Again, we may bring the $\mathbf{J}(\mathbf{r}')$ outside the integral since the region under consideration is small.

Moving the origin to \mathbf{r} so that $|\mathbf{r} - \mathbf{r}'| = R = |\mathbf{r}'|$, and changing to spherical coordinates, we see the unit dyad integral vanish as the limit is brought within the integral sign:

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \frac{e^{ikR}}{4\pi R} dV \\ &= \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \frac{e^{ikR}}{R} R^2 \sin \phi dR d\theta d\phi \\ &= \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \frac{e^{ikR}}{4} \sin \phi d\delta d\theta d\phi \\ &= 0 \end{aligned} \quad \begin{aligned} \delta &= 2R \\ \frac{d\delta}{dR} &= 2 \end{aligned}$$

To perform the $\nabla \nabla$ integral over the primed volume, we use $\nabla' = -\nabla$ and apply the equation

$$\int_V \nabla' \cdot \mathbf{a} dV' = \int_S \hat{\mathbf{u}}'_n \cdot \mathbf{a} dS'$$

to the outside and inside surfaces of the shaded volume in Figure 5, $V_\delta - V_\delta^D$ which contains no singularity.

$$\begin{aligned}
& - \lim_{\delta \rightarrow 0} \int_{V_\delta - V_\delta^D} \nabla' \left(\nabla \psi \right) dV' = \lim_{\delta \rightarrow 0} \left\{ - \int_{S_\delta} \hat{u}'_n \nabla \psi dS' + \int_{S_\delta^D} \hat{u}'_n{}^{D'} \nabla \psi dS' \right\} \\
& = k^2 \lim_{\delta \rightarrow 0} \int_{V_\delta - V_\delta^D} G_E(\mathbf{r}, \mathbf{r}') dV'
\end{aligned}$$

From Eq. (19), we know $\nabla \psi = \frac{e^{ikR}}{4\pi R} \left(ik - \frac{1}{R} \right) \hat{e}_r$, so we rewrite each surface integral as a sum and solve them all separately:

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \int_{S_\delta} \hat{u}'_n \nabla \psi dS' &= \lim_{\delta \rightarrow 0} \int_{S_\delta} \hat{u}'_n \hat{e}_r \frac{e^{ikR} ik}{4\pi R} dS' \\
- \lim_{\delta \rightarrow 0} \int_{S_\delta} \hat{u}'_n \hat{e}_r \frac{e^{ikR}}{4\pi R^2} dS' &
\end{aligned}$$

For the integral over the surface of V_δ , the surface element more than cancels the singularity.

$$\begin{aligned}
- \lim_{\delta \rightarrow 0} \int_{S_\delta} \hat{u}'_n{}^{A'} \nabla \psi dS' &= - \lim_{\delta \rightarrow 0} \frac{ik}{4\pi} \int_{S_\delta^A} \hat{u}'_n{}^{A'} \hat{e}_r \frac{e^{ikR}}{R} \frac{R^2 \sin \phi d\theta d\phi}{f(\theta, \phi)} \\
+ \lim_{\delta \rightarrow 0} \frac{1}{4\pi} \int_{S_\delta^A} \hat{u}'_n{}^{A'} \hat{e}_r \frac{e^{ikR}}{R^2} dS' &
\end{aligned}$$

Taking the limit, the first integral on the RHS vanishes with R , and $e^{ikr} \cong 1$, leaving

$$\frac{1}{4\pi} \int_{S_\delta^A} \hat{u}'_n{}^{A'} \frac{\hat{e}_r}{R^2} dS' = L_\delta.$$

For the surface integral over the disc principal volume, we write dS' in

cylindrical (polar) coordinates and take the limit as $\frac{\epsilon}{\delta} \rightarrow 0$, eliminating the contribution from the sides of the cylinder, as indicated in Figure 6.

$$\lim_{\epsilon \rightarrow 0} \int_{S_\delta^D} u_n^D \nabla \psi dS' = \lim_{\epsilon \rightarrow 0} \int_{S_\delta^D} u_n^D \hat{e}_r \frac{e^{ik\sqrt{\rho'^2 + \epsilon^2}}}{4\pi\sqrt{\rho'^2 + \epsilon^2}} \rho' d\rho' d\theta' \Big|_{\text{TOP} + \text{BOT}}$$

$$- \lim_{\epsilon \rightarrow 0} \int_{S_\delta^D} u_n^D \hat{e}_r \frac{e^{ik\sqrt{\rho'^2 + \epsilon^2}}}{4\pi(\rho'^2 + \epsilon^2)} \rho' d\rho' d\theta' \Big|_{\text{TOP} + \text{BOT}}$$

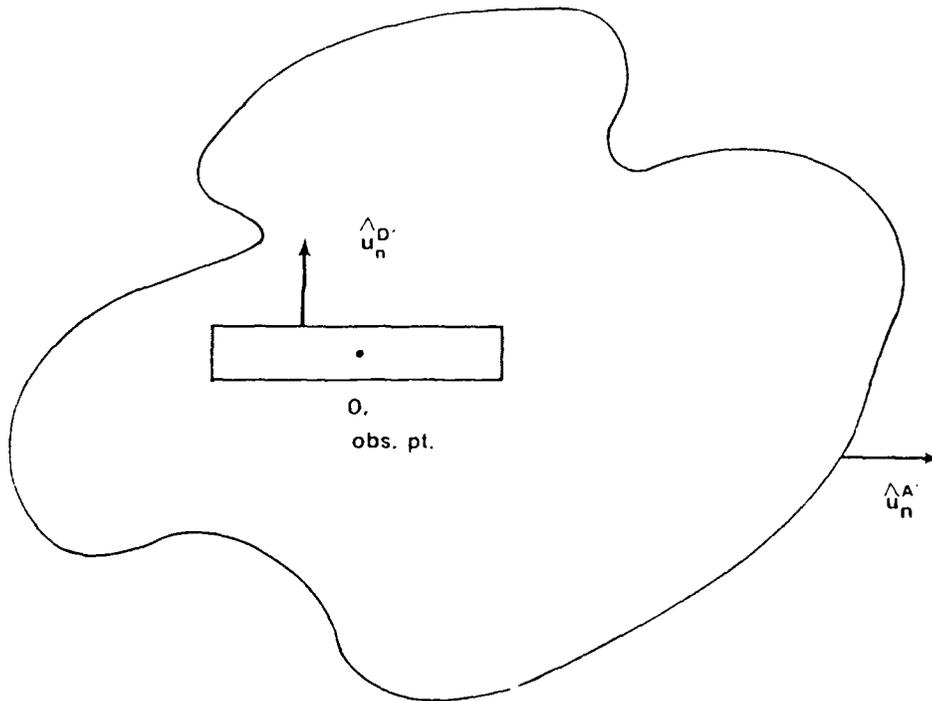


Figure 6. Unit Normal Directions for Disc and Arbitrary Principal Volumes.

To perform these integrals, \hat{e}_r and \hat{u}_n^D must be expressed in terms of ρ and ϵ , as shown in Figures 7 and 8, giving:

$$-\hat{e}_r = -b\hat{e}_z - a\hat{e}_\rho; \quad -\hat{e}_{\rho_{RHS}} = -(-\hat{e}_\rho)_{LHS}$$

so the ρ components cancel, leaving $\int \hat{e}_r \Rightarrow \int b\hat{e}_z$ over the cylinder ends. Using the ratio $\frac{-|\hat{e}_r|}{\sqrt{\rho^2 + \epsilon^2}} = \frac{-|\hat{e}_z| b}{\epsilon}$, we deduce that $b = \frac{\epsilon}{\sqrt{\rho^2 + \epsilon^2}}$. Obviously, $\hat{u}_n = \hat{z}$. Also we note that since \hat{e}_z and \hat{u}_n point in the same direction, the integrals over the ends of either cylinder add.

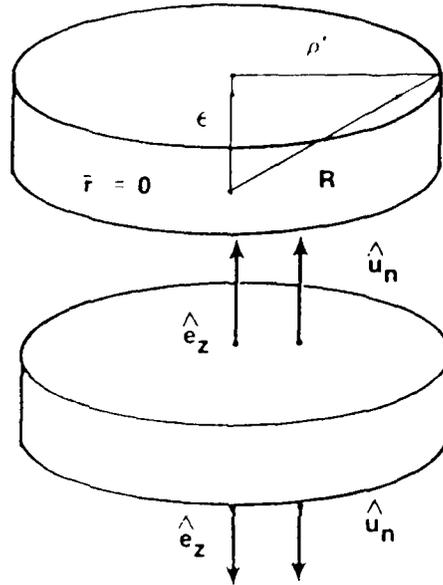


Figure 7. Cylindrical Polar Coordinate System and Unit Normal Directions Assumed in Deriving Source Dyadic, L_δ

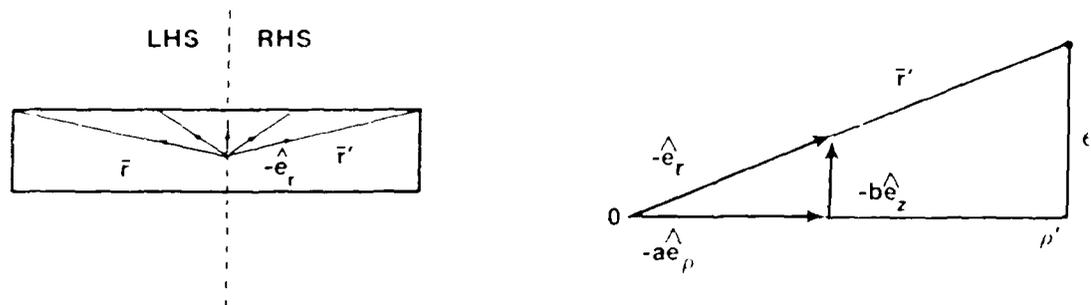


Figure 8. Side View and Cut of Disc Principal Volume Showing Directions of Position Unit Vector and Components

We now have, after performing the $d\theta$ integration, and setting $e^{ik\sqrt{\rho'^2 + \epsilon^2}} = 1$, (these integrals have already been solved):

$$\lim_{\epsilon \rightarrow 0} \int_{S_\delta^D} \hat{u}_n^D \nabla \psi dS' = ik \hat{z} \lim_{\epsilon \rightarrow 0} \int_{\rho=0}^{\delta/2} \frac{\rho' d\rho'}{(\rho'^2 + \epsilon^2)} - \hat{z} \lim_{\epsilon \rightarrow 0} \epsilon \int_{\rho=C}^{\delta/2} \frac{\rho' d\rho'}{(\rho'^2 + \epsilon^2)^{3/2}} = -\hat{z}.$$

Thus, returning to Eq. (27), we simply substitute $\frac{L_\delta - \hat{z}}{k^2}$ for $\lim_{\delta \rightarrow 0} \int_{v_\delta - v_\delta^D} G_E dV$ which gives

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \left\{ \lim_{\delta \rightarrow 0} \int_{v - v_\delta} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E(\mathbf{r}, \mathbf{r}') dV' + \mathbf{J}(\mathbf{r}) \cdot \frac{L_\delta - \hat{z}}{k^2} \right\} + \frac{\hat{z} \mathbf{J}_z(\mathbf{r})}{i\omega\epsilon_0}$$

or

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_0 \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_E(\mathbf{r}, \mathbf{r}') dV' + \frac{\mathbf{L}_\delta \cdot \mathbf{J}(\mathbf{r})}{i\omega\epsilon_0}.$$

This procedure of circumscribing a particular principal volume with another of arbitrary shape always results in two terms: \mathbf{L}_δ , and a term to exactly cancel the source dyadic representing the original principal volume. The complexity of our derivation of the disc source dyadic is thus somewhat redeemed by this simple connection to the general form. In fact, using this procedure, any accurate expression for the electric field in terms of the particular source dyadic for a sphere, disc, cube, ellipsoid, or any other principal volume shape can be transformed into the general formula, in terms of \mathbf{L}_δ , which in turn can be used to find the source dyadic for all other principal volume shapes⁵.

7. POLARIZATION AND MAGNETIZATION SOURCES

So far, we have considered only a source of electric current, \mathbf{J} . To include polarization and magnetization sources, we note that, in general, the "electric current", \mathbf{J}_E , includes both current and polarization, but the "magnetic current", \mathbf{J}_H , consists only of magnetization, written

$$\mathbf{J}_E = \mathbf{J} - i\omega\mathbf{P}$$

$$\mathbf{J}_H = 0 + i\omega\mu\mathbf{M}$$

We incorporate polarization by simply substituting for \mathbf{J}_E ; we incorporate magnetization by summing the field equations and their duals, (where $\mathbf{E} \cdot \mathbf{H}$, $\mathbf{H} \cdot \mathbf{E}$, $\epsilon \leftrightarrow \mu$) and substituting for \mathbf{J}_H . The total field expressions for an arbitrary principal volume, taking into account current, magnetization, and polarization, are thus:

$$\mathbf{E}(\mathbf{r}) = \lim_{\delta \rightarrow 0} \frac{1}{\omega \mu_0} \int_{V - V_\delta} \left(\mathbf{J}_E \cdot \mathbf{G}_E - \mathbf{M} \cdot \mathbf{G}_E \right) dV' + \frac{\mathbf{L}_\delta \cdot \mathbf{J}_E}{1\omega \epsilon_0}$$

$$\mathbf{H}(\mathbf{r}) = \lim_{\delta \rightarrow 0} \int_{V - V_\delta} \left(\mathbf{G}_H \cdot \mathbf{J}_E + k^2 \mathbf{G}_H \cdot \mathbf{M} \right) dV' - \mathbf{L}_\delta \cdot \mathbf{M}.$$

Having derived the fields using the 3-D Leibniz rule it is now easy to appreciate how the Leibniz additional terms form the source dyadic, \mathbf{L}_δ . Comparing Eq. (23) with Eq. (25).

$$\frac{\mathbf{L}_\delta \cdot \mathbf{J}}{1\omega \epsilon_0} = \mathbf{J}(\mathbf{r}) - \lim_{\epsilon \rightarrow 0} \hat{\mathbf{x}} \iint_{S_\delta} \gamma \epsilon J_x(\mathbf{x}', y', z) dS' - \lim_{\epsilon \rightarrow 0} \hat{\mathbf{y}} \iint_{S_\delta} \gamma \epsilon J_y(\mathbf{x}', y', z) dS'. \quad (28)$$

On the RHS are two additional Leibniz surface integrals which compensate for the \mathbf{r} dependence of V_δ .

If we had chosen in the beginning to evaluate \mathbf{E} and \mathbf{H} using a cavity volume, V_c , to exclude the observation point, the source dyadic would never result. Hence, the source dyadic $\frac{\mathbf{L}_\delta \cdot \mathbf{J}}{1\omega \epsilon_0}$ is the difference between the Maxwellian and cavity fields, that is, $\mathbf{E}_M - \mathbf{E}_C = \frac{\mathbf{L}_\delta \cdot \mathbf{J}}{1\omega \epsilon_0}$. It is easy to see why this occurs by considering how the equation above would read for a cavity volume rather than a principal volume. Since the cavity is formed by physically removing a small volume of current around the observation point \mathbf{r} , $\mathbf{J}(\mathbf{r}) = 0$. Because the location of the cavity is fixed during the differentiation of \mathbf{A} , V_c does not depend on \mathbf{r} . Obviously then, the integration limit, $V - V_c$ does not depend on the differentiation variable, \mathbf{r} . Thus, the regular Leibniz rule permits interchanging the differentiation and integration. Since the "generalized Leibniz rule", which involves variable integration limits, is not necessary for evaluating cavity fields, no additional terms [the surface integrals in Eq.(28)] arise, and $\frac{\mathbf{L}_\delta \cdot \mathbf{J}}{1\omega \epsilon_0} = \mathbf{J}(\mathbf{r}) = 0$.

8. CONCLUDING REMARKS

Using a principal volume approach, we have made a thorough study of how electromagnetic fields behave in the source region, by deriving and applying a generalized 3-D Leibniz rule to correctly differentiate the vector potential. Section 1 began by considering only electric current sources and emphasized that the difference between cavity and Maxwellian fields hinges on rules restricting interchanging differential and integral

operations. Then, a 3-D version of the generalized Leibniz rule was derived, to properly perform the interchange and was subsequently employed along with a disc principal volume to calculate both the magnetic and electric fields. In Section 5, the fields for an arbitrary principal volume were derived and found to be identical to those derived previously, using other techniques. Finally, the duality of Maxwell's equations was invoked to obtain the field expressions which include polarization and magnetization sources, in addition to current.

While this exercise reproduces previously known results for the fields, in doing so, it not only provides a clear and completely rigorous, straight-forward treatment of the problem, but an intuitive explanation as well, from both a mathematical and a physical viewpoint. Primarily, the extension to three dimensions demonstrates the breadth of Leibniz' fundamental rule, providing a glimpse of its potential to enhance precision in volume integral formulations.

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Appendix A

Proof of Validity of Exchanging the Pillbox Principal Volume for a Slab Principal Volume

Working in rectangular coordinates, the most logical principal volume shape for excluding a singularity of the scalar Green's function ψ is the disc, or pillbox, shown in Figure A1 below.

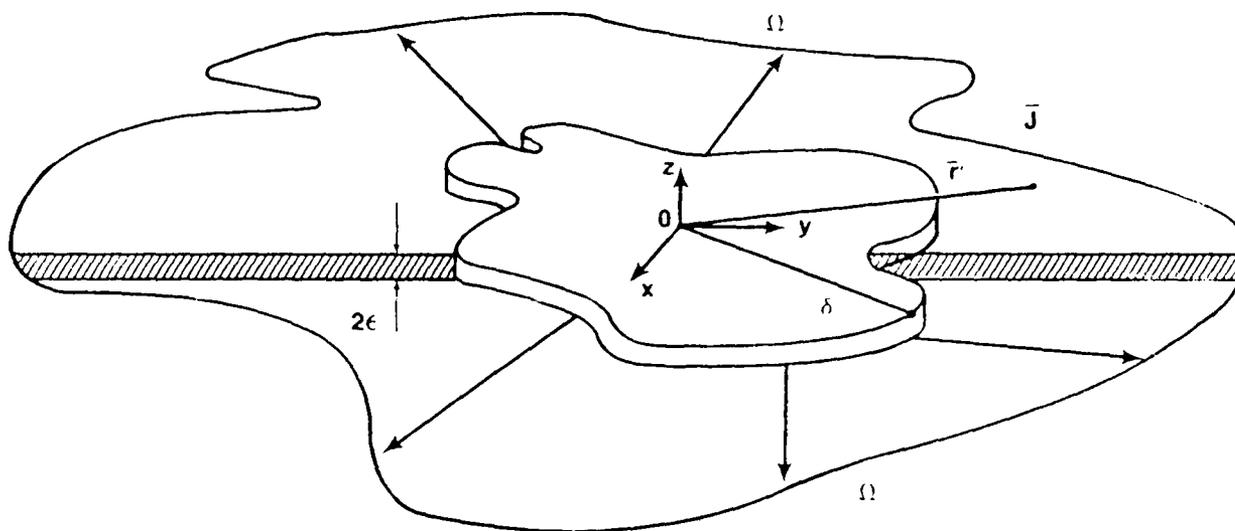


Figure A1. Extension of the Disc to a Slab

To specify which of the 36 additional terms in the generalized 3-D Leibniz rule actually contribute to the fields, it is advantageous to extend the x- and y- dimensions of the disc, forming a slab principal volume. We wish to show that the disc and slab can be used interchangeably as the limiting V_δ in the vector potential integral. To do this we simply show that the vector potential, integrated over the shaded region, vanishes as the thickness, 2ϵ , approaches zero:

$$\mathbf{A}_{\text{shaded}}(\mathbf{r}) = \lim_{\delta \rightarrow 0} \int_{V_{\text{slab}} - V_{\text{disc}}} \psi(|\mathbf{r} - \mathbf{r}'|) \mathbf{J}(\mathbf{r}') dV = 0 \quad (\text{A1})$$

or, in cylindrical coordinates,

$$\mathbf{A}_{\text{shaded}}(\mathbf{r}) = \lim_{\epsilon \rightarrow 0} \int_{z = -\epsilon}^{\epsilon} \int_{\delta}^{\Omega} \int_{\theta = 0}^{2\pi} \frac{e^{ik\sqrt{\rho'^2 + z'^2}}}{\sqrt{\rho'^2 + z'^2}} \mathbf{J}(\mathbf{r}') \rho' d\rho' d\theta' dz' \quad (\text{A2})$$

where we have moved the origin to \mathbf{r} . By definition of the disc geometry, $\frac{\epsilon}{\delta} \rightarrow 0$ so we have replaced $\delta \rightarrow 0$ with $\epsilon \rightarrow 0$. We assume \mathbf{J} is approximately constant over a thin Δz , and perform the z integral first, noting that the integrand is an even function, symmetric about $z' = 0$:

$$\lim_{\epsilon \rightarrow 0} 2 \int_{\delta}^{\Omega} \rho' d\rho' \int_0^{2\pi} d\theta' \mathbf{J}(\mathbf{r}') \int_{z=0}^{\epsilon} \frac{e^{ik\sqrt{\rho'^2 + z'^2}}}{\sqrt{\rho'^2 + z'^2}} dz' \quad (\text{A3})$$

Using the inequality

$$\left| \int_{z=0}^{\epsilon} \frac{e^{ik\sqrt{\rho'^2 + z'^2}}}{\sqrt{\rho'^2 + z'^2}} dz' \right| \leq \int_{z=0}^{\epsilon} \frac{dz'}{\sqrt{\rho'^2 + z'^2}} \quad (\text{A4})$$

we can evaluate the RHS as:

$$\lim_{\epsilon \rightarrow 0} J(r') \ln \left(z' + \sqrt{\rho'^2 + z'^2} \right) \Big|_{z'=0}^{\epsilon} \quad (\text{A5})$$

$$= \lim_{\epsilon \rightarrow 0} J(r') \left[\ln \left(\epsilon + \sqrt{\rho'^2 + \epsilon^2} \right) - \ln(\rho') \right]$$

$$= J(r') [\ln(\rho') - \ln(\rho')] = 0.$$

Therefore, $A_{\text{shaded}} = 0$ proving that it is, in fact, valid to use the disc and slab principal volumes interchangeably.

APPENDIX B

Proof That ψ Terms Cancel if They Are Equal and Opposite,
Despite Singularity in ψ

Recall that in the Maxwellian \mathbf{H} field calculation, we arrive at the following expression after considering the 3-D Leibniz rule:

$$\begin{aligned} \mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} & \left\{ \lim_{\delta \rightarrow 0} \int_{V-V_\delta} \nabla \times \mathbf{J}(\mathbf{r}') \psi(|\mathbf{r}-\mathbf{r}'|) d^3 r' \right. \\ & + \lim_{\epsilon \rightarrow 0} \left[\hat{\mathbf{x}} \iint \left(-J_y \psi_{\text{top}} + J_y \psi_{\text{bot}} \right) dx' dy' \right. \\ & \left. \left. + \hat{\mathbf{y}} \iint \left(J_x \psi_{\text{top}} - J_x \psi_{\text{bot}} \right) dx' dy' \right] \right\} \end{aligned} \quad (\text{B1})$$

Where ψ_{top} and ψ_{bot} are the values of $\psi(|\mathbf{r}-\mathbf{r}'|)$ on the upper and lower surfaces of the slab principal volume.

The functions ψ_{top} and ψ_{bot} are equal, ($\psi_{\text{top}} = \psi_{\text{bot}} = \psi$), yet still singular:

$$\Psi_{\text{top}} = \Psi \Big|_{z'=z+\epsilon} = \frac{e^{ik\sqrt{(x-x')^2 + (y-y')^2 + (-\epsilon)^2}}}{\sqrt{(x-x')^2 + (y-y')^2 + (-\epsilon)^2}} \quad \begin{array}{c} 2\epsilon \downarrow \quad \Psi(z'=z+\epsilon) \\ \bullet \\ \uparrow \quad \Psi(z'=z-\epsilon) \end{array} \quad (\text{B2})$$

$$\Psi_{\text{bot}} = \Psi \Big|_{z'=z-\epsilon} = \frac{e^{ik\sqrt{(x-x')^2 + (y-y')^2 + \epsilon^2}}}{\sqrt{(x-x')^2 + (y-y')^2 + \epsilon^2}}$$

We wish to prove that in spite of this singularity, the quantity in square brackets in equation (B1) vanishes. Since $\Psi_{\text{top}} = \Psi_{\text{bot}}$, we write

$$\lim_{\epsilon \rightarrow 0} \left[\hat{x} \iint \Psi (J_{y\text{bot}} - J_{y\text{top}}) dx'dy' + \hat{y} \iint \Psi (J_{x\text{top}} - J_{x\text{bot}}) dx'dy' \right] \quad (\text{B3})$$

We expand $J_x(z'=z \pm \epsilon)$ and $J_y(z'=z \pm \epsilon)$ in a Taylor series:

$$J_{\text{top}}(z'=z+\epsilon) = J(z) + \epsilon J'(z) + \frac{\epsilon^2}{2!} J''(z) + \dots$$

$$J_{\text{bot}}(z'=z-\epsilon) = J(z) - \epsilon J'(z) + \frac{\epsilon^2}{2!} J''(z) + \dots$$

Subtracting, we have

$$J_{y\text{bot}} - J_{y\text{top}} = -2\epsilon J'_z(z)$$

$$J_{x\text{top}} - J_{x\text{bot}} = 2\epsilon J'_z(z).$$

Substituting these expressions into (B3), we obtain for the bracketed quantity

$$\lim_{\epsilon \rightarrow 0} \left[2\epsilon J'_z(z) \left[-\hat{x} \iint \Psi dx'dy' + \hat{y} \iint \Psi dx'dy' \right] \right]$$

To integrate over ψ , we express the integrand in polar coordinates and impose an upper limit d on ρ' , since ψ is singular for only small values of ρ' (thus; we may approximate $e^{ik\sqrt{\rho'^2 + \epsilon^2}}$ as 1).

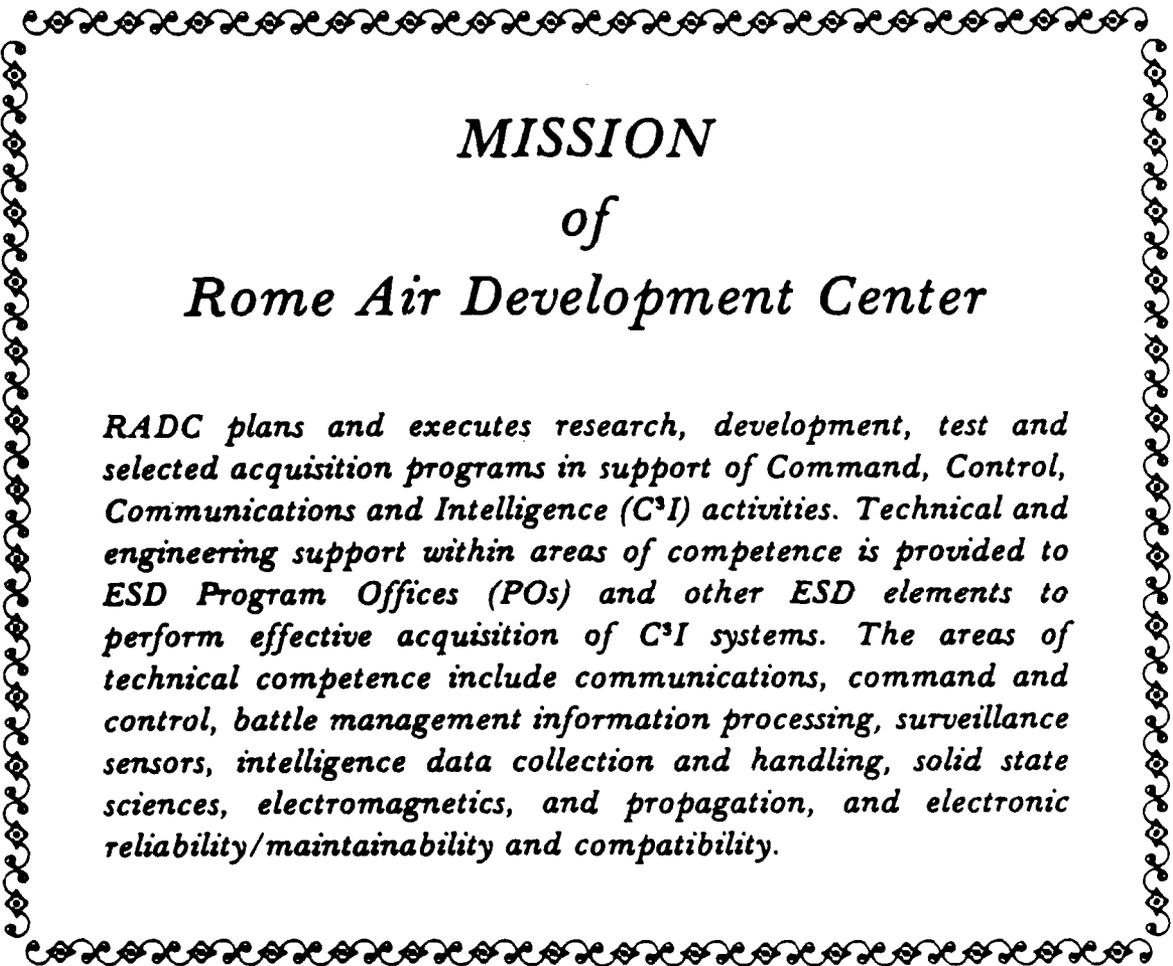
$$\iint \psi dx'dy' = \int_0^{2\pi} \int_0^d \frac{e^{ik\sqrt{\rho'^2 + \epsilon^2}}}{\sqrt{\rho'^2 + \epsilon^2}} \rho' d\rho' d\theta' \approx 2\pi \int_0^d \frac{\rho' d\rho'}{\sqrt{\rho'^2 + \epsilon^2}} = 2\pi \sqrt{d^2 + \epsilon^2} \quad (\text{B4})$$

Now, we take the limit to show that the bracketed quantity is zero:

$$\lim_{\epsilon \rightarrow 0} \left(2\epsilon J_z'(z) \right) \left(\left[\begin{array}{c} -\hat{x} + \hat{y} \\ \end{array} \right] \left[2\pi\sqrt{\rho'^2 + \epsilon^2} \right] \right) = 0$$

Thus, Eq. (B1) becomes simply,

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} \left\{ \lim_{\delta \rightarrow 0} \int_{V-V_\delta} \nabla \times \mathbf{J}(\mathbf{r}') \psi(|\mathbf{r}-\mathbf{r}'|) dV' \right\}$$



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