**Title:** Analysis of the Non-Planar Response of a Cantilever, with the Aid of Computerized Symbolic Manipulation

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ANALYSIS OF THE NON-PLANAR RESPONSE OF A CANTILEVER, 
WITH THE AID OF COMPUTERIZED SYMBOLIC MANIPULATION

M.R.M. Crespo da Silva*

ABSTRACT

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deformations of a cantilever with a tip mass are formulated 
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INTRODUCTION

Most of the work presented in the literature dealing 
with the response of structural elements such as beams 
is restricted to planar response. In addition, much 
of the work is also restricted to the cases where the 
deformations are very small and are analyzed by linear 
theory. In general, a structure can undergo flexure in any 
direction in space, and torsion. For infinitesimally small 
deformations, flexural and torsional motions are uncoupled. 
In this paper the flexural-flexural-torsional deformations 
of a cantilever with a tip mass are determined. The 
analysis makes use of computerized symbolic manipulation.

Computerized symbolic manipulation allows the analyst 
to relegate tedious algebra to a computer and, thus, to 
concentrate efforts in the basic steps of the formulation 
and analysis of problems. MACSYMA (Rand, 1984) is among 
the most powerful of such A.I. programs. It can perform 
differentiations and integrations in analytic form, Tay-
lor series expansions, solve algebraic and differential 
equations, etc. Here the nonlinear differential equations 
governing the flexural-flexural-torsional static response 
of a cantilever with a tip mass are formulated and analyzed 
with the aid of MACSYMA. The equations, and their boundary 
conditions, are formulated via Hamilton's principle. To 
consider the general case where the beam is able to bend 
in any direction in space and twist, a set of angles to 
describe the orientation of an arbitrary cross section of 
the beam is used. The equations are written in terms

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of the deflections relative to a set of inertial reference axes and of a torsional variable. Subsequently they are converted into a set of integro-partial differential equations in the bending deflections by eliminating the remaining elastic variables. For simplicity the material is assumed to be Hookean, and the beam's properties to be constants. The nonlinearities in the equations are geometric and include terms such as nonlinear contribution to the curvature expression. By expanding in Taylor series the nonlinearities in the equations, a set of equations with polynomial nonlinearities is generated. A perturbation analysis of the response is then performed using such equations to obtain an analytical approximation for the beam's deflections. This is also done with the aid of MACSYMA. All the tedious algebra involved in the formulation is relegated to the computer which displays the requested equations on the screen.

PROBLEM FORMULATION

Consider an initially straight and untwisted, clamped-free beam of length \( \ell \), mass \( m \) per unit length, and with a lumped mass \( M \), of weight \( Mg \), located at its tip \( s = 1 \). Here, \( s \) denotes arc-length along the beam, non-dimensionalized by \( \ell \). Figure 1a shows a beam element of length \( s \) in its deformed state. The position vector of the centroid \( C \) of the beam's cross section \( S \), normal to the tangent line at \( s = s \), is written in terms of the elastic deformation components of \( C, u(s), v(s) \) and \( w(s) \), as \( \mathbf{r}(s) = u(s)\hat{\mathbf{x}} + v(s)\hat{\mathbf{y}} + w(s)\hat{\mathbf{z}} \). The unit vectors \( (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \) are inertial. The \( \hat{\mathbf{z}} \) applied at the beam's tip at \( s = 1 \) is \( Mg[\cos \alpha + \sin \alpha \hat{\mathbf{z}}] \), where \( \alpha \) is the angle from the vertical axis to the direction defined as \( \hat{\mathbf{y}} = \hat{\mathbf{y}}(s = 0) \). The unit vectors \( (\hat{\mathbf{\epsilon}}, \hat{\mathbf{\eta}}, \hat{\mathbf{\zeta}}) \) are aligned with the principal axes of the beam's cross at \( s = s \). The orientation of the beam's cross section \( S \) at \( s = s \), \( \hat{\mathbf{\epsilon}}, \hat{\mathbf{\eta}}, \hat{\mathbf{\zeta}} \), relative to the inertial axes \( (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}) \), shown in Figure 1a, can be described by the three successive rotations \( (\psi, \theta, \phi) \) shown in Figure 1b. Here, \( (\psi, \theta, \phi) \) is the rotation of the element about the vertical axis, \( \psi \), then \( \theta \), and finally \( \phi \), the rotation of the element about the \( \psi \) axis. The additional rotations are then applied about the \( \psi \) and \( \theta \) axes, as shown in Figure 1b. The beam's principal bending stiffnesses, \( D_\psi, D_\theta, D_\phi \), denote the beam's properties, and \( D_\psi = \frac{D_\phi}{D_\psi} \) and \( D_\theta = \frac{D_\phi}{D_\theta} \). For simplicity it will be assumed that \( M > \frac{m}{\ell} \), so that the static deflection due to the weight of the beam is neglected. The differential equations of equilibrium can be formulated from hamilton's principle, with the constraint condition adjointed to the Lagrangean with a Lagrange multiplier \( \lambda \), as (Crespo da Silva and Glynn, 1978a)

\[
\delta I = \delta \int_0^1 \left\{ L(\psi, \theta, \phi, \psi', \theta', \phi') + \frac{\lambda}{2} \left[ 1 - (1 + u')^2 - v'^2 - w'^2 \right] \right\} ds
\]

\[+ \frac{Mg^2}{D_\psi} \left[ (\cos \alpha) \delta r + (\sin \alpha) \delta w \right]_{r=1} = 0 \]

(1)

The vector \( \hat{\rho} \) is obtained directly from Figure 1b as,

\[
\hat{\rho} = (\psi' - \psi \sin \theta) \hat{\mathbf{\epsilon}} + (\psi \cos \theta \sin \phi + \theta' \sin \phi) \hat{\mathbf{\eta}} + (\psi' \cos \theta \cos \phi + \theta' \sin \phi) \hat{\mathbf{\zeta}} \equiv \rho \hat{\mathbf{\epsilon}} + \rho \hat{\mathbf{\eta}} + \rho \hat{\mathbf{\zeta}} \]

(2)

By performing the variations indicated in Eqn.(1), the following differential equations are obtained,

\[
G_{\psi} \left[ A_{\psi} \frac{\partial^2 v}{\partial u'^2} + A_{\psi} \frac{\partial v}{\partial u'} + \lambda (1 + u') \right] = 0 \]

(3a)

\[
G_{\theta} \left[ A_{\theta} \frac{\partial^2 v}{\partial u'^2} + A_{\theta} \frac{\partial v}{\partial u'} + \lambda (1 + u') \right] = 0 \]

(3b)

\[
A_{\phi} = 0 \]

(3c)
where $A_k = (\partial L/\partial q^{'k}) - \partial L/\partial q$ for $k = v, \theta, \phi$. The following boundary condition equation is also obtained from Eqn. (1).

$$J_k (x') \delta \phi + G_k \delta u + (G_k - i P \cos \theta) \delta u + (G_k - i q) \sin \theta) \delta w$$

$$= \left[ H_k - \frac{U H_k}{1 + u'} \right] \delta x' - \left[ H_k - \frac{U H_k}{1 + u'} \right] \delta x' \bigg|_{x' = 0} = 0 \quad ; \quad i = 0, 1 \quad (4)$$

where $P = mg^2/d_n$, and $H_k = (\partial L/\partial v^{'k}) \partial \psi/\partial u' + (\partial L/\partial v) \partial \theta/\partial u'$ for $k = u, v, w$.

**RESPONSE ANALYSIS**

The differential equations (3a-c) are nonlinear and coupled. To find an approximate solution for the elastic deformations $v(s)$ and $w(s)$, and for the orientation angle $\phi(s)$, we will expand these equations in Taylor series about $v(s) = w(s) = 0$ and then solve the resulting equations by a perturbation technique. For this we let $k(s) = \phi(s)$, for $k = u, v, w$ and $\phi$, where $\epsilon$ is a small arbitrary parameter which is used for bookkeeping purposes only. Here we will obtain an approximate solution to the elastic deformations to order $\epsilon$. Since this involves a large amount of algebra, MACSYMA is used to develop all the various equations and to perform the perturbation analysis. MACSYMA is a very large symbolic manipulation program for applied mathematicians. It also includes a number of routines for numerical computation and for generating plots. MACSYMA runs on SUN, VAX and Symbolics computers, and on a few other machines. There are a number of other smaller programs that also run on these and other machines and, in some cases, on mainframes and personal computers. Among these are REDUCE (Hearn, 1973), MAPLE (Char, Geddes, Gonnet, Monagan and Watt, 1988) and muMATH (Wooff and Hodkinson, 1987). Many of MACSYMA commands are self-explanatory, such as (C1) TAYLOR$(\sin(\epsilon x))^*\sqrt{1 - \epsilon x^2}$, eps = 0, 10, for the Taylor series of the first argument, in EPS, about EPS=0, to EPS to the 10th power; (C2) DIFF$(\sin(\epsilon x))^*\sqrt{1 - \epsilon x^2}$, eps, 0, 1, for the derivative of the five derivative, with respect to x, of the given expression; (C3) INTEGRATING$(\sin(\epsilon x))^*\sqrt{1 - \epsilon x^2}$, x, 0, 1, for the integral of the given expression, in x, from x=0 to x=1; (C4) PART$(A1, A2, A3, 3)$, for the third part of the list [A1, A2, A3]; (C5) LS$\left(y^* x^2 - 3\right)$ for the left-hand side (or part 1), of the given equation; (C6) SOLVE$(EQN1, EQN2, \{x, y\})$ to obtain the solution of the system of equations EQN1=0 and EQN2=0 for the variables x and y; etc. For each Ci command, MACSYMA generates a Di output with the result of the computation. The computer generated equations are displayed as if they were written by a mathematician.

To analyze the problem considered here, we begin by telling MACSYMA that each angle in the list $[v, \theta, \phi]$ depends on the independent variable $\epsilon$, and then define the components of the vector $\delta$ as shown below. MACSYMA's response to the command C3 is shown in the line labeled B3.

(C1) list_of_angles : [psi, theta, phi]$
(C2)$ depends(list_of_angles, s)$
(C3) \{ \text{theta: diff(psi, s) \times \text{cos(theta)} \times \text{sin(phi)} + \text{diff(theta, s) \times \text{cos(phi)}}},$
\text{rho theta: diff(psi, s) \times \text{cos(theta)} \times \text{cos(phi)}} \times \text{diff(theta, s) \times \text{sin(phi)}} \}$

\[
(D3) \frac{d \phi}{ds} = \frac{d \psi}{ds} \text{cos(theta)} \times \text{sin(phi)} + \frac{d \theta}{ds} \text{cos(theta)} \times \text{sin(phi)}
\]

Next, the normalized Lagrangean $L$ is defined and the expressions for $\d$, defined in the previous section for $i = v, \theta$ and $\phi$, are developed as

(C4) $L := \text{beta} \times \text{theta} \times \text{rho theta} \times \text{beta} \times \text{rho theta} \times 2 \times \text{beta} \times \text{rho theta} \times 2 / 25$
(C5) for $i$ in list_of_angles do (A1: concat(A1, i), A1: diff(diff(L, diff(i, s)), s) - diff(L, i))$

As shown in the previous section, the nonlinear differential equations of equilibrium are obtained as $A_0$=0 and $G_0=0$, for $i = u, v$ and $w$. To let MACSYMA develop the expressions for $G_0$, the variables $u, v$ and $w$ are declared to be dependent on $s$, and the constraint relationships for the angles $v$ and $\theta$ are introduced as shown in CB:

(C6) list_of_vars: [u, v, w]$
(C7)$ depends(list_of_vars, s)$
(C8) \{ \psi: \sin(\psi, s) \times 1 + \text{diff(s, w)} \}$
\text{theta: atan(diff(w, s) \times \sqrt{1 + \text{diff(u, w)}}) + \text{diff(v, w)} \times 2 / 25}$$

To obtain a set of differential equations with polynomial nonlinearities, a small arbitrary bookkeeping parameter $\epsilon$ is now introduced so that $v=0(r), w=0(r)$ and $\phi=0(r)$. To order $\epsilon$, the inextensibility constraint yields a simple expression for $u$, as $u=\text{diff}(w, s) / 2$. This information is given to MACSYMA in the following lines, which will be used when generating the expanded forms of $G_0, G_0, \text{and } A_0$.

(C9) list_of_expr: [u = eps * 2 * w(s), w = eps * v(s), \text{psi: eps * psi(s)}]$
(C10)$ list_of_diff: [diff(u, s) =diff(v, s) \times 2 / 2, diff(w, s) \times 2 / 25]
The expanded forms of $G_u$, $G_v$, and $G_w$ can be generated with the following sequence of commands grouped in a loop.

(C11) for i in list_of_vars do ( print("evaluating", Gi:concat([G,i])), aux1:if i=u then ldiff(u,s) else diff(i,s), aux2:ev( Apsi*diff( psi, diff(i,s)) + Atheta*diff( theta, diff(i,s)) + lambda*aux1, aux2: ev(aux2, list1, list2, diff), print("doing the Taylor series expansion"), Gi:Taylor(aux2, eps0, 0.3) )$)

Since the Lagrange multiplier $\lambda$ is only needed to $O(e^2)$, as disclosed by Eqn. (3b), the differential equation $G_0=0$ is then solved for $\lambda$, with the boundary condition $G_0(s=1)=0$. This is simply done with the command C12 shown below. The expression obtained for $\lambda$ is shown in D12 (the symbols $\epsilon$, $\beta_0$ and $\beta_1$, instead of EPS, betay and betag, are used in the versions of the displays shown here).

(C12) lambda:rhs(part(taylor(solve(gu,lambda),eps0,0.2),1,1)):

\[(D12) \left( -\beta_0 \frac{d^2v(s)}{ds^2} - \frac{d^4w(s)}{ds^4} \right) e^2 + \ldots \]

The expanded form of $A_0$ is generated as shown below. The $O(e^3)$ expression for $A_0$ is shown in D14.

(C13) Ans:ev(Aphi)$
(C14) Aphi:Taylor(ev(Aphi, list1, list2, diff), eps0, 0.3):

\[(D14) -\beta_1 \frac{d^2\phi(s)}{ds^2} + \left( (\beta_0 - \beta_1 - 1) \frac{d^3v(s)}{ds^3} \frac{d^3w(s)}{ds^3} - \beta_1 \frac{d^4v(s)}{ds^4} \right) e^2 + \ldots \]

The above expression for $A_0$ discloses that $\phi=O(e^2)$. The expression for $A_0$ is then truncated to $O(e^2)$ -- command C15 -- and the differential equation $A_0=0$ is solved for $\phi$ as shown in C17 and C18. The resulting expression for $\phi$ is given in D16.

(C15) Aphi:subst([eps*e=0], Aphi)$
(C16) declare(integrate, additive)$
(C17) phi_double_prime:rhs(part(solve(Aphi, diff(phi(s), s, 2)), 1, 1))$\$
(C18) phi_of_s: integrate(integrate(phi_double_prime, s, 1, s), s, 0, s):

\[(D18) \left( \frac{\beta_0 - \beta_1 - 1}{\beta_1} \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds - \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds \right) \epsilon \]

The variable $\epsilon$ is now eliminated in the expressions obtained above for $G_v$ and $G_w$. This is done by re-evaluating those expressions with the MACSYMA command ev to replace $\phi$ by the expression phi_of_s given by D18, and forcing MACSYMA to carry out all the intermediate differentiations. As a result, two nonlinear integro-partial differential equations are now obtained in the variables $v(s)$ and $w(s)$. The integro-differential equations for $G_v$ and $G_w$ are obtained as shown below. The simplified form of the expression obtained for $G_v$ is displayed in D21.

(C19) Gv: expand( ev(gv, [phi(s) = phi_of_s], diff))$\$
(C20) Gw: expand( ev(gw, [phi(s) = phi_of_s], diff))$\$
(C21) ratsimp(Gv);

\[(D21) - (1/\beta_0 - (\beta_1 + 2)\beta_0 + \beta_1 + 1) \epsilon \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds \]

\[+ \beta_0 (1 - \beta_0) \epsilon \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds \]

\[+ (\beta_1^2 - 2\beta_0^2 - \beta_1^2 + 1) \epsilon \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds \]

\[+ \beta_0 \beta_1 \epsilon \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds \]

\[+ \beta_0 \beta_1 \epsilon \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds \]

\[+ \beta_0 \epsilon \int_{0}^{1} \int_{1}^{s} \frac{d^4v(s)}{ds^4} \frac{d^3w(s)}{ds^3} ds ds \]

A few integrations by parts in the new expressions for $G_v$ and $G_w$ disclose that they are the same as those obtained in (Crespo da Silva and Glynn, 1978b). Since the boundary conditions for the differential equations $G_v'=0$ and $G_w'=0$ are $G_v(s=1)=P\cos\alpha$ and $G_w(s=1)=P\sin\alpha$, it follows that $G_v(s)=P\cos\alpha$ and $G_w(s)=P\sin\alpha$. For $v=0(r)$ and $w=0(r)$, the $O(r)$ part of each of these equations is

(C22) [coeff(Gv=eps*P*cos(alpha), eps), 1]$\$
(C23) [coeff(Gw=eps*P*sin(alpha), eps), 1]:
COMPUTER UTILIZATION

\[ (D22) \quad \left[ -\beta_{y} \frac{d^2 w(s)}{ds^2} + P \cos \alpha, \quad -\frac{d^2 v(s)}{ds^2} = P \sin \alpha \right] \]

The solution to these \( O(\epsilon) \) differential equations is simply \( v(s) = -pv(s^3 - 3s^2)/6 \) and \( w(s) = -pw(s^3 - 3s^2)/6 \) where \( pv = P \cos \alpha/\beta_y \) and \( pw = P \sin \alpha \), i.e.,

\[ (C23) \quad [v_1 = pv(s^3 - 3s^2)/6, \quad w_1 = pw(s^3 - 3s^2)/6] \]

To obtain an approximate solution for \( v(s) \) and \( w(s) \) we then let \( v(s) = v_1 + v_2 \) and \( w(s) = w_1 + w_2 \). Substitute these expressions in the integro-differential equations of equilibrium and truncate the result to \( O(\epsilon) \) (see C27 and C28), and then do a series of integrations to solve for \( v_3 \) and \( w_3 \) (C29 through C32). This is done by MACSYMA as shown below. The solution obtained for \( v_3 \) is shown in D33.

\[ (C24) \quad \text{list4:}\{v(s) = v_1 + v_2 + v_3, \quad w(s) = w_1 + w_2 + w_3\} \]

\[ (C25) \quad \text{depends}\{(v_3, w_3, s), \quad \text{assume}\{\alpha > 0, \quad s > 1\}\} \]

\[ (C26) \quad \text{/* The following declaration and assignment are needed to force MACSYMA to truncate the desired expressions to } O(\epsilon^3) \quad \text{// (ratweight\{eps, 1\}, ratwrt\{v_1: 3\}\} ) \]

\[ (C27) \quad \text{eqn1:}\{(Gv - eps^2 \beta_y p v, \quad \text{list4, diff, expand, ratexpand}\text{)} \}

\[ (C28) \quad \text{eqn2:}\{(Gw - eps^2 \beta_y p w, \quad \text{list4, diff, expand, ratexpand}\text{)} \}

\[ (C29) \quad \text{// force MACSYMA to do all the integrations in EQN1 and EQN2//}

\[ \text{eqn1: ev\{eqn1\text{, integrate}\} \quad \text{, eqn2: ev\{eqn2\text{, integrate}\}\} } \]

\[ (C30) \quad \text{v3\_triple\_prime: } \text{rhs}\{\text{part\{solve\{eqn1\text{, diff\{v3, s, 3\}\}\}\}, \quad 1\}\}, \]

\[ \text{w3\_triple\_prime: } \text{rhs}\{\text{part\{solve\{eqn2\text{, diff\{w3, s, 3\}\}\}\}, \quad 1\}\} \]

\[ (C31) \quad \text{v3\_double\_prime: } \text{integrate\{v3\_triple\_prime, \quad s, \quad 1\}\}

\[ \text{w3\_double\_prime: } \text{integrate\{w3\_triple\_prime, \quad s, \quad 1\}\} \]

\[ (C32) \quad \text{v3\_prime: } \text{integrate\{v3\_double\_prime, \quad s, \quad 0\}\}

\[ \text{w3\_prime: } \text{integrate\{w3\_double\_prime, \quad s, \quad 0\}\} \]

\[ (C33) \quad \text{v3: } \]

\[ (D33) \quad \{(10\beta_6^2 + (23\beta_6 - 20)\beta_y - 5\beta_3 + 10)p v_5 + 18\beta_6 p v^3 \}

\[ + ((-70\beta_y^2 + (140 - 60\beta_3)\beta_y + 35\beta_3 - 70)p v_5 - 162\beta_6 \beta_y p v^3) \}

\[ + (210\beta_y^2 + 420\beta_3 - 1 - 126\beta_3 + 210)p v_5 + 294\beta_6 \beta_y p v^3) \}

\[ + ((-350\beta_y^2 + (700 - 490\beta_3)\beta_y + 280\beta_3 - 350)p v_5 - 210\beta_6 \beta_y p v^3) \}

\[ + (280\beta_y^2 + (280\beta_3 - 560)p v_5 - 280\beta_3 + 280)p v^3 \}

\[ + ((-168\beta_6 \beta_y p v_5 - 168\beta_6 \beta_y p v^3) - (5040\beta_6 / \beta_y) \}

Since the labelling \( v(s) \) and \( w(s) \) should be interchangeable (see Figure 1a), one would expect that if \( p v \) is replaced by \( p w \), \( \beta_6 \) is replaced by \( \beta_3 \), and \( p c \) is replaced by

\[ D_6 \text{ in the expression for } v_1 \text{ (and } v_3) \text{, the expression for } w_1 \text{ (and } w_3) \text{ should be obtained. This was verified to be true by re-evaluating } v_1 \text{ with } pv = p w, \quad \beta_6 = \beta_3 \text{ and } \beta_6 = \beta_3 \text{. The } O(\epsilon^3) \text{ approximate solution for } \phi(s) \text{ is obtained from } D16 \text{ with } v(s) = v_1 \text{ and } w(s) = w_1 \).

Figure 2 shows MACSYMA-generated plots of \( v(s = 1) \) and \( w(s = 1) \) versus \( \alpha \) for \( \beta_y = 15 \), \( \beta_y = 1.247 \) and for the two values of \( P \) indicated. The parameter values indicated match the values in (Hinnant and Hodges, 1987) for a rectangular aluminum beam, with a heavy tip mass, used in experiments reported in (Dowell, Traybar and Hodges, 1977). The value of \( P = 0.473 \) corresponds to a tip weight of \( 1/10 \), which is about 8 times higher than the weight of the beam. The analytical results shown in Figure 2 are nearly indistinguishable from the experimental data reported in those references and from the results of a finite element analysis presented in (Hinnant and Hodges, 1987).

\[ \text{Figure 2. Elastic Deformations at the Beam's Tip for } \beta_y = 15 \text{ and } \beta_y = 1.247 \text{ (--- } P = 0.473, \quad \text{--- } P = 1.419 \} \]

CONCLUSIONS

Computerized symbolic manipulation is a very useful and versatile tool for the analyst. It frees the analyst from the task of performing tedious and/or repetitive algebraic manipulations, and also allows for extending analysis capabilities beyond the level that would normally be feasible by pencil and paper for a number of problems. In this paper it has been used to assist in the formulation of the nonlinear equations of equilibrium of a beam, and in the analysis of the response of the beam. Its role in rotorcraft dynamics analysis has also been discussed and demonstrated in (Crespo da Silva and Hodges, 1986). As indicated in (Hodges, Crespo da Silva and Peters, 1987), problems of this nature have been addressed by other investigators, and a number of errors appear in their analysis due, in part, to a priori unwarranted approximations and shortcuts used in order to prematurely simplify the equations.
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