TECHNICAL REPORT
Office of Naval Research Contract No. N00014-86-K0029

SYSTEM SIZE AND REMAINING SERVICE IN M/G/1

by

Martin Krakowski

Report No. GMU/22474/114
August 15, 1989

Department of Operations Research and Applied Statistics
School of Information Technology and Engineering
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SYSTEM SIZE AND REMAINING SERVICE IN M/G/1

Abstract

Wishart [1961] and Takács [1963] derived the joint distribution of the size $N$ and residual service $R$ as encountered by a new arrival into a regular M/G/1. Wishart obtained the following expression for the generating function $\Pi(z, x) = \sum_{j=0}^{\infty} \Pi_j(x) z^j$, where $\Pi_j = \Pr\{R \leq x, N = j\}$:

$$
\Pi(z, x) = \frac{(1 - \rho) \lambda z (1 - z)}{\eta(\lambda - \lambda z - z)} \int_{0}^{\infty} e^{-\lambda(1-z)\xi} [H(\xi + x) - H(\xi)] d\xi.
$$

We exploit the fact that the system size $N$ is known and find the conditional r.v.'s $R_j = \text{residual service seen while } N = j$. Our method seems better suited to numerical work and we extend it to some variants of M/G/1: M/G/1/K, and then M/G/1 and M/G/1/K with state-dependent service and arrival rates.
Notation (for browsers; the symbols are also defined in context)

N = size of system;
N* = size of system provided server works;
λ = (poissonian) arrival frequency
λ_j = arrival frequency when N = j
x = service time;
\( x_j \) = (state-dependent) service time of a customer whose service starts when N = j;
\( \hat{B}(t) = \Pr(x \leq t) \);
\( \hat{B}_j(t) = \Pr(x_j \leq t) \);
\( P_j = \Pr(N = j) \);
\( P_* = \Pr(\text{server busy}) \);
\( \psi(x) \) is an arbitrary function of \( x \); \( \psi(x,y) \) is an arbitrary function of \( x,y \);
\( D_2 \psi(x,y) = \frac{\partial \psi(x,y)}{\partial y} \); partial derivative with respect to the second argument
\( f_{ij} \) = frequency of jumps "i → j"
\( c \ast x = x_1 + x_2 + \ldots + x_c \) where the \( x_i \) are free copies of \( x \) (i.i.d. r.v.'s equivalent to \( x \) and also independent of other variables within the argument)
\( \psi(Z) \) is an arbitrary function of \( Z \) such that \( E\psi(Z) \leq \infty \) (all terms entering analysis are required to have finite expectation, e.g. \( E\psi'(Z) \))
\( E\psi(Z) \) is called the omni-transform of \( \psi(Z) \). When \( \psi(Z) = \exp(-sZ) \) and \( Z \geq 0 \) then \( E\psi(Z) \) is the L-S transform of \( Z \)
Section 1 Service Residues as Conditioned on System Size in M/G/1

Find \( R_j \), the residual service time ("residue") of the ongoing service as seen when \( N = j, j \geq 1 \). Assume the \( P_j = \Pr(N = j) \) to be known (e.g. Gross and Harris 1985).

**Definition** Let \( \psi(X) \) be an arbitrary function of the process \( X \) with finite \( E\psi(X) \) and \( E\psi'(X) \). We call \( E\psi(X) \) the omni-transform of \( X \); if \( \psi(X) = e^{-sX} \) we get the L-S transform. The balance of the process \( \psi(X) \) is the equation \( E\psi'(X) = 0 \) for a random \( dt \).

The essence of the omni-method is to study the balance of \( \psi(X) \) rather than of \( X \). Among the method’s advantages are freedom to choose \( \psi \), notational ease in handling sums and mixtures of r.v.'s, and bypassing L-S transforms in many contexts.

**Definition**
\[
Z_j \overset{d}{=} R_j \text{ if } N = j \text{ and } Z_j \overset{d}{=} 0 \text{ if } N \neq j
\]

The process \( \psi(Z_j) \) varies by aging and jumps: \( j-1 \rightarrow j, j+1 \rightarrow j, j \rightarrow j-1 \text{ and } j \rightarrow j+1 \). We assume \( E\psi(Z_j) = 0 \) for a random \( dt \) and work out the balance. For a random \( dt \):

(a) aging adds: \( E\psi(Z_1) \big|_{\text{aging}} = P_1 E[\psi(R_1 - dt) - \psi(R_1)] = -dt P_1 E\psi'(R_1); dR_1 = -dt \)

(b) jumps "0→1" add: \( E\psi(Z_1) \big|_{0 \rightarrow 1} = dt f_{01} E[\psi(x) - \psi(0)]; f_{01} = \lambda P_0 \)

(c) jumps "2→1" add: \( E\psi(Z_1) \big|_{2 \rightarrow 1} = dt f_{21} E[\psi(x) - \psi(0)]; f_{21} = f_{12} = \lambda P_1 \)

(d) jumps "1→0" add: \( E\psi(Z_1) \big|_{1 \rightarrow 0} = dt f_{10} E[\psi(0) - \psi(0)] = 0 \)

(since \( Z_1 = R_1 = +0 \) just before "1→0" and \( Z_1 = 0 \) in any state other than "1")

(e) jumps "1→2" add: \( E\psi(Z_1) \big|_{1 \rightarrow 2} = dt f_{12} E[\psi(0) - \psi(R_1)]; f_{12} = \lambda P_1 \)

From (a) through (e) we get the balance of \( \psi(Z_1) \), i.e. \( E\psi'(Z_1) = 0 \),

\[
[P_1 E\psi'(R_1) + f_{12} E[\psi(R_1) - \psi(0)] = (f_{01} + f_{21}) E[\psi(x) - \psi(0)]]
\]

where \( f_{01} = f_{10} = \lambda P_0 \) and \( f_{12} = f_{21} = \lambda P_1 \). The right side of (1.1) is known.
Definition: The omni-convention calls for mentally applying the expectation operator $E$ to each side of an omni-equation; in case of ambiguity we retain $E$. (This convention is kin to summation convention in matrix and tensor calculus.) E.g. (1.1) becomes

$$p_1\psi'(R_1) + f_{12}[\psi(R_1) - \psi(0)] = (f_{01} + f_{21})[\psi(x) - \psi(0)]$$

(1.2)

Let us consider the changes in $E\psi(Z_j)$ for a $j \geq 2$. During a random $dt$:

(A) aging adds: $Ed\psi(Z_j)|_{aging} = P_j E[\psi(R_j) - dt - \psi(R_j)] = -dtP_jE\psi'(R_j)$; $dR_j = -dt$

(B) jumps “$j \rightarrow j$” add: $Ed\psi(Z_j)|_{j \rightarrow j} = dt f_{j-1,j} E[\psi(R_j) - \psi(0)]; f_{j-1,j} = \lambda_{P_{j-1}}$

(C) jumps “$j + 1 \rightarrow j$” add: $Ed\psi(Z_j)|_{j \rightarrow j+1} = dt f_{j+1,j} E[\psi(x) - \psi(0)]; f_{j+1,j} = f_{j,j+1} = \lambda_{P_{j}}$

(D) jumps “$j \rightarrow j - 1$” add: $Ed\psi(Z_j)|_{j \rightarrow j-1} = dt f_{j,j-1} E[\psi(0) - \psi(0)] = 0$

(E) jumps “$j \rightarrow j + 1$” add: $Ed\psi(Z_j)|_{j \rightarrow j+1} = dt f_{j,j+1}[\psi(0) - \psi(R_j)]; f_{j,j+1} = \lambda_{P_{j}}$

From (A) through (E) we get the balance of $\psi(Z_j)$

$$P_j\psi'(R_j) + f_{j,j+1}[\psi(R_j) - \psi(0)] = f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j}[\psi(x) - \psi(0)]$$

(1.3)

where $f_{j,j-1} = f_{j-1,j} = \lambda_{j-1}P_{j-1}$ and $f_{j,j+1} = f_{j+1,j} = \lambda_jP_j$. The right side of (1.3) is known when we solve for successive values of $j \geq 1$.

From (1.2) and (1.3) we get equations for moments; or L-S transforms; or tail distributions of $R_j$ by setting $\psi(R_j) = R_j^i$ for $i \geq 1$; or $\psi(R_j) = \exp(-sR_j)$; or $\psi(R_j) = \xi_j$ where $\xi_j = 1$ if $R_j > t$ and $\xi_j = 0$ if $R_j > t$ for then $\tilde{H}_j(t) \overset{d}{=} \Pr(R_j > t) = E\xi_j$.

Note: We can set $E\psi(R_j) = \Pr(R_j > t)$ if we know that $\Pr(R_j > t) = E\xi(R_j)$ for some $\xi(R_j)$. In linear omni-equations with constant coefficients, as in our paper, we can view $\psi$ as a general functional. We need not then find a $\xi(R_j)$ and need no omni-convention.

Let $\psi(R_j) = \tilde{H}_j(t) \overset{d}{=} \Pr(R_j > 0)$ in (1.1) and (1.2) and let $\psi(x_j) = \Pr(x_j > t) \overset{d}{=} \tilde{B}(t)$. Then

$$\psi'(R_j) = \lim_{-dt} \frac{\psi(R_j - dt) - \psi(R_j)}{-dt} = \lim_{-dt} \frac{\Pr(R_j - dt > t) - \Pr(R_j > t)}{-dt}$$
\[
\lim \frac{\tilde{H}_j(t+dt) - \tilde{H}_j(t)}{-dt} = -\tilde{H}'(t)
\]

and we get with \( \tilde{B}(t) \equiv Pr(x>t) \)

\[
j = 1 \quad \frac{-P_j \tilde{H}'_j(t) + \lambda P_j \tilde{H}_j(t)}{-dt} = (\lambda P_0 + \lambda P_1) \tilde{B}(t) \tag{1.4a}
\]

\[
j \geq 2 \quad \frac{-P_j \tilde{H}'_j(t) + \lambda P_j \tilde{H}_j(t)}{-dt} = \lambda P_{j-1} \tilde{H}_{j-1}(t) + \lambda P_j \tilde{B}(t) \tag{1.4b}
\]

\( \tilde{H}_j(t) \) can be found from (1.4a); from (1.4.b) we can derive \( \tilde{H}_j(t) \) if \( \tilde{H}_{j-1}(t) \) is known. Thus we can find the \( \tilde{H}_j(t) \) for successive \( j \). Equations (1.4) are easily verified for M/M/1 with \( \tilde{H}_j(t) = \tilde{B}(t) = e^{-\mu t} \) for each \( j \). Moreover (1.4a) implies

\[
j \geq 1 \quad \tilde{H}_j'(t) \to 0 \text{ as } t \to \infty, \text{ and } \quad P_j \tilde{H}_j'(t) \to -\lambda P_{j-1} \text{ as } t \to 0 \tag{1.5}
\]

From (1.1a) and (1.2a) we get the recursive relations for \( \bar{R}_j \)

\[
\lambda P_1 \bar{R}_1 = \rho P_0 - (1-\rho)P_1 \quad \text{and} \quad \lambda P_{j+1} \bar{R}_{j+1} = \lambda P_j \bar{R}_j - (1-\rho)P_{j+1} \tag{1.6}
\]

If we know all the conditional \( \psi(R_j) \) we can get the \( \psi(w) \). Clearly

\[
w_0 = 0 \quad \text{and} \quad w_j = \psi(w|N=j) = \psi((j-1)x + R_j), \quad j \geq 1 \tag{1.7}
\]

\[
\psi(w) = P_0 \psi(0) + P_1 \psi(R_1) + P_2 \psi(1x + R_2) + P_3 \psi(2x + R_3) + P_4 \psi(3x + R_3) + + \tag{1.8}
\]

where \( c \ast x \equiv x_1 + + x_c \); the \( x_i \) are free copies of \( x \) (i.i.d. copies of the generic \( x \) and independent of the \( R_j \)).

**Note:** Using the renewal relation (Krakowski 1987)

\[
\psi(Z) - \psi(0) = Z \mathbb{E} \psi'(\mathbb{R}Z), \quad Z \geq 0, \quad \mathbb{R}Z = \text{residue of } Z
\]

we can recycle (1.2) and (1.3) into (omni-convention still holds!)

\[
P_1 \psi(R_1) + f_1 \bar{R}_1 \psi(\mathbb{R} R_1) = (f_0 + f_2) x \psi(\mathbb{R} x) \tag{1.9a}
\]

\[
P_j \psi(R_j) + f_{j+1} \bar{R}_j \psi(\mathbb{R} R_j) = f_{j-1} R_j \bar{R}_{j-1} \psi(\mathbb{R} R_{j-1}) + f_{j+1} x \psi(\mathbb{R} x) \tag{1.9b}
\]

Equations (1.9) have no derivatives \( \psi' \) and thus in a sense are integrals of (1.2) and (1.3).

**But since both \( R_j \) and \( \mathbb{R} R_j \) are arguments in (1.9) there is no labor saved unless we take**
a special interest in the $\mathcal{R} R_j$ in addition to $R_j$.

Conjecture: $R_j \rightarrow$ residue of $x$ as $j \rightarrow \infty$
Section 2  \( N, R \) in \( M/G/1 \) with State Dependent Service

We modify \( M/G/1 \) as follows. A service which starts when \( N = j \) lasts \( x_j, j \geq 1 \). Our problem is to find the \( R_j \) for \( j \geq 1 \) assuming that the \( P_j \) are known (Harris 1967, Gross and Harris 1985, pp.289-292; Krakowski July 1986; a closely related vacation model was treated by Harris & Marchal 1988.)

Let \( Z_j \triangleq R_j \) if \( N = j \) and \( Z_j \triangleq 0 \) if \( N \neq j \); \( \psi(Z_j) \) is arbitrary except for \( E\psi(Z_j) < \infty \) and \( E\psi'\psi(Z_j) < \infty \). During a random \( dt \)

(a) aging adds: \( Ed\psi(Z_1)\big|_{aging} = P_1 E[\psi(R_1 - dt) - \psi(R_1)] = -P_1 dt E\psi'(R_1); dR_1 = -dt \)

(b) jumps "0→1" add: \( Ed\psi(Z_1)\big|_{0\to1} = dt f_{01} E[\psi(x_1) - \psi(0)]; f_{01} = \lambda P_0 \)

(c) jumps "2→1" add: \( Ed\psi(Z_1)\big|_{2\to1} = dt f_{21} E[\psi(x_1) - \psi(0)]; f_{21} = f_{12} = \lambda P_1 \)

(d) jumps "1→0" add: \( Ed\psi(Z_1)\big|_{1\to0} = dt f_{10} E[\psi(0) - \psi(0)] = 0 \)

(e) jumps "1→2" add: \( Ed\psi(Z_1)\big|_{1\to2} = dt f_{12} E[\psi(0) - \psi(R_1)]; f_{12} = \lambda P_1 \)

From (a) through (e) we get (mind the omni-convention!)

\[
\text{Balance of } \psi(Z_1) \quad P_1 \psi'(R_1) + f_{12} [\psi(R_1) - \psi(0)] = (f_{01} + f_{21})[\psi(x_1) - \psi(0)]
\]

where \( f_{01} = f_{10} = \lambda P_0 \) and \( f_{12} = f_{21} = \lambda P_1 \). The right side of (2.1) is known.

For \( j \geq 2 \), during a random \( dt \):

(A) aging adds: \( Ed\psi(Z_j)\big|_{aging} = P_j E[\psi(R_j - dt) - \psi(R_j)] = -dt P_j \psi'(R_j); dR_j = -dt \)

(B) jumps "j−1→j" add: \( Ed\psi(Z_j)\big|_{j-1\to j} = dt f_{j-1,j} [\psi(R_{j-1}) - \psi(0)]; f_{j-1,j} = \lambda P_{j-1} \)

(C) jumps "j+1→j" add: \( Ed\psi(Z_j)\big|_{j+1\toj} = dt f_{j+1,j} E[\psi(x_j) - \psi(0)]; f_{j+1,j} = f_{j,j+1} = \lambda P_j \)

(D) jumps "j→j−1" add: \( Ed\psi(Z_j)\big|_{j\toj-1} = dt f_{j,j-1} [\psi(0) - \psi(0)] = 0 \)

(E) jumps "j→j+1" add: \( Ed\psi(Z_j)\big|_{j\toj+1} = dt f_{j,j+1} E[\psi(0) - \psi(R_j)]; f_{j,j+1} = \lambda P_j \)

From (A) through (E) we get the balance of \( \psi(Z_j) \) (mind the omni-convention!)

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\[ P_j \psi'(R_j) + f_{j,j+1} \psi(R_j) - \psi(0) = f_{j-1,j} \psi(R_{j-1}) - \psi(0) + f_{j+1,j} \psi(x_j) - \psi(0) \]  

(2.2)

where \( f_{j,j-1} = f_{j-1,j} = \lambda_{j-1} P_{j-1} \) and \( f_{j,j+1} = f_{j+1,j} = \lambda_j P_j \). The right side of (2.2) is known when we solve for successive values of \( j \geq 1 \).

When all service lengths \( x_j \) are free copies of a generic \( x \), i.e. \( \psi(x_j) = \psi(x) \) for each \( j > 1 \), then (2.1) and (2.2) become (1.2) and (1.3) respectively – as they should.

We get \( \hat{\xi}_j(t) \equiv \Pr(R_j > t) \) by setting \( E(\psi(R_j) = \Pr(R_j > t) \).

Note: It is enough to know that \( E(\xi(R_j) = \Pr(R_j > t) \) for some function \( \xi(R_j) \) – we do not have to actually determine \( \xi((R_j) \).

It follows that

\[ j = 1 \quad P_j \hat{\xi}_j(t) - \lambda P_j \hat{\xi}_j(t) - \lambda P_0 \hat{\xi}_j(t) = \lambda P_0 - \lambda (P_0 + P_1) \hat{B}_j(t) \]  

(2.3a)

\[ j \geq 2 \quad P_j \hat{\xi}_j(t) - \lambda P_j \hat{\xi}_j(t) = \lambda P_{j-1} - \lambda P_{j-1} \hat{\xi}_{j-1}(t) - \lambda P_j \hat{B}_j(t) \]  

(2.3b)

The right side of (2.3a) is known; so is the right side of (2.3b) for each \( j \) when the \( \hat{\xi}_j(t) \) are known for \( i < j \). (2.3a,b) imply that for each \( j \geq 1 \),

\[ P_j \hat{\xi}_j(0) = \lambda P_{j-1} \]  

(2.4)

From (2.1) and (2.2) we get

\[ \lambda P_j R_j = (P_0 + P_1) \lambda x_1 - P_1 \quad \text{and} \quad \lambda P_{j+1} R_{j+1} = \lambda P_j R_j - (1 - \lambda x_{j+1}) P_{j+1} \]  

(2.5)

When, for each \( j \geq 1 \), \( \hat{B}_j(t) = \hat{B}(t) \), then (2.3a,b) become (1.4a,b), as they should.

\[ j = 1 \quad -P_j \hat{\xi}_j(t) + \lambda P_j \hat{\xi}_j(t) = (\lambda P_0 + \lambda P_1) \hat{B}(t) \]  

(1.4a)

\[ j \geq 2 \quad -P_j \hat{\xi}_j(t) + \lambda P_j \hat{\xi}_j(t) = \lambda P_{j-1} \hat{\xi}_{j-1}(t) + \lambda P_j \hat{B}(t) \]  

(1.4b)

The question arises, Can we derive the load or delay from (2.1) and (2.2)? Unfortunately, we see no fair way. In our model with state-dependent service the virtual load and delay (and kindred time lengths) depend on future arrivals; this makes them essentially more complex.
Section 3 The $R_j$ and the Load in M/G/1/K

Consider an M/G/1/K, i.e. where $N \leq K$; customers arriving while $N = K$ are lost. We consider the load (backlog, unfinished work) $L$; $L = w$, the virtual delay, when $N < K$ but for $N = K$ $w$ is not defined. We assume the $P_j$ to be known (Gross and Harris, p. 279-285, 1985).

Clearly

$$\psi(L) = P_0 \psi(0) + P_1 \psi(R_1) + P_2 \psi(1+x + R_2) + P_3 \psi(2+x + R_3) + \ldots + P_K \psi((K-1)x + R_3) \quad (3.1)$$

Define now $\psi(Z_j)$ as before: $\psi(Z_j) \equiv \psi(R_j)$ if $N = j$ and $\psi(Z_j) \equiv \psi(0)$ otherwise. The balance of $\psi(Z_j)$ clearly yields the same equations for $j = 1$ and for $1 < j < K$ as the balance for regular M/G/1. For $j = K$ during a random $dt$

(a) aging adds:

$$Ed\psi(Z_K)|_{aging} = P_K E[\psi(R_K) - dt] - \psi(R_K)] = -dt P_K \psi'(R_K); \ dR_K = -dt$$

(b) the jumps "K-1↔K" add:

$$Ed\psi(Z_K)|_{K-1↔K} = dt f_{K-1,K} E[\psi(R_{K-1}) - \psi(0)]; \ f_{K-1,K} = \lambda P_{K-1}$$

(c) the jumps "K↔K-1" add:

$$Ed\psi(Z_K)|_{K↔K-1} = dt f_{K↔K-1} E[\psi(0) - \psi(0)] = 0$$

Therefore

$$j = 1: \quad P_1 \psi'(R_1) + f_{1,2}[\psi(R_1) - \psi(0)] = (f_{01} + f_{21})[\psi(x) - \psi(0)] \quad (3.2a)$$

$$1 < j < K: \quad P_j \psi'(R_j) + f_{j,j+1}[\psi(R_j) - \psi(0)] =$$

$$= f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j}[\psi(x) - \psi(0)] \quad (3.2b)$$

$$j = K: \quad P_K \psi'(R_K) = f_{K-1,K}[\psi(R_{K-1}) - \psi(0)] \quad (3.2c)$$

The tail distributions $\tilde{H}_j(t) \equiv Pr(R_j > t)$ satisfy

$$j = 1 \quad -P_1 \tilde{H}_1'(t) - \lambda P_1 \tilde{H}_1(t) = (\lambda P_0 + \lambda P_0) \tilde{B}(t) \quad (3.3a)$$

$$1 < j < K \quad -P_j \tilde{H}_{j-1}'(t) + \lambda P_j \tilde{H}_j(t) = \lambda P_{j-1} \tilde{H}_{j-1}(t) + \lambda P_j \tilde{B}(t) \quad (3.3b)$$
\[
\begin{align*}
\text{j = K} & \quad P_K \tilde{R}_K'(t) = \lambda P_{K-1} \tilde{R}_{K-1}(t) \quad (3.3c)
\end{align*}
\]

Starting with \( j = 1 \) we can compute the successive \( \tilde{R}_j(t) \). Defining \( R = 0 \) for \( N = 0 \) (readers may favor other definitions) we have

\[
\psi(N, R) = P_0 \psi(0, 0) + P_1 \psi(1, R_1) + P_2 \psi(2, R_2) + \cdots + P_K \psi(K, R_K) \quad (3.4)
\]

For the load \( L \) (unfinished work, backlog) we have

\[
\psi(L) = P_0 \psi(0) + P_1 \psi(R_1) + P_2 \psi(x + R_2) + P_3 \psi(2x + R_3) + \cdots + P_K \psi((K-1)x + R_K) \quad (3.5)
\]

which appears to be new. The economic motivation must be strong to numerically evaluate the moments or the distribution of \( L \); but the complexity appears to inhere in the problem, not in our method.
Section 4  M/G/1 With State-Dependent Service And Arrival Rates

We extend now M/G/1 with state-dependent service lengths of Section 2 by allowing also state-dependent arrival rates. The service time is still determined at the instance in which a service begins and equals $x_i$, $i$ being the size of the system, including the customer to be served. The arrival rate is $\lambda_j$ when system size is $N=j$. Let us act as if the $P_j$ are known. So soon someone derives these $P_j$ our derivation of $N, R$ for this model will be completed.

Let, as in Section 2, $Z_j \overset{d}{=} R_j$ if $N=j$ and $Z_j \overset{d}{=} 0$ if $N \neq j$; $\psi(Z_j)$ is arbitrary but subject to $E\psi(Z_j) < \infty$ and $E\psi'(Z_j) < \infty$.

It is easy to see that equations (2.1a) and (2.2a) stay valid but their frequency coefficients reflect the changed arrival and service rates. Clearly we now have

\begin{align*}
P_1 \psi'(R_1) + f_{12} [\psi(R_1) - \psi(0)] &= (f_{01} + f_{21}) [\psi(x_1) - \psi(0)] \quad (2.1a) = (4.1) \\
P_j \psi'(R_j) + f_{j,j+1} [\psi(R_j) - \psi(0)] &= f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j} [\psi(x_j) - \psi(0)] \quad (2.2a) = (4.2)
\end{align*}

with $f_{01} = \lambda_0 P_0$; $f_{12} = f_{21} = \lambda_1 P_1$; $f_{j-1,j} = \lambda_{j-1} P_{j-1}$; $f_{j,j+1} = f_{j+1,j} = \lambda_j P_j$.

From (4.1) and (4.2) we get $\tilde{R}_j(t) \overset{d}{=} Pr(R_j > t)$ by setting $E\psi(R_j) = Pr(R_j > t)$. Thus

\begin{align*}
j = 1 \\
\tilde{R}_1(t) &= \lambda_0 P_0 - (\lambda_0 P_0 + \lambda_1 P_1) \tilde{B}_1(t) \quad (4.3a) \\
\tilde{R}_j(t) &= \lambda_j P_j \tilde{R}_j(t) = \lambda_{j-1} P_{j-1} \tilde{R}_{j-1}(t) + \lambda_j P_j \tilde{B}_j(t) \quad (4.3b)
\end{align*}

The right side is known for (4.3a), and for (4.3b) if we solve for successive $j \geq 1$. (4.3a) and (4.3b) imply that for each $j \geq 1$

\[ \tilde{R}_j'(0) = 0 \quad (4.4) \]

(4.1) and (4.2) imply

\[ \lambda_1 P_1 \tilde{R}_1 = (\lambda_0 P_0 + \lambda_1 P_1) x_1 - P_1 \quad \text{and} \quad \lambda_j P_j \tilde{R}_j = \lambda_{j-1} P_{j-1} \tilde{R}_{j-1} + \lambda_j x_j P_j \quad (4.5) \]
Section 5  M/G/1/K With State-Dependent Service and Arrival Rates

We extend M/G/1/K with state-dependent services of Section 4 by allowing state-dependent arrival rates; service time $x_i$ still depends on system size (including customer served) when service starts. The arrival rate is $\lambda_j$ when $N = j$. We act as if the $P_j$ are known, but there seems to be no analytical derivation yet available.

Define, as in Section 2, $Z_j \overset{d}{=} R_j$ if $N = j$, and $Z_j \overset{d}{=} 0$ if $N \neq j; j \geq 1$. Let $\psi(Z_j)$ satisfy $E\psi(Z_j) < \infty$, $E\psi'(Z_j) < \infty$, and be otherwise arbitrary; then

$$\psi(Z_j) = (1 - P_j)\psi(0) + P_j \psi(R_j) \quad (5.1)$$

It is clear that the equations (3.3a,b,c) are still valid so that

$$j = 1 \quad P_1 \psi'(R_1) + f_{12} [\psi(R_1) - \psi(0)] = (f_{01} + f_{21}) [\psi(x_1) - \psi(0)] \quad (3.3a) = (5.2a)$$

$$1 < j < K \quad P_j \psi'(R_j) + f_{j,j+1} [\psi(R_j) - \psi(0)] = f_{j-1,j} [\psi(R_{j-1}) - \psi(0)] + f_{j+1,j} [\psi(x_j) - \psi(0)] \quad (3.3b) = (5.2b)$$

$$j = K \quad P_K \psi'(R_K) = f_{K-1,K} [\psi(R_{K-1}) - \psi(0)] \quad (3.3c) = (5.2c)$$

$f_{01} = f_{10} = \lambda_0 P_0; f_{12} = f_{21} = \lambda_1 P_1; f_{j-1,j} = f_{j,j-1} = \lambda_{j-1} P_{j-1}; f_{j,j+1} = f_{j+1,j} = \lambda_j P_j$.

With the modified $f_{ij}$, the tail distributions $\tilde{H}_j(t) \overset{d}{=} \Pr(R_j > t)$ satisfy

$$j = 1 \quad -P_1 \tilde{H}_1'(t) + \lambda_1 P_1 \tilde{H}_1(t) = (\lambda_0 P_0 + \lambda_1 P_1) \tilde{H}_1(t) \quad (5.3a)$$

$$1 < j < K \quad -P_j \tilde{H}_{j-1}'(t) + \lambda_j P_j \tilde{H}_j(t) = \lambda_{j-1} P_{j-1} \tilde{H}_{j-1}(t) + \lambda_j P_j \tilde{H}_j(t) \quad (5.3b)$$

$$j = K \quad -P_K \tilde{H}_K'(t) = \lambda_{K-1} P_{K-1} \tilde{H}_{K-1}(t) \quad (5.3c)$$

We can compute all $\tilde{H}_j(t)$ for $j \geq 1$. Defining $(N,R) = (0,0)$ for $N = 0$, yields

$$\psi(N,R) = P_0 \psi(0,0) + P_1 \psi(1,R_1) + P_2 \psi(2,R_2) + \ldots + P_K \psi(K,R_K) \quad (4.4)$$

We cannot from the foregoing find (certainly not easily) the load or delay or any such variable because at any instant these durations depend also on future arrivals.
Appendix  Joint Treatment of System Size and Residual Service In M/G/1

We derive now a single global omni-equation equivalent to equations (1.2) and (1.3).

Definition $\psi(N, Z) \stackrel{d}{=} \psi(0, x)$ when $N = 0$ and $\psi(N, Z) \stackrel{d}{=} \psi(N_\ast, R)$ when $N \geq 1$; $\psi(N, Z)$ is defined at any time point.

Let $N_\ast$=system size provided $N \geq 1$. $R$ is defined only when $N \geq 1$ so it needs no asterisk. We assume that $\psi(N, Z)$ is a balanced r.v., so that $E\psi(N, Z) = 0$. Of course we have

$$\psi(N, Z) = P_0\psi(0, x) + P_\ast\psi(N_\ast, R) \quad P_0 = \lambda R \quad P_\ast = 1 - \lambda R \quad (A.1)$$

Let us consider the balance of $\psi(N, Z)$ during a random $dt$:

(a) aging, which goes on only while $N \geq 1$, adds

$$E\psi(N, Z)|_{aging} = P_\ast E[\psi(N_\ast, R - dt) - \psi(N_\ast, R)] = -dt P_\ast E\psi(N_\ast, R); \quad P_\ast = Pr(N \geq 1)$$

(b) arrivals "0→1" add $E\psi(N, Z)|_{0→1} = f_{01} E[\psi(1, x) - \psi(0, x)]; \quad f_{01} = \lambda P_0$

(c) arrivals while $N \geq 1$ add $E\psi(N, Z)|_{\ast\ast} = f_\ast E[\psi(N_\ast + 1, R_\ast) - \psi(N_\ast, R_\ast)]; \quad f_\ast = \lambda P_\ast$

A subscript $\ast\ast$ says that the r.v. is found by an arrival into a busy system.

Since true poissonian arrivals see, stochastically, what a continuous or random (poissonian) observer sees (cf. Wolff 1982), we have $\psi(N_\ast, R_\ast) = \psi(N_\ast, R)$ and

$$f_\ast E[\psi(N_\ast + 1, R_\ast) - \psi(N_\ast, R_\ast)] = f_\ast E[\psi(N_\ast + 1, R) - \psi(N_\ast, R)]; \quad f_\ast = \lambda P_\ast$$

(d) departures (perforce from a busy system) add $E\psi(N, Z)|_{d} = \lambda E[\psi(N_d, x) - \psi(N_d + 1, 0)]; \quad$ departure rate = arrival rate = $\lambda$

Subscripts $d$ say that a r.v. is seen by a just departed customer. (System size is $N_d + 1$ just before a departure, and is $N_d$ just after.) Since $\psi(N_d) = \psi(N_a) = \psi(N)$, and since $N_d$ and $x$ are independent,

$$E\psi(N, Z)|_{d} = \lambda E[\psi(N_d, x) - \psi(N_d + 1, 0)] = \lambda E[\psi(N, x) - \psi(N + 1, 0)]$$
From (a) through (d) we get (mind the omni-convention!)

\[-P_0 D_2 \psi(N_*, R) + f_0 \left[ \psi(1, x) - \psi(0, x) \right] + f_* \left[ \psi(N_* + 1, R) - \psi(N_*, R) \right] +
\]
\[+ \lambda [N, x] - \psi(N + 1, 0)] = 0 \tag{A.2}\]

where \( P_* = 1 - P_0 = \rho = \lambda \bar{x} ; \) \( f_0 = \lambda P_0 = \lambda (1 - \rho) ; \) and \( f_* = \) arrival rate while server works \( = \lambda P_* . \)

With \( \psi(N) = P_0 \psi(0) + P_* \psi(N_*) \) we get from (A.2) the equation

\[P_0 D_2 \psi(N_*, R) - \lambda P_* [\psi(N_* + 1, R) - \psi(N_*, R)] +
\]
\[+ \lambda P_* [\psi(N_* + 1, 0) - \psi(N_*, x)] = \lambda P_0 [\psi(1, x) - \psi(1, 0)] \tag{A.4}\]

*Equation (A.4) does not depend on the definition \( Z = x \) when \( N = 0. \)*

By specializing (A.4) to

\[\psi(N_*, R) = \Pr(N_* = j, R > t) = P_{*,j} \tilde{H}_j(t) = P_{,j} \tilde{H}_j(t)/P_* \tag{A.5}\]

where \( P_{*,j} = \Pr(\text{system size} = j | \text{server works}) = P_{,j}/P_* \) we get equations (1.4) = (A.6)

\[j = 1 \quad \boxed{-P_{,1} \tilde{H}_1'(t) + \lambda P_{,1} \tilde{H}_1(t) = (\lambda P_0 + \lambda P_{,1}) \tilde{B}(t)} \tag{1.4a} = \tag{A.6a}\]

\[j > 2 \quad \boxed{-P_{,j} \tilde{H}_j'(t) + \lambda P_{,j} \tilde{H}_j(t) = \lambda P_{,j-1} \tilde{H}_{j-1}(t) + \lambda P_{,j} \tilde{B}(t)} \tag{1.4b} = \tag{A.6b}\]

Equation (A.4) is equivalent to the infinite set (1.2) and (1.3). But (A.4), unlike (A.6), cannot be adapted, we think, to variants of M/G/1.
Bibliography


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