ON THE 2-EXTENDABILITY OF PLANAR GRAPHS

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ABSTRACT

Some sufficient conditions for the 2-extendability of k-connected k-regular ($k \geq 3$) planar graphs are given. In particular, it is proved that for $k \geq 3$, a k-connected k-regular planar graph with each cyclic cutset of sufficiently large size is 2-extendable.

1. Introduction and Terminology

All graphs in this paper are finite, undirected, connected and simple, although some parallel edge situations will occur after some contractions are made. However, any loops formed by these contractions will be deleted. Let ν and n be positive integers with $n \leq (\nu-2)/2$ and let G be a graph with ν vertices and ϵ edges having a perfect matching. The graph G is said to be n-extendable if every matching of size n in G lies in a perfect matching of G. (KR)

A graph G is called **cyclically m-edge-connected** if $|S| \geq m$ for each edge cutset S of G such that there are two components in $G - S$ each of which contains a cycle. Here S is called a **cyclic edge cutset**. The size of a minimum cardinality cyclic edge cutset is called the **cyclic edge connectivity** of G and is denoted by $c\lambda(G)$.

In [7], Plummer introduced the concept of an n-extendable graph and proved that a graph of large minimum degree is n-extendable. In [3] and [4], Holton and Plummer proved that some k-connected k-regular graphs ($k \geq 3$) of large cyclic edge connectivity are n-extendable, which lends support to the assertion by Thomassen [9] that a graph of large girth and minimum degree at least three shares many properties with a graph of large minimum degree.

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According to Plummer [8], no planar graph is 3-extendable. It is then natural to ask what kind of planar graphs are 2-extendable. Holton and Plummer [3] proved (see Theorem 1 below) that a 3-connected cubic planar graph G is 2-extendable when $c\lambda(G)$ is large enough.

Theorem 1. If G is a cubic 3-connected planar graph which is cyclically 4-edge-connected and has no faces of size 4, then G is 2-extendable.

Theorem 1 has the following immediate corollary.

Corollary 1. If G is a cubic 3-connected planar graph which is cyclically 5-edge-connected, then G is 2-extendable.

In this paper, we discuss the 2-extendability of k-connected k-regular planar graphs for $k = 4, 5$. All terminology and notation not defined in the paper can be found in [1] or [2]. In particular, if G is a graph and $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. If G is a plane graph, let f_i denote the number of faces of size i in the planar embedding of G and let ϕ denote the total number of faces in the embedding.

2. Preliminary Results

In this section, we present several lemmas and corollaries which will play an important role in the proofs of our main results. Note that we denote the number of odd components of $G - S$ by $o(G - S)$.

Lemma 1. If a k-connected k-regular graph G of even order is not 2-extendable (where $k \geq 2$), then there are two independent edges e_1 and e_2 which do not lie in any perfect matching and a set $S \subseteq V(G)$ such that $\{e_1, e_2\} \subseteq E(G[S])$ and $o(G-S) = |S|-2$.

Furthermore, if N is the number of edges from the components of $G - S$ to S, then $k(|S| - 2) \leq N \leq k|S| - 4$.

Proof. Suppose that G is not 2-extendable. Then there are two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ which do not lie in any perfect matching. Let $G' = G - \{u_1, v_1, u_2, v_2\}$. By Tutte's Theorem on perfect matchings, there is a set $S' \subseteq V(G')$ such that $o(G'-S') > |S'|$. By parity, $o(G' - S') = |S'| + 2r$, for some $r \geq 1$. Let $S = S' \cup \{u_1, v_1, u_2, v_2\}$ and let N be the number of edges from the components of $G - S$ to S. By the k-regularity, $N \leq k|S| - 4$. By the k-connectedness, $N \geq k(o(G' - S')) = k(|S'| + 2r)$. If $r \geq 2$, then $N \geq k(|S'| + 4) = k|S|$, contradicting the fact that $N \leq k|S| - 4$. So $r = 1$ and $o(G - S) = o(G' - S') = |S'| + 2 = |S| - 2$. Then we have $k(|S| - 2) \leq N \leq k|S| - 4$. ∎

Lemma 2. Let G be a connected plane graph with all vertices of degree k except for r vertices. Let the degrees of the r exceptional vertices be $d_1, d_2, \ldots, d_r$. Then the following equation holds:

$$4f_2 + (6-k)f_3 = 2[(2-r)k + \sum_{i=1}^{r} d_i] + \sum_{j\geq 4}[(k-2)j - 2k]f_j,$$

where f_j is the number of faces of size j.

Proof. Let G be a connected plane graph satisfying the hypotheses of the lemma. We then have

$$\nu k - rk + \sum_{i=1}^{r} d_i = 2\epsilon = \sum_{j \geq 2} j f_j.$$

Then

$$\nu = \left(2\epsilon + rk - \sum d_i\right)/k \text{ and } \epsilon = \left(\sum j f_j\right)/2.$$

Substituting into Euler's Formula for plane graphs, we get

$$\left((2\epsilon + rk - \sum d_i)/k\right) - \epsilon + \phi = 2$$

$$(2-k)\epsilon + k\phi = (2-r)k + \sum d_i$$

$$[(2-k)\sum j f_j]/2 + k \sum f_j = (2-r)k + \sum d_i$$

$$\sum [(2-k)j + 2k] f_j = 2[(2-r)k + \sum d_i]$$

and hence

$$4f_2 + (6-k)f_3 = 2[(2-r)k + \sum d_i] + \sum_{j \geq 4} [(k-2)j - 2k] f_j,$$

as claimed. ∎

3. 2-extendability of 5-connected 5-regular planar graphs

Planar graphs which are 5-connected and 5-regular have, in a sense, sufficiently large minimum degree for 2-extendability. In the next result, we see that all such graphs are 2-extendable.

Theorem 2. Every 5-connected 5-regular planar graph G is 2-extendable.

Proof. Assume that G is not 2-extendable and let $e_i = u_i v_i$ $(i = 1, 2)$ be two independent edges in G which cannot be extended to a perfect matching. Let S and N be as in Lemma 1 and let $r = |E(G[S])|$. Then $N = 5|S| - 2r$ and $r \geq 2$. Since $o(G - S) \geq 2$, S is a cutset (and hence $|S| \geq 5$). Let $C_1, \ldots, C_m$ be the components of $G - S$. Let G'' be the graph obtained from G by contracting $C_1, \ldots, C_m$ to single vertices (retaining multiple edges, but discarding any loops formed). (Note that from this point on in this paper, when we contract such a component C_i to a singleton, we will denote the resulting singleton by $\hat{C}_i$.) Then by Lemma 2, we have

$$4f_2'' + f_3'' = 2(10 - 5m + \sum_{i=1}^{m} d_i) + \sum_{j \geq 4}(3j - 10)f_j''$$
$$\geq 2(10 + \sum_{i=1}^{m}(d_i - 5)),$$

where d_i is the degree of C_i in G'' and f_j'' is the number of faces of size j in G''. Since every triangular face of G'' uses an edge in $G[S]$, $f'' \leq 2r$. Let $\delta_i = d_i - 5$ $(1 \leq i \leq m)$. Since all the digons result from contraction of C_i's, $f_2'' \leq \sum_{i=1}^{m} \delta_i$. Therefore, $4\sum_{i=1}^{m} \delta_i + 2r \geq 2(10 + \sum_{i=1}^{m} \delta_i)$ or $\sum_{i=1}^{m} \delta_i \geq 10 - r$. On the other hand, by Lemma 1, $m \geq |S| - 2$ and

$$5|S| - 2r = N = \sum_{i=1}^{m} d_i = \sum_{i=1}^{m}(\delta_i + 5) = \sum_{i=1}^{m} \delta_i + 5m$$
$$\geq 10 - r + 5m \geq 10 - r + 5(|S| - 2) = 5|S| - r.$$

This is a contradiction. ∎

4. 2-extendability of 4-connected 4-regular planar graphs

For 4-connected 4-regular planar graphs the problem of determining when 2-extendability holds is more difficult as the degree of the graphs is not "large enough" and the cyclic edge connectivity is not larger than six because there is always a triangle in a connected 4-regular planar graph. If a 4-regular graph G has as a subgraph the graph shown in Figure 1, (this five-vertex graph will be called a **butterfly**), then G is clearly not 2-extendable. So it makes sense to study only those 4-connected 4-regular planar graphs which do not contain a butterfly. These we will call **butterfly-free** 4-connected 4-regular planar graphs.

Figure 1

Theorem 3. Let G be a butterfly-free 4-connected 4-regular planar graph. If every cyclic edge cutset has size greater than six, except those incident with a triangle, then G is 2-extendable.

Proof. Assume that G is not 2-extendable and let $e_i = u_i v_i$ $(i = 1, 2)$, N and S be as in the proof of Theorem 2. Again contract the m components of $G - S$ to singletons and call the resulting graph G''. Then by Lemma 2, we have

$$4f_2'' + 2f_3'' = 2(8 - 4m + \sum_{i=1}^{m} d_i) + \sum_{j\geq 4}(2j-8)f_j''$$

$$= 16 - 8m + 2\sum_{i=1}^{m} d_i + \sum_{j\geq 4}(2j-8)f_j''$$

$$= 16 + 2\sum_{i=1}^{m}(d_i - 4) + \sum_{j\geq 4}(2j-8)f'' = 16 + 2\sum_{i=1}^{m}\delta_i + \sum_{j\geq 4}(2j-8)f''$$

$$\geq 2(8 + \sum_{i=1}^{m}\delta_i),$$

or

$$2f_2'' + f_3'' \geq 8 + \sum_{i=1}^{m}\delta_i,$$

where $\delta_i = d_i - 4$.

Again, since $f_3'' \leq 2r$ and $f_2'' \leq \sum_{i=1}^{m}\delta_i$, we have $2r + 2\sum_{i=1}^{m}\delta_i \geq f_3'' + 2f_2'' \geq 8 + \sum_{i=1}^{m}\delta_i$ or $\sum_{i=1}^{m}\delta_i \geq 8 - 2r$. Furthermore, $m \geq |S| - 2$. Therefore, $4|S| - 2r = N = \sum_{i=1}^{m} d_i = \sum_{i=1}^{m}(\delta_i + 4) = \sum_{i=1}^{m}\delta_i + 4m \geq 8 - 2r + 4m \geq 8 - 2r + 4(|S| - 2) = 4|S| - 2r$. But then equality must hold in each inequality above. This means

(a) $f_j'' = 0$ for $j \geq 5$,
(b) $G - S$ has no even components,
(c) $f_3'' = 2r$, and
(d) $f_2'' = \sum_{i=1}^{m}\delta_i = 8 - 2r$. (In particular, $r \leq 4$.)

We now treat the three possible values of N.

Case 1. $N = 4|S| - 4$.

By parity, there are now two subcases to consider.

(1.1) There are eight edges from S to C_1 and exactly four edges from S to C_i, for $i = 2, \ldots, |S| - 2$.

Now $C_2, \ldots, C_{|S|-2}$ are all singletons, for if not, it is easy to show that a cyclic cutset of size four must exist and that would contradict the cyclic connectivity hypothesis of this theorem.

Recall from above that $f_3'' = 4$. Thus each edge e_i lies on exactly two different triangles by 4-connectedness. So let w_1 and w_2 be the two distinct vertices adjacent to both u_1 and v_1 in G'' and let w_3 and w_4 be the two distinct vertices adjacent to both u_2 and v_2 in G''. (Recall that none of these four w_i's can lie in S. Also note that we may have $\{w_1, w_2\} \cap \{w_3, w_4\} \neq \emptyset$.)

First assume $w_1 \neq \hat{C}_1$ and $w_2 \neq \hat{C}_1$ in G''. Then if $\{w_1, w_2\} \cap \{w_3, w_4\} \neq \emptyset$, there is a butterfly in G, contradicting one of the hypotheses of this theorem. On the other hand, if $\{w_1, w_2\} \cap \{w_3, w_4\} = \emptyset$, then the induced subgraph $H_1 = G[\{w_1, w_2, u_1, v_1\}]$ is a component different from a triangle in $G - T$ where T is the set of all edges from H_1 to $G - H_1$. However, T is a cyclic edge cutset of size six in G, contradicting an hypothesis of the theorem. The case in which $w_3 \neq \hat{C}_1$ and $w_4 \neq \hat{C}_1$ in G'' is similar.

So we may assume by symmetry that $w_1 = \hat{C}_1 = w_3$. Then, because $f''_2 = 4$, u_1, v_1, u_2 and v_2 are the only vertices in S adjacent to vertices of C_1 in G and C_1 contains all the neighbors of u_1, v_1, u_2 and v_2 in $G - \{u_1, v_1, u_2, v_2\}$, except w_2 and w_4. So $\{w_2, w_4\}$ is a cutset of G separating $F = G[V(C_1) \cup \{u_1, v_1, u_2, v_2\}]$ and $G - F - \{w_2, w_4\}$, contradicting the 4-connectedness of G.

(1.2) There are six edges from S to each of C_1 and C_2 and there are exactly four edges from S to each C_j, for $j = 3, 4, \ldots, |S| - 2$.

As in Case (1.1) we may assume that each C_j is a singleton, for $j = 3, 4, \ldots, |S| - 2$. Contracting C_1 and C_2, we obtain graph G''. By (d), (c) and (a) above, we know that each of $\hat{C}_1$ and $\hat{C}_2$ is incident with two digons and there are exactly four vertices in S adjacent to vertices of C_i for $i = 1, 2$.

If either C_1 or C_2 is not a triangle, the hypothesis concerning cyclic edge cutsets is contradicted. Hence both C_1 and C_2 are triangles.

Let x_1, x_2 and x_3 be the vertices of C_1. As there are two digons in G'' incident with $\hat{C}_1$, there is a vertex u in S adjacent to two vertices of C_1. Let $H = G[\{u, x_1, x_2, x_3\}]$. Then there is a cyclic cutset of size at most six separating H from $G - V(H)$ and once more we have a contradiction.

Case 2. $N = 4|S| - 6$.

Again, relabeling the C_i's if necessary, by parity we may assume that there are exactly six edges from S to C_1 and there are exactly four edges from S to each C_j, for $j = 2, 3, \ldots, |S| - 2$. As before, we may assume that each $C_j, j = 2, 3, \ldots, |S| - 2$, is a singleton. Moreover, there are exactly three edges $e_i = u_i v_i, i = 1, 2, 3$ in $G[S]$. Recall from (d), (c) and (a) above that $f''_2 = 2, f''_3 = 6$ and $f''_j = 0$, for all $j \geq 5$ in G''. As $f''_2 = 2$, there are exactly four vertices in S adjacent to vertices of C_1. Let w_{2i-1} and w_{2i} be the vertices adjacent to both u_i and v_i for $i = 1, 2, 3$ in G''. But then it is easy to check that $\hat{C}_1$ cannot be simultaneously in $\{w_1, w_2\}, \{w_3, w_4\}$ and $\{w_5, w_6\}$.

Without loss of generality, assume C_1 is not adjacent to both u_1 and v_1. Let $H = G[\{w_1, w_2, u_1, v_1\}]$. Then there is a cyclic cutset of size at most six separating H from $G - V(H)$, again a contradiction since H is not a triangle.

Case 3. $N = 4|S| - 8$.

There are exactly four edges from S to C_j, for $j = 1, 2, \ldots, |S| - 2$. Once again, as in Case (1.1), we may assume that C_j is a singleton for $j = 1, \ldots, |S| - 2$. But from (d), (c) and (a) above, $f_2 = 0$, $f_3 = 8$ and $f_j = 0$ for all $j \geq 5$.

Let w_1 and w_2 be adjacent to both u_1 and v_1. Let $H = G[\{w_1, w_2, u_1, v_1\}]$. Then once again we have a cyclic cutset of size at most six separating H from $G - V(H)$, contradicting an hypothesis of the theorem. ∎

Figure 2 gives a 2-extendable 4-connected 4-regular planar graph satisfying the hypotheses of Theorem 3.

Figure 2

Indeed, an infinite family of such graphs can be constructed (of which the graph in Figure 2 is the smallest) as follows. Let C_{12} denote the twelve-vertex configuration shown in Figure 3(a). Take $s \geq 2$ copies of C_{12} and join them in a ring-like fashion as indicated in Figure 3(b). It is routine to show that the resulting graphs satisfy the properties claimed above.

not satisfy the hypotheses of Theorem 3. Figure 4 shows one such example.

Figure 4

In the next theorem, we present an infinite family of such graphs. A graph isomorphic to the graph in Figure 5 is called a **JT** (for "joined triangles".)

Figure 5

As an immediate corollary of our next theorem, we note that every 4-connected 4-regular planar graph consisting of some vertex-disjoint JT's and some other edges joining them is always 2-extendable.

First, however, we will have need of the following result.

Lemma 3. Suppose G is a 4-regular 4-connected butterfly-free planar graph in which each vertex lies in a JT. Then any 2 JT's in G are either identical or vertex disjoint.

Proof. Suppose JT_1 and JT_2 are 2 JT's in G and that $JT_1 \neq JT_2$. Let $V(JT_i) = \{u_i, v_i, x_i, y_i\}$ and $E(JT_i) = \{x_iu_i, x_iv_i, y_iu_i, u_iv_i\}$. Let $A = V(JT_1) \cap V(JT_2)$.

(1) If $|A| = 1$, we get a butterfly and hence a contradiction.

(2) Next suppose that $|A| = 2$.

(2.1) First suppose that $A = \{x_1, v_1\}$. By symmetry, there are three cases to consider.

(2.1.1) Suppose $x_1 = x_2$ and $v_1 = v_2$. Then we get a butterfly.

(2.1.2) If $x_1 = u_2$ and $v_1 = v_2$, then $\deg_G v_1 \geq 5$, a contradiction.

(2.1.3) So suppose $x_1 = x_2$ and $v_1 = y_2$. But then again we have that $\deg_G v_1 \geq 5$, a contradiction.

(2.2) Next suppose that $A = \{x_1, y_1\}$. By symmetry, there is only one case we have not yet treated. Suppose $x_1 = x_2$ and $y_1 = y_2$. But then we have a butterfly.

(2.3) So next we suppose that $A = \{u_1, v_1\}$. By symmetry, there remains only one untreated case. Suppose $u_1 = u_2$ and $v_1 = v_2$. But then $\deg_G u_4 \geq 5$, a contradiction.

(3) Finally, suppose $|A| = 3$.

(3.1) First, suppose $A = \{x_1, u_1, v_1\}$. But by symmetry, this can happen in essentially only two different ways.

(3.1.1) Suppose first that $A = \{x_2, u_2, v_2\}$.

(3.1.1.1) If $x_1 = x_2$, $u_1 = u_2$ and $v_1 = v_2$, we get a separating triangle by planarity, a contradiction of 4-connectedness.

(3.1.1.2) On the other hand, if $x_1 = u_2$, $u_1 = x_2$, and $v_1 = v_2$, then we get a butterfly.

(3.1.2) So suppose $A = \{x_2, y_2, u_2\}$. But this too can happen in essentially only two different ways.

(3.1.2.1) If $x_1 = x_2$, $u_1 = y_2$ and $v_1 = u_2$, we get a butterfly.

(3.1.2.2) On the other hand, if $x_1 = u_2$, $u_1 = x_2$ and $v_1 = y_2$, then we also get a butterfly.

(3.2) So suppose $A = \{x_1, u_1, y_1\} = \{x_2, u_2, y_2\}$. Once again, we employ symmetry to point out that this can happen in only two fundamentally different ways.

(3.2.1) Suppose $x_1 = x_2$, $u_1 = u_2$ and $y_1 = y_2$. We then get a butterfly.

(3.2.2) Finally, suppose that $x_1 = u_2$, $u_1 = x_2$ and $y_1 = y_2$. Yet again we obtain a butterfly and the proof of the Lemma is complete. ∎

Now we are prepared to state and prove the final result of this paper.

Theorem 4. Let G be a butterfly-free 4-connected 4-regular planar graph. If every vertex lies in a subgraph isomorphic to a JT and if the four endvertices of no two independent edges separate G into two odd components, then G is 2-extendable.

Proof. Suppose G is not 2-extendable. Then there are two independent edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ which do not lie in any perfect matching of G. Let S and N be as in Lemma 1. However, this time among all such sets S, choose one of *minimum cardinality*. Again, let $C_1, \ldots, C_{|S|-2}$ be the odd components of $G - S$. Let $w_1, \ldots, w_4$ be as before as well.

If there are exactly four edges joining one of the C_i's to S, and C_i is not a singleton, by 4-regularity C_i has at least five vertices and the four edges from C_i to S must be independent. By hypothesis, every vertex in C_i lies in a JT which must therefore lie wholly within C_i. But since G contains no butterfly, each pair of these JT's must be vertex disjoint and hence component C_i is even, a contradiction. So if exactly four edges join a C_i to S, that particular C_i must be a singleton.

Let G'' be the graph resulting from G by contracting all non-singleton components of $G - S$ to single vertices. Exactly as in the proof of Theorem 3, we obtain the facts (a), (b), (c) and (d) listed there for graph G''. Also as in the proof of Theorem 3, there are three cases to consider.

Case 1. $N = 4|S| - 4$.

(1.1) Suppose first that there are eight edges from S to C_1 and so there are exactly four edges from S to each C_j, for $j = 2, \ldots, |S| - 2$. Hence each C_j, for $j = 2, \ldots, |S| - 2$ must be a singleton and e_1 and e_2 are the only edges in $G[S]$.

Contracting C_1, we obtain graph G'' which has $f_2'' = 4$ by (d) and so there are exactly four vertices in S adjacent to vertices of C_1. Let $X_1 = \{x_1, x_2, x_3, x_4\}$ be this set of four vertices in S.

If there is a vertex v in $S - \{x_1, \ldots, x_4, u_1, v_1, u_2, v_2\}$, then v does not lie on any triangle in G and hence is not in any JT in G, contrary to hypothesis. So no such v exists and hence $S = \{x_1, x_2, x_3, x_4\} \cup \{u_1, v_1, u_2, v_2\}$.

If there is an odd component C_i different from C_1, w_1, w_2, w_3 and w_4, it too cannot lie in any JT, again contrary to hypothesis. So no such odd components exist. Hence $G - S$ has at most five odd components and therefore $|S| \leq 7$.

Let $U = \{u_1, v_1, u_2, v_2\}$. Suppose there is an x_i in $X_1 - U$ from which there is just one edge to C_1. Then x_i cannot lie in any triangle and hence in any JT, contrary to hypothesis. If there is an $x_i \in X_1 - U$ from which there are three edges to C_1, the fourth edge from x_i must go to some C_j, where $j \neq 1$. But then $C_1' = G[V(C_1) \cup \{x_i\} \cup V(C_j)]$ has an odd number of vertices and thus $S'' = S - x_i$ is a smaller set than S, $o(G - S'') = |S''| - 2$ and e_1 and e_2 lie in S''. This contradicts the minimality of S. Thus any x_i in $X_1 - U$ has an *even* number of edges joining it to C_1 (i.e., either two or four). But none can send four edges to C_1, for then the remaining three x_i's would be a cutset in G, contradicting 4-connectedness. Thus any $x_i \in X_1 - U$ sends *exactly two* edges to C_1.

(1.1.1) Suppose $|S| = 7$.

Then without loss of generality we may assume that $x_1 = u_1$. Suppose x_1 is adjacent to C_1. If x_1 sends exactly one edge to C_1, then some $x_i, i = 2, 3, 4$ must send three edges to C_1, a contradiction. So x_1 sends two edges to C_1 and hence the degree of x_1 is at least five, a contradiction.

(1.1.2) Suppose $|S| = 6$.

Without loss of generality, assume $X_1 - U = \{x_3, x_4\}$ and also that w_1 is in C_1. So $\{x_1, x_2\} = \{u_1, v_1\}$. But since x_3 and x_4 each send exactly two edges to C_1, x_1 and x_2 send two each also. Thus $\{x_3, x_4, w_2\}$ is a cutset in G, a contradiction.

(1.1.3) Suppose $|S| = 5$.

Without loss of generality, assume $X_1 - U = \{x_4\}$. Then also without loss of generality, assume $x_1 = u_1, x_2 = v_1$ and $x_3 = u_2$.

Since v_2 is not adjacent with any vertex in C_1, it must be that $\{w_3, w_4\} \cap V(C_1) = \emptyset$. Thus we may assume that $C_2 = w_3$ and $C_3 = w_4$. Hence $\{w_1, w_2\} \subseteq V(C_1)$. Since $\deg x_3 = 4$, there is exactly one edge from x_3 to C_1. Hence one of x_1 and x_2 sends three edges to C_1 and the other sends two. But then counting edges from S to $G - S$, we have that v_2 must send parallel edges to one of w_3 or w_4 and this contradicts the assumption that G has no digons.

(1.1.4) Suppose $|S| = 4$.

Assume, without loss of generality, that $w_4 \notin V(C_1)$. But since deg $w_4 = 4$, it is adjacent to both x_1 and x_2. But then we have a butterfly and a contradiction.

(1.2) Suppose there are six edges from C_1 to S and six from C_2 to S. Hence there are exactly four from S to each of the C_j, for $j = 3, \dots, |S|-2$. But then each of $C_3, \dots, C_{|S|-2}$ must be a singleton.

Contracting C_1 and C_2, we obtain a graph G'' in which, by (d), (c) and (a) respectively, we have $f''_2 = 4, f''_3 = 4$ and $f''_j = 0$ for $j \geq 5$.

Hence by 4-connectedness, each of $\hat{C}_1$ and $\hat{C}_2$ is incident with exactly two digons in G'' and hence each of C_1 and C_2 is joined to exactly four vertices of S. For $i = 1, 2$, denote the neighbors of C_i in S by $X_i = \{x_{4i-3}, x_{4i-2}, x_{4i-1}, x_{4i}\}$. (Note that X_1 and X_2 are not necessarily disjoint.) Let $X = X_1 \cup X_2$. Finally, let $X' = X - U$. As in Case (1.1), if there is a vertex v in $S - \{x_1, \dots, x_8, u_1, v_1, u_2, v_2\}$, it cannot lie on a triangle and we have a contradiction. Also as in Case (1.1), there can be no odd component of $G - S$ different from $C_1, C_2, w_1, w_2, w_3, w_4$.

Hence $o(G - S) \leq 6$ and therefore $|S| \leq 8$.

If there is a vertex v in $S - \{u_1, v_1, u_2, v_2\}$ from which there is at most one edge to each of C_1 and C_2, then v cannot lie in a triangle and again we have a contradiction. In particular, then, $S = X \cup U$. If there is a vertex v in $S - \{u_1, v_1, u_2, v_2\}$ with three edges to C_1 or C_2–without loss of generality, say C_1–then v is adjacent to only one other C_i. So it follows that $C'_1 = G[V(C_1) \cup \{v\} \cup V(C_i)]$ is an odd component of $G - S'$, where $S' = S - \{v\}$. This contradicts the minimality of S.

Also for every vertex $v \in S - \{u_1, v_1, u_2, v_2\}$, if v is joined to C_1 by four edges, then there must be a cutset of size three, a contradiction. Similarly for C_2. So for every vertex v in $S - \{u_1, v_1, u_2, v_2\}$, if it is joined to C_1 at all, it must be by *exactly two* edges. Similarly for C_2.

(1.2.1) Suppose $|S| = 8$. (So $|X'| = 4$.)

By the symmetry between C_1 and C_2, we need only consider the following three cases.

First, suppose that $|X' \cap X_1| = 4$, that is, $X' = \{x_1, x_2, x_3, x_4\}$. But then by the remark above, there must be eight edges from C_1 to S, a contradiction.

Next, suppose that $|X' \cap X_1| = 3$; so without loss of generality, we may assume $X' = \{x_1, x_2, x_3, x_5\}$. Then each of x_1, x_2 and x_3 is joined to C_1 by two edges and hence $\{x_1, x_2, x_3\}$ is a 3-cutset in G, a contradiction.

Finally, suppose $|X' \cap X_1| = 2$; so without loss of generality we may suppose $X' = \{x_1, x_2, x_5, x_6\}$.

Now each of x_1 and x_2 are joined by exactly two edges to C_1. If the fifth and sixth edges joining C_1 to S are adjacent (in S or in C_1), we can find a 3-cut for G containing x_1, x_2 and this vertex of adjacency. So we have a contradiction. Hence the fifth and sixth edges from C_1 to S are independent. Thus at most two different JT's join vertices of C_1 to S.

If x_1 is joined to C_2, it must have exactly two edges to C_2. Hence $\{x_1, x_5, x_6\}$ is a

3-cut in G, again a contradiction. Thus x_1 is not joined to C_2. By symmetry, x_2 is joined to no vertex of C_2 as well (and neither of x_5 and x_6 is joined to any vertex of C_1).

Now, and henceforth, let us denote by $JT(v)$ the JT covering vertex v, for all $v \in V(G)$.

Suppose $JT(x_1) = JT(x_2)$. Then $JT(x_1)$ covers exactly two vertices in C_1 and all other JT's covering vertices of C_1 lie entirely in C_1. Thus C_1 is even, contradicting (b).

So, by Lemma 3, we may suppose $JT(x_1)$ and $JT(x_2)$ are vertex disjoint. But then each must cover exactly three vertices in C_1 and together they cover six vertices in C_1. Thus again C_1 is even and again we have a contradiction.

Note that if $|X' \cap X_1| = 1$, then $|X' \cap X_2| = 3$, and if $|X' \cap X_1| = 0$, then $|X' \cap X_2| = 4$ and we repeat the above arguments on X_2 and C_2 in place of X_1 and C_1.

(1.2.2) Suppose $|S| = 7$ and hence $|X'| = 3$.

First suppose that $|X' \cap X_1| = 3$. Without loss of generality, assume that $X' \cap X_1 = \{x_1, x_2, x_3\}$. But then each of these three vertices sends two edges to C_1 and hence they form a 3-cut of G, a contradiction.

Now suppose that $|X' \cap X_1| = 2$. Without loss of generality, assume that $|X' \cap X_1| = \{x_1, x_2\}$. Since C_1 sends exactly six edges to S and since G is 4-connected, it follows that both x_3 and x_4 are in U. As in Case (1.2.1), the fifth and sixth edges from C_1 to S must be independent.

Let the one vertex of $X' - (X_1 \cup U)$ be x_8, since it must be a neighbor of C_2 and not a neighbor of C_1. So x_8 is adjacent to exactly two vertices in C_2, none in C_1, and hence to two of the singleton odd components C_3, C_4 and C_5. Say, without loss of generality, that x_8 is adjacent to C_3 and C_4.

Suppose both x_1 and x_2 are adjacent to C_2. Then $\{x_1, x_2, x_8\}$ is a 3-cut in G, a contradiction. So at most one of x_1 and x_2 is adjacent to C_2. Without loss of generality, assume that x_1 is not adjacent to C_2.

First assume that x_2 is not adjacent to C_2 either.

Now if $JT(x_1) = JT(x_2)$, then each joins C_1 to X and as before, no other JT can join C_1 to S. So $|V(C_1) \cap V(JT(x_1)) \cap V(JT(x_2))| = 2$ and again it follows that C_1 is even, a contradiction.

So we may assume that $JT(x_1)$ and $JT(x_2)$) are vertex disjoint. So they jointly cover six vertices of C_1 and once more C_1 is even, a contradiction.

So suppose that x_1 is not adjacent to C_2, but that x_2 *is* adjacent to C_2. But now x_2 is adjacent to both C_1 and C_2 by two edges to each. Thus $G[C_1 \cup C_2 \cup \{x_2\}]$ is an odd component of $G - (S - x_2)$ and hence $G - (S - x_2)$ has $|S| - 3 = |S - x_2| - 2$ odd components, contradicting the minimality of S.

Next suppose that $|X' \cap X_1| = 1$. But then $|X' \cap X_2| = 2$ and we proceed as in the above case for $|X' \cap X_1| = 2$, except we replace X_1 with X_2 and interchange the roles of C_1 and C_2 in that argument.

Finally, if $|X' \cap X_1| = 0$, it follows that $|X' \cap X_2| = 3$ and hence that $X' \cap X_2$ is a 3-cut, a contradiction.

(1.2.3) Suppose $|S| = 6$ and so $|X'| = 2$.

(1.2.3.1) First suppose that $|X' \cap X_1| = 2$. Let $X' \cap X_1 = \{x_1, x_2\}$. As before, each of x_1 and x_2 sends two edges to C_1. Suppose x_1 is adjacent to C_2. Then, $G[C_1 \cup C_2 \cup \{x_1\}]$

is an odd component of $G - (S - x_1)$ and this contradicts the minimality of S. So assume that x_1 is not adjacent to C_2 and by symmetry, that x_2 is not adjacent to C_2 as well. Thus $JT(x_1)$ has three vertices in C_1 as does $JT(x_2)$. But then C_1 is even, a contradiction.

(1.2.3.2) Next, suppose $|X' \cap X_1| = 1$. Denote $X' \cap X_1$ by $\{x_1\}$.

Since $|X' \cap X_2| = 1$, denote $X' \cap X_2$ by $\{x_8\}$. As before, x_1 sends exactly two edges to C_1 and x_8 sends exactly two edges to C_2.

Now $\{x_2, x_3, x_4\} \subseteq U$. Without loss of generality, assume that there are two edges from C_1 to x_2 and one each from C_1 to x_3 and x_4. As before, by 4-connectedness, the two edges to x_3 and x_4 must be independent. Also we now know that x_8 is not adjacent to C_1 and so x_8 is adjacent to both C_3 and C_4.

By symmetry, at this point there are essentially two different ways we can have edges e_1 and e_2 in U. First, without loss of generality, assume $x_2 = u_1$. Then, again without loss of generality, we need only treat two subcases.

(1.2.3.2.1) Suppose $v_1 = x_3$.

Without loss of generality, let $u_2 = x_4$. Now each of C_3 and C_4 lies on a JT. Of course, again by Lemma 3, they are the same or vertex disjoint. Moreover, each of these JT's must use one of e_1 and e_2.

(1.2.3.2.1.1) Suppose $JT(C_3) = JT(C_4)$.

Then $JT(C_3)$ cannot use edge e_1 since $\deg_G x_2 = 4$, so we may assume it uses e_2. Then the fourth edge from v_2 must go to C_2. Now since all edges incident with x_4, v_2 and x_8 are accounted for, there must be three edges from C_2 to $\{x_1, x_2, x_3\}$. But then there is a homeomorph of $K_{3,3}$ in G'' with sets of principal vertices $\{x_4, v_2, x_8\}$ and $\{\hat{C}_2, C_3, C_4\}$, a contradiction.

(1.2.3.2.1.2) So suppose $JT(C_3)$ and $JT(C_4)$ are vertex disjoint. But each uses one of e_1 and e_2. Without loss of generality, suppose $JT(C_3)$ uses e_1 and $JT(C_4)$ uses e_2. Thus u_1 and v_1 are adjacent to some common vertex $y_1 \in V(C_1)$, since $\deg u_1 = 4$. Moreover, C_4 is adjacent to u_2 and v_2. Now $JT(x_1)$ is vertex disjoint from $JT(x_2) = JT(C_3)$, so $JT(x_1)$ has exactly three vertices or no vertices in component C_1. If it has three vertices in C_1, then it follows that C_1 is even, a contradiction.

So $JT(x_1)$ has no vertices in C_1 and hence either two or three vertices in C_2.

(1.2.3.2.1.2.1) Suppose $JT(x_1)$ has exactly two vertices in C_2. Then the fourth vertex of $JT(x_1)$ must be x_8. But since C_2 is odd, we must have $JT(C_4)$ containing one vertex of C_2; call it y_2. But then $\{x_1, x_8, y_2\}$ is a 3-cut in G, a contradiction.

(1.2.3.2.1.2.2) So suppose that $JT(x_1)$ has exactly three vertices in C_2. So $JT(C_4)$ must use exactly one vertex y_2 of C_2. But then again $\{x_1, x_8, y_2\}$ is a 3-cut in G, a contradiction.

(1.2.3.2.2) So suppose that $v_1 \notin \{x_3, x_4\}$. So $\{x_3, x_4\} = \{u_2, v_2\}$; without loss of generality, suppose $x_3 = u_2$ and $x_4 = v_2$. Without loss of generality, we may assume that $JT(C_3)$ uses edge e_1. But then $\deg x_2 = 4$ implies that $JT(C_3)$ meets C_1. But that is impossible, since v_1 is not adjacent to any vertex in C_1.

(1.2.3.3) Suppose $|X' \cap X_1| = 0$. Then $|X' \cap X_2| = 2$. So we proceed as in Case (1.2.3.1), except we interchange the roles of X_1 and X_2 and those of C_1 and C_2.

(1.2.4) So suppose $|S| = 5$. Thus $|X'| = 1$.

Without loss or generality, suppose $X' = \{x_1\}$. So as before, we have exactly two edges from x_1 to C_1. Suppose x_1 is adjacent to C_2 and hence to exactly two vertices in C_2. Then $S' = S - \{x_1\}$ has the property that $G - S'$ has two odd components (one of which is $G[V(C_1) \cup V(C_2) \cup \{x_1\}]$ and the other is C_3). So $G - S'$ has $|S'| - 2$ odd components and $\{e_1, e_2\} \subseteq E(G[S'])$. Thus once again we contradict the minimality of the choice of set S.

(1.2.5) Suppose $|S| = 4$. So $X' = \emptyset$ and $S = U$. But then the endvertices of e_1 and e_2 separate G into two odd components, a contradiction.

This completes Case 1.

Case 2. $N = 4|S| - 6$.

We may assume that there are six edges from S to C_1 and exactly four edges from S to each of $C_2, C_3, \ldots, C_{|S|-2}$. So each of $C_2, \ldots, C_{|S|-2}$ is a singleton. Also there are exactly three edges $e_i = u_i v_i$, $i = 1, 2, 3$ in $G[S]$. Let $U = \{u_1, u_2, u_3, v_1, v_2, v_3\}$.

Upon contracting component C_1 to a single vertex we obtain graph G'' in which $f_2'' = 2, f_3'' = 6$ and $f_j'' = 0$ for $j \geq 5$, by (d), (c) and (a) respectively.

Since $f_2'' = 2$, there are exactly four vertices of attachment for C_1 in S. Again denote them by x_1, x_2, x_3 and x_4. Let $w_1, \ldots, w_6$ be as in the proof of Case 2 of Theorem 3.

Let $v \in S - U$. Since v must lie on a triangle in G, v must be adjacent to at least two vertices of C_1. If v is adjacent to three vertices in C_1, it is adjacent to precisely one of $C_2, \ldots, C_{|S|-2}$. Suppose it is C_i. Then, if $C_1' = G[V(C_1) \cup V(C_i) \cup \{v\}]$, then if $S' = S - \{v\}$, set S' contains edges e_1, e_2 and e_3, graph $G - S'$ has $|S| - 3 = |S'| - 2$ odd components (one of which is C_1') and this contradicts the minimality of S.

If v is adjacent to four vertices in C_1, then there must be a 3-cut in G separating C_1 from the rest of G and this is a contradiction of the 4-connectedness of G.

Hence, if v is any vertex in $S - U$ which is adjacent to C_1, it must send *exactly* two edges to C_1.

Also, since $f_3'' = 6$, if there is any singleton odd component C_i different from $w_1, \ldots, w_6$, it cannot lie on a JT. So it follows that $o(G - S) \leq 7$ and hence that $|S| \leq 9$. Also since $f_3'' = 6$, each of e_1, e_2 and e_3 lies on exactly two triangles. But then by 4-regularity, it follows that these three edges are vertex disjoint and thus that $|S| \geq 6$.

(2.1) Suppose $|S| = 9$.

Now since every vertex of $S - U$ lies on a triangle, it is adjacent to C_1 and, therefore, by the above remark, it sends exactly two edges to component C_1. But then $S - U$ is a 3-cut in G, a contradiction.

(2.2) Suppose $|S| = 8$.

Without loss of generality, assume that $S - U = \{x_1, x_2\}$. Since each of the singleton odd components $C_2, \ldots, C_{|S|-2}$ must lie on a triangle in G'' and each of these triangles must contain one of the edges e_i, we may assume without loss of generality that the two triangles containing e_1 also contain vertices $\hat{C}_1$ and C_2, the two containing edge e_2 contain C_3 and C_4 and the two containing edge e_3 contain C_5 and C_6.

But now back in the parent graph G, vertices C_3, C_4, C_5 and C_6 lie in *unique* JT's which must be spanned by C_3, C_4 and the two ends of edge e_1 and by C_5, C_6 and the two ends of edge e_2 respectively. Also C_2 lies in a unique triangle consisting of C_2 and the two ends of edge e_1. But then C_2 lies in a unique JT which must use the two edges to C_1 which are not incident with vertices x_1 and x_2. Call these two edges f_1 and f_2. But

Figure 6

then f_1 and f_2 have a common endvertex y in C_1. But then $\{x_1, x_2, y\}$ is a 3-cut in G, a contradiction.

(2.3) Suppose $|S| = 7$.

Then at most two $JT(C_i)$'s ($i \geq 2$) send edges to C_1, since the total number of edges into C_1 from S is six. Suppose that two $JT(C_i)$'s ($i \geq 2$) send edges into C_1. Then G has a 3-cut consisting of x_1 and two vertices in C_1.

Suppose next that exactly one $JT(C_i)$ ($i \geq 2$)–say $JT(C_2)$–has a vertex in C_1. Then relabeling if necessary, we may assume that $JT(C_4) = JT(C_5)$ uses edge e_3 and then odd component C_3 lies in no JT, a contradiction.

So suppose that no $JT(C_i)$ ($i \geq 2$) has vertices in C_1. Without loss of generality, we may suppose that $JT(C_2) = JT(C_3)$ uses edge e_2 and that $JT(C_4) = JT(C_5)$ uses edge e_3. Then the JT using edge e_1 has exactly two vertices in C_1; call them α and β. But then $\{x_1, \alpha, \beta\}$ is a 3-cut in G, a contradiction.

(2.4) Suppose $|S| = 6$.

Since the $JT(C_i)$, $i \geq 2$, must be vertex disjoint, at most one of them uses two of C_2, C_3 and C_4.

First suppose exactly one $JT(C_i)$ uses two of C_2, C_3 and C_4. Relabeling, if necessary, we may assume that $JT(C_3) = JT(C_4)$ and that $JT(C_3)$ uses edge e_3. Then $JT(C_2)$ uses edge e_2 say, and one vertex y_1 of C_1.

Now consider $JT(u_1)$ and $JT(v_1)$.

First suppose that $JT(u_1) = JT(v_1)$. Also suppose first that $JT(u_1)$ uses edge e_1. Then $\{u_1, v_1, y_1\}$ is a 3-cut in G. So suppose that $JT(u_1)$ does not use edge e_1. Then $JT(u_1)$ uses exactly two vertices in C_1 and again $\{u_1, v_1, y_1\}$ is a 3-cut in G.

So suppose that $JT(u_1)$ and $JT(v_1)$ are vertex disjoint. Then neither uses edge e_1 and so together they use six distinct vertices in C_1. But then again $\{u_1, v_1, y_1\}$ is a 3-cut in G.

So suppose that no $JT(C_i)$, ($i \geq 2$), uses two of the vertices C_2, C_3 and C_4. Then $JT(C_2), JT(C_3)$ and $JT(C_4)$ are vertex disjoint and each uses a different vertex of C_1, say C_i uses y_i, for $i = 2, 3, 4$. But then either $\{y_2, y_3, y_4\}$ is a 3-cut in G, which is impossible, or $|V(C_1)| = 3$. But if C_1 has only three vertices, no vertex in C_1 can be covered by a JT, a contradiction.

Case 3. $N = 4|S| - 8$.

Note that there are exactly four edges from each of $C_1, \ldots, C_{|S|-2}$ to S and $G[S]$ contains four edges $e_i = u_i v_i$ for $i = 1, \ldots, 4$. So all of $C_1, \ldots, C_{|S|-2}$ are singletons.

Note that by (c) and (a) respectively, we have $f_3 = 8$ and $f_j = 0$, $j \geq 5$. Hence each e_i lies in exactly two triangles in G.

(3.1) Suppose two of the e_i's share a vertex; without loss of generality suppose $e_1 = ab$ and $e_2 = bc$.

(3.1.1) Suppose also that a is adjacent to c, say $ac = e_3$. So e_1 lies on triangle $abca$ and one other triangle which uses one of the C_i's–say C_1. If c is adjacent to C_1, then acC_1a must be a separating triangle in G, a contradiction. So we may assume that c is not adjacent to C_1. So let the second triangle using e_2 be abC_2a, where $C_2 \neq C_1$. But then $\{a, b, c, C_1, C_2\}$ must span a butterfly, contrary to hypothesis.

So, by symmetry, no three of the e_i's can form a triangle.

(3.1.2) So assume that a is not adjacent to c. Let the two triangles using edge e_1 be abC_1a and abC_2a. Since G contains no butterfly, we have that c is adjacent to neither C_1 nor C_2. But then since $\deg_G b = 4$ and $N(b) = \{a, c, C_1, C_2\}$, edge e_2 cannot lie on a triangle in G, a contradiction.

(3.2) So we may assume that no two e_i's share a vertex; that is, $\{e_1, e_2, e_3, e_4\}$ are vertex disjoint. Since every vertex of G must lie on a triangle, if follows that every vertex of S must be an end vertex of one of the e_i's. Thus $|S| = 8$ and hence $o(G - S) = 6$.

Now all triangles–and hence all JT's–in G each must use exactly one of the e_i's and hence two of the singleton odd components $C_1, \ldots, C_6$. But this is clearly impossible and the proof of the theorem is complete. ∎

5. Concluding Remarks

Let us close by offering a few remarks as to the sharpness of the results in this paper.

Remark 1. According to [5], there are non-2-extendable k-connected k-regular graphs, for $k = 3, 4, 5$, with cyclic edge connectivity arbitrarily large. So in this sense, planarity is necessary in the hypotheses of Theorems 1, 2 and 3. Figure 7 shows a non-planar graph in which e_1 and e_2 cannot be extended to a perfect matching, which shows that Theorem 4 also requires planarity in the hypothesis.

Remark 2. Figure 8 shows a cyclically 6-edge-connected butterfly-free non-2-extendable 4-connected 4-regular planar graph in which each cyclic edge cutset has size greater than six, except the edges incident with a triangle and the edges incident with a JT. Hence this graph shows the sharpness of Theorem 3 with respect to the cyclic edge connectivity assumption in the hypothesis.

Remark 3. Figure 8 also shows the sharpness of Theorem 4 with respect to the JT covering assumption, as every vertex in the graph lies in a JT, with the exception of exactly two.

Remark 4. Figure 9 shows how to build a cyclically 4-edge-connected non-2-extendable planar graph which consists of disjoint triangles and some other edges. Substituting the graph in Figure 9(b) for each of C_1 and C_2 in Figure 9(a) by identifying edges as shown, we get a non-2-extendable 4-connected 4-regular planar graph in which edges e_1 and e_2 do not lie in any perfect matching. So in the hypothesis of Theorem 4 we cannot change the demand that G be vertex partitionable into JT's to say instead that G be vertex partitionable into triangles.

Remark 5. Figure 6 shows a butterfly-free 4-connected 4-regular planar graph in which every vertex lies in a subgraph isomorphic to a JT. However, the four endvertices of edges e_1 and e_2 separate G into two odd components and hence e_1 and e_2 lie in no perfect matching in G. This graph shows, in particular, that the last hypothesis in Theorem 4 is not derivable from the others.

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Figure 7

Figure 8

Figure 9

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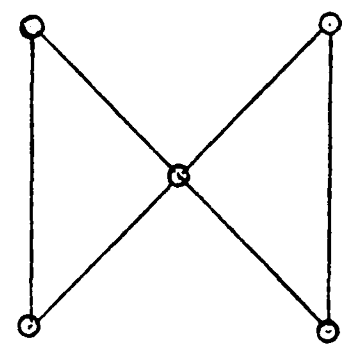


Figure 1

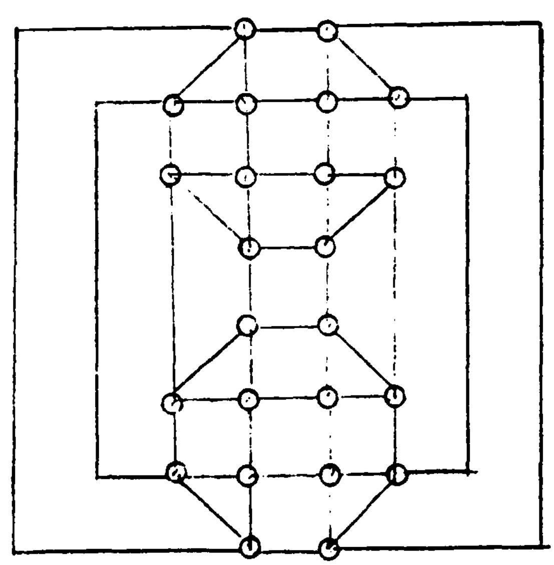
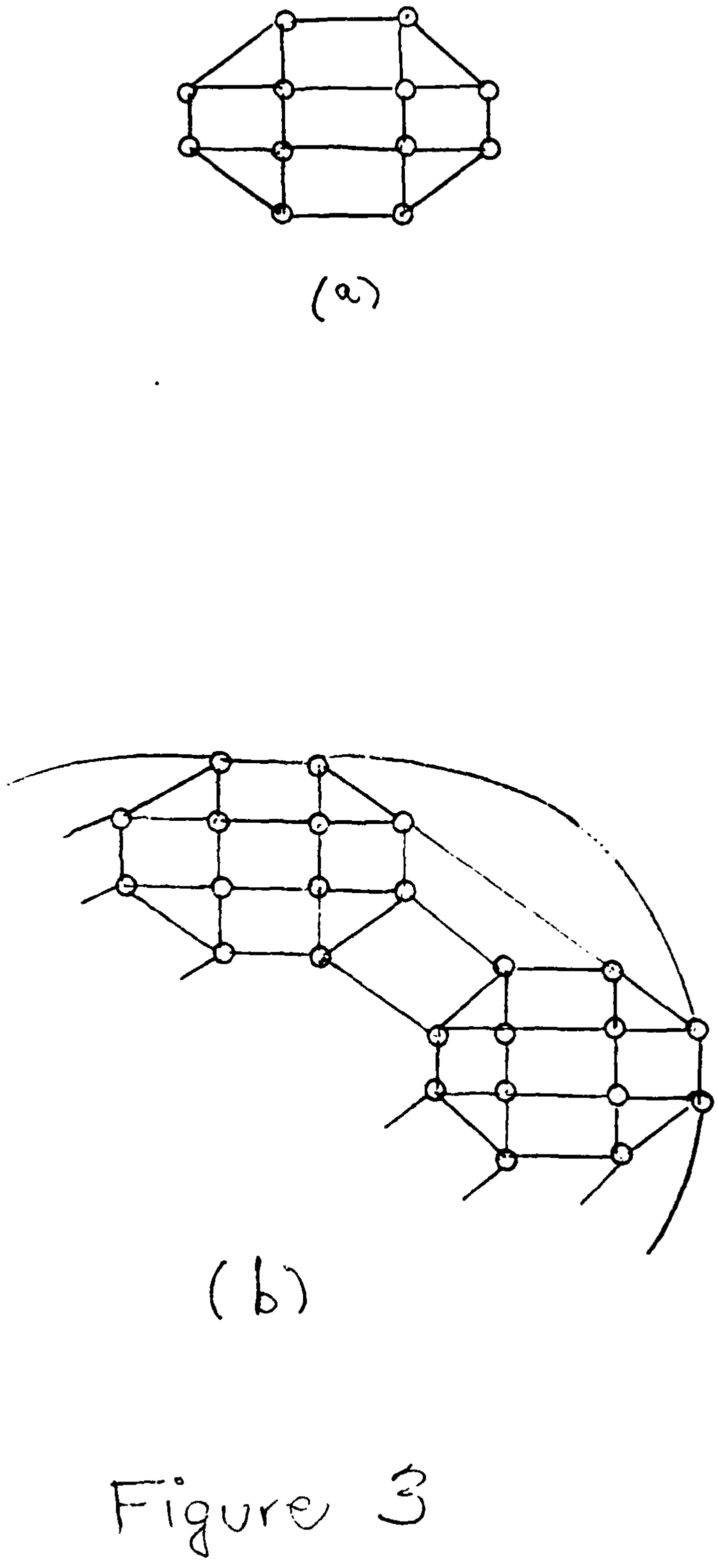


Figure 2



(a)

(b)

Figure 3

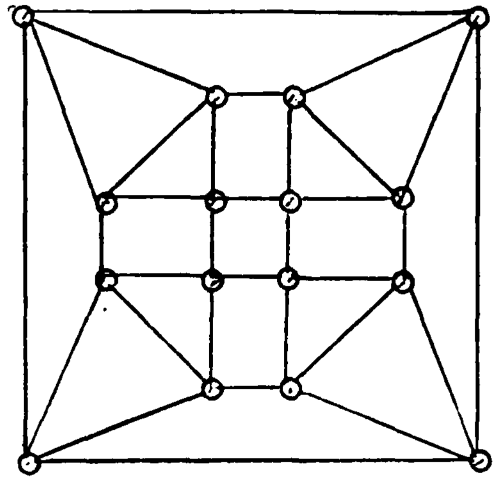


Figure 4

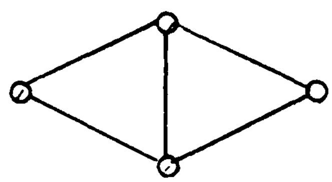


Figure 5

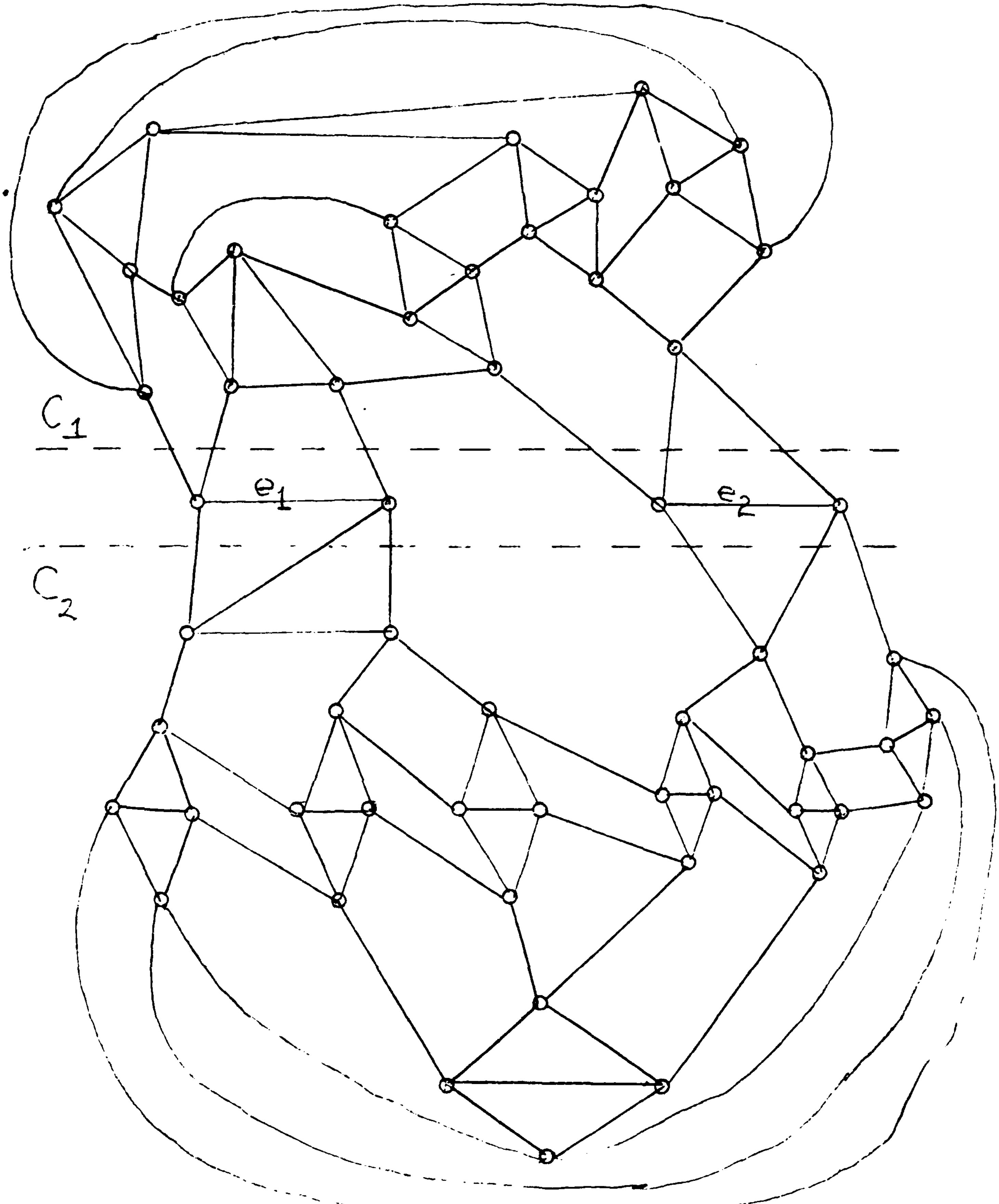


Figure 6

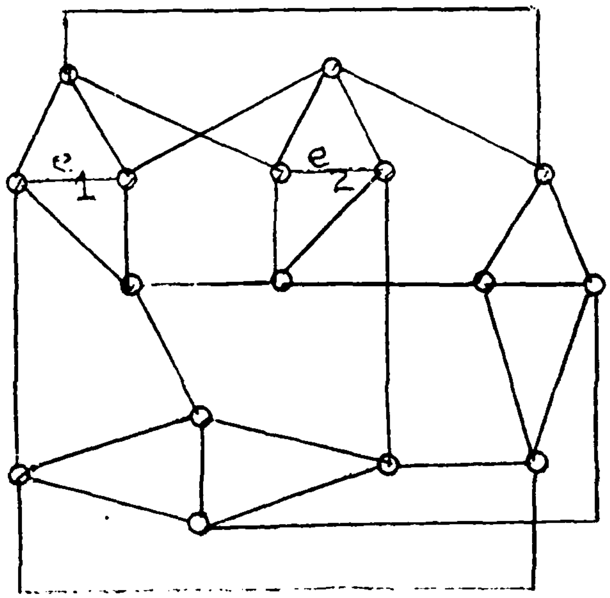


Figure 7

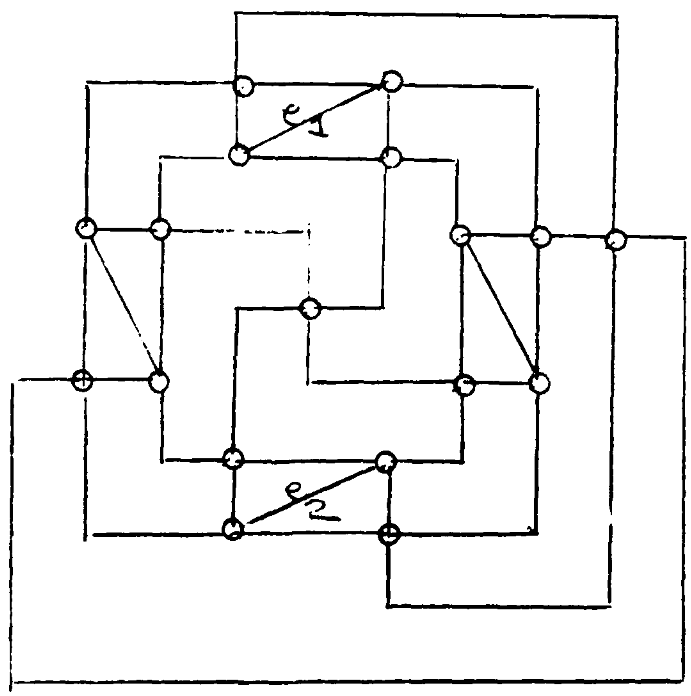
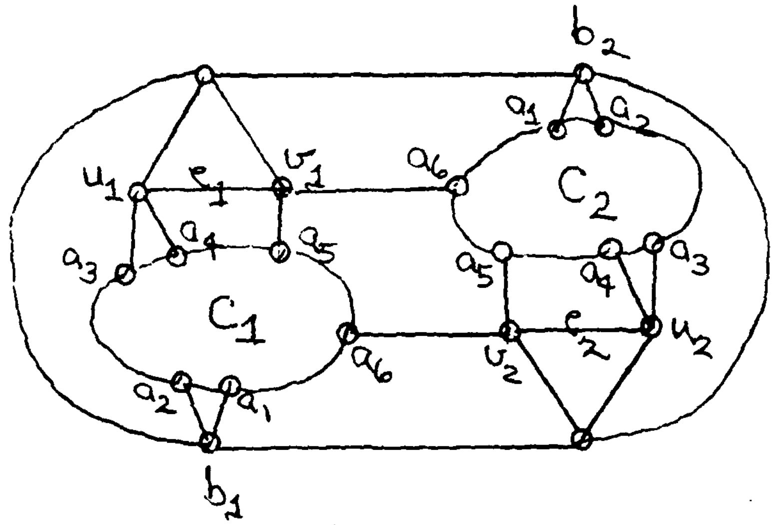
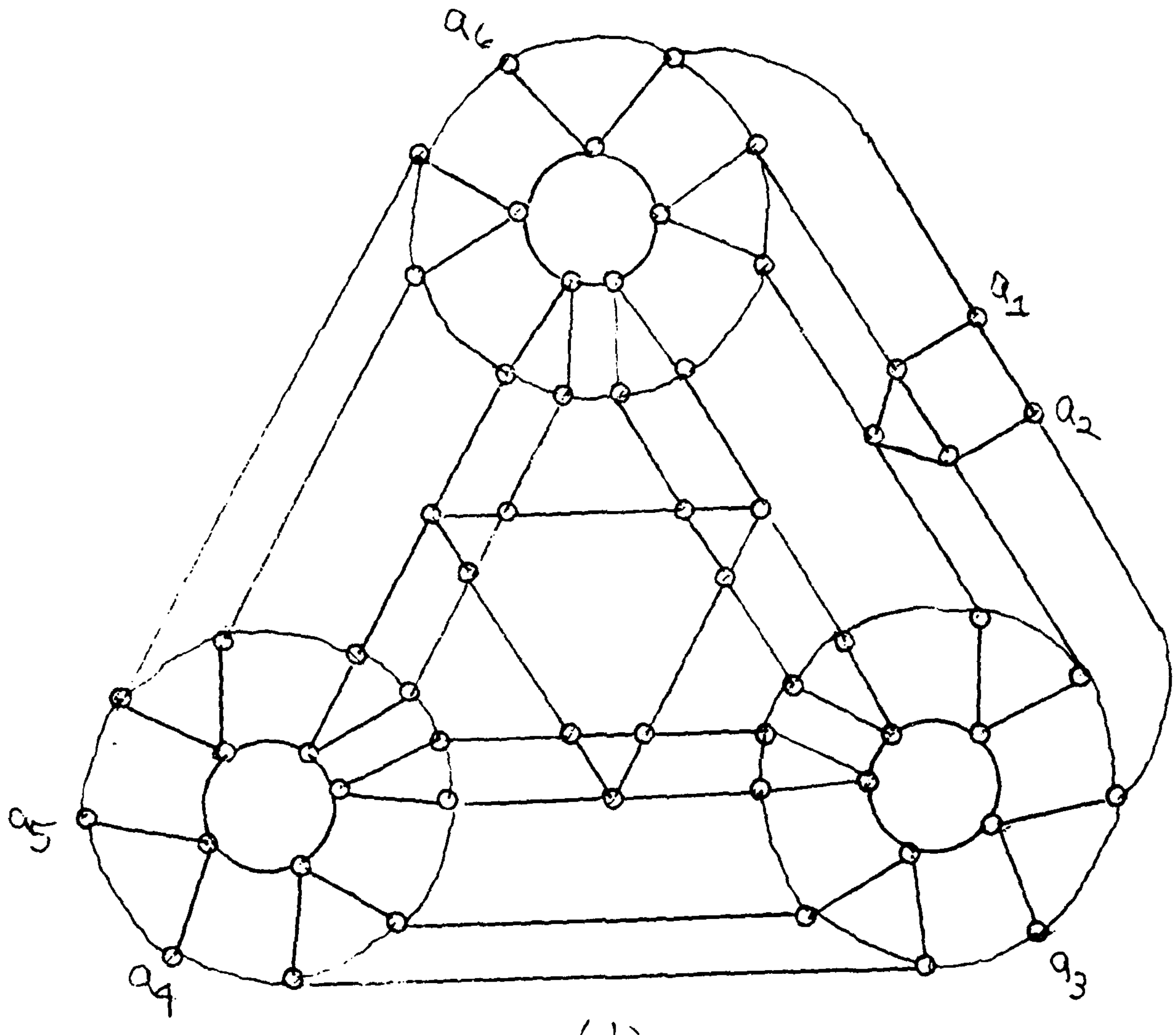


Figure 8



b_2
a_1
a_2
u_1
e_1
v_1
a_6
C_2
a_4
a_5
a_3
C_1
a_6
a_5
a_4
a_3
e_2
v_2
u_2
a_2
a_1
b_1

(a)



a_6
a_1
a_2
a_5
a_4
a_3

(b)

Figure 9