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THE PRINCIPAL PIVOTING METHOD REVISITED

by Richard W. COTTLE

ABSTRACT

The Principal Pivoting Method (PPM) for the Linear Complementarity Problem (LCP) is shown to be applicable to the class of LCPs involving the newly identified class of sufficient matrices.

1. Background

The classes of row sufficient and column sufficient matrices were recently introduced in a paper by Cottle, Pang, and Venkateswaran [6]. It was shown there that such matrices provide answers to natural theoretical questions concerning the linear complementarity problem (LCP). Further, on the algorithmic side, it was noted that Lemke's Method (Scheme 1) [9] for the LCP can "process" any problem in which the matrix is row sufficient. In fact, by a theorem of Aganagić and Cottle [1], the latter is true for any $\mathbf{Q}_0$-matrix having nonnegative principal minors, and row sufficient matrices are of this sort. These observations prompt one to ask whether the principal pivoting method (PPM) [3], [7], [5], [3] is also applicable to this class of LCPs. This question is especially relevant inasmuch as the kinds of matrices that the principal pivoting method *can* handle have heretofore been limited to **P**-matrices and positive semi-definite (**PSD**-) matrices, both of which types are row sufficient as well as column sufficient. Thus, a demonstration that the PPM can process LCPs with row sufficient matrices amounts to a unification of the existing theory of the PPM and an extension of its scope. Such is the main goal of the present paper.

Let us begin by fixing notation and reviewing some terminology. Given a column vector $q \in R^n$ and a matrix $M \in R^{n \times n}$, the pair (q, M) specifies a linear complementarity problem (of order n) through the system

$$
\begin{aligned}
z &\geq 0, & (1)\\
q + Mz &\geq 0, & (2)\\
z^{\mathrm{T}}(q + Mz) &= 0. & (3)
\end{aligned}
$$

An alternate formulation of (q, M) is to find vectors w, z satisfying

$$
w = q + Mz, \qquad w \geq 0,\ z \geq 0, \qquad z^{\mathrm{T}}w = 0.
$$

This system involves n pairs of *complementary variables* w_i and z_i for $i = 1, \ldots, n$. The members of a complementary pair of variables are said to be *complements* of each other.

To *process* the linear complementarity problem (q, M) means to obtain a solution (i.e., a vector z satisfying (1) – (3)) or to demonstrate that the problem has no solution. Discussions of the processing capabilities of various linear complementarity algorithms usually focus on properties of matrix classes. For instance, it is known that the principal pivoting method will process any LCP (q, M) when M is a **P**-matrix (i.e., has positive principal minors) or when M is **PSD**, (i.e., $x^TMx \geq 0$ for all x). In the former case, (q, M) must always have a unique solution—regardless of what q equals—and the PPM will find it. In the latter case, the LCP will always have a solution provided the constraints (1) and (2) are consistent. (The matrices having this property form a class denoted $\mathbf{Q}_0$.) When M is a positive semi-definite matrix, the PPM will find a solution of any LCP (q, M) or detect that the corresponding inequalities (1) and (2) are inconsistent.

The matrix classes **P** and **PSD** are *complete* in the sense that they contain all principal submatrices of all their members. Furthermore, the matrix classes **P** and **PSD** are distinct but not disjoint. Consequently (by the completeness property), if M' is a **P**-matrix that is not positive semi-definite, and M'' is a positive semi-definite matrix that does not belong to **P**, then their direct sum, the block matrix

$$M = \begin{pmatrix} M' & 0 \\ 0 & M'' \end{pmatrix},$$

belongs to neither of these classes, yet the PPM will process the LCP (q, M) where

$$q = \begin{pmatrix} q' \\ q'' \end{pmatrix}$$

because it decomposes into the problems (q', M') and (q'', M'') each of which can be processed by the PPM. So, in a sense, it is incorrect to think of the PPM as being limited to LCPs (q, M) with $M \in$ **P** or $M \in$ **PSD**.

Although their basic definitions are quite different, the classes **P** and **PSD** are subclasses of $\mathbf{P}_0$ the elements of which are the matrices with nonnegative principal minors. Unfortunately, the class $\mathbf{P}_0$ is too large for purposes of LCP theory or algorithms. In this paper, our attention will center on a class of matrices that contains **P** and **PSD**, yet is contained in $\mathbf{P}_0 \cap \mathbf{Q}_0$. (The subclass $\mathbf{P}_0 \cap \mathbf{Q}_0$ was characterized in [1].) This intermediate class consists of the "row sufficient" matrices whose definition we recall in the next section—along with the definitions of two related matrix classes.

The plan for the remainder of this paper is as follows. Section 2 contains the definitions of row and column sufficient matrices and an example. Section 3 gives some of their elementary properties. Section 4 focuses on the operation known as principal pivoting and establishes some invariance theorems needed for the PPM. Section 5 presents the principal pivoting method for LCPs with row sufficient matrices.

2. Definitions and an example

For ease of reference, we recall what is meant by row (and column) sufficient matrices.

Definition. The matrix $M \in R^{n \times n}$ is

(i) *row sufficient* if

$$x_i(M^{\mathrm{T}}x)_i \leq 0 \text{ for all } i = 1, \ldots, n \quad \Longrightarrow \quad x_i(M^{\mathrm{T}}x)_i = 0 \text{ for all } i = 1, \ldots, n, \tag{4}$$

(ii) *column sufficient* if

$$x_i(Mx)_i \leq 0 \text{ for all } i = 1, \ldots, n \quad \Longrightarrow \quad x_i(Mx)_i = 0 \text{ for all } i = 1, \ldots, n, \tag{5}$$

(iii) *sufficient* if it is row and column sufficient.

In dealing with the above properties, it is sometimes handy to use the notion of the Hadamard product of vectors (or matrices). If $u \in R^n$ and $v \in R^n$, their *Hadamard product* is the vector $u * v \in R^n$ defined by

$$(u * v)_i = u_i \cdot v_i \quad i = 1, \ldots, n.$$

To apply this notion to the definition of a column sufficient matrix, we let $u = x$ and $v = Mx$. Then the defining condition is

$$x * (Mx) \leq 0 \quad \Longrightarrow \quad x * (Mx) = 0.$$

In the case of a row sufficient matrix, the defining condition is

$$x * (M^{\mathrm{T}}x) \leq 0 \quad \Longrightarrow \quad x * (M^{\mathrm{T}}x) = 0.$$

Example 1. The 3×3 matrix

$$M = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{pmatrix}$$

is *not* in the class **P** since it has zeros on its diagonal. These same zeros and the nonzero elements in their rows and columns prevent the matrix from being row or column adequate (in the sense of Ingleton [8]). Furthermore, the matrix is not **PSD**, for whenever such a positive semi-definite matrix has a zero (say, m_{ii}) on its diagonal, the sum of corresponding off-diagonal entries (m_{ij} and m_{ji}) must equal zero, which is not true in this case. It is clear that the matrix M is not the direct sum of matrices of these three kinds, either.

The matrix M is sufficient, however. To see this, suppose $x \in R^3$ is a vector such that

$$x * (Mx) \leq 0.$$

Then

$$-x_1x_2 + 2x_1x_3 \leq 0$$

$$2x_1x_2 - 2x_2x_3 \leq 0$$

$$-x_1x_3 + x_2x_3 \leq 0$$

These inequalities imply

$$2x_2x_3 \leq 2x_1x_3 \leq x_1x_2 \leq x_2x_3.$$

Hence

$$\max\{x_1x_2, x_1x_3, x_2x_3\} \leq 0.$$

Each of these three products being nonpositive, the variables must, pairwise, be of opposite sign. But there are only two signs to share among three variables, so it follows that at least one of x_1, x_2, x_3 equals zero. It is now easy to verify that for $i = 1, 2, 3$

$$x_i = 0 \Longrightarrow \prod_{j \neq i} x_j = 0 \Longrightarrow x * (Mx) = 0.$$

This shows that M is *column sufficient.* The same type of argument applied to M^{T} can be used to demonstrate that M is *row sufficient.* Accordingly, M is sufficient as asserted.

This example shows that sufficient matrices are different from **P**-matrices, adequate matrices, and **PSD**-matrices. But could they be *positively scaled* versions of such things? That is, do there exist *diagonal* matrices Λ and Ω with positive diagonal elements such that $\bar{M} := \Lambda M \Omega$ belongs to one of the aforementioned classes? The answer is clearly in the negative for the classes of **P**-matrices and adequate matrices. We shall now show that the **PSD** case can also be ruled out.

For the matrix M given above, suppose there exist positive definite diagonal matrices $\Lambda = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\Omega = \mathrm{Diag}(\omega_1, \omega_2, \omega_3)$ such that $\bar{M} := \Lambda M \Omega \in \mathbf{PSD}$. Then

$$\bar{M} = \begin{pmatrix} 0 & -\lambda_1\omega_2 & 2\lambda_1\omega_3 \\ 2\lambda_2\omega_1 & 0 & -2\lambda_2\omega_3 \\ -\lambda_3\omega_1 & \lambda_3\omega_2 & 0 \end{pmatrix}.$$

The assumption that $\bar{M}$ is positive semi-definite implies its corresponding off-diagonal elements must add to zero, so

$$2\lambda_2\omega_1 = \lambda_1\omega_2$$

$$2\lambda_1\omega_3 = \lambda_3\omega_1$$

$$2\lambda_2\omega_3 = \lambda_3\omega_2$$

The product of the left-hand sides equals the product of the right-hand sides, so we have

$$8\lambda_1\lambda_2^2\omega_1\omega_3^2 = \lambda_1\lambda_3^2\omega_1\omega_2^2.$$

Dividing both sides by $\lambda_1\omega_1$ and factoring, we get

$$(2\lambda_2\omega_3)(4\lambda_2\omega_3) = (\lambda_3\omega_2)(\lambda_3\omega_2).$$

Substituting via the necessary conditions of positive semi-definiteness (above) and cancelling the left-hand factors, we obtain the contradiction

$$4\lambda_2\omega_3 = \lambda_3\omega_2 = 2\lambda_2\omega_3.$$

Accordingly, M cannot be positively scaled to be positive-semi definite.

3. Elementary properties of row (and column) sufficient matrices

To avoid being overly tiresome, we state the proofs of the following simple propositions rather tersely.

Proposition 1. Let $u,\ v \in R^n$ be arbitrary and let P be an arbitrary $n \times n$ permutation matrix. Then

$$P^{\mathrm{T}}(u * v) = (P^{\mathrm{T}}u) * (P^{\mathrm{T}}v).$$

Proof. This is obvious. ■

Proposition 2. Let $M \in R^{n\times n}$ be arbitrary and let P be an arbitrary $n \times n$ permutation matrix. Then, for all $x \in R^n$

$$P^{\mathrm{T}}(x * (Mx)) = (P^{\mathrm{T}}x) * ((P^{\mathrm{T}}MP)(P^{\mathrm{T}}x)).$$

Proof. This follows at once from Proposition 1 and the fact that $PP^{\mathrm{T}} = I$. ■

By a *principal rearrangement* of $M \in R^{n\times n}$ we mean a matrix of the form $P^{\mathrm{T}}MP$ where P is a permutation matrix.

Proposition 3. Every principal rearrangement of a row (column) sufficient matrix is row (column) sufficient.

Proof. This is clearly a consequence of Proposition 2 and the definition of row (column) sufficiency. ■

Proposition 4. Let $M \in R^{n\times n}$ be arbitrary and let $D = \mathrm{Diag}(\delta_1, \ldots, \delta_n)$ be any $n \times n$ diagonal matrix. Then for all $x \in R^n$

$$x * ((DMD)x) = (Dx) * (M(Dx)).$$

Proof. We show that the i-th component of the vector on each side is the same. Indeed, for all $i = 1, \dots, n$

$$\begin{aligned} [x * ((DMD)x)]_i &= x_i(DMDx)_i \\ &= x_i \left(\delta_i \sum_{j=1}^{n} m_{ij}\delta_j x_j \right) \\ &= (\delta_i x_i) \sum_{j=1}^{n} m_{ij}(\delta_j x_j) \\ &= [(Dx) * (M(Dx))]_i . \end{aligned}$$

■

Proposition 5. If $M \in R^{n \times n}$ is row (column) sufficient, then so is DMD where D is a conformable diagonal matrix.

Proof. This is immediate from Proposition 4 and the definitions. ■

Proposition 6. Each nonempty principal submatrix of a row (column) sufficient matrix is row (column) sufficient.

Proof. Let $M \in R^{n \times n}$ be row (column) sufficient. If $\alpha = (\alpha_1, \dots, \alpha_k)$ is an index set contained in $\{1, \dots, n\}$, the corresponding principal submatrix of M is denoted $M_{\alpha\alpha}$; it consists of the rows and columns of M whose indices belong to α. Now suppose there exists a vector $y \in R^k$ such that $y * (M_{\alpha\alpha}^{\mathrm{T}} y) \leq 0$. Then define $x \in R^n$ such that $x_\alpha = y$, and $x_{\bar{\alpha}} = 0$. Then $x * (M^{\mathrm{T}} x) \leq 0$. Hence when M is row sufficient, it follows that $x * (M^{\mathrm{T}} x) = 0$, but

$$\left(x * (M^{\mathrm{T}} x) \right)_\alpha = y * (M_{\alpha\alpha}^{\mathrm{T}} y).$$

The same sort of argument does the job for column sufficiency. ■

This proposition implies that the classes of row sufficient matrices and column sufficient matrices (and hence sufficient matrices) are *complete* in the sense given above. The are also subclasses of $\mathbf{P}_0$.

Proposition 7. Every row (column) sufficient matrix has nonnegative principal minors.

Proof. This was shown in [6]. ■

Proposition 8. Let a and b denote arbitrary real numbers whose product is negative. Then the 2×2 matrix

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

is both row and column sufficient. The matrix M does not belong to $\mathbf{P}$ and if $a + b \neq 0$, then M is also not positive semi-definite.

Proof. The fact that M is both row and column sufficient is obvious from the definitions. It is also obvious that M cannot be a **P**-matrix. If M were positive semi-definite, the condition $a + b = 0$ would follow from the fact that it has a zero diagonal entry. ■

Proposition 9. Let a and b be real numbers such that $a \geq 0$ and $b \neq 0$. Then a matrix of the form

$$M = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

cannot be row sufficient. Its transpose cannot be column sufficient.

Proof. For such a matrix it is possible to find a vector x such that $x * (M^{\mathrm{T}}x) \leq 0$ and $x_2(M^{\mathrm{T}}x)_2 < 0$. Hence M cannot be row sufficient. By the same token, M^{T} cannot be column sufficient. ■

The next proposition is noteworthy for algorithmic reasons.

Proposition 10. Let $M \in R^{n \times n}$ be row sufficient. If, for some i, $m_{ii} = 0$ and $m_{ji} \geq 0$ for all $j = 1, \ldots, n$, then $m_{ij} \leq 0$ for all $j = 1, \ldots, n$.

Proof. By Proposition 6, it suffices to prove this assertion for the case where $n = 2$. By Proposition 3, it is not restrictive to assume that $i = 1$. The case where $m_{21} = 0$ is ruled out by Proposition 9. Thus, $m_{21} > 0$. By Proposition 7, the condition $m_{12} > 0$ cannot hold, so the desired conclusion follows. ■

Section 1 contains an allusion to the fact that row sufficient matrices also belong to the class $\mathbf{Q}_0$. This was proved in [6]. Although column sufficient matrices also enjoy an interesting property with respect to the LCP, they *do not* form a subclass of $\mathbf{Q}_0$. This can be seen from the example

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that this matrix is column sufficient (in fact, column adequate). But the union of the corresponding complementary cones is not convex, and hence $M \notin \mathrm{Q}_0$. This observation may explain why in this paper we devote more attention to row sufficient matrices.

4. Principal pivoting

As its name suggests, the principal pivoting method is based upon the algebraic process of principal pivoting which we shall briefly review. (See also [5], [4].) Once this is done, we can develop theorems on the invariance of row and column sufficiency under principal pivoting. Such results are essential for the application of the PPM to linear complementarity problems of this sort.

Consider an affine transformation of the form $z \mapsto w = q + Mz$ where $M \in R^{n\times n}$ and $q \in R^n$. For this discussion, the only special property required of M is that for some index set $\alpha \subset \{1,\ldots,n\}$ the principal submatrix $M_{\alpha\alpha}$ be nonsingular. For notational ease, we also assume that $M_{\alpha\alpha}$ is a *leading* principal submatrix of M. This is not a restrictive assumption, as it can be brought about by relabeling. Now consider the equation $w = q + Mz$ in partitioned form:

$$\begin{aligned} w_\alpha &= q_\alpha + M_{\alpha\alpha}z_\alpha + M_{\alpha\bar\alpha}z_{\bar\alpha} \\ w_{\bar\alpha} &= q_{\bar\alpha} + M_{\bar\alpha\alpha}z_\alpha + M_{\bar\alpha\bar\alpha}z_{\bar\alpha} \end{aligned} \tag{6}$$

In this representation, the z-variables are ***nonbasic*** (independent) and the w-variables are *basic* (dependent).

Since $M_{\alpha\alpha}$ is nonsingular by hypothesis, we may exchange the roles of w_α and z_α thereby obtaining a system of the form

$$\begin{aligned} z_\alpha &= q'_\alpha + M'_{\alpha\alpha}w_\alpha + M'_{\alpha\bar\alpha}z_{\bar\alpha} \\ w_{\bar\alpha} &= q'_{\bar\alpha} + M'_{\bar\alpha\alpha}w_\alpha + M'_{\bar\alpha\bar\alpha}z_{\bar\alpha} \end{aligned} \tag{7}$$

where

$$\begin{array}{lll} q'_\alpha = -M_{\alpha\alpha}^{-1}q_\alpha & M'_{\alpha\alpha} = M_{\alpha\alpha}^{-1} & M'_{\alpha\bar\alpha} = -M_{\alpha\alpha}^{-1}M_{\alpha\bar\alpha} \\ q'_{\bar\alpha} = q_{\bar\alpha} - M_{\bar\alpha\alpha}M_{\alpha\alpha}^{-1}q_\alpha & M'_{\bar\alpha\alpha} = M_{\bar\alpha\alpha}M_{\alpha\alpha}^{-1} & M'_{\bar\alpha\bar\alpha} = M_{\bar\alpha\bar\alpha} - M_{\bar\alpha\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\bar\alpha} \end{array} \tag{8}$$

Definition. The system (7) is said to be obtained from (6) by a ***principal pivotal transformation*** on the matrix $M_{\alpha\alpha}$. In this process, the matrix $M_{\alpha\alpha}$ is called the ***pivot block***.

This terminology also applies to the matrices M and M' alone. To indicate that M' is a principal pivotal transform of M with respect to the index set α (and the nonsingular principal submatrix $M_{\alpha\alpha}$), we write

$$M' = \wp_\alpha(M).$$

Another notion we shall need is that of a ***sign-changing matrix***. Once again, let α and $\bar\alpha$ denote complementary index sets in $\{1,\ldots,n\}$. Let $S_{\bar\alpha}$ be the diagonal matrix $\mathrm{Diag}(\sigma_1,\ldots,\sigma_n)$ such that for $i = 1,\ldots,n$

$$\sigma_i = \begin{cases} 1 & i \in \alpha \\ -1 & i \in \bar\alpha \end{cases}$$

These two notations come together in the following result.[1]

Theorem 1. Let $M \in R^{n\times n}$ have the nonsingular principal submatrix $M_{\alpha\alpha}$. Then

$$(\wp_\alpha(M))^{\mathrm{T}} = S_{\bar\alpha}(\wp_\alpha(M^{\mathrm{T}}))S_{\bar\alpha}. \tag{9}$$

Proof. This formula follows from an essentially routine calculation that makes use of the following facts:

1. Pre- and postmultiplication by $S_{\bar\alpha}$ changes the signs of the off-diagonal blocks but not the diagonal blocks.
2. $(M^{\mathrm{T}})_{\alpha\alpha} = (M_{\alpha\alpha})^{\mathrm{T}} =: M^{\mathrm{T}}_{\alpha\alpha}$.
3. $(M^{-1}_{\alpha\alpha})^{\mathrm{T}} = (M^{\mathrm{T}}_{\alpha\alpha})^{-1}$.
4. $(M_{\alpha\beta})^{\mathrm{T}} = (M^{\mathrm{T}})_{\beta\alpha}$.

∎

We are mainly interested in the effect of principal transformation on *row* sufficient matrices, but it is convenient to treat the column sufficient case first.

Theorem 2. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of $M \in R^{n\times n}$. If M is column sufficient and $M' = \wp_\alpha(M)$, then M' is also column sufficient.

Proof. As remarked earlier, it is not restrictive to assume that the pivot block is a leading principal submatrix of M. Let $y = M'x$ and suppose $x * y \le 0$. We may write

$$\begin{pmatrix} y_\alpha \\ y_{\bar\alpha} \end{pmatrix} = \begin{pmatrix} M'_{\alpha\alpha} & M'_{\alpha\bar\alpha} \\ M'_{\bar\alpha\alpha} & M'_{\bar\alpha\bar\alpha} \end{pmatrix} \begin{pmatrix} x_\alpha \\ x_{\bar\alpha} \end{pmatrix}.$$

The condition $x * y \le 0$ means

$$\begin{pmatrix} x_\alpha \\ x_{\bar\alpha} \end{pmatrix} * \begin{pmatrix} y_\alpha \\ y_{\bar\alpha} \end{pmatrix} = \begin{pmatrix} x_\alpha * y_\alpha \\ x_{\bar\alpha} * y_{\bar\alpha} \end{pmatrix} = \begin{pmatrix} y_\alpha * x_\alpha \\ x_{\bar\alpha} * y_{\bar\alpha} \end{pmatrix} \le 0.$$

Since $M' = \wp_\alpha(M)$, we have

$$\begin{pmatrix} x_\alpha \\ y_{\bar\alpha} \end{pmatrix} = \begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\bar\alpha} \\ M_{\bar\alpha\alpha} & M_{\bar\alpha\bar\alpha} \end{pmatrix} \begin{pmatrix} y_\alpha \\ x_{\bar\alpha} \end{pmatrix}.$$

[1] This may be known, but I don't recall a source for it

But M is column sufficient, so it follows that

$$\begin{pmatrix} y_\alpha \\ x_{\bar\alpha} \end{pmatrix} * \begin{pmatrix} x_\alpha \\ y_{\bar\alpha} \end{pmatrix} = 0.$$

Accordingly, $x * y = 0$ which implies that M' is column sufficient. ■

We now come to the result we really want.

Theorem 3. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of $M \in R^{n\times n}$. If M is row sufficient and $M' = \wp_\alpha(M)$, then M' is also row sufficient.

Proof. It is obvious from first principles that a matrix is row sufficient if and only if its transpose is column sufficient. Thus, it suffices to prove that $(M')^{\mathrm{T}}$ is column sufficient. Our hypothesis implies that M^{T} must be so. Theorem 2 implies that $\wp_\alpha(M^{\mathrm{T}})$ is column sufficient. By the definition of M' and by equation (9) we have

$$(M')^{\mathrm{T}} = (\wp_\alpha(M))^{\mathrm{T}} = S_{\bar\alpha}(\wp_\alpha(M^{\mathrm{T}}))S_{\bar\alpha}.$$

The result now follows from Proposition 5. ■

In light of Theorems 2 and 3, we say that column and row sufficient matrices are *invariant under principal pivoting.* In other words, when one performs a principal pivot operation on a row (column) sufficient matrix the resulting matrix is again row (column) sufficient. These invariance theorems generalize early results on principal pivotal transforms of **P**-matrices and **PSD**-matrices. (See [10], [3].) There is no counterpart for adequate matrices since it is not true.

We close this section with a generalization of two technical results [4, Theorem 4 and Theorem 4′] that have very important bearing on a version of the PPM.

Theorem 4. Let A be 2×2 matrix with the following properties:

(*i*) $a_{11} \le 0$;

(*ii*) $a_{21} \le 0$;

(*iii*) $a_{11} + a_{21} < 0$;

(*iv*) if $a_{11} < 0$, then

$$A_1 := \frac{1}{a_{11}} \begin{pmatrix} -a_{12} & 1 \\ a_{11}a_{22} - a_{12}a_{21} & a_{21} \end{pmatrix}$$

is row sufficient;

(v) if $a_{21} < 0$, then

$$A_2 := \frac{1}{a_{21}} \begin{pmatrix} a_{11} & a_{12}a_{21} - a_{11}a_{22} \\ 1 & -a_{22} \end{pmatrix}$$

is column sufficient.

Then A must have the following properties:

(vi) $a_{12} \geq 0$;

(vii) $a_{22} \geq 0$;

($viii$) $a_{12} + a_{22} > 0$.

Proof. Suppose $a_{11} < 0$. Then as A_1 is row sufficient, $a_{12} \geq 0$ and $\det A_1 \geq 0$. Hence

$$\left(\frac{-1}{a_{11}}\right)(-a_{12}a_{21} + a_{12}a_{21} - a_{11}a_{22}) = a_{22} \geq 0.$$

If $a_{12} = a_{22} = 0$, then A_1 has a zero column and hence cannot be row sufficient, a contradiction. Thus, $a_{12} + a_{22} > 0$. Now suppose $a_{21} < 0$. We may (and do) assume $a_{11} = 0$. Since A_2 is column sufficient, it follows that $a_{22} \geq 0$ and $\det A_2 \geq 0$. Thus,

$$\left(\frac{-1}{a_{21}}\right)(-a_{12}a_{21}) \geq 0.$$

This implies $a_{12} \geq 0$. If $a_{12} = a_{22} = 0$, then A_2 cannot be column (or row) sufficient, again a contradiction. This means that $a_{12} + a_{22} > 0$. ■.

5. The Principal Pivoting Method

As matters presently stand, there are two versions of the principal pivoting method—symmetric and asymmetric—both of which can be applied to linear complementarity problems (q, M) with either a **PSD**-matrix or a **P**-matrix. The latter case is much simpler than the former because it does not require the use of certain precautions.These differences will become apparent in due course.

Like numerous other algorithms, the PPM works with pivotal transforms of the system

$$w = q + Mz. \tag{10}$$

In the development below, we use the superscript ν as an iteration counter. The initial value of ν will be 0, and the system shown in (10) will be written as

$$w^0 = q^0 + M^0 z^0. \tag{11}$$

In general, after ν principal pivots, the system will be

$$w^\nu = q^\nu + M^\nu z^\nu. \tag{12}$$

Generically, the vectors w^ν and z^ν, which represent the system's basic and nonbasic variables, respectively, may each be composed of w and z variables. Principal rearrangements can be used to make $\{w_i^\nu, z_i^\nu\} = \{w_i, z_i\}$ $i = 1, \ldots, n$.

The systems (12) can also be represented in the familiar tableau form

$$\begin{array}{c|c|ccc|}
 & 1 & z_1^\nu & \cdots & z_n^\nu \\
\hline
w_1^\nu & q_1^\nu & m_{11}^\nu & \cdots & m_{1n}^\nu \\
\vdots & \vdots & \vdots & & \vdots \\
w_n^\nu & q_n^\nu & m_{n1}^\nu & \cdots & m_{nn}^\nu \\
\hline
\end{array}$$

This way of presenting the PPM is just an expository convenience. Tableaux are not essential; the algorithm can make use of a "revised simplex approach," analogous to what has been done in an implementation Lemke's method.

The symmetric version of the PPM uses principal pivotal transformations (of order 1 or 2) in order to achieve one of two possible terminal sign configurations in the tableau. The first is a nonnegative "constant column", that is, $q_i^\nu \geq 0$ for all $i = 1, \ldots, n$. The other is a row of the form

$$q_r^\nu < 0 \quad \text{and} \quad m_{rj}^\nu \leq 0 \quad j = 1, \ldots, n.$$

The first sign configuration signals the discovery of a solution to (q, M). The second sign configuration indicates that the problem has no feasible solution. The PPM (as originally conceived) does not actually check for this condition. It cannot occur when M is a **P**-matrix. When M is **PSD**, it can be inferred from the condition

$$q_r^\nu < 0, \quad m_{rr}^\nu = 0 \quad \text{and} \quad m_{ir}^\nu \geq 0 \quad \forall i \neq r,$$

which *is* checked in the "minimum ratio test." The key observation is that the same inference can be made when M is (row) sufficient. (See Propositions 6,7, and 10 and Theorem 3.)

The PPM consists of a sequence of *major cycles*, each of which begins with the selection of a *distinguished* variable whose value is currently negative. That variable remains the one and only distinguished variable throughout the major cycle. The object during the major cycle is to make the value of the distinguished variable increase to zero, if possible. Each iteration involves the increase of a nonbasic variable in an effort to drive the distinguished variable up to zero. This increasing nonbasic variable is called the *driving variable*. According to the rules of the method, all variables whose values are currently nonnegative must remain so. The initial trial solution is $(w^0, z^0) = (q^0, 0)$, hence at least n of the variables must be nonnegative. For those variables w_i^0 whose initial value is $q_i^0 < 0$, we impose a *negative lower bound* λ where

$$\lambda < \min_{1 \le i \le n} \{q_i^0\}.$$

This artifice is used in all cases except where $M \in \mathbf{P}$. Then, in addition to requiring all variables with currently nonnegative values to remain so, the PPM also demands that the variables currently having a negative value remain at least as large as λ. To accommodate this feature, we broaden the notion of ***basic solution*** by allowing the nonbasic variables to have the value 0 or λ. (See [2], [3], [4].) We also say that a solution of the system (10) is *nondegenerate* if at most n of its $2n$ variables have the value 0 or λ. Otherwise, the solution is called *degenerate*.

In the interest of clarity, it will help to introduce the following notations. We want to distinguish between the names of variables and their values. To this end, we use bars over the generic variable names w_i^ν and z_i^ν to indicate definite values of these variables. At the beginning of a major cycle in which negative lower bounds λ are in use, we will have $\bar{z}_i^\nu = 0$ or $\bar{z}_i^\nu = \lambda$ $i = 1, \ldots, n$. Next, we use the notation

$$W^\nu(z^\nu) = q^\nu + M^\nu z^\nu.$$

The definition of the mapping W^ν is the same as that of w^ν, but it emphasizes the argument z^ν.

A simple example will help to motivate preceding ideas, especially the need for the negative lower bounds, λ. Consider the LCP of order 2 in which

$$q = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}.$$

The matrix M is sufficient, i.e., row ***and*** column sufficient. At the outset we have $(\bar{w}^0, \bar{z}^0) = (-3, -2, 0, 0)$. Suppose we choose w_1^0 as the initial distinguished variable. Then z_1^0 would be used as the initial driving variable. If only nonnegative variables are required to remain nonnegative, there would be no limit to the allowable increase of the driving variable. Under ordinary circumstances, such an outcome would indicate that the problem is unsolvable (at least by this method). But notice that this LCP has the solution $(\bar{w}, \bar{z}) = (1, 0, 0, 2)$. Hence some sort of modification is needed.

If, at the outset of a major cycle, the selected distinguished variable is basic, the first driving variable is the *complement* of the distinguished variable. Thus, if w_r^ν is the distinguished variable for the current major cycle, then z_r^ν is the first driving variable. The distinguished variable need not be a basic variable, however. Thus, with the broader definition of basic solution (given above), the current solution $(\bar{w}^\nu, \bar{z}^\nu)$ may have $\bar{z}_r^\nu = \lambda < 0$ at the beginning of a major cycle. In such circumstances, z_r^ν can be the distinguished variable as well as the driving variable. In this event, the increase of the driving variable will always be blocked, either when a basic variable reaches its (current) lower bound (0 or λ) or when z_s^ν reaches zero (in which case the major cycle ends).

The following is a formal statement of this algorithm.

Symmetric PPM with Nondegeneracy Assumption in Force

Step 0. Set $\nu = 0$; define $(\bar{w}^0, \bar{z}^0) = (q^0, 0)$. Let λ be any number less than $\min_i q_i^0$.

Step 1. If $q^\nu \geq 0$ or if $(\bar{w}^\nu, \bar{z}^\nu) \geq (0,0)$, stop; $(\bar{w}^\nu, \bar{z}^\nu) := (q^\nu, 0)$ is a solution. Otherwise[2], determine an index r such that $\bar{z}_r^r = \lambda$ or (if none such exist) an index r such that $\bar{w}_r^\nu < 0$.

Step 2. Let ζ_r^ν be the largest value of $z_r^\nu \geq \bar{z}_r^\nu$ satisfying the following conditions:

(i) $z_r^\nu \leq 0$ if $\bar{z}_r^\nu = \lambda$.

(ii) $W_r^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \leq 0$ if $\bar{w}_r^\nu < 0$.

(iii) $W_i^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \geq 0$ if $\bar{w}_i^\nu > 0$.

(iv) $W_i^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \geq \lambda$ if $\bar{w}_i^\nu < 0$.

Step 3. If $\zeta_r^\nu = +\infty$, stop. No feasible solution exists. If $\zeta_r^\nu = 0$, let $\bar{z}_i^{\nu+1} = 0$, $\bar{z}_i^{\nu+1} = \bar{z}_i^\nu$ for all $i \neq r$, and let

$$\bar{w}^{\nu+1} = W^{\nu+1}(\bar{z}^{\nu+1}) = W^\nu(\bar{z}^{\nu+1}).$$

Return to Step 1 with ν replaced by $\nu + 1$. If $0 < \zeta_r^\nu < +\infty$, let s be the unique index determined by the conditions (ii), (iii), and (iv) in Step 2.

Step 4. If $m_{ss}^{\nu+1} > 0$, perform the principal pivot $\langle w_s^\nu, z_s^\nu \rangle$. Let

$$\bar{z}_s^{\nu+1} = W_s^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, \zeta_r^\nu, \bar{z}_{r1}^\nu, \ldots, \bar{z}_n^\nu) \quad \text{and} \quad \bar{w}^{\nu+1} = W^{\nu+1}(\bar{z}^{\nu+1}).$$

If $s = r$, return to Step 1 with ν replaced $\nu + 1$. If $s \neq r$, return to Step 2 with ν replaced $\nu + 1$. If $m_{ss}^\nu = 0$, perform the principal pivot $\{\langle w_s^\nu, z_r^\nu \rangle, \langle w_r^\nu, z_s^\nu \rangle\}$. Put $\bar{w}_r^{\nu+1} = \bar{z}_s^\nu, \bar{w}_s^{\nu+1} = \zeta_r^\nu, \bar{z}_i^{\nu+1} = \bar{z}_i^\nu$ for all $i \notin \{r,s\}$, and then $\bar{w}_i^{\nu+1} = W_i^{\nu+1}(\bar{z}^{\nu+1})$ for all $i \notin \{r,s\}$. Return to Step 2 with ν replaced by $\nu + 1$ and r replaced by s.

Discussion

Here we wish to discuss what algorithm does and why it actually processes any LCP with a row sufficient matrix, M.

[2] At the beginning of a major cycle, for each index r, at most one of w_r^ν, z_r^ν can be negative.

All major cycles of the PPM begin at Step 1 were the algorithm checks whether it is possible to terminate with a solution. This will be the case if $(\bar{w}^{\nu}, \bar{z}^{\nu}) \geq (0,0)$ since $(\bar{w}^{\nu}, \bar{z}^{\nu})$ must then be a nonnegative solution of (10) with $\bar{z}^{\nu} = 0$. As illustrated in Example 2 (below), it can happen that the constant column q^{ν} becomes nonnegative before z^{ν} does. In such a case, resetting z^{ν} to zero yields a solution. If neither of these forms of termination occurs, there is an index r such that $\bar{z}_r^{\nu} < 0$ or $\bar{w}_r^{\nu} < 0$ and it becomes the ***distinguished variable*** for the current major cycle.

For a linear complementarity problem (q, M) of order n, there are $2n$ variables in equation (10). The number of negative components in a solution of (10) is called its *index of infeasibility.* The conditions imposed in Step 2 of the Symmetric PPM prevent this number from increasing. Furthermore, with each return to Step 1, the algorithm produces a basic solution having a smaller index of infeasibility than its predecessor. Since there are at most a finite number of basic solutions, there can be at most finitely many returns to Step 1. The proof of finiteness therefore boils down to showing that each major cycle consists of at most a finite number of steps.

Termination can also occur in Step 3. In this event, $\zeta_r^{\nu} = +\infty$. For this to happen, the distinguished variables must be w_r^{ν}; it must also be true that

$$m_{rr}^{\nu} = 0 \quad \text{and} \quad m_{ir}^{\nu} \geq 0 \quad \forall i \neg r.$$

From Proposition 10, it follows that $m_{rj}^{\nu} \leq 0 \; j = 1, \ldots, n$. Now, since $\bar{z}_j^{\nu} \leq 0$ for all j and

$$\bar{w}_r^{\nu} = q_r^{\nu} + \sum_{j=1}^{n} m_{rj}^{\nu} \bar{z}_j^{\nu} < 0,$$

it follows that $q_r^{\nu} < 0$, so that the r-th equation

$$w_r^{\nu} = q_r^{\nu} + \sum_{j=1}^{n} m_{rj}^{\nu} z_j^{\nu} < 0$$

has no nonnegative solution. Another outcome in Step 3 is that $\zeta_r^{\nu} = 0$ in which case (by nondegeneracy) the distinguished variable and the driving variable must have been z_r^{ν} which increased to zero. This brings the major cycle to a close without necessitating a pivot. The remaining possibility $0 < \zeta_r^{\nu} < +\infty$ means that *some* basic variable blocked the increase of z_r^{ν}.

The various alternatives that arise in the latter situation are addressed in Step 4. If $m_{ss}^{\nu} > 0$, the indicated principal pivot is executable. If $s = r$, the distinguished variable must have increased to 0. This brings about a return to Step 1 and a reduction in the index of infeasibility by at least one. If $s \neq r$, the principal pivot is made and the increase of the driving variable continues in accordance with the rules of Step 2. If $m_{ss}^{\nu} = 0$, then $s \neq r$. The fact that w_s^{ν} blocked z_r^{ν} means $m_{sr}^{\nu} < 0$. The principal pivot of order 2 is executable because the row sufficiency of

$$\begin{pmatrix} m^{\nu}_{rr} & m^{\nu}_{rs} \\ m^{\nu}_{sr} & m^{\nu}_{ss} \end{pmatrix}$$

and the negativity of m^{ν}_{sr} implies that $m^{\nu}_{rs} > 0$. (See Proposition 9.) The values of the variables immediately after the pivot are those they had when blocking occurred. At the return to Step 2, the variable w^{ν}_{r} becomes $z^{\nu+1}_{s}$; a principal rearrangement to restore the natural order of subscripts would be possible.

As noted above, the argument that the algorithm will process any nondegenerate LCP with a row sufficient matrix comes down to showing that there can be at most finitely many returns to Step 2. But this is clear from the fact that there are only finitely many principal transformations of the system and finitely many ways to evaluate the nonbasic variables z^{ν}_{i} $(i \neq r)$. As for z_r, its value and that of its complement w^{ν}_{r} increase monotonically and their sum increases strictly throughout the major cycle. Hence the definition of ζ^{ν}_{r} and $\zeta^{\nu+\kappa}_{r}$ $(\kappa > 0)$ make it impossible to have $z^{\nu}_{i} = z^{\nu+\kappa}_{i}$ $(i = 1, \ldots, n)$ and $\bar{z}^{\nu}_{i} = \bar{z}^{\nu+\kappa}_{i}$ $(i \neq r)$ as would have to be the case with infinitely many steps within a major cycle.

Example 2. Consider the LCP (q, M) where

$$q = \begin{pmatrix} -3 \\ 6 \\ -1 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

The PPM applies to this problem because the matrix M is sufficient (as shown in Example 1). It is easy to verify that (q, M) has the solution $(\bar{w}; \bar{z}) = (2, 0, 0; 0, 1, 3)$. The discussion below illustrates how this solution can be obtained by the symmetric version of the PPM. The reader is advised that, for simplicity, the superscripts (iteration counters) and bars (denoting fixed values of variables) are omitted.

For this choice of data, the problem (q, M) has the tabular form

	1	z_1	z_2	z_3
w_1	−3	0	−1	2
w_2	6	2	0	−2
w_3	−1	−1	1	0

The number $\lambda = -4$ will serve as the negative lower bound for the initial negative basic variables w_1 and w_3. Choose w_1 as the distinguished variable and its complement z_1 as the driving variable. The blocking variable is w_3 which decreases and reaches its lower bound -4 when z_1 increases to 3. Since the corresponding diagonal entry m_{33} equals 0, it is necessary to perform a principal pivot of order 2: $\langle w_3, z_1 \rangle$ and $\langle w_1, z_3 \rangle$. The new tableau is

$$
\begin{array}{c|c|ccc|c}
 & 1 & w_3 & z_2 & w_1 & \\
\hline
z_3 & \frac{3}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
w_2 & 1 & -2 & 1 & -1 & 12 \\
z_1 & -1 & -1 & 1 & 0 & 3 \\
\hline
 & & -4 & 0 & -3 &
\end{array}
$$

Now the distinguished variable w_1 is nonbasic and can be increased directly as the driving variable. In this case, the driving variable blocks itself. Thus, the first major cycle ends with the tableau

$$
\begin{array}{c|c|ccc|c}
 & 1 & w_3 & z_2 & w_1 & \\
\hline
z_3 & \frac{3}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
w_2 & 1 & -2 & 1 & -1 & 9 \\
z_1 & -1 & -1 & 1 & 0 & 3 \\
\hline
 & & -4 & 0 & 0 &
\end{array}
$$

For the next major cycle, the only possible distinguished variable is w_3 which is nonbasic at value -4. This becomes the driving variable and is blocked when it reaches -1 and z_1 decreases to 0. Once again a principal pivot of order 2 is needed. It leads to

$$
\begin{array}{c|c|ccc|c}
 & 1 & z_1 & z_2 & z_3 & \\
\hline
w_1 & -3 & 0 & -1 & 2 & 0 \\
w_2 & 6 & 2 & 0 & -2 & 3 \\
w_3 & -1 & -1 & 1 & 0 & -1 \\
\hline
 & & 0 & 0 & \frac{3}{2} &
\end{array}
$$

Here the driving variable is the z_3 which starts from the value $\frac{3}{2}$; it is blocked when it reaches 3 and w_2 decreases to 0. This time the algorithm performs a different principal pivot of order 2: $\langle w_2, z_3 \rangle$ and $\langle w_3, z_2 \rangle$. This yields

$$
\begin{array}{c|c|ccc|c}
 & 1 & z_1 & w_3 & w_2 & \\
\hline
w_1 & 2 & 1 & -1 & -1 & 3 \\
z_3 & 3 & 1 & 0 & -\frac{1}{2} & 3 \\
z_2 & 1 & 1 & 1 & 0 & 0 \\
\hline
 & & 0 & -1 & 0 &
\end{array}
$$

The distinguished variable is still w_3 whose current value is -1. If used as the driving variable, it will block itself and a solution will be obtained. Another option is to observe that the "constant column" is positive. In such a case the negative basic variable(s) can be set equal to zero. Either way, the solution found is $(\bar{w}; \bar{z}) = (2, 0, 0; 0, 1, 3)$.

The asymmetric version of the PPM also consists of a major cycles. Instead of executing only principal pivots of order 1 or 2, each major cycle involves a sequence of "simple pivots" whose effect may be a principal pivot of larger order. The rules governing blocking are the same as those in the Symmetric PPM (nonnegative variables are bounded below by 0, and negative variables are bounded below by λ. A negative driving variable is bounded above by 0.) The main difference between the two versions of the algorithm is that in the asymmetric one entails pivotal exchanges between the driving variable and the driving variable and then takes the new driving variable to be the complement of the blocking variable. Just as in the positive semi-definite case [4], the distinguished variable and the driving variable increase monotonically and their sum increases strictly. This assertion is can be proved by using Theorem 4. We omit further details and simply point out th .t the argument used to justify the aysmmetric version of the PPM for **PSD** matrices carries over *mutatis mutandis* to the row sufficient case.

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The Principal Pivoting Method (PPM) for the Linear Complementarity Problem (LCP) is shown to be applicable to the class of LCPs involving the newly identified class of sufficient matrices.

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