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Representation of shift invariant operators on L^2 by H^∞ transfer functions: an elementary proof, a generalization to L^p and a counterexample for L^∞

by

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March 1989 LCDS#89-5

Lefschetz Center for Dynamical Systems
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Representation of shift invariant operators on L^2 by H^∞ transfer functions: an elementary proof, a generalization to L^p and a counterexample for L^∞

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Abstract. We give an elementary proof of the well known fact that shift invariant operators on $L^2[0,\infty)$ are represented by transfer functions which are bounded and analytic on the right *open half-plane.* We prove a generalization to Banach space-valued L^p-functions, where $1 \leq p < \infty$. We show that the result no longer holds for $p = \infty$.

1. The scalar case

In this section we give an elementary and short proof of the well known fact that any shift invariant (and hence causal) linear operator on $L^2[0,\infty)$ is represented by an H^∞ transfer function (for the precise statement see Theorem 1.3 below). This proof will serve as a model for a more difficult proof in Section 2, where we show that the result remains valid for Banach space-valued p-integrable functions, where $1 \leq p < \infty$ (see Theorem 2.3). For $p = \infty$ the result no longer holds, not even for scalar functions (in spite of claims to the contrary in the literature), as we shall see in Section 3.

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Notations. We denote by $\mathbb{C}_0$ the right open half-plane in $\mathbb{C}$ and by H^∞ the space of bounded analytic functions on $\mathbb{C}_0$, with the sup-norm. For $u \in L^2[0,\infty)$, $\hat{u}$ will denote the Laplace transform of u, i.e.

$$\hat{u}(s) = \int_0^\infty e^{-st}u(t)dt , \qquad \forall\, s \in \mathbb{C}_0 . \tag{1.1}$$

For any $\lambda \in \mathbb{C}$, e_λ will denote the function defined on $[0,\infty)$ by $e_\lambda(t) = e^{\lambda t}$. Adjoints of operators on $L^2[0,\infty)$ will be considered in the linear (not antilinear) sense, i.e., with respect to the bilinear form

$$\langle u, v\rangle = \int_0^\infty u(t)v(t)dt . \tag{1.2}$$

In particular, $\hat{u}(s) = \langle u, e_{-s}\rangle$, for any $s \in \mathbb{C}_0$. For $\tau \geq 0$, $\mathbf{S}_\tau$ will denote the operator of right shift by τ on $L^2[0,\infty)$. Its adjoint, $\mathbf{S}_\tau^*$ is the left shift by τ.

We shall need the following simple remark concerning left shifts.

Remark 1.1. Let $s \in \mathbb{C}_0$. If $z \in L^2[0,\infty)$ satisfies

$$\mathbf{S}_\tau^* z = e^{-s\tau} z , \qquad \forall\, \tau \geq 0 , \tag{1.3}$$

then there is an $a \in \mathbb{C}$ such that $z = ae_{-s}$.

We leave the proof of this remark to the reader, with the following hint: $z \in L^1[0,\infty)$ and the continuous function $Z(t) = \int_t^\infty z(\sigma)d\sigma$, defined for $t \geq 0$, satisfies (1.3).

Definition 1.2. Let $\mathcal{F}$ be a bounded linear operator from $L^2[0,\infty)$ to itself. We say that $\mathcal{F}$ is *shift invariant* if for any $\tau \geq 0$, $\mathcal{F}$ commutes with the right shift $\mathbf{S}_\tau$ (i.e., $\mathcal{F}\mathbf{S}_\tau = \mathbf{S}_\tau\mathcal{F}$).

Theorem 1.3. *If $\mathcal{F}$ is a shift invariant operator on $L^2[0,\infty)$, then there is a (unique) $\mathbf{H} \in H^\infty$ such that for any $u \in L^2[0,\infty)$, denoting $y = \mathcal{F}u$,*

$$\hat{y}(s) = \mathbf{H}(s)\hat{u}(s) , \qquad \forall\, s \in \mathbb{C}_0 . \tag{1.4}$$

Proof. Let $s \in \mathbb{C}_0$ be fixed. We have, using that $\mathcal{F}^*$ commutes with left shifts $\mathbf{S}_\tau^*$,

$$\mathbf{S}_\tau^*\mathcal{F}^* e_{-s} = e^{-s\tau}\mathcal{F}^* e_{-s} , \qquad \forall\, \tau \geq 0 .$$

So $z = \mathcal{F}^* e_{-s}$ satisfies (1.3), whence by Remark 1.1 there exists a number $\mathbf{H}(s)$ such that

$$\mathcal{F}^* e_{-s} = \mathbf{H}(s) e_{-s} . \tag{1.5}$$

We form the product of both sides in (1.5) with $u \in L^2[0,\infty)$ (with respect to $\langle \cdot, \cdot \rangle$ defined in (1.2)) and get

$$\langle \mathcal{F} u, e_{-s} \rangle = \mathbf{H}(s) \langle u, e_{-s} \rangle ,$$

which is exactly (1.4).

The above argument being valid for any $s \in \mathbb{C}_0$, we have obtained a function $\mathbf{H}$ defined on $\mathbb{C}_0$. The unicity of $\mathbf{H}$ is obvious. To see that $\mathbf{H}$ is analytic, apply (1.4) to $u = e_{-1}$, so $\hat{u}(s) = \frac{1}{s+1}$, and use that $\hat{y}$ (like any Laplace transform) is analytic. Finally, $\mathbf{H}$ is bounded because, according to (1.5), $\mathbf{H}(s)$ is an eigenvalue of $\mathcal{F}^*$, so $|\mathbf{H}(s)| \leq \|\mathcal{F}^*\| = \|\mathcal{F}\|$. □

The function $\mathbf{H}$ appearing in the above theorem is called the *transfer function* associated with $\mathcal{F}$.

Remark 1.4. Suppose $\mathbf{H}$, u and y are as in Theorem 1.3. Then $\mathbf{H}$, $\hat{u}$ and $\hat{y}$ have nontangential limits in almost every point $i\omega$ of the imaginary axis, which we denote by $\mathbf{H}(i\omega)$, $\hat{u}(i\omega)$ and $\hat{y}(i\omega)$, respectively. The functions $\omega \to \hat{u}(i\omega)$ and $\omega \to \hat{y}(i\omega)$ (defined for a.e. $\omega \in \mathbb{R}$) are the Fourier transforms of u and y, respectively, multiplied by $\sqrt{2\pi}$. Theorem 1.3 implies

$$\hat{y}(i\omega) = \mathbf{H}(i\omega)\hat{u}(i\omega) , \qquad \text{for a.e. } \omega \in \mathbb{R} . \tag{1.6}$$

Remark 1.5. Suppose $\mathcal{T}$ is a time invariant and causal bounded linear operator on $L^2(-\infty,\infty)$ (i.e., $\mathcal{T}$ commutes with shifts and $L^2[0,\infty)$ is $\mathcal{T}$-invariant). Let $\mathcal{F}$ be the restriction of $\mathcal{T}$ to $L^2[0,\infty)$ and let $\mathbf{H}$ be the transfer function associated with $\mathcal{F}$. The extension of $\mathbf{H}$ to a.e. point of the imaginary axis is defined as in Remark 1.4. Let $u \in L^2(-\infty,\infty)$, $y = \mathcal{T}u$ and let the functions $\hat{u}$ and $\hat{y}$ be defined on the imaginary axis such that $\omega \to \hat{u}(i\omega)$ and $\omega \to \hat{y}(i\omega)$ are the Fourier transforms of u and y, respectively, multiplied by $\sqrt{2\pi}$.

Then again (1.6) holds. However, in this situation analytic extension outside the imaginary axis may be impossible. The idea of the proof is to approximate u with functions having support bounded to the left and to apply Remark 1.4. This is possible because for any real τ, the restriction of $\mathcal{T}$ to $L^2[\tau,\infty)$ is isomorphic (via a shift) to $\mathcal{F}$.

Actually, (1.6) remains valid without the causality assumption on $\mathcal{T}$, but this requires a different proof.

Remark 1.6. The converse of Theorem 1.3 holds (as is well known): If $\mathbf{H} \in H^\infty$ then (1.4) defines a shift invariant (and hence causal) operator $\mathcal{F}$ on $L^2[0,\infty)$ and $\|\mathcal{F}\| \leq \|\mathbf{H}\|$. This follows easily from the Paley-Wiener theorem characterizing Fourier transforms of elements of $L^2[0,\infty)$, see e.g. Rudin [14, p. 405]. Together with the last inequality in the proof of Theorem 1.3, we obtain that $\|\mathcal{F}\| = \|\mathbf{H}\|$.

Bibliographical Notes. Theorem 1.3 is due to Fourés and Segal [4], who actually proved a more general theorem, concerning Hilbert space-valued functions defined on $\mathbf{R}^n$ (instead of our one-dimensional time). They defined causality with respect to a cone C (such that for $x \in \mathbf{R}^n$, $x + C$ is the "future" with respect to x). They considered also unbounded causal operators. By particularizing the n-dimensional version of their result to Green's operators corresponding to partial differential operators with constant coefficients, Fourés and Segal obtained a simple causality criterion for such operators (see also Kannai [10]).

The discrete time version of Theorem 1.3 (i.e., concerning shift invariant operators on l^2) is due to Hartman and Wintner [8, p. 880], who obtained it as a consequence of a related result of O. Toeplitz. For a modern proof of the discrete time version see Halmos [6, Problem 147]. For another proof of the discrete time version (for Hilbert space-valued functions), which is closer in spirit to our proof of Theorem 1.3, see Rosenblum and Rovnyak [13, p. 15].

The proof in [4], even after reducing it to the particular situation described in Theorem 1.3 above, is rather involved. Several new approaches have since been proposed. The proof of Harris and Valenca [7, p. 83] is related to our proof. They also obtained representation theorems for shift invariant operators on several other function spaces, and a mistaken one for L^∞ (see also Remark 3.6). Logemann [11, Chapter II] showed that Theorem 1.3 can be obtained as a consequence of the corresponding result for discrete time. Logemann also obtained a characterization of shift invariant and closed (but not necessarily densely defined) operators on $L^2[0,\infty)$ (see [11, p. 71]). Helton [9, p. 10] gave an intuitive argument for Theorem 1.3, using "delta functions". Salamon [16] proposed two new proofs for Theorem 1.3 (the version for Hilbert space-valued functions). The first one (his Proposition 4.1) is based on approximating $\mathcal{F}$ with shift invariant operators which are in a certain sense smoothing. However, this proof contains two nontrivial steps (which are left to the reader). The second proof is to construct a state space realization for $\mathcal{F}$ (which is in fact the aim of the paper) and then write down the corresponding transfer function. Results related to Theorem 1.3 appeared in Freedman, Falb and Anton [5], Masani [12] and Zemanian [18, Chapter 6].

2. Banach space-valued L^p-functions

Theorem 1.3 can be extended without difficulty to Hilbert space-valued functions, as is well known. In this section we show that actually it can be extended to Banach space-valued functions of class L^p, where $1 \le p < \infty$.

Notations. If X is a Banach space and $p \in [1,\infty)$, the Laplace transform of any $u \in L^p([0,\infty),X)$ is defined and denoted as in (1.1). Let $q \in (1,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For $u \in L^p([0,\infty),X)$ and $v \in L^q([0,\infty),X^*)$, we denote

$$\langle u,v\rangle \;=\; \int_0^\infty \langle u(t),v(t)\rangle dt \,.$$

With respect to this (bilinear) product, $L^q([0,\infty),X^*)$ is a closed subspace of $L^p([0,\infty),X)^*$ (they may be equal), see Diestel and Uhl [3, p. 97]. For $\tau \ge 0$, the operator of right shift by τ on $L^p([0,\infty),X)$ will be denoted by $\mathbf{S}_\tau$, as in the scalar case. If U and Y are Banach spaces and $\mathcal{F}$ is a bounded linear operator from $L^p([0,\infty),U)$ to $L^p([0,\infty),Y)$, then we say that $\mathcal{F}$ is *shift invariant* if for any $\tau \ge 0$, $\mathcal{F}\mathbf{S}_\tau = \mathbf{S}_\tau\mathcal{F}$.

The following lemma is a generalization of Remark 1.1.

Lemma 2.1. *Let U be a Banach space, $p \in [1,\infty)$ and $s \in \mathbb{C}_0$. If $z \in L^p([0,\infty),U)^*$ satisfies*

$$\mathbf{S}_\tau^* z \;=\; e^{-s\tau} z \,, \qquad \forall\, \tau \ge 0 \,, \tag{2.1}$$

then there is a $\mathrm{v} \in U^*$ *such that* $z = e_{-s}\mathrm{v}$.

Proof. Let the bilinear operator $\phi : U \times L^p([0,\infty),U)^* \to L^p[0,\infty)^*$ be defined by

$$\langle \alpha, \phi(h,y)\rangle \;=\; \langle \alpha h, y\rangle \,,$$

for any $\alpha \in L^p[0,\infty)$, any $h \in U$ and any $y \in L^p([0,\infty),U)^*$. The definition makes sense since $\alpha h \in L^p([0,\infty),U)$, and it is clear that ϕ is bounded (actually $\|\phi\| = 1$). We identify $L^p[0,\infty)^*$ with $L^q[0,\infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $h \in U$ and any $y \in L^q([0,\infty),U^*)$ we have

$$\phi(h,y)(t) = \langle h, y(t)\rangle , \qquad \text{for a.e. } t \geq 0 , \tag{2.2}$$

as is easy to verify.

The operator ϕ has the following useful property: for any $h \in U$ and any $y \in L^p([0,\infty),U)^*$

$$\phi(h, \mathbf{S}_\tau^* y) = \mathbf{S}_\tau^* \phi(h,y) , \qquad \forall\, \tau \geq 0 . \tag{2.3}$$

Indeed, for any $\alpha \in L^p[0,\infty)$

$$\begin{aligned} \langle \alpha, \phi(h, \mathbf{S}_\tau^* y)\rangle &= \langle \alpha h, \mathbf{S}_\tau^* y\rangle \\ &= \langle \mathbf{S}_\tau \alpha h, y\rangle \\ &= \langle \mathbf{S}_\tau \alpha, \phi(h,y)\rangle \\ &= \langle \alpha, \mathbf{S}_\tau^* \phi(h,y)\rangle . \end{aligned}$$

Formulae (2.1) and (2.3) imply that for any $h \in U$

$$\mathbf{S}_\tau^* \phi(h,z) = e^{-s\tau} \phi(h,z) , \qquad \forall\, \tau \geq 0 .$$

By the version of Remark 1.1 for $L^q[0,\infty)$ (instead of $L^2[0,\infty)$) it follows that for any $h \in U$ there is an $a \in \mathbb{C}$ such that

$$\phi(h,z) = a e_{-s} .$$

The map $h \to a$ is obviously a bounded linear functional on U, and we denote it by v. So we have $\phi(h,z) = \langle h, \mathrm{v}\rangle e_{-s}$, or equivalently, using (2.2)

$$\phi(h,z) = \phi(h, e_{-s}\mathrm{v}) , \qquad \forall\, h \in U . \tag{2.4}$$

Let $E \subset [0,\infty)$ be a bounded interval, let α be the characteristic function of E, let $h \in U$ and put $u = \alpha h$. Then an elementary computation using the definition of ϕ and (2.4) implies

$$\langle u, z\rangle = \langle u, e_{-s}\mathrm{v}\rangle . \tag{2.5}$$

Clearly (2.5) remains valid if we replace u by a finite linear combination of functions constructed like u, i.e., for U-valued step functions. Since the step functions are dense in $L^p([0,\infty),U)$, it follows that (2.5) holds for any $u \in L^p([0,\infty),U)$, whence $z = e_{-s}\mathrm{v}$. $\square$

Remark 2.2. The proof of the previous lemma can be made much easier if U is such that

$$L^p([0,\infty),U)^* = L^q([0,\infty),U^*) .$$

A sufficient condition for this is that U is reflexive, see [3, p. 76 and 98].

Theorem 2.3. *Suppose U and Y are Banach spaces, $1 \leq p < \infty$ and $\mathcal{F}$ is a shift invariant bounded linear operator from $L^p([0,\infty),U)$ to $L^p([0,\infty),Y)$. Then there is a (unique) bounded analytic $\mathcal{L}(U,Y)$-valued function $\mathbf{H}$ defined on $\mathbb{C}_0$ such that for any $u \in L^p([0,\infty),U)$, denoting $y = \mathcal{F}u$, (1.4) holds and*

$$\sup_{s\in\mathbb{C}_0} \|\mathbf{H}(s)\| \leq \|\mathcal{F}\| . \tag{2.6}$$

Proof. Let $s \in \mathbb{C}_0$ be fixed and let $\mathrm{w} \in Y^*$. Then $e_{-s}\mathrm{w} \in L^q([0,\infty),Y^*)$, so $\mathcal{F}^* e_{-s}\mathrm{w} \in L^p([0,\infty),U)^*$. Since $\mathcal{F}$ is shift invariant, we have

$$\mathbf{S}_\tau^* \mathcal{F}^* e_{-s}\mathrm{w} = e^{-s\tau}\mathcal{F}^* e_{-s}\mathrm{w} , \qquad \forall\, \tau \geq 0 .$$

By Lemma 2.1 there is a unique $\mathrm{v} \in U^*$ such that $\mathcal{F}^* e_{-s}\mathrm{w} = e_{-s}\mathrm{v}$. It is clear that the map $\mathrm{w} \to \mathrm{v}$ is linear and bounded, and we denote it by $G(s)$, so

$$\mathcal{F}^* e_{-s}\mathrm{w} = e_{-s}G(s)\mathrm{w} , \qquad \forall\, \mathrm{w} \in Y^* . \tag{2.7}$$

Forming the product of both sides in (2.7) with $u \in L^p([0,\infty),U)$, we get

$$\langle \hat{y}(s), \mathrm{w} \rangle = \langle \hat{u}(s), G(s)\mathrm{w} \rangle , \qquad \forall\, \mathrm{w} \in Y^* . \tag{2.8}$$

Since $G(s) \in \mathcal{L}(Y^*,U^*)$, we have $G^*(s) \in \mathcal{L}(U^{**},Y^{**})$. Let $\mathbf{H}(s)$ denote the restriction of $G^*(s)$ to U, then (2.8) is equivalent with (1.4). This shows that the range of $\mathbf{H}(s)$ is contained in Y, so $\mathbf{H}(s) \in \mathcal{L}(U,Y)$.

As in the proof of Theorem 1.3, we have obtained a unique function $\mathbf{H}$ defined on $\mathbb{C}_0$, satisfying (1.4). For any $h \in U$, taking $u = e_{-1}h$, we get that the function $s \to \mathbf{H}(s)h$ is analytic. By a standard argument (see e.g. Zemanian [18, p. 18]) we get that $\mathbf{H}$ is analytic in norm. Finally, let us show that $\mathbf{H}$ is bounded. Let $s \in \mathbb{C}_0$. Using (2.7) we get that for any $\mathrm{w} \in Y^*$ with $\mathrm{w} \neq 0$

$$\frac{\|G(s)\mathrm{w}\|}{\|\mathrm{w}\|} = \frac{\|e_{-s}G(s)\mathrm{w}\|_{L^p}}{\|e_{-s}\mathrm{w}\|_{L^p}} = \frac{\|\mathcal{F}^* e_{-s}\mathrm{w}\|_{L^p}}{\|e_{-s}\mathrm{w}\|_{L^p}} \leq \|\mathcal{F}^*\| ,$$

whence $\|G(s)\| \leq \|\mathcal{F}^*\| = \|\mathcal{F}\|$. By the definition of $\mathbf{H}(s)$ we get (2.6). □

Remark 2.4. The converse of Theorem 2.3 is generally not true, unless U and Y are Hilbert spaces and $p = 2$. In that latter case, the converse follows from the version of the Paley-Wiener theorem which concerns Hilbert space-valued functions, see e.g. Rosenblum and Rovnyak [13, p. 91], and again $\|\mathcal{F}\| = \|\mathbf{H}\|$ holds (as in Remark 1.6).

Remark 2.5. Our proof of Theorem 2.3 is related to a part of the proof of a result in Curtain and Weiss [2, Theorem 4.2].

3. Nonrepresentable operators on L^∞

In this section we show by means of a counterexample that in Theorem 2.3 we can not replace $p < \infty$ by $p \leq \infty$, not even in the simplest case $U = Y = \mathbb{C}$. (In fact, the term counterexample might be inappropriate, since our construction uses the axiom of choice.)

Notations. We denote by G the circle group

$$G = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\} ,$$

and by λ the normalized Haar measure on G (i.e., $\lambda(G) = 1$). $L^\infty(G)$ is the space of bounded λ-measurable complex valued functions on G, with the usual identification of functions equal almost everywhere. An *interval* in G is, by definition, the image of a real interval through the map $t \to e^{it}$. If $J \subset G$ is an interval, then P_J denotes the projection of $L^\infty(G)$ onto $L^\infty(J)$, i.e.,

$$(\mathrm{P}_J f)(\zeta) = \begin{cases} f(\zeta) , & \text{for } \zeta \in J , \\ 0 , & \text{for } \zeta \notin J . \end{cases}$$

For any $g \in G$, the shift operator $\mathbf{S}_g \in \mathcal{L}(L^\infty(G))$ is defined by

$$(\mathbf{S}_g f)(\zeta) = f(\zeta g^{-1}), \qquad \forall\, \zeta \in G .$$

A functional $\beta \in L^\infty(G)^*$ will be called ***shift invariant*** if $\mathbf{S}_g^* \beta = \beta$ holds for any $g \in G$. The simplest example of a shift invariant functional on $L^\infty(G)$ is given by the Haar integral, which we denote by $\tilde{\lambda}$:

$$\langle f, \tilde{\lambda} \rangle = \int_G f \, d\lambda .$$

A function $f : G \to \mathbb{C}$ is called a *step function* if G can be partitioned into a finite number of intervals on which f is constant. A function $f : G \to \mathbb{C}$ is called *regulated* if it is a uniform limit of step functions, or equivalently, if it has lateral limits in each point of G. We denote by $C(G)$ the space of continuous functions on G and by $\mathrm{Reg}(G)$ the space of regulated functions on G. Clearly $C(G) \subset \mathrm{Reg}(G) \subset L^\infty(G)$.

In the proof of the next lemma we shall need the following two remarks.

Remark 3.1. Suppose J and K are nonoverlapping intervals in G and $\beta \in L^\infty(G)^*$. Then obviously $\mathrm{P}_{J \cup K}^* \beta = \mathrm{P}_J^* \beta + \mathrm{P}_K^* \beta$, and moreover

$$\|\mathrm{P}_{J \cup K}^* \beta\| = \|\mathrm{P}_J^* \beta\| + \|\mathrm{P}_K^* \beta\| .$$

The proof of this remark is left to the reader.

Remark 3.2. If $\beta \in L^\infty(G)^*$ is shift invariant, then for any interval $J \subset G$

$$\|\mathrm{P}_J^* \beta\| = \lambda(J) \|\beta\| . \tag{3.1}$$

Indeed, if β is shift invariant, then $\|\mathrm{P}_J^* \beta\|$ depends only on $\lambda(J)$. Let $n \in \mathbb{N}$ and let $J \subset G$ be an interval of measure $\frac{1}{n}$. Since G can be partitioned into n translates of J (up to sets of measure zero, which don't matter) Remark 3.1 implies $\|\mathrm{P}_J^* \beta\| = \frac{1}{n} \|\beta\|$. This implies, using again Remark 3.1, that if $J \subset G$ is an interval of measure $\frac{k}{n}$, where $k \in \{0, 1, \ldots n\}$, then $\|\mathrm{P}_J^* \beta\| = \frac{k}{n} \|\beta\|$. Thus we have proved (3.1) for rational $\lambda(J)$. Using the fact that the map $\lambda(J) \to \|\mathrm{P}_J^* \beta\|$ is nondecreasing, it follows that (3.1) holds for any $\lambda(J) \in [0, 1]$.

Lemma 3.3. *There is a shift invariant functional $\beta \in L^\infty(G)^*$ with $\|\beta\| = 1$, such that $\langle f, \beta \rangle = 0$ for any $f \in \mathrm{Reg}(G)$.*

Proof. Let $D \subset G$ be a dense open set with $\lambda(D) < 1$. Then there is a nonzero and shift invariant functional $\beta_0 \in L^\infty(G)^*$ which is supported in

D, in the following sense: if $f \in L^\infty(G)$ and $f(\zeta) = 0$ for any $\zeta \in D$, then $\langle f, \beta_0 \rangle = 0$. This follows from Rudin [15, Examples 2.2 and Theorem 3.4]. The restriction of β_0 to continuous functions is of the form

$$\langle f, \beta_0 \rangle = k \int_G f \, d\lambda , \qquad \forall f \in C(G) , \tag{3.2}$$

for some fixed $k \in \mathbb{C}$. This follows from the uniqueness of the Haar measure (up to a constant factor) as a shift invariant regular Borel measure on G, see e.g. Cohn [1, p. 309]. We put

$$\beta_1 = \beta_0 - k\tilde{\lambda} .$$

Then β_1 is shift invariant, because both β_0 and $\tilde{\lambda}$ are shift invariant. For $f \in C(G)$ we have $\langle f, \beta_1 \rangle = 0$, according to (3.2) and the definition of β_1. Further, $\beta_1 \neq 0$ because $\beta_1 = 0$ would mean $\beta_0 = k\tilde{\lambda}$, which would contradict the fact that β_0 is supported in D. This enables us to define

$$\beta = \frac{1}{\|\beta_1\|} \beta_1 .$$

Obviously $\|\beta\| = 1$ and (like β_1) β is shift invariant and is zero on $C(G)$. Let us show that $\langle f, \beta \rangle = 0$ for regulated f. For this, it will be enough to show that $\langle f, \beta \rangle = 0$ if f is a step function. If f is a step function and $\epsilon > 0$, then it is not difficult to construct a continuous function $\tilde{f}$ which approximates f in the following sense: $\|\tilde{f}\| \leq \|f\|$ and $\tilde{f} - f$ is supported on the union of a finite number of intervals in G of total measure $\leq \epsilon$. Using Remarks 3.1 and 3.2 we can show that $|\langle \tilde{f} - f, \beta \rangle| \leq \epsilon \|f\|$. Letting $\epsilon \to 0$ and using that $\langle \tilde{f}, \beta \rangle = 0$, we get $\langle f, \beta \rangle = 0$. □

We introduce two more notations: $CB_0[0, \infty)$ is the space of continuous and bounded functions $u : [0, \infty) \to \mathbb{C}$ such that $u(0) = 0$, and $\mathrm{Reg}[0, \infty)$ is the space of regulated functions on $[0, \infty)$ (i.e., uniform limits of step functions). Clearly

$$CB_0[0, \infty) \subset \mathrm{Reg}[0, \infty) \subset L^\infty[0, \infty) .$$

Shift invariant operators on $L^\infty[0, \infty)$ are defined exactly as on $L^2[0, \infty)$, and of course they are causal.

Theorem 3.4. *There is a nonzero shift invariant bounded linear operator* $\mathcal{M} : L^\infty[0, \infty) \to CB_0[0, \infty)$ *such that* $\mathcal{M}u = 0$ *for any* $u \in \mathrm{Reg}[0, \infty)$.

Proof. Let $u \in L^\infty[0,\infty)$. We extend u to $(-\infty,\infty)$ by putting $u(t) = 0$ for $t < 0$. For any $t \in \mathbf{R}$, u_t will denote the element of $L^\infty(G)$ defined by

$$u_t(e^{2\pi i\sigma}) = u(t-\sigma)\,, \qquad \forall\, \sigma \in [0,1)\,.$$

Obviously $\|u_t\| \leq \|u\|$ and if $u \in \mathrm{Reg}[0,\infty)$ then $u_t \in \mathrm{Reg}(G)$. We define the function $\mathcal{M}u$ by

$$(\mathcal{M}u)(t) = \langle u_t, \beta\rangle\,, \qquad \forall\, t \in \mathbf{R}\,,$$

where β is the functional introduced in Lemma 3.3. Clearly $(\mathcal{M}u)(t) = 0$ for $t \leq 0$ and $\mathcal{M}u = 0$ if $u \in \mathrm{Reg}[0,\infty)$. It is also easy to see that for any $t \in \mathbf{R}$, $\|(\mathcal{M}u)(t)\| \leq \|u\|$ and

$$(\mathcal{M}\mathbf{S}_\tau u)(t) = (\mathcal{M}u)(t-\tau)\,, \qquad \forall\, \tau \geq 0\,.$$

Because of $\beta \neq 0$ there are $u \in L^\infty[0,\infty)$ for which $(\mathcal{M}u)(1) \neq 0$. It only remains to show that $\mathcal{M}u$ is continuous. Let $\epsilon \in (0,1)$ and put $g = e^{2\pi i\epsilon}$ and $J = \{e^{2\pi i\sigma} \mid \sigma \in [0,\epsilon)\}$. Then a simple reasoning shows that for any $t \in \mathbf{R}$, $u_{t+\epsilon} - \mathbf{S}_g u_t$ is supported in J, i.e.,

$$u_{t+\epsilon} - \mathbf{S}_g u_t = \mathrm{P}_J\left(u_{t+\epsilon} - \mathbf{S}_g u_t\right)\,.$$

Forming the product with β and using that β is shift invariant, we get

$$(\mathcal{M}u)(t+\epsilon) - (\mathcal{M}u)(t) = \langle u_{t+\epsilon} - \mathbf{S}_g u_t\,,\, \mathrm{P}_J^*\beta\,\rangle\,.$$

By (3.1) and the fact that $\|\beta\| = 1$, the above equality implies

$$\begin{aligned}|(\mathcal{M}u)(t+\epsilon) - (\mathcal{M}u)(t)| &\leq \lambda(J)\|u_{t+\epsilon} - \mathbf{S}_g u_t\| \\ &\leq 2\epsilon\|u\|\,.\end{aligned}$$

This shows that $\mathcal{M}u$ is actually Lipschitz continuous. □

The following proposition shows that $\mathcal{M}$ can not be described by any transfer function.

Proposition 3.5. *Let $\mathcal{M}$ be the operator introduced in Theorem* 3.4. *Then there is no function $\mathbf{H} : \mathbb{C}_0 \to \mathbb{C}$ such that for any $u \in L^\infty[0,\infty)$, denoting $y = \mathcal{M}u$,* (1.4) *should hold.*

Proof. Suppose there is such a function $\mathbf{H}$. Let $u = e_{-1}$, then $\hat{u}(s) = \frac{1}{s+1}$, in particular $\hat{u}(s) \neq 0$ for any $s \in \mathbb{C}_0$. But $\mathcal{M}u = 0$, whence $\mathbf{H}(s) = 0$ for any $s \in \mathbb{C}_0$. This implies $\mathcal{M} = 0$, which is a contradiction. □

Remark 3.6. In Harris and Valenca [7] the version for $p = \infty$ and $U = Y = \mathbb{C}$ of Theorem 2.3 is stated (their Theorem A.3 on p. 116). However, a careful reading reveals an error in the proof of another theorem (their Theorem A.1 on p. 113), which is used in the proof of the first mentioned theorem. The error occurs on p. 114, in the sentence "Let us order the intervals in such a way that $a_{k+1} \geq b_k$, ..." (there is nothing that assures the existence of such an ordering).

Remark 3.7. Theorem 3.4. enables us to answer a question raised in Weiss [17, Problem 3.10], in the sense that the state space representation theorem proved in that paper for input functions of class L^p, where $1 \leq p < \infty$, no longer holds for $p = \infty$. Details on this will be given elsewhere.

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