In this report we present a block diagonalization theorem which is designed to study the stability and bifurcation of rotating systems, or more generally, of relative equilibria. The context of the discussion is the energy-momentum method for mechanical systems with symmetry proposed by Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989] discovered crucial special cases of the block diagonalization theorem for uniformly rotating systems, including general nonlinear elasticity and geometrically exact rods. Our purpose is to abstract these examples and prove a general geometric theorem. We expect these general results will be important for rotating gravitational fluid masses as well.
§1 Introduction

In this report we present a block diagonalization theorem which is designed to study the stability and bifurcation of rotating systems, or more generally, of relative equilibria. The context of the discussion is the energy-momentum method for mechanical systems with symmetry. Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989] discovered crucial special cases of the block diagonalization theorem for uniformly rotating systems, including general nonlinear elasticity and geometrically exact rods. Our purpose is to abstract these examples and prove a general geometric theorem. We expect these general results will be important for rotating gravitational fluid masses as well.

For rotating systems the result says that a splitting of coordinates can be explicitly found on a linearized level which represent the rotational and internal vibrational modes. In these coordinates, the second variation of an augmented Hamiltonian is block diagonal. Of course coordinates can always be found in principle to do this, but we are able to do it explicitly enough to give useful stability and, we also believe, bifurcation criteria.

On the other hand, the symplectic form does not block diagonalize, indicating that the rotational and internal modes are in fact dynamically coupled. However, for purposes of the stability calculation, block diagonalization of the augmented energy is what is important. The off diagonal terms in the symplectic form (sometimes called Coriolis coupling terms) are, however, sufficiently simple that they should be useful for studying the dynamic interaction of the rotational and internal vibrational modes.

For rotating pseudo-rigid bodies, Lewis and Simo [1989] noticed that the computation of the definiteness of the second variation is considerably simplified by our result - in this case the simplification saves considerable computation time. In fact the symbolic and numerical manipulation required one to test a full $14 \times 14$ matrix for definiteness; block diagonalization reduces this to testing a $6 \times 6$ matrix for nonisotropic bodies and to a $3 \times 3$ matrix for the isotropic case.

§2 The Energy-Momentum Method

We begin our work in the context of standard mechanical systems with symmetry before any reductions have taken place. In other words, we begin with a symplectic manifold $(P, \Omega)$
rather than a Poisson manifold. In fact, shortly we shall specialize to the case of \( P = T^*Q \) and a Hamiltonian of the form kinetic plus potential.

Let \( G \) be a Lie group acting symplectically on \( P \) with an equivariant momentum mapping

\[
J : P \to \mathfrak{g}^* \tag{1}
\]

(see Abraham and Marsden [1978], Marsden [1981] or Marsden et al. [1982] for the standard definitions and results used here).

Let \( H : P \to \mathbb{R} \) be a given \( G \)-invariant Hamiltonian. A point \( z_e \) in \( P \) is called a relative equilibrium if there is a \( \xi \in \mathfrak{g} \), the Lie algebra of \( G \), such that for all \( t \in \mathbb{R} \),

\[
z(t) = \exp(t\xi) z_e, \tag{2}
\]

where \( z(t) \) is the dynamical orbit of \( X_H \), the Hamiltonian vector field of \( H \), with \( z(0) = z_e \).

The energy-momentum method rests on the following result.

### 2.1 Relative Equilibrium Theorem

A point \( z_e \) is a relative equilibrium iff there is a \( \xi \in \mathfrak{g} \) such that \( z_e \) is a critical point of \( H_\xi : P \to \mathbb{R} \), where

\[
H_\xi(z) = H(z) - \langle J(z) - \mu_e, \xi \rangle \tag{3}
\]

and \( \mu_e = J(z_e) \).

In (3), the Lie algebra element \( \xi \in \mathfrak{g} \) may be regarded as a Lagrange multiplier. Since \( J \) is conserved by the flow of \( X_H \), the set \( J - \mu_e = 0 \) is preserved, so one may regard it as a (non-holonomic) constraint set. It also follows that \( \xi \in \mathfrak{g}_{\mu_e} \), the isotropy algebra of \( \mu_e \) (with respect to the coadjoint action). Thus,

\[
\delta H_\xi(z_e) = 0 \tag{4}
\]

may be regarded as a (constrained) variational principle for relative equilibria.

The relative equilibrium theorem is readily verified. Of course it has a long history, going back to Lagrange and Poincaré for rotating systems. Like many basic results, it has been rediscovered in a number of contexts by various authors. Early references in our context are Arnold [1966], Smale [1970] and Marsden and Weinstein [1974]. As we shall state below, the
relative equilibrium theorem sometimes specializes to the *principle of symmetric criticality* (Palais [1979]).

The energy-momentum method proceeds as follows (see Holm et al. [1985] for the meaning of formal stability and related references).

**Energy-Momentum Method**

To test a relative equilibrium \( z_e \in P \) for formal stability:
1. Choose \( \xi \in g \) such that \( \delta H_\xi(z_e) = 0 \)
2. Choose a linear subspace \( S \subseteq T_{z_e}P \) such that
   i. \( S \subseteq \ker dJ(z_e) \) and
   ii. \( S \) complements \( T_{z_e}(G_{\mu_e}z_e) \) in \( \ker TJ(z_e) \), where \( G_{\mu_e} \subseteq G \) is the isotropy subgroup of \( \mu_e \).
3. Test
   \[
   \delta^2 H_\xi(z_e) 
   \]
   for definiteness as a bilinear form on \( S \).

The energy-momentum method "covers" the energy-Casimir method (Holm et al. [1985]) in the sense that if the latter applies and gives formal stability, so does the former. One difficulty with the energy-Casimir method is that on the reduced space \( P/G \), there may not be enough Casimirs to make the method effective, even to get the analogue \( \delta(H + C) = 0 \) of (4). This difficulty is genuine for the case of geometrically exact rods, for instance. See Simo, Posbergh and Marsden [1989] for further details.

The fact that \( \delta^2 H_\xi(z_e) \) drops to the reduced space follows from the next lemma.

### 2.2 Gauge Invariance Lemma

\[
\delta^2 H_\xi(z_e)(\eta_p(z_e), \delta z) = 0 \tag{5}
\]

for all \( \delta z \in \ker TJ(z_e) \) and \( \eta \in g \), where \( \eta_p \) denotes the infinitesimal generator of the group action on \( P \).
This follows readily from invariance of \( H \) and equivariance of \( J \). One can view (5) as a block diagonalization result on the unconstrained tangent space \( T_{z_e} P \), but it does not yield block diagonalization within the constrained subspace \( S \) in the energy-momentum method. It is the latter that we are concerned with.

One can identify any choice of \( S \) with the tangent space to the reduced space

\[
P_{\mu_e} = J^{-1}(\mu_e)/G_{\mu_e}
\]

at \([z_e]\) (assuming, as we shall, that \( \mu_e \) is a regular and generic value; c.f. Weinstein [1984]). However, it is easier to do our analysis directly on \( T_{z_e} T^* Q \) rather than on the quotient space. This is the usual situation found in constrained optimization problems. However, dropping the calculations to the quotient space at the appropriate point will play a useful role.

§3 Simple Mechanical Systems

We now specialize to the systems we will be studying. Let \( Q \) be a configuration manifold and \( P = T^* Q \) with its canonical symplectic structure and cotangent coordinates \((q', p)\) in the finite dimensional case. (Whenever we use coordinates, we assume \( Q \) is finite dimensional, although the results are not restricted to this case.) Coordinates on the velocity phase space \( TQ \) are similarly denoted \((q^i, \dot{q}^i)\).

Let \( g \) denote a Riemannian metric on \( Q \); in coordinates we write the components of \( g \) as \( g_{ij} \) as usual, and we write \( g^{ij} \) for the inverse tensor. Let \( K : TQ \to \mathbb{R} \) denote the corresponding kinetic energy, i.e.,

\[
K(q, \dot{q}) = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j.
\]

Let \( V : Q \to \mathbb{R} \) be a given potential.

Assume \( G \) acts on \( Q \) (by a left action) and hence on \( T^* Q \) by the cotangent lift, so the equivariant momentum map is given by

\[
\langle J, \xi \rangle(\alpha_q) = \langle \alpha_q, \xi_Q(q) \rangle.
\]

In coordinates, we define the action coefficients \( A^i_\alpha(q) \) by writing
\[ [\xi_Q(q)]^i = A^i_a(q) \xi^a \] (3)

where \( a, b, c, \ldots \) denote coordinate indices for the Lie algebra \( \mathfrak{g} \). Thus (2) becomes

\[ J_a(q, p) = p_i A^i_a(q). \] (4)

We assume that \( \mathcal{G} \) acts on \( Q \) by isometries and that the potential \( V \) is \( \mathcal{G} \)-invariant. For elasticity, for instance, this is the requirement of material frame indifference. Note that (3) of §2 reads

\[ H_\xi(q, p) = \frac{1}{2} g^{ij} p_i p_j + V(q) - p_i A^i_a(q) \xi^a. \] (5)

Define the moment of inertia tensor \( I \) for the system locked at \( q \in Q \) by

\[ I_{ab}(q) = g_{ij}(q) A^i_a(q) A^j_b(q) \] (6)

(alternatively, in terms of the \( q \)-dependent inner product \( \langle \xi, \eta \rangle := \langle \xi_Q(q), \eta_Q(q) \rangle \) on \( \mathfrak{g} \), we have \( \langle \xi, \eta \rangle = I_{ab}(q) \xi^a \eta^b \), and define the augmented potential \( V_\xi \) by

\[ V_\xi(q) = V(q) - \frac{1}{2} I_{ab}(q) \xi^a \xi^b. \] (7)

One can readily verify the following (see Abraham and Marsden [1978] and Palais [1979]) by writing out the conditions \( \delta H_\xi = 0 \) in 2.1. A more elegant argument is, however, given below.

3.1 Principle of Symmetric Criticality A point \( z_e = (q^i, p_j) \) is a relative equilibrium if and only if there is a \( \xi \in \mathfrak{g}_{q_e} \) such that

I \( p_i = g_{ij} A^j_a \xi^a \) (i.e., \( p_e \) is the Legendre transform of \( \xi_Q(q_e) \)) \hspace{1cm} (8a)

and

II \( q^i \) is a critical point of \( V_\xi \) (i.e., \( \frac{\delta}{\delta q} V_\xi \bigg|_{q = q_e} = 0. \)) \hspace{1cm} (8b)

This is useful for carrying out the computations that follow. We also observe that \( V_\xi \) is \( \mathcal{G}_\xi \)-invariant, and so induces a function on \( Q/\mathcal{G}_\xi \).

Define the one-form \( A_\xi \) on \( Q \) by
or abstractly, \( \mathbf{A} (\mathbf{q}) = [\xi_{Q}(\mathbf{q})]^{b} \), where \( b \) denotes the index lowering operation with respect to the metric \( g_{ij} \). In other words, \( \mathbf{A} (\mathbf{q}) \) is the Legendre transform of \( \xi_{Q}(\mathbf{q}) \). We remark that \( \mathbf{A} \) may be viewed as a G-connection for the bundle \( Q \to Q/G \) and that this connection plays an important role in Berry's phase; cf. Marsden, Montgomery and Ratiu [1988]. Now notice that at equilibrium, (8a) says

\[
\mathbf{p}_e = \mathbf{A} (\mathbf{q}_e).
\]

(10)

Also note that

\[
\mathbf{H}_{\xi}(\mathbf{q}, \mathbf{p}) = \mathbf{K}_{\xi}(\mathbf{q}, \mathbf{p}) + \mathbf{V}_{\xi}(\mathbf{q}, \mathbf{p}) + \langle \mu_e, \xi \rangle
\]

(11)

where \( \mathbf{K}_{\xi}(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \| \mathbf{p} - \mathbf{A} (\mathbf{q}) \|^2 \), and \( \mathbf{V}_{\xi} \) is given by (7). By (10), \( \mathbf{K}_{\xi} \) has a critical point at \( \mathbf{z}_e \). Thus, (8b) is a direct consequence of the relative equilibrium theorem and (11).

In the energy-momentum method we shall use a special choice of \( \mathcal{S} \), namely

\[
\mathcal{S} = \{ \mathbf{v}_{\mathbf{z}_e} \in T_{\mathbf{z}_e} T^*Q \mid T\pi_Q \cdot \mathbf{v}_{\mathbf{z}_e} \text{ is g-orthogonal to } T(G_{\mathbf{z}_e} \cdot \mathbf{q}_e) \text{ and } \mathbf{v}_{\mathbf{z}_e} \in \ker[TJ(z_e)] \}.
\]

(12)

Letting coordinates on \( TT^*Q \) be denoted

\[
(q^i, p_i, \delta q^i, \delta p_i),
\]

(12) reads, with the help of (8a),

\[
\mathcal{S} = \{(q^i, p_i, \delta q^i, \delta p_i) \mid g_{ij}(\delta q^j) A^i_a \chi^a = 0 \text{ for all } \chi \in \mathfrak{g}_{\mu_e} \text{ and }
(\delta p)^i A^i_a + g_{ij} A^j_b \frac{\partial A^i_a}{\partial q^k} (\delta q)^k = 0 \}
\]

(12')

§4 Rigid Variations

One version of the cotangent bundle reduction theorem (see Abraham and Marsden [1978] and Kummer [1981], Montgomery [1986] and references therein) states that the reduced space
(T*Q)_{\mu_e} is a symplectic bundle over $T^*(Q/G)$ with fiber the coadjoint orbit through $\mu_e$. Thus there is an isomorphism

$$T_{[x_e]}(T^*Q)_{\mu_e} \cong g/\mathfrak{g}_e \times T_{[x_e]}(T^*(Q/G)) \cong g/\mathfrak{g}_{\mu_e} \times (U_{\text{INT}} \times U^*_{\text{INT}})$$

where $U_{\text{INT}}$ is a model space for $Q/G$. For $G = SO(3)$, $U_{\text{INT}}$ models the configuration space for the internal modes, while $g/\mathfrak{g}_e \cong T_{\mu_e}O_{\mu_e}$ models the phase space for rigid modes. Our goal is to realize this decomposition explicitly, in such a way that $\delta^2H_\xi(z_e)$ block diagonalizes. The bundle $(T^*Q)_{\mu} \to T^*(Q/G)$ with fiber $O_\mu$ also has a natural connection (Montgomery [1986]) and our decomposition should be related in some way to the horizontal-vertical split for this connection. However we proceed directly here; see also the comments in §5 below.

We will define two subspaces $S_{\text{RIG}}$ and $S_{\text{INT}}$ of $S$ and further subspaces $U_{\text{INT}}$ and $U^5_{\text{INT}}$ of $S_{\text{INT}}$ such that

$$S = S_{\text{RIG}} \oplus S_{\text{INT}} \cong S_{\text{RIG}} \oplus (U_{\text{INT}} \oplus U^5_{\text{INT}})$$

relative to which $\delta^2H_\xi(z_e)$ will be block diagonal. As above, the first component $S_{\text{RIG}} \cong g/\mathfrak{g}_{\mu_e}$ of $S$ is isomorphic to the tangent space to the coadjoint orbit through $\mu_e$. As we shall see, this component will also carry the coadjoint orbit symplectic structure. This first component is defined in terms of rigid variations as follows: Let

$$g_Q = \{ \eta_Q(q) \in TQ \mid \eta \in g \text{ and } q \in Q \}$$

and let $Tg_Q \subset TTQ$ be its tangent bundle.

4.1 Definition Let $V_{\text{RIG}} = s(Tg_Q)$ where $s : T^2Q \to T^2Q$ is the canonical involution. Alternatively, $V_{\text{RIG}}$ consists of double tangents of curves denoted by $\Delta q$ (identified with velocity variations of superposed rigid body motions in the case of $SO(3)$)

$$\Delta q = \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{dt} \right|_{eu=0} \exp(\varepsilon \eta(t)) q(t)$$

where $\eta(t)$ is a curve in $g$ with $\eta(0) = \eta$ and $q(t)$ is a curve in $Q$. (The canonical involution in effect swaps the order of differentiation.)
In coordinates, if we write elements of $V_{\text{RIG}}$ as

$$(q^i, \dot{q}^i, \Delta q^i, \Delta \dot{q}^i),$$  \hspace{1cm} (3a)$$

then we find that

$$\Delta q^i = A_i^a \eta^a \quad \text{and} \quad \Delta \dot{q}^i = \frac{\partial A_i^a}{\partial q^k} \dot{q}^k \eta^a + A_i^a \zeta^a.$$  \hspace{1cm} (3b)$$

Now let $\mathcal{F}L: TQ \to T^*Q$ be the Legendre transform given by

$$p_i = g_{ij} \dot{q}^j$$  \hspace{1cm} (4)$$

and let $\mathcal{T} \mathcal{F}L : TTQ \to TT^*Q$ be its tangent map. Set

$$\mathcal{S}_{\text{RIG}} = \mathcal{T} \mathcal{F}L \cdot V_{\text{RIG}} \cap \mathcal{S},$$  \hspace{1cm} (5)$$

where $\mathcal{S}$ is defined by (12) of §3. If we let $\mathfrak{g}^\perp_{\mu_e}$ denote the (q-dependent) orthogonal complement of $\mathfrak{g}_{\mu_e}$ in the metric $I_{ab}$, then one finds that $\mathcal{S}_{\text{RIG}}$ is parametrized by elements $\eta \in \mathfrak{g}^\perp_{\mu_e}$ as follows: we write elements of $\mathcal{S}_{\text{RIG}}$ as

$$(q^i, p_j, \Delta q^i, \Delta p_j)$$  \hspace{1cm} (6a)$$

where

$$\Delta q^i = A_i^a \eta^a$$  \hspace{1cm} (6b)$$

and

$$\Delta p_i = - \frac{\partial A^a_{jk}}{\partial q^l} \eta^a p_k + g_{ij} A^a \xi^a$$  \hspace{1cm} (6c)$$

where $\eta \in \mathfrak{g}^\perp_{\mu_e}$ and where $\xi \in \mathfrak{g}$; the condition that (6a) belongs to ker($T_{\xi}J$) is equivalent to the relation

$$\zeta^a = I^{ab} (\text{ad}_{\eta}^* \mu_e)_b$$  \hspace{1cm} (7)$$

i.e., $\xi^b = \text{ad}_{\eta}^* \mu_e$, so $\zeta$ is determined by $\eta$. One checks that $\eta \in \mathfrak{g}^\perp_{\mu_e}$ as well.
§5 The Internal Vibration Space

Now we define a complement to $S_{\text{RIG}}$ in $S$. We will do this by a constructive procedure that can be effectively carried out in examples. As we have mentioned, this complement appears to be not the same as, but related to the complement to the vertical space relative to a natural connection on the coadjoint orbit bundle $(T^*Q)_\mu \to T^*(Q/G)$. In this regard we note that the metric naturally induced on $Q/G$ is Wilson's $G$-matrix (see Wilson, Decius and Cross [1955]). Our decomposition appears to be finer than the one proposed by Guichardet [1984] and discussed by Iwai [1988]. Notice that we have connections on all levels of this tower of bundles

$$T^*Q \supset J^{-1}(\mu) \to (T^*Q)_\mu \to T^*(Q/G)$$

where $J^{-1}(\mu) \to (T^*Q)_\mu$ is regarded as a $G_\mu$ bundle and $(T^*Q)_\mu \to T^*(Q/G)$ is regarded as an $O_\mu$ bundle, where $O_\mu$ is the coadjoint orbit through $\mu$.

The Guichardet-Iwai results appear to be largely concerned with the bundle $J^{-1}(\mu) \to (T^*Q)_\mu$; the fact that the reduced space $(T^*Q)_\mu$ still has the factor $O_\mu$ seems to be the reason the connection on the $G_\mu$ bundle $J^{-1}(\mu) \to (T^*Q)_\mu$ is not sufficient to completely isolate the vibrational modes from the rotational ones. We believe that the $O_\mu$ bundle fills this gap. These remarks aside, we turn to the explicit construction of $S_{\text{INT}}$. To do this, we first describe $U_{\text{INT}}$.

Recall that the augmented potential $V_\xi$ is given by

$$V_\xi = V + L_\xi$$

(1a)

where

$$L_\xi(q) = -\frac{1}{2} \langle \xi Q(q), \xi Q(q) \rangle.$$  

(1b)

For mechanical systems undergoing stationary rotations about $\xi$, i.e., $G = SO(3)$ and $G_{\mu_\xi}$ = rotations about the axis $\mu_\xi$ which is parallel to $\xi$, we note that $L_\xi$ gives the potential of the centrifugal force. Now define $U_{\text{INT}}$ as the subspace on which $V_\xi$ or equivalently $L_\xi$ looks objective in the sense of nonlinear elasticity (cf. Marsden and Hughes [1984]). More precisely:
5.1 Definition

\[ U_{\text{INT}} = \{ \delta q \in T_\xi Q \mid \langle \delta q, (L_{\eta Q} dC_\xi)(q) \rangle = 0 \text{ for all } \eta \in \mathfrak{g}_{\mu}^\perp \text{ and } \langle \delta q, \chi_Q(q) \rangle = 0 \text{ for all } \chi \in \mathfrak{g}_{\mu} \} \]  

(2)

where the first pairing is the natural pairing between vectors and one forms while the second is the metric inner product.

Since \( V_\xi \) has a critical point at \( q_e \) (by the principal of symmetric criticality) and \( V \) is \( G \)-invariant, we find that

\[ \langle \delta q, (L_{\eta Q} dC_\xi)(q) \rangle = \delta^2 V_\xi(q_e)(\delta q, \eta_Q(q)) \]

(3)

and so we see that the geometric condition (2) is exactly what is needed to block diagonalize \( \delta^2 V_\xi(q_e) \) within \( S \). In coordinates, the first condition on \( \delta q \) defining \( U_{\text{INT}} \) is the geometric condition

\[ \delta q^i \eta^a \xi^b \xi^c \frac{\partial}{\partial q^i} \left[ A^k_a \frac{\partial}{\partial q^k} (A_b^c \eta^m g_{tm}) \right] = 0; \]

(2')

the second condition is just the defining condition on \( S \). Now we are ready to define \( S_{\text{INT}} \).

5.2 Definition \( S_{\text{INT}} = \{ \delta z \in T_\xi T^*Q \mid \delta q \in U_{\text{INT}} \text{ and } \delta z \in \ker[TJ(z_e)] \} \subset S \).  

(4)

5.3 Proposition \( S = S_{\text{RIG}} \oplus S_{\text{INT}} \).

This is easy to check. The idea is that \( S_{\text{RIG}} \cap S_{\text{INT}} = \{ 0 \} \), that \( \dim S_{\text{RIG}} = \dim(\mathfrak{g}/\mathfrak{g}_\mu) \) and that \( S_{\text{INT}} \) is determined by \( \dim(\mathfrak{g}/\mathfrak{g}_\mu) \) equations. Also, we write

\[ S_{\text{INT}} = U_{\text{INT}} \oplus U_{\text{INT}}^\xi \]

(5)

where \( U_{\text{INT}}^\xi = \{ \delta p - A^\xi(q_e) \mid \delta p \in U_{\text{INT}}^* \} \) is the dual space with a momentum shift by \( A^\xi \) (see equation (10) of §4). The relation (5) is really a coordinate description; to do it intrinsically, we use the metric connection to split \( T_\xi T^*Q = T_{q_e} Q \oplus (T_{q_e} Q)^* \) (this split is in fact nothing more than what we do in coordinates to identify accelerations and momenta with vectors) then we take the
horizontal and vertical splitting of \( U_{\text{INT}} \) in \( S_{\text{INT}} \) with the vertical component followed by the momentum shift by \( A^\xi(q_e) \).

We remark here that even if \( G \) is abelian (for instance, \( G = S^1 \) in the case of planar coupled rigid bodies) then the decompositions are not trivial: while \( S_{\text{RIG}} = \{0\} \) in this case, \( S_{\text{INT}} = U_{\text{INT}} \oplus U_{\text{INT}}^\xi \) is still not a trivial decomposition.

Now \( H_\xi = K_\xi + V_\xi + (\mu_\xi, \xi) \) and we have arranged for \( V_\xi \) to be block diagonal. As far as \( K_\xi \) is concerned, we compute in coordinates that

\[
K_\xi = \frac{1}{2} \epsilon^{ijkl} (p_i - g_{ik} A^k \xi_a)(p_j - g_{jm} A^m \xi_b).
\]

Thus, since \( p_i = g_{ik} A^k \xi_a \) at equilibrium, we get

\[
\delta^2 K_\xi(z_\varepsilon) \cdot (\delta z, \delta z') = g^{ij} \delta p_i \delta p_j.
\]

It is clear that \( \delta^2 K_\xi \) block diagonalizes from \( \delta^2 V_\xi \) within \( U_{\text{INT}} \oplus U_{\text{INT}}^\xi \) by construction. Regarding the block diagonalization of \( \delta^2 K_\xi \) on \( S_{\text{RIG}} \oplus S_{\text{INT}} \), we shall use some further interesting identities.

First, here is an equivalent characterization of \( U_{\text{INT}} \) in terms of superposed motions:

5.4 Proposition Let \( q_e \in Q \) be a curve tangent to \( \delta q \) at \( q_e \), let \( \eta \in g_{\mu_e}^\perp \) and let \( \eta_\varepsilon = \text{Ad}_{\exp(\varepsilon \xi)}(\eta) \). Then \( U_{\text{INT}} \) is characterized by those \( \delta q \) orthogonal to \( T_{q_e}(G_{\mu_e} \cdot q_e) \) and satisfying

\[
\left. \frac{d}{d\varepsilon} \langle \xi_Q(q_e), (\eta_\varepsilon)_Q(q_e) \rangle \right|_{\varepsilon=0} = 0
\]

or, equivalently,

\[
\left. \frac{d}{d\varepsilon} \langle \xi_Q(\exp(\varepsilon \xi) q_e), (\eta_\varepsilon)_Q(\exp(\varepsilon \xi) q_e) \rangle \right|_{\varepsilon=0} = 0.
\]

This is verified by a direct coordinate calculation. We can lift this expression to get an alternative characterization of \( S_{\text{INT}} \). We consider the momentum map \( J \) restricted to \( g_{\mu_e}^\perp \) and regarded as a function on \( TQ \). In other words, for \( \zeta \in g_{\mu_e}^\perp \), set
\[
J(\zeta)(\delta q) = \langle \zeta Q(q), \delta q \rangle = g_{ij} A^i_\zeta \xi^*(\delta q)^j. \tag{9}
\]

Now consider the condition

\[
\frac{d}{dt} J(\zeta)(\delta q) \bigg|_{t=0} = 0 \tag{10}
\]

where \( \zeta \) is to evolve as \( \dot{\zeta} = [\xi, \zeta] \) which is consistent with (8a) and \( \zeta \in \mathfrak{g}_{ue} \); here \( \xi \) is the Lie algebra element giving the relative equilibrium. Equation (10) defines a condition on \( T(TQ) \). We shall regard it as a condition on \( T_q(T^*Q) \) via the Legendre transform. For simplicity we still write the resulting condition as \( J = 0 \).

5.5 Proposition

\[
\mathcal{S}_{\text{INT}} = \{ J(z_e) = 0 \} \cap \mathcal{S}. \tag{11}
\]

The condition \( T J(z_e) \cdot \delta z = 0 \) reads

\[
\delta p_i A^i_\zeta(q) + p_i \frac{\partial A^i_\zeta}{\partial q^k} dq^k = 0 \tag{12}
\]

and using this, one can express the conditions defining \( \mathcal{S}_{\text{INT}} \) entirely in terms of \( \delta q \). This recovers the space \( U_{\text{INT}} \), which models \( T_{\zeta_e}(Q/G) \), and then one gets, as before,

\[
\mathcal{S}_{\text{INT}} = U_{\text{INT}} \oplus U^\xi_{\text{INT}}.
\]

§ 6 Block Diagonalization

The block diagonalization results for \( \delta^2 H_\zeta \) follow from two basic formulas:

6.1 Proposition Let \( \Delta z \in \mathcal{S}_{\text{RIG}} \) and \( \delta z \in T_{\xi_e} P. \) Then

\[
\delta^2 H_\zeta(z_e)(\Delta z, \delta z) = \frac{d}{dt} \langle \zeta Q(q), \delta q \rangle - \langle [\xi, \eta], \delta J(z_e) \cdot \delta z \rangle. \tag{1}
\]
where \( \Delta z \) has associated \( \eta \) and \( \zeta \) as in (3b) and (7) of §4.

### 6.2 Proposition

Let \( \delta z_1 \) and \( \delta z_2 \in S_{\text{INT}} \); then

\[
\delta^2 H_\xi(z_e)(\delta z_1, \delta z_2) = \delta^2 K_\xi(z_e) \cdot (\delta z_1, \delta z_2) + \delta^2 V_\zeta(q_e)(\delta q_1, \delta q_2)
\]  

(2)

Proposition 6.1, which is proved by direct calculation, shows that \( \delta^2 H_\xi(z_e) \) block diagonalizes on \( S_{\text{RIG}} \oplus S_{\text{INT}} \), i.e., if \( \Delta z \in S_{\text{RIG}} \) and \( \delta z \in S_{\text{INT}} \), then

\[
\delta^2 H_\xi(z_e)(\Delta z, \delta z) = 0.
\]  

(3)

Proposition 6.2 then follows from our earlier calculations. It also follows that if \( \Delta z \in S_{\text{RIG}} \) and \( \Delta \bar{z} \in S_{\text{RIG}} \), then

\[
\delta^2 H_\xi(z_e)(\Delta z, \Delta \bar{z}) = \frac{d}{dt} \langle \zeta_Q(q), \zeta_Q \rangle
\]  

(4)

which is a generalization of the rigid body second variation formula for motion on the coadjoint orbit \( O_{qe} \) with the metric \( I_{ab} \). We summarize:

### 6.3 Theorem

The relative equilibrium \( z_e \) is formally stable (with \( \delta^2 H_\xi(z_e) \) on \( S \) positive definite) if and only if

1. \( \frac{d}{dt} \langle \zeta_Q(q), \zeta_Q(q) \rangle \) is positive definite on \( S_{\text{RIG}} \)

and

2. \( \delta^2 V_\xi(q_e) \) is positive definite on \( U_{\text{INT}} \).

We note that \( \delta^2 V_\xi(q_e) \) separates (in coordinates on \( U_{\text{INT}} \)) into \( \delta^2 V(q_e) \) plus a term quadratic in \( \xi \). Thus, if I is implied by a condition of the form \( \| \xi \| \leq \sqrt{\lambda_{\text{min}}} \), where \( \| \| \) is a suitable norm and \( \lambda_{\text{min}} \) is the minimum (non-zero) eigenvalue of \( \delta^2 V(q_e) \); one has to take care here that \( V \) itself does not have a critical point at \( q_e \), so \( \delta^2 V(q_e) \) does not make intrinsic sense.

To see how this works in examples, see Simo, Posbergh and Marsden [1989] and Lewis and Simo [1989].

As far as the symplectic form \( \Omega \) is concerned, we have
6.4 Theorem Let $\Delta z \in S_{\text{RIG}}$ and $\delta z \in T_{z_e} \mathcal{P}$. Then

$$\Omega(z_e)(\Delta z, \delta z) = -\langle \eta, \delta J(z_e) \cdot \delta z \rangle + \langle \zeta_{\mathcal{Q}}(q_e), \delta q \rangle. \quad (5)$$

Notice that in an appropriate sense, $\delta^2 H_{\xi}(z_e)$ on $S_{\text{RIG}} \times T_{z_e} \mathcal{P}$ is the time derivative of the symplectic form $\Omega$!

From (5) and (7) of §4 one finds that on $S_{\text{RIG}} \times S_{\text{RIG}}$, $\Omega$ gives the coadjoint orbit symplectic form

$$\Omega(z_e)(\Delta z, \Delta \bar{z}) = -\langle \mu_e, [\eta, \pi] \rangle, \quad (6)$$

while on $S_{\text{RIG}} \times S_{\text{INT}}$ we have the cross terms

$$\Omega(z_e)(\Delta z, \delta z) = \langle \zeta_{\mathcal{Q}}(q_e), \delta q \rangle, \quad (7)$$

which depend on the $\delta q$ components alone.

We can summarize the situation with the following matrices:

\[
\begin{bmatrix}
\text{Generalized} \\
\text{Rigid Body} \\
\text{Second Variation}
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & \delta^2 K_{\xi}(z_e)
\end{bmatrix}
\]

for the amended energy, and

\[
\begin{bmatrix}
0 & d^2 V_{\xi}(z_e)
\end{bmatrix}
\]

for the

...
For information on the "magnetic" term, and its interpretation as a curvature, we refer the reader to Kummer [1981]. Also, the coupling terms can be interpreted in terms of the curvature of the connection on the coadjoint bundle $T^*Q \to T^*(Q/G)$; see Montgomery [1986] and Lewis, Marsden, Montgomery and Ratiu [1986].

References


