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ROYAL SIGNALS & RADAR ESTABLISHMENT

TWISTONS

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Abstract

The twiston is a new kind of particle defined in the context of non-relativistic quantum mechanics. It possesses mass, charge, spin and magnetic moment. If its gyromagnetic ratio is given by the Dirac value ($g = 2$), the classical analogue is that of a rapidly rotating, charged black hole with naked singularity. However, this paper is concerned with (first-) quantised twistons and with spin 1/2. The electromagnetostatic field of such a particle is shown to be non-singular except at a single point. Further, it contains finite self-energy. For a twiston of charge Ze, the self-energy contributes a fraction of its total rest mass that approximates to $4Z^2\alpha/g$, e being the charge of the electron while α is the fine structure constant. Its electrostatic potential violates the law "like charges repel, unlike charges attract" at small distances. Interactions between twistons of spin 1/2 are then considered. Identical twistons in the triplet state, with their spins aligned parallel, are found to repel strongly, there being a "brick wall" potential at small distances. Finally the existence of a non-linear relationship between the twiston field and vector potential, suggestive of a non-Abelian gauge theory, is revealed.

Contents

1 INTRODUCTION

I want to introduce a new kind of particle called a *twiston*. The underlying framework will be that of non-relativistic quantum mechanics though ultimately this must give way to a relativistic theory. Indeed twistons have their roots in Penrose's theory of *twistors* and in general relativity. However, I do not propose to discuss those theories in any depth apart from stating some of the pertinent results that motivate this paper.

A twiston has mass, spin, electric charge and a magnetic dipole moment parallel to its spin. A classical twiston that has Dirac gyromagnetic ratio is the Euclidean analogue of a charged, rapidly rotating, black hole. Indeed the field it produces possesses the same ring singularity. However, quantising the spin and choosing it to be 1/2, removes the singularity.

Section 2, which may be omitted by the reader, provides a brief outline to the historical development that motivates the introduction of twistons. The hypotheses that emerge from research over the last twenty years is that the centres of mass and charge of a particle with spin are located not in real space-time but rather in a space-time with four complex dimensions. Further, the imaginary displacements of these centres lie parallel to the spin vector.

Section 3 inherits the philosophy of the previous section but the framework is that of Euclidean space with *three* complex dimensions. A spinning, charged particle is deemed to be located at a complex point of this space, the *complex centre of charge*, its imaginary displacement being proportional to its intrinsic angular momentum vector. The classical Coulomb potential, generated from the complex source point, is then presented. When examined at real field points, its real and imaginary parts are interpreted respectively as electrostatic and magnetostatic potentials. At large distances, these yield the standard expressions for an electric monopole and magnetic dipole respectively. The particle's gyromagnetic ratio is determined by the magnitude of the imaginary displacement of the source and equals the Dirac value when the complex centres of mass and charge coincide.

At small distances the potentials are singular on a ring analogous to that of the naked singularity of a rapidly rotating black hole. However, when the spin is quantised according to the standard prescription and chosen to be 1/2, it is found that the ring singularity disappears. Henceforth, spin 1/2 is assumed and the potentials evaluated explicitly. In particular, the electrostatic potential is finite everywhere, deviating from the standard $1/r$ form at around the Compton wavelength. Further, a potential that is repulsive at large distances becomes attractive at small distances and *vice versa*.

Section 4 presents the classical, electromagnetic energy density associated with the complex potential derived in the previous section. It too possesses a ring singularity and indeed the integrated self-energy is infinite. Quantising the energy density, however, and specialising to spin 1/2, yields a finite answer. For a twiston of mass m, charge Ze and gyromagnetic ratio g, the self-energy approximates to $(4Z^2/g)\ \alpha\ mc^2$, e being the charge of the electron and α being the fine structure constant. For $g = 2$ and $Z = 1$, roughly 1.5 % of the twiston's mass is electromagnetic in origin.

Section 5 begins by investigating the interaction between a classical twiston and a classical, external electro-magnetostatic field. The field is represented by a complex potential, assumed analytic, which in turn is extended analytically into complexified Euclidean three space. The product of the twiston's charge with the external potential is then evaluated at the complex point defined by the twiston. The real part is then taken to be the energy of coupling; and it is seen that this includes the coupling between the magnetic moment of the twiston and the external magnetic field.

The classical calculation is then used to motivate an expression for the energy of interaction between a pair of spinning twistons. Quantisation is then performed for twistons of spin 1/2. The cases of identical particles and distinct particles are both considered. In each case three interactions are derived, two for the triplet state and one for the singlet. For identical particles with their spins aligned parallel a quantum potential is found which has the behaviour of a "brick wall" at a certain finite range (the Compton wavelength for $g = 2$).

Section 6, with an eye on relativistic extensions, considers the possibility of defining a quantum, vector potential for the twiston. By applying a certain well known commutation relation, an expression is derived for the field in terms of the potential that comprises a pair of terms:

- a term linear in the potential that is effectively the *curl*;
- a term quadratic in the potential that involves the structure constants of the rotation group.

This is suggestive of a non-Abelian behaviour at short distances but the theme is not developed here.

2 MOTIVATION

The purpose of this section is to motivate the twiston concept. There have been several related developments in the literature that stem from the early 1960s:

1. The work of Newman and colleagues, within the framework of general relativity, relates spinning black holes to spinless black holes, whether charged or uncharged, via complex coordinate transformations. (See eg [1] for discussion and references.) That is to say, methods exist that generate new solutions to Einstein's equations from old by means of an imaginary displacement a. If the old solution describes a spinless system of mass m then the new solution possesses an intrinsic spin of mca (c being the speed of light)

2. Newman and Winicour [2] regard real Minkowski space as a subspace of complexified Minkowski space and consider the kinematics of a charged, spinning particle in uniform motion. Working with four complex dimensions leads naturally to the concept of the *complex centre of mass* displaced from the real space by an amount $i\mathbf{J}/mc$, ($i = \sqrt{(-1)}$), where $\mathbf{J}$ is the intrinsic spin vector [1]. In a similar manner, they define the complex *centre of charge* and this is displaced from the real space by an amount $i\mathbf{M}/q$, q being the charge and $\mathbf{M}$ being the magnetic moment. It follows that the complex centres of mass and charge coincide if and only if the particle has gyromagnetic ratio given by the Dirac value ($g = 2$).

[1] Pauli-Lubanski vector

(This result also holds [3] for stationary, isolated systems in general relativity. Curved space-time gives rise to a canonical 4-parameter family of null hypersurfaces isomorphic to Minkowski space-time. The momentum, spin, centres of mass etc are all defined with respect to this asymptotic *observation space.*)

3. The theory of *twistors* introduced by Penrose [4] describes massless particles by one-index twistors (Z^α) and massive particles by two-index twistors ($A^{\alpha\beta}$). In complexified Minkowski space these correspond to a pair of spinor fields, satisfying certain conformally invariant differential equations and representing the angular momenta of the respective particles. The (complex) centre of mass of a particle is defined by those points for which its angular momentum spinor vanishes. For a massless particle this yields a complex two-dimensional surface each of whose tangent vectors is null (light-like) while a massive particle gives rise to a geodesic parallel to the momentum vector and displaced into the complex by an amount $i\mathbf{J}/mc$. Twistors may also be used in an active mode to generate solutions to Maxwell's equations and to Einstein's equations. Use of $A^{\alpha\beta}$ for a spinless, massive particle, for example, generates the standard Coulomb fields singular on the particle's real world line. When spin is introduced, classically, the fields are the specially relativistic analogues of rotating black holes and exhibit ring singularities.

I hope that enough has been said to motivate the following sections without immersing the reader in unnecessary detail. The interested reader may of course study the relevant literature if so desired. In fact, I shall henceforth restrict attention to Euclidean space, albeit with three complex dimensions. I shall be concerned not so much with the centre of mass but rather with the centre of charge; and, to allow for the fact that the gyromagnetic ratio g may differ from the Dirac value, I shall assume that the centre of charge has an imaginary displacement $ig\mathbf{J}/2mc$.

3 COULOMB POTENTIAL OF A TWISTON

3.1 CLASSICAL POTENTIAL

Consider real, Euclidean three-space, R^3, embedded within complexified Euclidean three-space C^3 and choose real origin O. I shall examine the effects in R^3 of sources in C^3. So, consider a real charge q located at the "source point" Y whose displacement from O has imaginary part $i\mathbf{y}$, $\mathbf{y}$ being a real three-vector.

Let X be the real "field point" whose complex displacement YX from Y is given by

$$\mathbf{s} = \mathbf{x} - i\mathbf{y}, \tag{1}$$

$\mathbf{x}$ being real. Then the potential at X will depend upon the quantity

$$\mathbf{s}^2 = (r^2 - y^2 - 2i\, \mathbf{x}.\mathbf{y}), \tag{2}$$

where $r = |\mathbf{x}|$ and $y = |\mathbf{y}|$. Notice, regarding $\mathbf{y}$ as fixed but allowing $\mathbf{x}$ to range over R^3, that expression (2) vanishes on the *Kerr ring*:

$$\text{RING} = (\,(r = y) \wedge (\mathbf{x}.\mathbf{y} = 0)\,). \tag{3}$$

For future reference, I shall refer to the plane, two-dimensional surface orthogonal to $\mathbf{y}$, containing and bound by RING as the *Kerr disc*:

$$\text{DISK} = (\ (r \leq y) \wedge (\mathbf{x}.\mathbf{y} = 0)\ ). \tag{4}$$

Next, avoiding RING, write

$$\mathbf{z}^2 = R^2 \ \exp -i\lambda, \tag{5}$$

where the real quantities R and λ satisfy:

$$\begin{aligned} R &> 0, & (6)\\ R &= (\ (r^2 - y^2)^2 + 4(\mathbf{x}.\mathbf{y})^2\ )^{1/4}, & (7)\\ \cos\lambda &= (r^2 - y^2)/R^2, & (8)\\ \sin\lambda &= 2\ \mathbf{x}.\mathbf{y}/R^2. & (9)\end{aligned}$$

Further, choosing

$$-\pi < \lambda \leq \pi \tag{10}$$

allows us to give meaning to $\mid \mathbf{z} \mid$, ie to $\mid \mathbf{x} - i\mathbf{y} \mid$. Specifically, there is a unique expression, analytic in $\mathbf{x}$ on $R^3 -$ DISK, that behaves like r for large r:

$$\mid \mathbf{z} \mid = R \ \exp(-i\lambda/2). \tag{11}$$

The point is that λ, though defined on $R^3 -$ RING is not analytic there. It flips from $-\pi$ to $+\pi$ as we proceed in the direction $\mathbf{y}$ along the axis of symmetry through the centre of the ring. Specifically, its behaviour may be summarised as follows:

1. $\mathbf{x}.\mathbf{y} = 0$
   λ is π for $r < y$ (ie on DISK) and 0 for $r > y$.

2. $\mathbf{x}.\mathbf{y} > 0$
   λ decreases monotonically as r increases.
   $\lambda \to (\pi, 0)$ as $r \to (0, \infty)$, taking the value $\pi/2$ on $r = y$.

3. $\mathbf{x}.\mathbf{y} < 0$
   λ increases monotonically as r increases.
   $\lambda \to (-\pi, 0)$ as $r \to (0, \infty)$, taking the value $-\pi/2$ on $r = y$.

We now define the classical potential of the twiston outside the forbidden disk to be:

$$\begin{aligned} V_{class} &= q/\mid \mathbf{z} \mid & (12)\\ &= (q/R)\ \exp(i\lambda/2). & (13)\end{aligned}$$

Splitting this into its real and imaginary parts and examining it for large r, it follows that:

$$\begin{aligned} V_{class} &= \phi_{class} + i\ \psi_{class} & (14)\\ &= q\ (1/r + i\ \mathbf{x}.\mathbf{y}/r^3) + O(1/r^3). & (15)\end{aligned}$$

Clearly, ϕ behaves like an electrostatic potential for a monopole while ψ behaves like the magnetostatic, scalar potential [5] for a system with magnetic dipole moment $q\mathbf{y}$. This is consistent with the association

$$\mathbf{E} + i\mathbf{B} = -\nabla V. \tag{16}$$

Choosing $\mathbf{y} = g\mathbf{J}/2mc$, for a particle of mass m and (intrinsic) spin $\mathbf{J}$, leads to a gyromagnetic ratio of g, the centres of mass and charge coinciding if and only if $g = 2$.

3.2 QUANTISED POTENTIAL

Consider now the two-body system comprising a twiston and a spinless, test particle with unit charge. In the spirit of the preceding section, let the displacement of the test particle from the twiston be

$$\mathbf{s} = \mathbf{x} - ig\mathbf{J}/2mc. \tag{17}$$

If the system be quantised using the standard methods of non-relativistic quantum mechanics [6], we must consider the set of observables:

$$\text{OBSERVABLES } = \{q, g, m, \mathbf{x}, \mathbf{J}\}. \tag{18}$$

These do not form a complete commuting set but, recalling that $\mathbf{J}$ is the *intrinsic* spin of the twiston, the only non-trivial commutation relations involve $\mathbf{J}$ with itself. Indeed a complete commuting set is given by

$$\text{COMMUTING OBSERVABLES } = \{q, g, m, \mathbf{x}, \mathbf{J}^2, \mathbf{J}.\mathbf{x}\}; \tag{19}$$

and in what follows I shall assume $g > 0$. Note that the direction of quantisation for $\mathbf{J}$ is tied to the geometry of the system by choosing it parallel to the outward pointing radial vector $\mathbf{x}$. Specialising to spin 1/2 yields

$$\mathbf{J}^2 = 3/4\ \hbar^2. \tag{20}$$

Letting $\mathbf{x}$ have eigenvalues $\mathbf{x}'$ that range over $R^3 - \{O\}$, $\mathbf{J}.\mathbf{x}$ has eigenvalues $\pm 1/2\ \hbar\ r'$. Indeed, regarding q, g and m as fixed, the system's eigenstates form the union of a pair of disjoint sets,

$$\{|\, \mathbf{x}' \rightarrow\rangle\} \cup \{|\, \mathbf{x}' \leftarrow\rangle\}. \tag{21}$$

Think of the spin states here as being *out* and *in* rather than up and down; for they lie respectively parallel and antiparallel to $\mathbf{x}$.

I now repeat the analysis of the previous section but using quantum instead of classical observables. First, to exploit the use of dimensionless quantities, let:

$$b = g\hbar/2mc, \tag{22}$$

$$s = r/b, \tag{23}$$

$$\sigma = 2\,\mathbf{J}.\mathbf{x}/\hbar r. \tag{24}$$

Note that the operator σ is defined only for those states of expression (21) for which $\mathbf{x}'$ is non-zero. Thus, excluding the origin, σ has eigenvalues ± 1. The square of the relative displacement of the two bodies is then given by:

$$\mathbf{s}^2 = b^2(s^2 - 3/4 - is\sigma). \tag{25}$$

In turn we may rewrite this as:

$$\mathbf{s}^2 = b^2\ S^2\ \exp -i\beta\sigma, \tag{26}$$

where the real operators S and β have eigenvalues satisfying:

$$S' \geq 0, \tag{27}$$

$$S' = (\ (s'^2 - 1/4)^2 + 1/2\ )^{1/4}, \tag{28}$$

$$\cos\beta' = (s'^2 - 3/4)/S'^2, \tag{29}$$

$$\sin\beta' = s'/S'^2 \tag{30}$$

Note first that S' never vanishes. As s' increases from 0 to 1/2 so S' decreases from $\sqrt{3}/2$ to its minimum value of $(1/2)^{1/4}$. Thereafter it increases monotonically, behaving like s' for large s'. Further,

$$S' = (\sqrt{3}/2)\,(\,1 - 2s'^2/9 + O(s'^4)\,), \text{ for small } s'; \tag{31}$$
$$s'(\,1 - 1/8s'^2 + O(1/s'^4)\,), \text{ for large } s'. \tag{32}$$

As regards β', we may choose it to satisfy

$$0 < \beta' \leq \pi. \tag{33}$$

β' is now globally defined, analytic on the whole of R^3 and monotonically decreases as s' increases. As s' increases from 0 to $(3/4)^{1/2}$, β' decreases from π to $\pi/2$ while as $s' \to \infty$, $\beta' \to 0$. Further,

$$\beta' = \pi - 4s'/3 + O(s'^3), \text{ for small } s', \tag{34}$$
$$1/s' + 5/12s'^3 + O(1/s'^5) \text{ for large } s'. \tag{35}$$

Now define $|\,\mathbf{z}\,|$ by choosing the unique square root of equation (26) whose eigenvalues behave like r' for large r':

$$|\,\mathbf{z}\,| = b\ S\ \exp(-i\beta\sigma/2). \tag{36}$$

(In fact there are three other square roots.) It follows that the quantum potential is given by:

$$V_{quant} = q/\,|\,\mathbf{z}\,| \tag{37}$$
$$= (q/bS)\ \exp(i\beta\sigma/2). \tag{38}$$

As before, splitting this into real and imaginary parts yields:

$$V_{quant} = \phi_{quant} + i\ \psi_{quant} \tag{39}$$
$$= (q/bS)\,(\cos\beta/2 + i\sigma\sin\beta/2). \tag{40}$$

Note that V_{quant} is well defined for states for which $\mathbf{x}' \neq \mathbf{0}$. In fact the electrostatic potential energy, ϕ_{quant}, contributing to the two-body interaction is defined for all states, its behaviour being determined by a certain function from the real line to itself. For, in any state, whether spin out or spin in, ϕ_{quant} will have eigenvalues

$$(q/b)\,(1/S')\ \cos(\beta'/2). \tag{41}$$

Ignoring the factor q/b, it is this function of s' that is illustrated in figure 1 for $s' \leq 2.4$. Its behaviour for small and large s' is given by:

$$(1/S')\ \cos(\beta'/2) = (4\sqrt{3}/9)\ s' + O(s'^3), \text{ for small } s', \tag{42}$$
$$1/s' + O(1/s'^5), \text{ for large } s'. \tag{43}$$

Furthermore, there is a maximum of ~ 0.78 when $s' \sim 0.98$.

The magnetostatic potential may best be understood after identifying the magnetic moment $\mathbf{M}$ according to

$$\mathbf{M} = q\ \mathbf{y} = qb\ \mathbf{J}/mc. \tag{44}$$

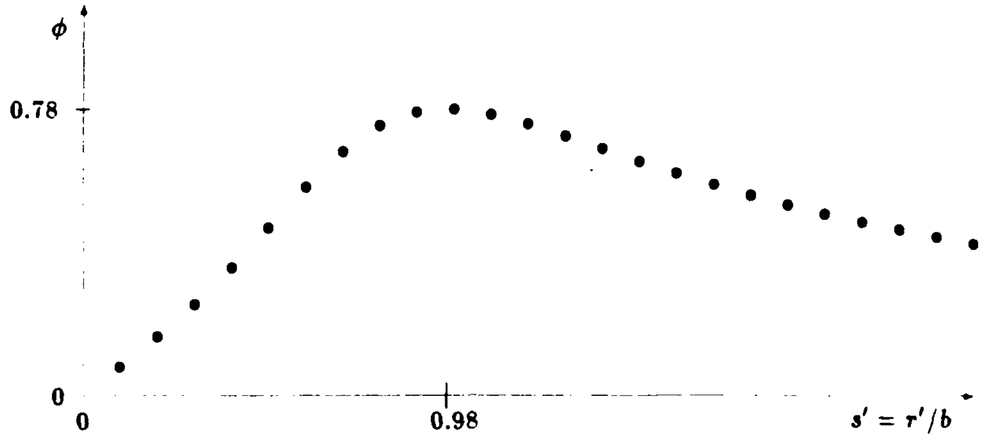

Figure 1: The electrostatic interaction between a spin 1/2 twiston and a spinless point particle, normalised to behave like $1/s'$ for large s'.

Then it is straightforward to show that

$$\psi_{quant} = (\mathbf{M}.\mathbf{x}/b^3)\ (2/sS)\sin(\beta/2). \tag{45}$$

In particular, its asymptotic behaviour is determined by

$$(2/s'S')\ \sin(\beta'/2) \ = \ (4/\sqrt{3})\ (1/s' + O(s')),\ \text{for small s'}, \tag{46}$$
$$1/s'^3 + 1/2s'^5 + O(1/s'^7),\ \text{for large s'}. \tag{47}$$

Consider next the issue of the charge distribution implied by the potential illustrated in figure 1. It is clear that the eigenvalues of ϕ_{quant} do not satisfy Laplace's equation, even excluding certain regions. In particular, the twiston's charge is localised neither at a single point nor on a ring. Indeed, the charge is smeared out over all space. An expression for the charge $Q(r)$ within a given radius may be obtained by exploiting the spherical symmetry of the potential:

$$Q(r) = -r^2\ \partial\phi_{quant}/\partial r. \tag{48}$$

Ignoring an overall factor of e, this is plotted as a function of the eigenvalue for $s' \leq 2.4$ in figure 2. It turns out that over 99.9 % of the total charge resides within a radius of 10 b.

To conclude this section consider the implications of the preceding analysis for the interaction between a spin 1/2 twiston with mass m, charge $Z_1 e$ and gyromagnetic ratio g and a spinless particle with charge $Z_2 e$. Effectively this entails replacing q by $Z_1 Z_2 e^2$. Further, writing

$$e^2/b \ = \ (2/g)\ \alpha\ mc^2, \tag{49}$$
$$\text{with } \alpha \ = \ e^2/\hbar c \sim 1/137, \tag{50}$$

and reverting to dimensional quantities leads to the following.

- For small r', the potential behaves like $(4\sqrt{3}/9)\ Z_1 Z_2\ e^2 r'/b^2$. Note that, at distances small compared with b, like (unlike) charges encounter a radial attractive (repulsive) force.

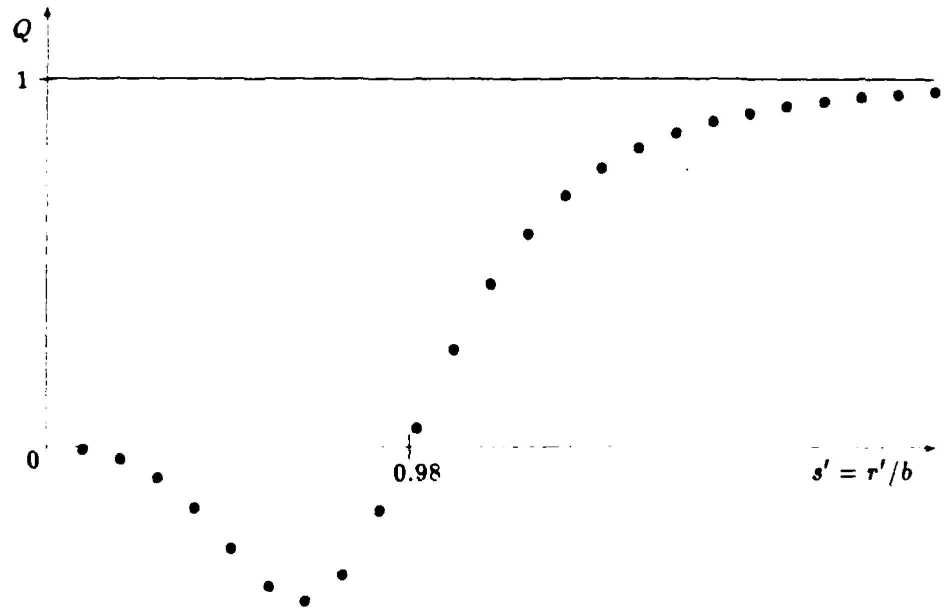

Figure 2: The electric charge of a spin 1/2 twiston plotted as a function of radius and normalised to behave like 1 for large s'.

- For large r', the potential behaves like $Z_1 Z_2\, e^2/r + O(1/r^5)$.
- If Z_1 and Z_2 have the same (opposite) signs, the potential has a barrier of height (well of depth) $\sim (1.55/g)\, Z_1 Z_2\, \alpha\, mc^2$ occurring when $r \sim 0.98b$.

Of course, the alert reader will have noted from the definition in equation (22) that when g takes the (Dirac) value 2, b amounts to the Compton wavelength of the twiston $\hbar/mc$. For a particle having the mass of the electron, this approximates to 4×10^{-11} cm while the mass of a nucleon yields 2×10^{-14} cm.

On a final note of caution, it should be remembered that the Schrodinger equation for the two body system that has been discussed will involve not only the electrostatic scalar potential but also the magnetic vector potential. For states that have no orbital angular momentum, the contribution from the latter involves its square and is usually neglected in the textbook literature. However, the possibility that it is significant at short range should not be overlooked.

4 THE TWISTON SELF-ENERGY

4.1 CLASSICAL TWISTON

The problem of the self-energy of a classical point charge is well known. The potential behaves like $1/r$ while the field behaves like $1/r^2$. In turn, the energy density E_{class} behaves like $1/r^4$. It follows that, owing to its behaviour near $r = 0$, the integral of the energy density over Euclidean three-space cannot be defined. Even if we remove from the space a small sphere with origin as centre, the integral diverges as the sphere is made progressively smaller.

The situation for a classical twiston with spin is no better. Here, the energy density, defined on R^3 - DISK, diverges as the Kerr ring is approached. To see this note that the field is given by:

$$\begin{aligned}(\mathbf{E} + i\mathbf{B})_{class} &= -\nabla V_{class} &(51)\\ &= q\ (\mathbf{x} - i\mathbf{y})/\mid \mathbf{x} - i\mathbf{y} \mid^3 . &(52)\end{aligned}$$

Defining $DENSITY$ by

$$DENSITY(\mathbf{E} + i\mathbf{B}) = (\mathbf{E}^2 + \mathbf{B}^2)/8\pi, \qquad (53)$$

it follows that

$$\begin{aligned}E_{class} &= DENSITY(\mathbf{E} + i\mathbf{B}) &(54)\\ &= q^2\ (r^2 + y^2)/8\pi R^6. &(55)\end{aligned}$$

with R defined in equation (7). If we remove from the space a sphere centre the origin and radius $a > y$, then the integrated energy density diverges as a approaches y. To see this, write

$$\mathbf{x}.\mathbf{y} = r\ y\ u \qquad (56)$$

and use

$$\int_{-1}^{+1} du/(p + qu^2)^{3/2} = 2/(p\ (p+q)^{1/2}). \qquad (57)$$

Hence,

$$\begin{aligned}\int_{r \geq a} d^3\mathbf{x}\ E_{class} &= 2\pi \int_a^\infty r^2\ dr \int_{-1}^{+1} du\ E_{class} &(58)\\ &= (q^2/2) \int_a^\infty dr\ r^2/(r^2 - y^2)^2 &(59)\\ &= (q^2/8)\ (\ 2a/(a^2 - y^2) + (1/y)\log(a + y)/(a - y)\ ) &(60)\end{aligned}$$

4.2 QUANTUM TWISTON

I now show that quantisation and the choice of spin 1/2 removes the infinity. Before proceeding, however, it will be wise to spend a few moments reflecting on issues of factor ordering. We have supposed that, given the potential V, application of the derivative $-\nabla$ generates the field $\mathbf{E} + i\mathbf{B}$. Further, applying the map $DENSITY$ yields the energy density E. Finally, given a classical operator, there is a prescription (call it Q), for defining the corresponding quantum operator. The problem is that these operations do not all commute. *A priori*, there appears to be at least three distinct ways of quantising the energy density leading to three distinct operators:

1. $E_{quant1} = Q(DENSITY(-\nabla V_{class}))$;
2. $E_{quant2} = DENSITY(Q(-\nabla V_{class}))$;
3. $E_{quant3} = DENSITY(-\nabla(Q(V_{class})))$.

The first of these is derived directly from equation (55) yielding:

$$E_{quant1} = (q^2/8\pi b^4)\ (s^2 + 3/4)/S^6. \tag{61}$$

Note that an eigenstate of $\mathbf{x}$ is also an eigenstate of E_{quant1}. Also, since S' never vanishes, we may associate a field energy density with each (eigen)value $\mathbf{x}'$. Integrating over R^3 then yields an expression which may be interpreted as the self-energy. A numerical approximation based on the trapezium rule yields

$$\text{SELF-ENERGY1} = 1.09\ q^2/b. \tag{62}$$

However, the logic leading to this result seems a little unnatural; for it entails reverting to the classical potentials having quantised them in section 3.2. To vary the moment of quantisation depending on whether we are calculating the interaction energy or the self energy would appear to be inconsistent and I shall therefore abandon this approach.

The second option involves quantising expression (52) for the classical field. However, owing to the ambiguity of factor ordering, this is not well defined and shall therefore be dismissed.

The third possibility is closest to the spirit of section 3.2. The potentials are quantised first. After all, interaction *potentials* certainly appear in Schrodinger's equation and these must therefore be quantum operators. The quantised fields are then defined, for states for which $\mathbf{x}' \neq 0$, by taking the gradient. Thus, using equation (38):

$$(\mathbf{E} + i\mathbf{B})_{quant} = (q/2bS^2)\ (\ (2\ \nabla S - i\ \sigma\ S\ \nabla\beta)\ \exp(i\beta\sigma/2) + 2i\ S\ \sin\beta/2\ \nabla\sigma\ ) \tag{63}$$

which leads to

$$\mathbf{E}_{quant} = (q/2bS^2)\ (\ 2\ \nabla S\ \cos\beta/2 + S\ \nabla\beta\ \sin\beta/2\ ) \tag{64}$$

$$\mathbf{B}_{quant} = (q/2bS^2)\ (\ (2\ \nabla S\ \sin\beta/2 - S\ \nabla\beta\ \cos\beta/2)\ \sigma + 2\ S\ \sin\beta/2\ \nabla\sigma\ ). \tag{65}$$

The meaning of $\nabla\sigma$ follows from the definition (24):

$$\nabla\sigma = (2/\hbar r)\ (\mathbf{J} - \mathbf{J}.\mathbf{x}\ \mathbf{x}/r^2) \tag{66}$$

whence

$$\mathbf{x}.\nabla\sigma = 0 \tag{67}$$

$$\text{and}\ \mid\nabla\sigma\mid^2 = 2/r^2. \tag{68}$$

It follows that

$$(\mathbf{E}^2 + \mathbf{B}^2)_{quant} = (q^2/4b^2S^4)\ (\ 4 \mid\nabla S\mid^2 + \mid S\ \nabla\beta\mid^2 + 8\ (S/sv\ \sin\beta/2)^2\ ). \tag{69}$$

A similar analysis starting from equation (26) yields:

$$\mathbf{x}^2 + \mathbf{y}^2 = (b^4S^2/4)\ (\ 4 \mid\nabla S\mid^2 + \mid S\ \nabla\beta\mid^2 + 2\ (S/sb\ \sin\beta)^2\ ). \tag{70}$$

It follows that the energy density, defined for states for which $\mathbf{x}' \neq 0$, is given by:

$$E_{quant3} = (q^2/8\pi b^4)\ (\ (s^2 + 3/4)/S^6 + 2/s^2S^2\ \sin^4\beta/2\ ). \tag{71}$$

Integrating the eigenvalues, this time over $R^3 - \{O\}$, and using a numerical approximation as before, yields an answer almost double that obtained previously:

$$\text{SELF-ENERGY3} = 2.01\ q^2/b. \tag{72}$$

For a particle of mass m, charge Ze and gyromagnetic ratio g this gives

$$\text{SELF-ENERGY3} = 2.01\ (2/g)\ Z^2\alpha\ mc^2. \tag{73}$$

In particular, for $Z = 1$ and $g = 2$ (the Dirac value), roughly 1.5 % of the particle's rest mass is electromagnetic in origin. More generally, if we demand that the self-energy be bounded by mc^2, then

$$4.02\ Z^2\alpha \leq g. \tag{74}$$

A final remark concerns the nature of removal of the infinite self-energy. The point is that the classical field diverges on the Kerr ring, the plane of the ring being defined by:

$$\text{RING PLANE} = (\mathbf{J}.\mathbf{x} = 0). \tag{75}$$

For a quantised system with spin 1/2, however, this is meaningless: $\mathbf{J}.\mathbf{x}$ has no zero eigenvalue, its eigenvalues being $\pm\hbar/2$; and this is the genesis of the singularity removal.

5 SPIN-SPIN INTERACTION

5.1 CLASSICAL TREATMENT

So far I have examined interactions between spinning twistons and spinless point particles. In section 3 the two-body potentials, both classical and quantum, were presented while in section 4 the quantised interaction was used to calculate the self-energy of the spin 1/2 twiston field.

I now seek the interaction between a pair of spinning twistons. Of course, classically this is equivalent to asking for the potential energy of one twiston in the field of the other. For spinless point particles this led to an expression of the form:

$$q_1\phi_2 = q_1q_2/\mid \mathbf{x}_1 - \mathbf{x}_2 \mid = q_2\phi_1. \tag{76}$$

So, we shall need to extend this definition to twistons with spin.

First, however, consider the interaction between a classical, spinning twiston and a classical external field determined by the complex potential V. If the twiston has charge q and centre of charge $\mathbf{x} + i\mathbf{y}$ it seems natural that it samples the external potential at that point. Thus, define the interaction to be:

$$\text{EXTERNAL FIELD COUPLING} = q\ \Re\ V(\mathbf{x} + i\mathbf{y}). \tag{77}$$

(Taking the real part is entirely equivalent to identifying with the twiston a pair of charge centres $\mathbf{x} \pm i\mathbf{y}$, the one left-handed, the other right-handed, and averaging the values of the potential at those points. Connoisseurs of twistor theory may prefer this!)

Note that expression (77) is entirely consistent with expression (12) that describes the classical interaction between a spinning twiston and a spinless, unit, test charge. Though we considered the test charge in the external field of the twiston, we could equally have considered the twiston in the Coulomb field of the test charge.

To understand definition (77), write

$$V = \phi + i\,\psi \tag{78}$$

and expand in powers of $\mathbf{y}$. This yields:

$$\begin{aligned} \text{EXTERNAL FIELD COUPLING} &= q(\,\phi(\mathbf{x}) - \mathbf{y}.\nabla\psi(\mathbf{x})\,) + O(y^2) && (79)\\ &= q\phi + q\mathbf{y}.\mathbf{B} + O(y^2). && (80) \end{aligned}$$

The second term is just the interaction energy of a particle with magnetic moment $q\mathbf{y}$ in an external magnetic field $\mathbf{B}$.

Extending the argument logically leads to a natural expression for the interaction between a pair of classical twistons with charges q_1 and q_2, located respectively at complex points $\mathbf{z}_1$ and $\mathbf{z}_2$:

$$\text{CLASSICAL TWISTON-TWISTON INTERACTION} = \Re\, q_1q_2/\mid \mathbf{z}_1 - \mathbf{z}_2 \mid \tag{81}$$

The meaning of this expression may be inferred from the discussion in section 3.1 on writing:

$$\begin{aligned} \mathbf{z}_k &= \mathbf{x}_k + i\mathbf{y}_k, && (82)\\ \mathbf{x}_1 - \mathbf{x}_2 &= \mathbf{x}, && (83)\\ \mathbf{y}_2 - \mathbf{y}_1 &= \mathbf{y}. && (84) \end{aligned}$$

In this case we get an expression that is well-defined outside the forbidden disk:

$$\text{CLASSICAL TWISTON-TWISTON INTERACTION} = \Re\, q_1q_2/\mid \mathbf{x} - i\mathbf{y} \mid \tag{85}$$

5.2 QUANTISATION OF DISPLACEMENT

To approach the quantisation of the twiston-twiston interaction I proceed by analogy with section 3.2 by identifying the imaginary displacement of each twiston with a vector proportional to the corresponding spin vector:

$$\mathbf{y}_i = g_i\mathbf{J}_i/2m_ic. \tag{86}$$

Further, it will be convenient to define:

$$\begin{aligned} b_i &= g_i\hbar/2m_ic, && (87)\\ \text{and } \sigma_i &= 2\,\mathbf{J}_i.\mathbf{x}/\hbar r. && (88) \end{aligned}$$

Then,

$$\mathbf{y} = (1/\hbar)\,(b_2\mathbf{J}_2 - b_1\mathbf{J}_1). \tag{89}$$

Now the complex displacement of 1 from 2 is given by:

$$\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2 = \mathbf{x} - i\mathbf{y}. \tag{90}$$

Hence, for twistons of spin 1/2, the square of their relative displacement is given by:

$$\mathbf{s}^2 = r^2 - 3/4\ (b_1 + b_2)^2 + b_1 b_2\ (\mathbf{J}_1 + \mathbf{J}_2)^2/\hbar^2 - ir\ (b_2\sigma_2 - b_1\sigma_1). \tag{91}$$

Further, defining

$$\mathbf{j} = (\mathbf{J}_1 + \mathbf{J}_2)/\hbar, \tag{92}$$

$$\sigma_\pm = (\sigma_1 \pm \sigma_2)/2, \tag{93}$$

gives:

$$\mathbf{s}^2 = r^2 - 3/4\ (b_1 + b_2)^2 + b_1 b_2\ \mathbf{j}^2 + ir\ (\ (b_1 - b_2)\sigma_+ + (b_1 + b_2)\sigma_-\ ). \tag{94}$$

Following convention, consider the complete commuting set of observables

$$\text{COMMUTING OBSERVABLES} = \{\mathbf{x}, \mathbf{j}^2, \sigma_+\} \tag{95}$$

whose simultaneous eigenstates form four disjoint sets:

$$\{|\ \mathbf{x}'; 1\ 1\rangle\} \cup \{|\ \mathbf{x}'; 1\ -1\rangle\} \cup \{|\ \mathbf{x}'; 1\ 0\rangle\} \cup \{|\ \mathbf{x}'; 0\ 0\rangle\}. \tag{96}$$

Readers will note that the first three sets comprise the so called *triplet* state corresponding to total spin 1 while the fourth comprises the *singlet* state having spin 0. Using $\rightarrow$ and $\leftarrow$ respectively to depict spins parallel and antiparallel to $\mathbf{x}$ (defined to point from 2 to 1) yields the following graphic equivalence in terms of eigenstates of σ_2 and σ_1:

$$|\ 1\ 1\rangle = |\rightarrow\rightarrow\rangle, \tag{97}$$

$$|\ 1\ -1\rangle = |\leftarrow\leftarrow\rangle, \tag{98}$$

$$|\ 1\ 0\rangle = (|\rightarrow\leftarrow\rangle + |\leftarrow\rightarrow\rangle)/\sqrt{2}, \tag{99}$$

$$|\ 0\ 0\rangle = (|\rightarrow\leftarrow\rangle - |\leftarrow\rightarrow\rangle)/\sqrt{2}. \tag{100}$$

Surprisingly, in what follows, the four sets fall into two natural pairs, the states with spins aligned forming one pair and the third member of the triplet forming an unusual association with the singlet. Associated with this pairing are the natural projection operators $\sigma_\pm^2$ satisfying:

$$\sigma_+^2 + \sigma_-^2 = 1, \tag{101}$$

$$\sigma_+\ \sigma_- = 0, \tag{102}$$

$$\text{and } \sigma_\pm(\sigma_\pm^2 - 1) = 0. \tag{103}$$

Also:

$$\mathbf{j}^2(\mathbf{j}^2 - 2) = 0. \tag{104}$$

Using the projection operators allows us to rewrite expression (94) as the sum of two distinct pieces:

$$\begin{aligned} \mathbf{s}^2 = &(\ r^2 - 3/4\ (b_1 + b_2)^2 + 2\ b_1 b_2 + ir\ (b_1 - b_2)\sigma_+\ )\ \sigma_+^2 + \\ &(\ r^2 - 3/4\ (b_1 + b_2)^2 + b_1 b_2\ \mathbf{j}^2 + ir\ (b_1 + b_2)\sigma_-\ )\ \sigma_-^2\ . \end{aligned} \tag{105}$$

This expression will be examined in the sections that follow and potentials derived for each of three spin states.

5.3 IDENTICAL PARTICLES

To generate a potential from expression (105) entails defining an inverse square root appropriately; and this is complicated by the fact that the real and imaginary parts of the operator do not commute owing to the presence of the σ_- term.

To understand what is involved and to examine a potential of particular interest, consider the case of identical particles with spin 1/2 and charges q. This allows us to set

$$b_1 = b_2 = b(>0), \tag{106}$$
$$s = r/b. \tag{107}$$

(As before, g is assumed positive.) Expression (105) now simplifies to:

$$\mathbf{z}^2 = b^2 \left((s^2 - 1)\,\sigma_+^2 + (s^2 - 3 + \mathbf{j}^2 + 2is\,\sigma_-)\,\sigma_-^2 \right). \tag{108}$$

To examine this further, consider the operator τ defined by

$$\tau = 2s\,\sigma_- - i\,(\mathbf{j}^2 - 1) \tag{109}$$

which satisfies

$$(\tau^2 + 1 - 4s^2)\,\sigma_-^2 = 0, \tag{110}$$

use having been made of the anti-commutation relation

$$\left((\mathbf{j}^2 - 1)\sigma_- + \sigma_-(\mathbf{j}^2 - 1) \right) \sigma_-^2 = 0. \tag{111}$$

We therefore seek an appropriate inverse square root of the operator

$$\mathbf{z}^2 = b^2 \left((s^2 - 1)\,\sigma_+^2 + (s^2 - 2 + i\,\tau)\,\sigma_-^2 \right). \tag{112}$$

That is to say we seek

$$V = V_+\,\sigma_+^2 + V_-\,\sigma_-^2 \tag{113}$$

such that

$$V_+^2\,(s^2 - 1) = 1/b^2, \tag{114}$$
$$V_-^2\,(s^2 - 2 + i\,\tau) = 1/b^2, \tag{115}$$
$$\text{with } V \sim 1/r \text{ for states with large } r'. \tag{116}$$

Of course the real part is needed for the interaction potential which we aim to define at least for the following subset of states:

$$\{|\,\mathbf{x}';1\ \pm 1\rangle : s' > 1\} \cup \{|\,\mathbf{x}';1\ 0\rangle : s' > 1/2\} \cup \{|\,\mathbf{x}';0\ 0\rangle : s' > 1/2\}. \tag{117}$$

Clearly V_+ presents no problem. For states having $s' > 1$ it is given by:

$$V_+ = (1/b)\,(s^2 - 1)^{-1/2}, \tag{118}$$

the positive root being taken. (For $s' < 1$, the solution is ambiguous in sign though, being pure imaginary, it vanishes when the real part is taken.)

To define V_- for states having $s' > 1/2$ first note that if f is a function of s,

$$\exp(if\ \tau(4s^2-1)^{-1/2})\ \sigma_-^2 = (\ \cos f + i\ \tau\ (4s^2-1)^{-1/2}\ \sin f\ )\ \sigma_-^2. \tag{119}$$

It therefore follows that, for $s' > 1/2$,

$$(s^2 - 2 + i\ \tau)\ \sigma_-^2 = S^2\ \exp(i\beta\tau(4s^2-1)^{-1/2})\ \sigma_-^2. \tag{120}$$

where the eigenvalues of the real operators S and β satisfy:

$$S' > 0, \tag{121}$$
$$S' = (s'^4+3)^{1/4}, \tag{122}$$
$$\cos\beta' = (s'^2-2)/S'^2, \tag{123}$$
$$\sin\beta' = (4s'^2-1)^{1/2}/S'^2, \tag{124}$$
$$0 < \beta' < \pi. \tag{125}$$

β's behaviour may be summarised as follows:

- β' decreases as s' increases on the range $s' > 1/2$.
- As $s' \to 1/2$, $\beta' \to \pi$.
- When $s' = 1$, $\beta' = 2\pi/3$.
- When $s' = \sqrt{2}$, $\beta' = \pi/2$.
- As $s' \to \infty$, $\beta' \to 0$.

It now follows that, at least for states for which $s' > 1/2$,

$$V_- = (1/bS)\ \exp(-i\beta\tau(4s^2-1)^{-1/2}/2), \tag{126}$$

an expansion of which may be obtained on using expression (119). (For states having $s' < 1/2$ the non-uniqueness of square root fails to disappear on taking the real part.)

The interaction potential for a pair of identical spin 1/2 twistons may now be written

$$\phi = \Re\ q^2\ (V_+\ \sigma_+^2 + V_-\ \sigma_-^2). \tag{127}$$

with V_+ satisfying (118) and V_- satisfying (126). It follows that the potential for the triplet state with spins aligned and $s' > 1$ is given by:

$$\phi_+ = (q^2/b)\ (s^2-1)^{-1/2}; \tag{128}$$

while the potentials for the remaining spin states, with $s' > 1/2$, are given by:

$$\phi_- = (q^2/b)\ (s^4+3)^{-1/4}\ (\ \cos\beta/2 - (\mathrm{j}^2-1)\ (4s^2-1)^{-1/2}\ \sin\beta/2\ ). \tag{129}$$

Manifestly there are three separate potential functions involved here. Suppressing an overall factor of q^2/b, these are depicted below and illustrated in figure 3 for $s' \leq 2.4$:

- Triplet potential for states with spins aligned and having $s' > 1$:

$$\phi_{|1\ \pm 1\rangle} = (s^2 - 1)^{-1/2}. \tag{130}$$

This diverges positively as $s' \to 1$ thus behaving like a "brick wall" there.

- Triplet potential with spins not aligned, having $s' > 1/2$:

$$\phi_{|1\ 0\rangle} = (s^4 + 3)^{-1/4} \left(\cos\beta/2 - (4s^2 - 1)^{-1/2} \sin\beta/2 \right). \tag{131}$$

This possesses a maximum of ~ 0.34 when $s' \sim 1.98$. It vanishes when $s' = 1$ and diverges negatively as $s' \to 1/2$.

- Singlet potential when $s' > 1/2$:

$$\phi_{|0\ 0\rangle} = (s^4 + 3)^{-1/4} \left(\cos\beta/2 + (4s^2 - 1)^{-1/2} \sin\beta/2 \right). \tag{132}$$

This diverges positively as $s' \to 1/2$.

5.4 DISTINCT PARTICLES

Having explored the interaction between identical twistons, consider now the case of distinct spin 1/2 twistons, with charges q_1 and q_2, whose masses and gyromagnetic ratios satisfy

$$b_1 > b_2 > 0. \tag{133}$$

(If the gyromagnetic ratios are the same, this means that $m_2 > m_1$.) Thus set

$$\begin{aligned} b_1 &= b, && (134)\\ b_2 &= \epsilon b, && (135)\\ 0 &< \epsilon < 1, && (136)\\ s &= r/b. && (137) \end{aligned}$$

It follows that expression (105) now becomes

$$\begin{aligned} \mathbf{s}^2/b^2 \;=\; & \left(s^2 - 3/4\,(1-\epsilon)^2 - \epsilon + is\,(1-\epsilon)\sigma_+ \right) \sigma_+^2 + \\ & \left(s^2 - 3/4\,(1+\epsilon)^2 + \epsilon + i\tau \right) \sigma_-^2, \end{aligned} \tag{138}$$

with τ now given by

$$\tau = s(1+\epsilon)\,\sigma_- - i\epsilon(\mathbf{j}^2 - 1). \tag{139}$$

Writing

$$d = \left(s^2(1+\epsilon)^2 - \epsilon^2 \right)^{1/2}, \tag{140}$$

it follows that, for states for which $s' > \epsilon/(1+\epsilon)$,

$$(\tau^2 - d^2)\,\sigma_-^2 = 0. \tag{141}$$

The aim now is to use the techniques of sections 3.2 and 5.3 to rewrite the expression for $\mathbf{s}^2$ as the sum of two pieces involving exponentials. The inverse square root will then

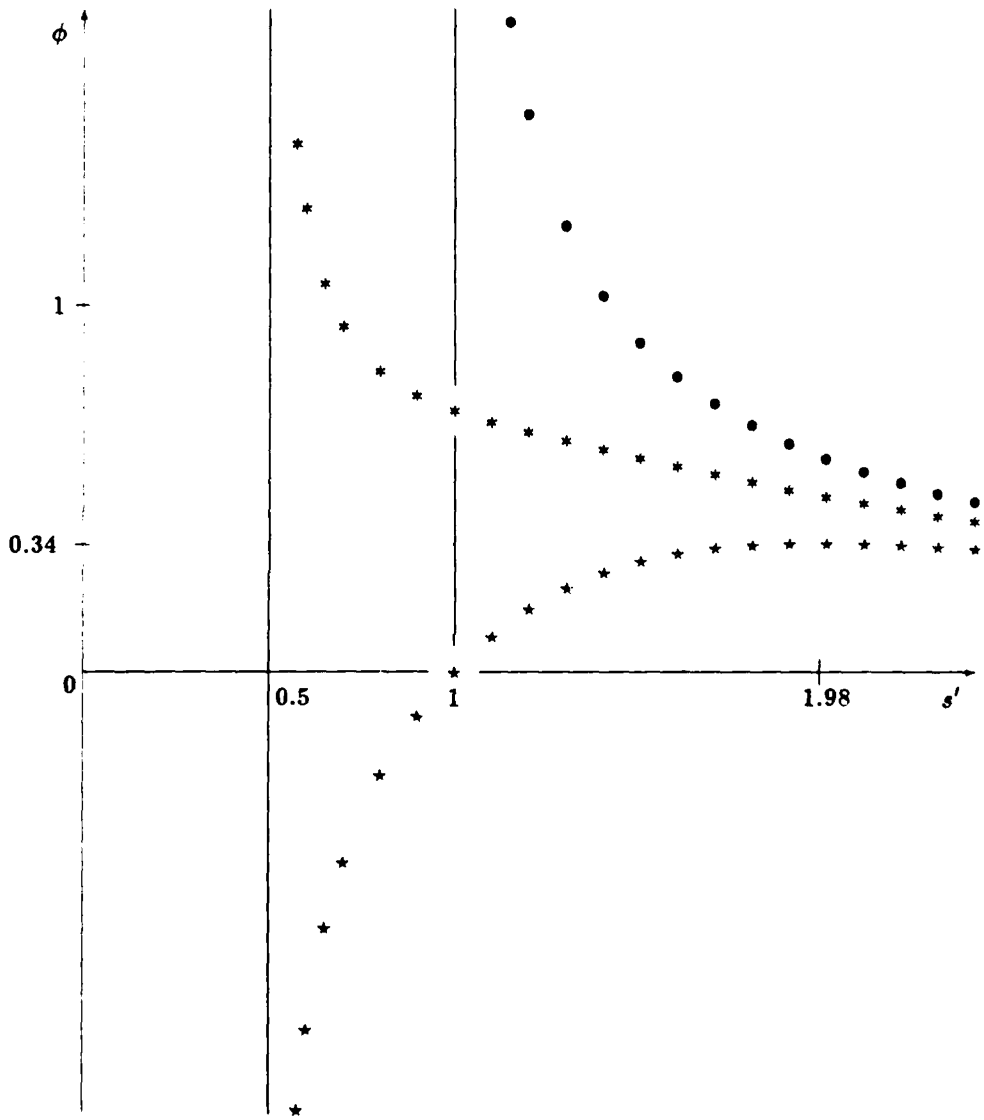

Figure 3: The interaction between identical spin 1/2 twistons, normalised to behave like $1/s'$ for large s'. The potentials for spin states $|\ 1\ \pm 1\rangle$, $|\ 1\ 0\rangle$ and $|\ 0\ 0\rangle$ are depicted by $\bullet$, $\star$ and $*$ respectively.

be chosen to behave like $1/r'$ for large r'. The set of states for which this will be defined is given by:

$$\{|\,\mathbf{x}';1\ \pm 1\rangle : s' > 0\} \cup \{|\,\mathbf{x}';1\ 0\rangle : s' > \epsilon/(1+\epsilon)\} \cup \{|\,\mathbf{x}';0\ 0\rangle : s' > \epsilon/(1+\epsilon)\}. \quad (142)$$

Thus,

$$\mathbf{z}^2 = b^2 \left(\ S_+^2 \exp(i\beta_+\sigma_+)\sigma_+^2 + S_-^2 \exp(i\beta_-\tau/d)\sigma_-^2\ \right), \quad (143)$$

where the real operators $S_\pm$ and $\beta_\pm$ have eigenvalues that satisfy:

$$\begin{aligned}
S'_+ &> 0, && (144)\\
S'^4_+ &= (s'^2 - 1/4\ (1+\epsilon)^2)^2 + 1/2\ (1+\epsilon^2)(1-\epsilon)^2, && (145)\\
\cos\beta'_+ &= (s'^2 - 3/4\ (1-\epsilon)^2 - \epsilon)/S'^2_+, && (146)\\
\sin\beta'_+ &= s'(1-\epsilon)/S'^2_+, && (147)\\
0 &< \beta'_+ < \pi; && (148)
\end{aligned}$$

and

$$\begin{aligned}
S'_- &> 0, && (149)\\
S'^4_- &= (s'^2 - 1/4\ (1-\epsilon)^2)^2 + 1/2\ (1+2\epsilon+2\epsilon^3+\epsilon^4), && (150)\\
\cos\beta'_- &= (s'^2 - 3/4\ (1+\epsilon)^2 + \epsilon)/S'^2_-, && (151)\\
\sin\beta'_- &= d'/S'^2_-, && (152)\\
0 &< \beta'_- < \pi. && (153)
\end{aligned}$$

As before, the interaction between the two particles is defined by

$$\phi = \Re\ q_1 q_2\ (V_+\ \sigma_+^2 + V_-\ \sigma_-^2), \quad (154)$$

where

$$\begin{aligned}
V_+ &= (1/bS_+)\ \exp(-i\beta_+\sigma_+/2) && (155)\\
\text{and } V_- &= (1/bS_-)\ \exp(-i\beta_-\tau/2d). && (156)
\end{aligned}$$

It follows that the potential for triplet states with spins aligned is given by

$$\phi_+ = (q_1q_2/bS_+)\ \cos(\beta_+/2), \quad (157)$$

while the potentials for the remaining states are given by

$$\phi_- = (q_1q_2/bS_-)\ (\ \cos(\beta_-/2) - \epsilon/d\ (\mathbf{j}^2 - 1)\sin(\beta_-/2)\ ). \quad (158)$$

As for the case of identical particles, there are separate potentials which, suppressing an overall factor of q_1q_2/b, are given by:

- Triplet potential for states with spins aligned and having $s' > 0$:

$$\phi_{|1\ \pm 1\rangle} = (1/S_+)\ \cos(\beta_+/2). \quad (159)$$

- Triplet potential for states with spins not aligned and having $s' > \epsilon/(1+\epsilon)$:

$$\phi_{|1\ 0\rangle} = (1/S_-)\ (\ \cos(\beta_-/2) - \epsilon/d\ \sin(\beta_-/2)\ ). \tag{160}$$

- Singlet potential when $s' > \epsilon/(1+\epsilon)$:

$$\phi_{|0\ 0\rangle} = (1/S_-)\ (\ \cos(\beta_-/2) + \epsilon/d\ \sin(\beta_-/2)\ ). \tag{161}$$

The potentials for distinct spin 1/2 twistons are illustrated in figure 4, for $s' \leq 2.4$, when ϵ takes the value 0.1. (Equal charges have been taken.) The salient features of these potentials may be summarised as follows:

- For large s' all three potentials behave like $1/s'$.
- The $|1\ \pm 1\rangle$ potential reaches a maximum of ~ 0.83 at $s' \sim 0.95$ and thence decreases to 0 at $s' = 0$.
- The $|1\ 0\rangle$ potential reaches a maximum of ~ 0.68 at $s' \sim 1.08$; thence decreases to 0 at $s' \sim 0.36$; and finally diverges negatively as $s' \rightarrow \sim 0.09$.
- The $|0\ 0\rangle$ potential reaches a local maximum of ~ 0.79 at $s' \sim 0.93$; thence decreases to a local minimum of ~ 0.57 at $s' \sim 0.35$; and finally diverges positively as $s' \rightarrow \sim 0.09$.

The behaviour of these potentials for very small ϵ is qualitatively similar. It is remarkable that three distinct regimes emerge corresponding to distinct features at three different distances: b, $\sqrt{\epsilon}b$ and ϵb. To fix our ideas, suppose we choose parameters for the twistons that correspond to those for the proton (particle 2) and electron (particle 1). (Of course this means changing the overall sign of the interaction.) Then using

$$m_p/m_e \sim 1836 \tag{162}$$
$$\text{and } g_p/g_e \sim 2.79, \tag{163}$$

and recalling that

$$m_1/m_2 = \epsilon\ g_1/g_2, \tag{164}$$

it follows that

$$\epsilon \sim 1.52 \times 10^{-3} \text{ and } \sqrt{\epsilon} \sim 3.90 \times 10^{-2}. \tag{165}$$

Thus the three length scales are given by:

$$b_e \sim 4 \times 10^{-11}\ cm, \tag{166}$$
$$\sqrt{\epsilon}b_e \sim 2 \times 10^{-12}\ cm, \tag{167}$$
$$\text{and } b_p = \epsilon b_e \sim 6 \times 10^{-14}\ cm. \tag{168}$$

The point is that for ϵ very small the potentials behave in the following manner as s' decreases:

- For large s' all three potentials behave like $-1/s'$.

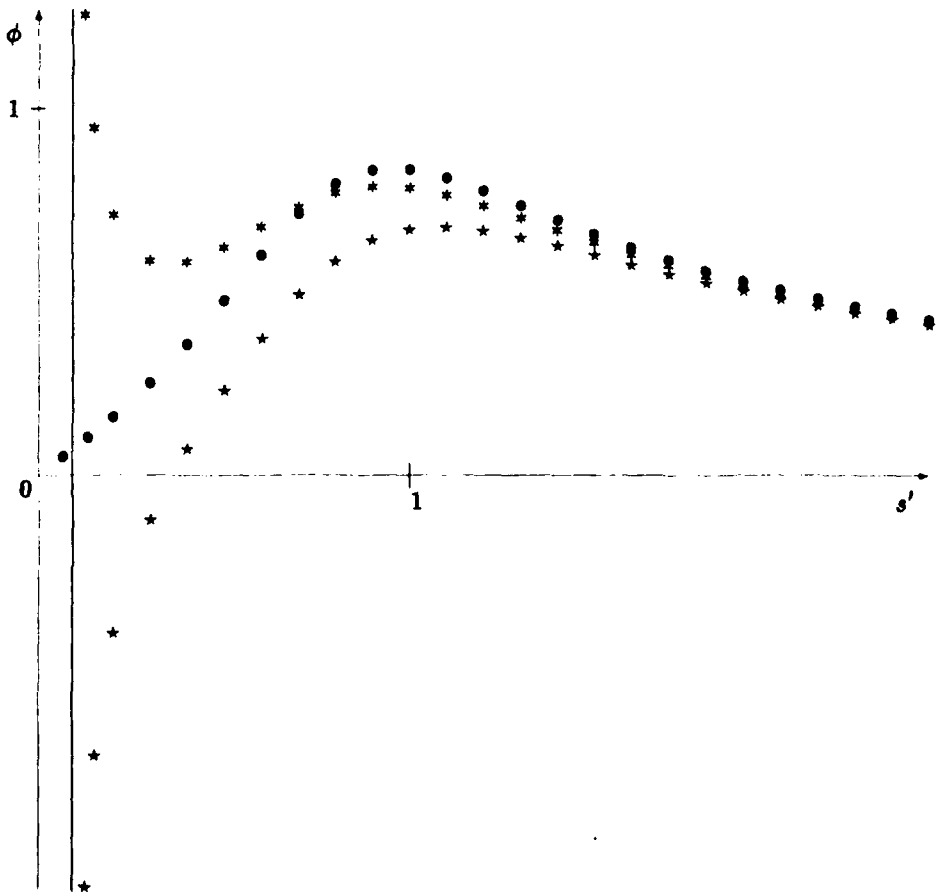

Figure 4: The (normalised) interaction between distinct spin 1/2 twistons. $\epsilon = 0.1$. The potentials for spin states $|\,1\ \pm 1\rangle$, $|\,1\ 0\rangle$ and $|\,0\ 0\rangle$ are depicted by $\bullet$, $\star$ and $*$ respectively.

- All three potentials reach a minimum of ~ -0.78 at $s' \sim 0.98$ (a local minimum for the singlet potential).
- The $|1\ \pm 1\rangle$ potential then increases to 0 as s' decreases to 0, its slope there approximating to $-4\sqrt{3}/9$.
- The $|1\ 0\rangle$ potential, having reached its minimum, increases to 0 at $s' \sim \sqrt{3\epsilon}/2$, continues increasing and finally diverges positively as $s' \rightarrow\sim \epsilon$.
- The $|0\ 0\rangle$ potential, having reaches its local minimum, increases to a local maximum of $\sim -4\sqrt{2\epsilon/3}$ at $s' \sim \sqrt{3\epsilon/2}$; thence it decreases and finally diverges negatively as $s' \rightarrow\sim \epsilon$.

In particular, the behaviour of the $|1\ 0\rangle$ and $|0\ 0\rangle$ potentials near $\sqrt{\epsilon}$ may be seen by writing $s = x\sqrt{\epsilon}$, so that

$$\begin{aligned} S_- &= \sqrt{3}/2 + O(\epsilon), & (169)\\ d &= x\sqrt{\epsilon}\,(1 + O(\epsilon)), & (170)\\ \beta_- &= \pi - 4x/3\,\sqrt{\epsilon}\,(1 + O(\epsilon)). & (171) \end{aligned}$$

Thus, near $\sqrt{\epsilon}$, the potentials $\phi_{|1\ 0\rangle}$ and $\phi_{|0\ 0\rangle}$ behave respectively like

$$(2/\sqrt{3})\,(2x/3 \mp 1/x)\,\sqrt{\epsilon}\,(1 + O(\epsilon)). \qquad (172)$$

All three potentials are illustrated in figure 5 for $s' \leq 0.2$.

5.5 NEARLY IDENTICAL PARTICLES

Having examined the interaction between twistons for small ϵ it seems natural to examine it for ϵ close to unity. It turns out that the potentials for spin states $|1\ 0\rangle$ and $|0\ 0\rangle$ are qualitatively similar to those depicted in figure 3. However, the behaviour of the triplet state with spins aligned is different; and this is depicted in figure 6 for $s' \leq 2.4$.

The large but finite barrier (or well for opposite charges) is a feature for all values of ϵ close to 1. To see this, put $\epsilon = 1 - \delta$ and $s = 1 + x\delta$, with δ small and positive. Then it is not hard to show that

$$\begin{aligned} S_+ &= ((2x+1)^2 + 1)^{1/4}\,\sqrt{\delta}\,(1 + O(\delta)), & (173)\\ \cos\beta_+ &= (2x+1)/((2x+1)^2+1)^{1/2} + O(\delta), & (174)\\ \text{with } 0 &< \beta'_+ < \pi. & (175) \end{aligned}$$

Hence the potentials $\phi_{|1\ \pm1\rangle}$ have behaviour that is $O(1/\sqrt{\delta})$. In fact, they reach a maximum of $\sim 0.81/\sqrt{\delta}$ for $s' \sim 1 - 0.21\delta$.

The behaviour at very short range may be obtained from the following expansions that omit terms in δ^2.

$$\begin{aligned} S'_+ &= (1 - \delta/2) + O(s^2), & (176)\\ \text{and } \beta'_+ &= \pi - s\delta + O(s^3) & (177) \end{aligned}$$

Thus, near the origin, the potential behaves like $s\ \delta/2$.

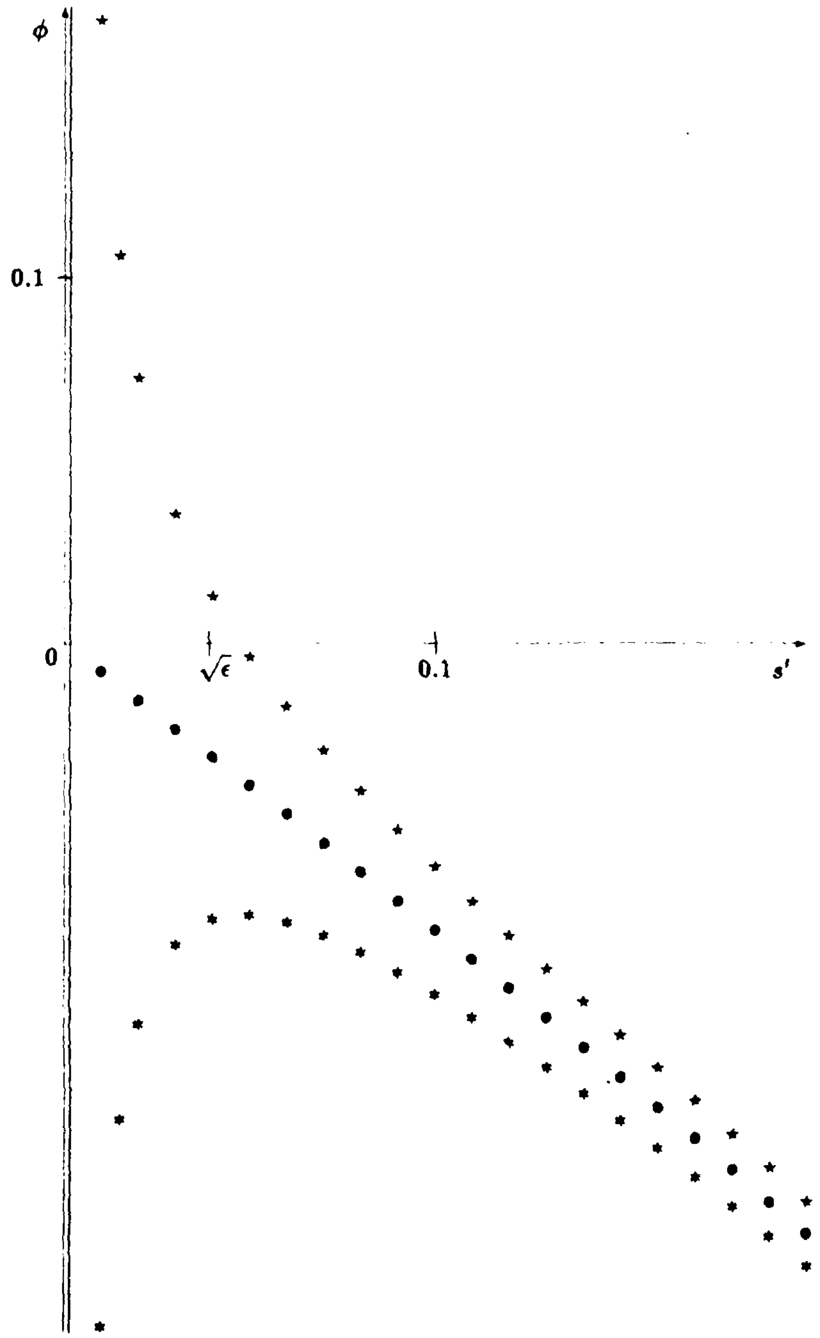

Figure 5: The (normalised) interaction between twistons whose parameters correspond to those of the proton and electron. $\epsilon = 0.00152$. The states $|\,1\ \pm 1\rangle$, $|\,1\ 0\rangle$ and $|\,0\ 0\rangle$ are depicted by •, ⋆ and ∗ respectively.

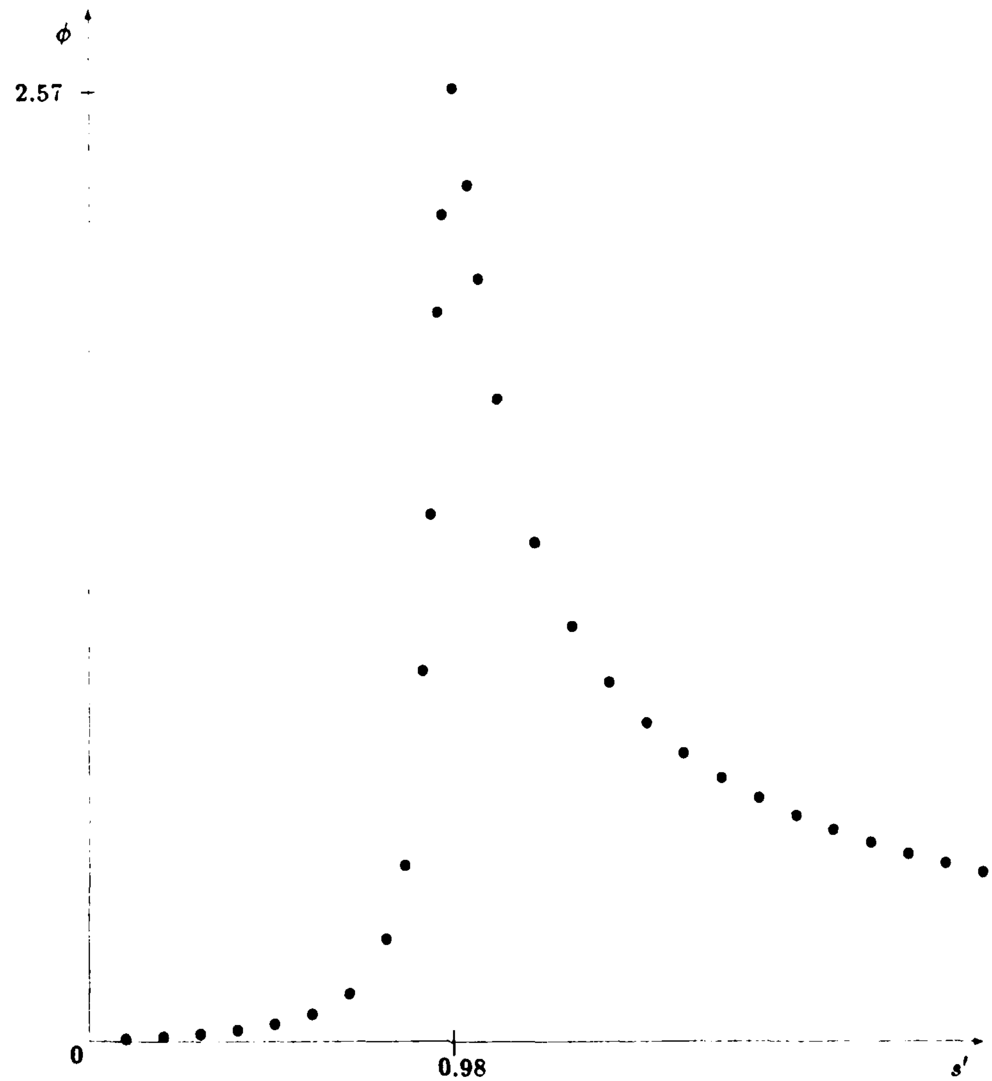

Figure 6: The (normalised) interaction between nearly identical twistons. $\epsilon = 0.9$. Only the triplet state with spins aligned is shown.

6 THE VECTOR POTENTIAL

So far I have said nothing about possible applications and in particular the consequences of using the potentials, so far described, in the (non-relativistic) Schrodinger equation. For the purpose of this section, consider a small, point test particle of charge e in the field of a spin 1/2 twiston of the same charge. Then, the principle of minimal coupling yields an equation of the form:

$$(H - e\,\phi)\,|state\rangle = 1/2\mu\,(\mathbf{p} + e/c\,\mathbf{A})^2\,|state\rangle. \tag{178}$$

Here, ϕ is the interaction potential of equation (41) illustrated in figure 1. However, $\mathbf{A}$ is the magnetic vector potential. Classically, this is related to the magnetic field by

$$\mathbf{B}_{class} = \nabla \wedge \mathbf{A}_{class}. \tag{179}$$

For a stationary source, we also have the option of defining $\mathbf{B}$ by means of the scalar potential ψ:

$$\mathbf{B}_{class} = -\nabla\psi_{class}; \tag{180}$$

and it is this form that arose naturally in section 3. Thus, before employing Schrodinger's equation we need to derive an expression for $\mathbf{A}$ in terms of ψ. It might appear that we should use equations (179) and (180). However, this would imply that ψ satisfies Laplace's equation and although this is true classically it is not true quantum mechanically. (Neither does ϕ satisfy Laplace's equation. The model to which we have been led indicates a charge distribution, illustrated in figure 2, that is smeared over all space. Similarly, although the total magnetic monopole vanishes, there is a non-zero monopole distribution.)

In short $\nabla.\mathbf{B} = 0$ breaks down at short range. Consequently, equation (179) must be abandoned. In fact there is another compelling reason for challenging equation (179) when things are quantised. This has to do with the method for introducing electromagnetic couplings via the theory of gauge fields. Specifically, if the connection ∇ be replaced by

$$\mathbf{D} = \nabla - i\,e/\hbar c\,\mathbf{A}, \tag{181}$$

then the associated curvature F, defined by the commutator

$$F_{ij}\,\Phi = [\mathbf{D}_i, \mathbf{D}_j]\,\Phi \tag{182}$$

yields not $\mathbf{B} = \nabla \wedge \mathbf{A}$ but rather

$$\mathbf{B} = \nabla \wedge \mathbf{A} - i\,e/\hbar c\,\mathbf{A} \wedge \mathbf{A}. \tag{183}$$

The final term, reminiscent of a non-Abelian gauge theory [7, 8], does not always vanish if $\mathbf{A}$ is a quantum-mechanical operator. For example

$$\mathbf{J} \wedge \mathbf{J} = i\hbar\,\mathbf{J}. \tag{184}$$

It follows that, we need to solve the equation

$$\nabla \wedge \mathbf{A} - i\,e/\hbar c\,\mathbf{A} \wedge \mathbf{A} = -\nabla\psi. \tag{185}$$

It proves instructive to rescale to dimensionless quantities by means of the natural length b defined in equation (22):

$$\begin{aligned} \mathbf{x} &= b\mathbf{s}, &(186)\\ \mathbf{A} &= e/b\,\Gamma, &(187)\\ \psi &= e/b\,\Psi, &(188)\\ \mathbf{J} &= 1/2\,\hbar\,\mathbf{j}, &(189)\\ \text{and } \nabla &= 1/b\,\nabla_s. &(190)\end{aligned}$$

Then the equation for $\mathbf{A}$ becomes

$$\nabla_s \wedge \Gamma - i\,\alpha\,\Gamma \wedge \Gamma = -\nabla_s \Psi \quad (191)$$

α being the fine structure constant. For the particular interaction postulated, Ψ has the form

$$\Psi = f(s)\,\mathbf{j}.\mathbf{s}., \quad (192)$$

where, from equation (40),

$$f(s) = (1/sS)\,\sin(\beta/2). \quad (193)$$

So, exploiting the fact that the spin is 1/2, put

$$\Gamma = F(s)\,\mathbf{j} + G(s)\,\mathbf{j}.\mathbf{s}\,\mathbf{s} + H(s)\,\mathbf{s}\wedge\mathbf{j} + K(s)\,\mathbf{s}. \quad (194)$$

In passing (if we were to use Schrodinger's equation) note that

$$\begin{aligned} \nabla_s.\Gamma &= (F'/s + G's + 4G)\,\mathbf{j}.\mathbf{s} + (K's + 3K) &(195)\\ \text{and } \Gamma.\Gamma &= (F + Gs^2)^2 + 2(F^2 + H^2 s^2) + 2K(F + Gs^2). &(196)\end{aligned}$$

Recalling that, whenever $\mathbf{u}$ and $\mathbf{v}$ commute with $\mathbf{j}$,

$$\mathbf{j}.\mathbf{u}\,\mathbf{j}.\mathbf{v} = \mathbf{u}.\mathbf{v} + i\,\mathbf{j}.(\mathbf{u}\wedge\mathbf{v}), \quad (197)$$

it will be seen that the expansion (194) is sufficiently general. Terms polynomial in $\mathbf{j}$ may always be reduced to an affine expression. (The term in K represents a gauge freedom.)

Substituting expansions (192) and (194) into equation (191) and equating coefficients appropriately yields:

$$\begin{aligned} \partial F/\partial s - sG + 2\alpha\, sH\,(F + s^2 G) &= 0, &(198)\\ -s\partial H/\partial s - 2H + 2\alpha\,F\,(F + s^2 G) &= -f, &(199)\\ \partial H/\partial s + 2\alpha\,s\,(H^2 - FG) &= -\partial f/\partial s. &(200)\end{aligned}$$

Letting $'$ indicate $\partial/\partial s$, a tedious calculation leads to the following relations for $s \neq 0$:

$$\begin{aligned} FG &= (H' + f' + 2\alpha s\,H^2)/2\alpha s, &(201)\\ F^2 &= 2H - (sf)' - 2\alpha\,s^2 H^2)/2\alpha &(202)\\ \text{and } fH &= -(sf'' + 4f')/4\alpha s. &(203)\end{aligned}$$

Note the freedom $(F,G) \to (-F,-G)$. Equally puzzling is the fact that the solution to these equations for large s seems to bear little relation to standard, linear theory for which $f = 1/2s^3$ and $\Gamma = -\mathbf{s}\wedge\mathbf{j}/2s^3$. However, a deep investigation of the implications of all this lies outside the scope of this paper.

7 CONCLUSION

The purpose of this paper has been to examine the consequences of fusing two pieces of disparate, yet established theory, established that is within two quite separate communities. On the one hand Newman and Penrose, in turn influenced by Kerr and Robinson, found that the centres of mass and charge of spinning particles were located at points of complexified space-time, at least from the viewpoint of the fields that they generate. On the other hand, there is non-relativistic quantum mechanics. Fusion implies that we must extract the non-relativistic essence of the Newman-Penrose ideology. The result is non-trivial because the particles in question have imaginary displacements proportional to their spins; and quantum mechanics is non-trivial when it comes to spin! I have called *these particles twistons in deference to Penrose's theory of twistors.*

Only spin 1/2 has been considered in this paper. Many surprising features have emerged though it is not clear whether all, some or none will survive the transition to a theory consistent with special relativity. Nor is it clear whether twistons represent anything in the physical world. The following summarises the main results of this paper:

1. The twiston behaves as an extended body, albeit with an extension that is pure imaginary. *Prior* to quantisation its (stationary) electro-magnetic field possesses the well-known ring singularity akin to that of a rapidly rotating, charged black hole.

2. The act of quantisation, removes the singularity, yielding an electrostatic potential that is spherically symmetric and finite everywhere. Further the usual Coulomb potential breaks down as a certain critical radius b is approached, b being half the product of the Compton wavelength and the gyromagnetic ratio of the particle. For charge q, the potential behaves like $q/r + O(1/r^5)$ for large r but like $4\sqrt{3}/9\ qr/b^2$ for small r. Further, there is a maximum/minimum of $\sim 0.78\ q/b$ at a range of $\sim 0.98\ b$.

3. The magnetic scalar potential of a twiston of magnetic moment $\mathbf{M}$ behaves like $\mathbf{M.x}/r^3 + O(1/r^5)$ for large r but like $4/\sqrt{3}\ \mathbf{M.x}/rb^2$ for small r.

4. The effective charge distribution of the twiston is distributed over all space but with 99.9 % lying within a range 10 b. Also, there is a non-zero magnetic monopole density though the total monopole vanishes. This means that in the quantum theory the magnetic field is not divergence free at short range. Indeed the relation between the field and the vector potential appears to resemble that of a non-Abelian gauge theory.

5. The electromagnetic self-energy of a twiston is finite, approximating to 2.01 q^2/b. For a twiston with the charge and gyromagnetic ratio of an electron, the self-energy amounts to 2.01 $\alpha\ mc^2$ thus contributing 1.5 % to the rest mass. For a twiston with atomic number Z, the self-energy contributes the whole of the rest mass if the gyromagnetic ratio approximates to 4.02 $Z^2\alpha$.

6. The interaction between a pair of (spin 1/2) twistons depends upon their relative spin state. Choosing the direction of quantisation parallel to the relative radial vector, three potentials emerge: triplet with spins aligned, $|1\ \pm 1\rangle$; triplet with spins non-aligned, $|1\ 0\rangle$; and singlet, $0\ 0\rangle$. All three behave like $1/r$ for large r. In what follows, I shall assume charges of identical sign and shall omit an overall factor of q_1q_2.

7. For identical twistons, the $|1 \ \pm 1\rangle$ potential encounters an infinite barrier at $r = b$ while the $0\ 0\rangle$ potential encounters a similar feature at $r = b/2$. On the other hand, the $|1\ 0\rangle$ potential vanishes at $r = b$ and encounters an infinite well at $r = b/2$.
8. For distinct twistons, characterised by length scales b and ϵb, with ϵ very small, all three potentials behave like the electrostatic potential for a single twiston when $r \gg \sqrt{\epsilon}b$, reaching a maximum of $\sim 0.78/b$ when $r \sim 0.98\ b$. The $|1\ \pm 1\rangle$ potential then decreases with r, behaving like $4\sqrt{3}/9\ r/b^2$ for r/b small. The remaining potentials behave differently. As r decreases from 0.98 b to $\sqrt{3\epsilon/2}\ b$ both decrease, the $|1\ 0\rangle$ potential decreasing to 0 with the $|0\ 0\rangle$ potential encountering a local minimum of about $4\sqrt{2\epsilon/3}/b$. As r continues to decrease the $|1\ 0\rangle$ potential decreases while the $|0\ 0\rangle$ potential increases. As $r = \epsilon b$ is approached, an infinite well and an infinite barrier are encountered respectively. (In the main text, opposite charges are chosen so that the terms barrier and well are interchanged.) It is most interesting to observe the emergence of three length scales here: b, $\sqrt{\epsilon}b$ and ϵb. For twistons with the characteristics of the electron and proton, these scales are respectively $4 \times 10^{-11}\ cm$, $2 \times 10^{-12}\ cm$ and $6 \times 10^{-14}\ cm$.
9. For nearly identical twistons, characterised by scales b and $(1 - \delta)b$, with δ small, the behaviour of the $|1\ 0\rangle$ and $|0\ 0\rangle$ potentials follows that for identical twistons. The $|1\ \pm 1\rangle$ potential however is radically different. Instead of finding an infinite barrier at range b, a finite barrier of height $\sim 0.81/(b\sqrt{\delta})$ is encountered at a range of $\sim (1 - 0.21\delta)b$. As r decreases, the potential then decreases rapidly. For very small r, it behaves like $r\delta/2b^2$.

It is difficult at this early stage to assess the implications of all this. Some very strong electromagnetic forces are predicted at short range and for identical particles in certain spin states a force of exclusion appears to emerge. At the same time the existence of a relation between field and potential, suggestive of behaviour more usually claimed for nuclear forces, is intriguing. Are nuclear forces electromagnetic in origin?

On a less speculative level, the removal of the Coulomb singularity is encouraging. It is worth recalling that the existence of sources with point singularities led to some of the divergences of quantum field theory; so there is the hope that the smearing out process, arising naturally in twiston theory, might ultimately cure such problems.

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Abstract The twiston is a new kind of particle defined in the context of non-relativistic quantum mechanics. It possesses mass, charge, spin and magnetic moment. If its gyromagnetic ratio is given by the Dirac value ($g = 2$), the classical analogue is that of a rapidly rotating, charged black hole with naked singularity. However, this paper is concerned with (first-) quantised twistons and with spin 1/2. The electromagnetostatic field of such a particle is shown to be non-singular except at a single point. Further, it contains finite self-energy. For a twiston of charge Ze, the self-energy contributes a fraction of its total rest mass that approximates to $4Z^2a/g$,e being the charge of the electron while a is the fine structure constant. Its electrostatic potential violates the law "like charges repel, unlike charges attract" at small distances.

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CONTINUATION

ABSTRACT

Interactions between twistons of spin 1/2 are then considered. Identical twistons in the triplet state, with their spins aligned parallel, are found to repel strongly, there being a "brick wall" potential at small distances. Finally the existence of a non-linear relationship between the twiston field and vector potential, suggestive of a non-Abelian gauge theory, is revealed.