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MPD thrusters are examples of distributed parameter systems that have temporal and spatial variations. In order to establish an intelligent diagnostic system for MPD thrusters fundamental theorems had to be derived for the general case of diagnostics of distributed systems. Advanced mathematical techniques of semigroups and groups, equivalent norms, invariant principle, Lyapunov functional, etc., have been integrated with control schemes in order to develop criterion for stability, controllability and observability of distributed parameter systems. A simple model of the MPD thruster was developed and used in evaluation of the resulted theorems. Instability conditions for MPD thrusters are derived and stabilizability inputs are presented. This research has led to new understanding for the general problem of distributed parameter systems.

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DIAGNOSTICS FOR INTELLIGENT CONTROL OF MPD ENGINES

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EXECUTIVE SUMMARY

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Research Summary:

Plasma-dynamic thrusters designed for high speed low thrust space missions, are nonlinear distributed parameter systems with very complex dynamics. The main objective of this study was to analyze MPD thrusters' controllability and observability without the need to solve the engines governing partial differential equations. This analysis falls into the category of stabilizability of distributed parameter systems (DPS), which is the most challenging problem for the control community. During the three-year period of this investigation, several new concepts were derived and developed. Following is the highlights of the main accomplishments in stabilizability and observability of the MPD thursters.

1. Integration of advancements in the mathematical theory of partial differential equations, dynamical systems, and advanced feedback control theorems.
2. Derivation and application of Lyapunov theorems for the case of distributed parameter systems.
3. Application of Lyapunov's stability theorems to MPD systems.
4. Derivation of the characteristics of a general, nonlinear hyperbolic systems represented by partial differential equations, and their relations to controllability of the MPD systems.
5. Stability analysis and stabilizability of MPD thrusters.
6. Controllability and observability of distributed parameter systems.

This research has a general focus on the analysis of the DPS, but the results at every stage is implemented on an MPD thruster model. Therefore, the new foundations and theorems resulted from this study have potentials for applications in many other distributed systems than MPD, such as flexible structures, jet engines, integration of controls for frame vibration and combustion process in an aircraft engine, etc.

CHAPTER 1 - INTRODUCTION

Distributed parameter systems (DPS), also regarded as infinite dimensional systems, are often described by a set of linear or nonlinear partial differential, integro-differential or differential-delay equations. Conversely, lumped parameter or finite dimensional systems are represented by a set of ordinary differential equations. For distributed systems, state variables are functions of time and another set of parameters, whereas lumped parameter systems are only dependent on time. In many physical systems the states of DPS are functions of time and spatial coordinates.

In general, all physical systems are intrinsically distributed in nature. However, in many instances the system's spatial energy distribution is sufficiently concentrated such that an approximate lumped parameter description may be an "adequate" representation for the system. On the other hand, the spatial energy distributions of many practical systems are widely dispersed and require controls of some or all of the spatially distributed states. In general, such systems have to be represented by distributed parameter models. Typical examples of DPS are combustion processes in engines, boilers, and furnaces; heat exchangers; distillation processes; nuclear and chemical reactors; gas dynamic thrusters and lightly damped flexible structures. Another example of DPS with a widely dispersed distribution of electro magnetic energy is a magneto-plasma dynamic accelerator. Investigation of the dynamics of distributed parameter models pertinent to magneto-plasma dynamic accelerators is of interest to the Air Force for future space vehicles and has been one of the motivations for this research.

Often a nominal (approximate) model for a system is considered in order to provide numerical estimates about the characteristics of DPS, as in determining the

system response to an excitation and/or design a control input to achieve a desired system performance. These approximate models are often derived by means of some modal reduction techniques. The most common methods for derivation of finite order models are discretization and modal truncation. Due to the reduction of the system order, some of the information about the dynamic characteristics of the system is neglected. This implies that there will be an error signal between the exact and approximate solutions, based on the unmodeled dynamics of the system. Usually, if an upper bound for such approximations exists, then that bound grows with the reduction of system order. The higher the approximation bound, the lower the precision in the system representation. To increase accuracy, the number of discretized points (nodes) needed may reach several hundred. Application of these large order models in control design would require a large number of sensors and actuators, otherwise interferences of uncontrolled modes would affect the system performance. This condition is often impractical for economical reasons.

One of the main concerns of control theory is whether or not a system is stable. For finite dimensional systems, Lyapunov stability theory have become an important vehicle in the stability analysis of a system. This approach attempts to make statements about a dynamic system's stability of motion without explicit knowledge of the solutions to its governing equations. Although the development of Lyapunov's stability theory for ordinary differential equations has been widely investigated, its application to solutions of partial differential equations, namely distributed parameter systems, has been limited. Stability results for distributed parameter systems have been derived based on spectral analysis and frequency domain characteristics of the system. Such analysis provides necessary conditions for stability, in contrast to the Lyapunov method that results in

sufficient conditions. In order to obtain sufficient conditions for stability based on spectral analysis of DPS, other system properties are required. Moreover, frequency domain analysis of DPS is often used to provide an input-output type of stability which do not necessarily imply internal stability of DPS. These requirements are in addition to the complexities associated with the spectral analysis. In general, spectral analysis requires an explicit knowledge of solutions to system equations. Furthermore, approximate models cannot be used to imply sufficient conditions for stability of DPS. Considering these issues indicates that the Lyapunov method may be a more suitable alternative for stability analysis of DPS.

In cases where system performance is not satisfactory one needs to determine whether the application of an external control input will provide the desired results. To obtain such information, the system must be "observable", namely, one must be able to estimate its characteristics from measurements of its states. Moreover, the system must be "controllable". In qualitative terms, based on the knowledge obtained from observations, one must be able to manipulate the system to accomplish a desired performance. These concepts for finite dimensional linear time invariant systems have been derived and are well known. Extension of these results to time variant and nonlinear finite dimensional systems can be established in the local sense by use of implicit function theorem. However, the technical complexity in dealing with controllability and observability of DPS arises from the fundamental differences between properties of finite and infinite dimensional spaces. Hence, the approximate models have subtle shortcomings to answer questions about controllability and observability of DPS.

In cases where the system is not stable or asymptotically stable, one must know how to make the system stable or asymptotically stable. This process, called

stabilization, in turn depends on the system characteristics, i.e., whether or not it can be "stabilized." Although this property for DPS is related to observability/controllability, the relationship is not similar to the way these three properties are coupled in the finite dimensional systems. Therefore, the approximate and reduced order models of DPS are usually insufficient for stability analysis and the design of stabilizing controllers. Control actions based on such models may have interactions with uncontrolled modes of the system which may have adverse effects on stabilization of the system.

The phenomenon of excitation of unmodeled dynamics, or "spillover", is an example of such adverse effects. Problems of this nature can be avoided either by means of distributed, i.e., "body force" type control, or by finite order controls based on a distributed model of the system. In applications of finite order (not spatially distributed) controls where actuators are concentrated at certain finite number of locations, stabilization of the system is not always guaranteed unless the points of measurement and actuation of DPS are strategic points. Often, the boundary of DPS, or parts of it, has strategic point properties. This leads to boundary control for stabilization of DPS. These issues are the current research topics in the control area. This study attempts to develop answers for these questions for the case of hyperbolic and parabolic types. The following section describes some of the previous work in this area.

Literature Review

The first significant work towards control of DPS was initiated by Butkovskii and Lerner [1]* Their work has been concentrated on the derivation of maximum principle

applied to a class of DPS. Subsequently, [2] studied the optimal control problem of a specific linear diffusion system. A complete description of all relevant issues involved in modeling, stability, control and optimization of DPS was first reported by Wang [3]. The studies of distributed parameter systems, as systems of partial differential equations, have been the result of the developments in the mathematical theory of partial differential equations (PDE) and dynamical systems. In fact, the progress made in the developments related to DPS are due to the concurrence of three fields of science: controls, PDE and dynamical systems. The theory of PDE, which was developed to deal with problems and phenomenon of continua, is associated with the names of many great mathematicians. More modern and abstract approaches to this field are due to Hadamard, Lax, Sobolev and Hilbert [4-9]. The theory of dynamical systems was initiated by the pioneering work of Zubov [10]. He tried to make mathematical abstraction of physical systems, called dynamical systems, in order to distinguish general properties of these systems, such as stability and asymptotic behavior. With this categorization he could extend stability results of Lyapunov [11] to systems of PDE. Sufficient conditions for the stability of equilibrium solutions for a system of PDE were derived by Massera [12]. The application of Lyapunov stability theorem based on the work of Zubov has been investigated by several authors. Hsu [13] applied this theory to a nuclear reactor system. Wang [3] considered the stability of those evolution equations whose solutions involve a semigroup property. There are also many other applications which utilize Lyapunov functions directly to study special problems [14,15,16]. A completely rigorous and abstract approach to the theory of Lyapunov stability for

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* Argument of [ ] denotes the reference number.

infinite dimensional systems was studied by [17] and [18]. Stability study of an equilibrium solution of a magneto-plasma dynamic (MPD) system for the special case, where the plasma equilibrium velocity is zero was addressed in [19]. In cases where a system does not have an equilibrium state, the invariance principle provides information about its asymptotic behavior. The invariance principle was first introduced by LaSalle [20] for finite dimensional systems. The generalization of this principle for abstract dynamical systems has been accomplished by Hale [21]. Applications and contributions in the asymptotic behavior of an abstract dynamical system have been investigated by [22-26].

Other developments in the theory of DPS have come from advances in the control theory applied to DPS. Historically, for finite dimensional systems, the concept of controllability was introduced in the early 1960s by Kalman [27,28]. The concept of controllability was extended to DPS initially by Fattorini [20-31], who investigated controllability analysis of a heat equation. Subsequently, Triggiani [35] has developed controllability and observability concepts for general DPS. Applications of these results to the hyperbolic boundary value problems are reported in [36-40]. Other applications of observability analysis for DPS of parabolic type are reported by [41,43]. In the process of the evolution of controllability concepts for DPS, the theory of time optimal control was extended to DPS. This theory was developed based on investigation of many researchers, especially the work of Butkovskii and Lions [44,45]. Lions' work is a landmark in this category. Also, with more emphasis being placed on hyperbolic boundary value problems, investigations of [46,47] can be recalled. Moreover, due to the nature of DPS, the optimal control problem can be defined in terms of optimal locations for sensors and actuators. Measurements at certain points in the spatial domain of the

system may yield more information about the system than other points. Since the number of sensors is generally governed by economical considerations, it is desirable to locate the given number of measurement sensors at points that lead to the best estimates of the system. The existence theorem for the solution of this optimal location problem was proven by Bensoussan [48]. An algorithm for derivation of suboptimal sensor locations based on eigenfunction expansion representation of linear DPS was developed by Yu and Seinfeld [49]. The dual problem of optimal actuator location for DPS has been investigated by Amouroux and Barbary [50].

Often, a control problem is concerned with stabilization as well as the aforementioned performance optimization. Research on the problem of stabilizability in infinite dimensional spaces was first initiated by Slemrod [51,52]. His work was primarily motivated by hyperbolic systems, and he used a generalization of the invariance principle to study stabilization of infinite dimensional systems in Hilbert spaces. This problem was also treated using the spectral theory of unbounded operators for stabilization of systems in Banach spaces by Triggiani [53]. Stabilization of DPS based on [52,53] has been applied to several classes of systems [54-56].

Finite order control problems for the stabilization of DPS has been studied by Balas and Slemrod [57-59]. Balas [57] studied a flexible structure with pointwise sensor and actuator to control a modal truncated model of the system, based on the original DPS. The truncated model is not always controllable and/or observable. It is only controllable when an estimate is determined to prevent spillover problems. This estimate is related to the initial perturbations. In [58,50] boundary control of the wave equation and a flexible beam subject to the boundary controls were investigated, respectively.

This study considers stability of two general models of DPS based on Lyapunov's direct methods. The controllability and observability concepts based on abstract dynamical system are analyzed and their relations to stabilization by distributed and finite order controllers are derived. Although the basics of stabilization by spectral theory of unbounded operator is described, the fundamental approach for stabilization is based on [52] and the invariance principle.

CHAPTER 2 - MATHEMATICAL PRELIMINARIES

This chapter provides the mathematical bases and preliminaries for the following chapters of this study to provide the reader with a quick review of the required definitions and theorems. Additional details on these topics can be found in references [60-65].

Vector Spaces

Definition 2.1. A real (or complex) vector space is a set X such that:

1. given any $x,y \epsilon X$, there is an element $x + y$ in X satisfying
   (a) $x + y = y + x, \ \forall x,y \epsilon X$
   (b) $x + (y+z) = (x+y) + z, \ \forall x,y,z \epsilon X$
   (c) there is an element $0 \epsilon X$ such that $0 + x = x, \ \forall x \epsilon X$
   (d) given an $x \epsilon X$, there is an element $-x$ in X such that $x + (-x) = 0$;
2. given any $x \epsilon X$ and any number $\alpha \epsilon F$ (F being a real or complex field), there is an element $\alpha x \epsilon X$ such that
   (e) $\alpha(\beta x) = (\alpha\beta)x$, for any $\alpha, \beta \epsilon F$ and $x \epsilon X$
   (f) $(\alpha+\beta)x = \alpha x + \beta x, \ \forall \alpha, \beta \epsilon F$ and $x \epsilon X$
   (g) $\alpha(x+y) = \alpha x + \alpha y, \ \forall \alpha \epsilon F$ and $x,y \epsilon X$
   (h) $1x = x, \ \forall x \epsilon X$

Definition 2.2. A subset W of vector space X is a subspace if it is a vector space with the operations of addition and multiplication defined on X.

Definition 2.3. A set of vectors $\{x_1, x_2, ..., x_n\}$ are linearly dependent if there exists a set of scalars $\alpha_1, \alpha_2, ..., \alpha_n \epsilon F$ not all zero, such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$. If no such set of scalars exists, then $x_1, x_2, ..., x_n$ are linearly independent.

Definition 2.4. A linear combination of a finite set of vectors $x_1, x_2, ..., x_n$ is a vector of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, where α_i's ϵ F.

Definition 2.5. Let W be any set of vectors in a vector space X. The set of all linear combinations of elements of W is called the subspace spanned (generated) by W.

Definition 2.6. A set of $\{x_1, x_2, ..., x_n\}$ is called a finite basis for a vector space X if it is linearly independent and it spans X.

Definition 2.7. A vector space is said to be n-dimensional if it has a finite basis consisting of n elements. A vector space with no finite basis is said to be infinite dimensional. For example, the space of continuous functions on the domain [a,b] (called C[a,b]) is infinite dimensional.

Normed Linear Spaces

Definition 2.8. Let X be a vector space over a real or complex field F. A norm on X, denoted by $\|\cdot\|$, is a real-valued function on X with the following properties:

a) $\|x\| > 0$ if $x \neq 0$ and $\|x\| = 0$ for $x = 0$, for all $x \epsilon X$

b) $\|\alpha x\| = vb\,\alpha\,vb\,\|x\|$, for $\alpha \epsilon F$

c) $\|x + y\| \leq \|x\| + \|y\|$, for x and $y \epsilon X$

Norms can be constructed in different ways.

If $$x = \{x_i \text{ vb } i=1,2...,n.\}$$

then for infinite dimensional space X, $n \rightarrow \infty$, the following is defined:

$$\ell_p \text{ norm} \triangleq \|\cdot\|_{\ell_p} = \left(\sum_{i=1}^{n} x_i^P \right)^{1/P} < \infty, \text{ and } P \geq 1$$

$$\ell_2 \text{ norm} \triangleq \|\cdot\|_{\ell_2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} < \infty$$

$$\ell_\infty \triangleq \|\cdot\|_{\ell_\infty} = \operatorname{Sup}_i \text{ vb } x_i \text{ vb}$$

$$L_2 \text{ norm} \triangleq \|f\|_{L_2} = \left(\int_t \text{vb } f(t) \text{ vb}^2 \, dt \right)^{1/2}, \; f(t) \epsilon \text{functional space X}$$

L_p, L_∞ norm are defined correspondingly.

Definition 2.9. A linear vector space with a norm $\|\cdot\|_x$ is called a normed linear space and is denoted by $(X, \|\cdot\|_x)$.

For example, the sets of real (and complex) numbers R^n (and C^n) are linear normed spaces with norm:

$$\|x\| = \left(\sum_{1}^{n} \text{vb } x_i \text{ vb}^2 \right)^{1/2} \quad \forall x \epsilon R^n \text{ (or } x \epsilon C^n, \text{ respectively)}$$

The previously mentioned vector space C[a,b] of continuous functions on the interval [a,b] can be turned into a normed linear space by defining the norm

$$\|f\| = \left(\int_a^b \text{vb } f(t) \text{ vb}^2 \, dt \right)^{1/2} \qquad \forall f \epsilon \, C[a,b]$$

or

$$\|f\| = \text{Sup}\,\{\,\text{vb}\,f(t)\,\text{vb}:\ a \le t \le b\,\} \qquad \forall f \in C[a,b]$$

where each norm can be proven to satisfy the properties (a), (b), (c) in definition 2.8.

Lemma 2.1. For vectors $x = \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_n\}$,

$$\text{vb}\sum_{i=1}^{n} x_i \bar{y}_i\,\text{vb}^2 \le \left(\sum_{i=1}^{n} \text{vb}\,x_i\,\text{vb}^2\right)\left(\sum_{i=1}^{n} \text{vb}\,y_i\,\text{vb}^2\right) = \|x\|^2\ \|y\|^2$$

where $\bar{y}$ denotes complex conjugate of y. This inequality is called Cauchy-Schwartz inequality.

This same inequality is true for infinite dimensional spaces

$$\text{vb}\int_a^b f(t)\,\overline{g(t)}\,dt\,\text{vb}^2 \le \left(\int_a^b \text{vb}\,f(t)\,\text{vb}^2\,dt\right)\left(\int_a^b \text{vb}g(t)\,\text{vb}^2\,dt\right)$$

$$\le \|f(t)\|^2\ \|g(t)\|^2$$

Definition 2.10. A subset S of a normed linear space X is called bounded if there is a number M such that $\|x\| \le M$ for all $x \in S$.

Definition 2.11. A map F: $X \longrightarrow Y$ between two normed linear spaces $(X, \|\cdot\|_x)$ and $(Y, \|\cdot\|_y)$ is said to be continuous at $x_o \in X$ if for a given $\epsilon > 0$ if there exists a $\delta > 0$ $(\delta = \delta(\epsilon, x_o))$ such that

$$\|F(x) - F(x_o)\|_y < \epsilon$$

whenever $\|x - x_o\| < \delta$.

Definition 2.12. A sequence $\{x_n\}$ in a normed linear space $(X, \|\cdot\|_x)$ converges to x_o if

$$\lim_{n\to\infty} \|x_n - x_o\|_x = 0\,.$$

Continuity and convergence are closely related and a map $F : X \longrightarrow Y$ is continuous if and only if (iff)

$$\lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n)$$

Example 2.1. The sequence $\{f_n(t)\}$, where $f_n = e^{-nt}$ and $f_n \in C[0,1]$, converges to

$$F(t) = \begin{cases} 0 \text{ for } t \neq 0 \\ 1 \text{ for } t = 0 \end{cases}$$

in both L_∞ and L_2 norms. It is clear that $F(t) \notin C[0,1]$. If one considers $F_1(t) = 0$ for all $t\ [0,1]$, then $\{f_n(t)\}$ converges to $F_1(t) \in C[0,1]$ in the L_2 norm but $\{f_n\}$ does not converge to $F_1(t)$ in L_∞ norm.

Definition 2.13: Open and Closed Set. A set A in a normed linear space X is closed if all convergent sequences in A have their limit points in A.

A set A is open if its algebraic complement is closed. Alternatively, a set A is open if for any point $x \epsilon A$, there is $\epsilon > 0$, such that the set $\{y : \|x-y\| < \epsilon\}$ is wholly contained in A. The closure of a set can be formed by adding all limit points of sequences in A to A, and is denoted by $\bar{A}$.

Definition 2.14: Cauchy Sequence. A sequence $\{x_n\}$ of elements in a normed linear space $(X, \|\cdot\|_x)$ is termed Cauchy if

$$\|x_n - x_m\|_x \to 0 \quad \text{as } m, n \to \infty .$$

Definition 2.15. A normed linear space is complete if every Cauchy sequence defined on that space converges to a limit point in the space. A complete normed linear space is a Banach space.

Theorem 2.1. The space $C[a,b]$ with norm $\|f\| = \text{Sup}\{\text{vb}\, f(t)\, \text{vb} : a \leq t \leq b\}$ is complete.

The space of continuous function with the square integral norm (L_2) is an incomplete space. This fact is clear from example 2.1. The space of $L_2[a,b]$ (square integrable functions) provides the completion of $C[a,b]$ with respect to L_2 norm. Thus, $L_2[a,b]$ contains all functions which are the limits of continuous functions in the sense of mean square (L^2) convergence. This property of $L^2[a,b]$ will be more elaborated under the topic of distributions and Sobolev spaces.

Definition 2.16. A subspace S of a normed linear space X is called dense in X if its closure with respect to the norm is equal to X.

Equivalent Norms

Definition 2.17. Let $\|\cdot\|_1$ and $\|\ \ \|_2$ be two different norms on the same vector space X. A norm $\|\ \|_1$ is called equivalent to $\|\cdot\|_2$ if there are positive numbers a and b such that

$$a\ \|x\|_1 \leq \|x_2\| \leq b\ \|x\|_1\ , \quad \forall x \epsilon X$$

It is easy to show that if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$, then $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$ with bound coefficients replaced by b^{-1} and a^{-1}.

Theorem 2.2. If a sequence converges with respect to one norm, then it converges with respect to any norm equivalent to it.

Corollary 2.1. If a space is complete with respect to one norm, then it is complete with respect to any norm equivalent to it.

Theorem 2.3. In a finite dimensional space, all norms are equivalent.

Operators on Vector Spaces

Definition 2.18. Let X and Y be subsets of two vector spaces M and N, respectively. An operator, mapping or transformation $T : X \rightarrow Y$ is a rule which associates any given $x \epsilon X$ with an element of Y and is denoted by Tx or T(x). This mapping may also be implied by x vb⟶ Tx, which implies "x is mapped into Tx".

The sets X and Y are called the domain and range of T, respectively. The vector Tx is called the image of x under T.

Definition 2.19. A mapping $T : X \rightarrow X$, where X is a subset of a normed linear space N, is called a contraction mapping if there is a positive number $\alpha < 1$ such that

$$\| Tx - Ty \| \leq \alpha \| x - y \| \qquad \text{for all } x, y \epsilon X$$

Theorem 2.4. If $T : X \rightarrow X$ is a contraction mapping of a closed subset X of a Banach space, then there is exactly one $x \epsilon X$ such that $Tx = x$. The point x in this transformation is called the fixed point. This theorem gives proof of existence and uniqueness and a computational method for finding the solution (fixed point) of differential and/or integral equations.

Compactness

A bounded region in a finite dimensional space has a finite volume. Successive applications of an operator T to a point provides a sequence of infinite points. Since these infinite number of points are contained in a bounded volume, then the sequence of operation converges to a limit point, which is the fixed point of T. But in an infinite dimensional space, even a bounded region is so large, due to the infinite number of dimensions into which it extends, that a sequence can wander through the region indefinitely. Therefore, the sequence never converges and never reaches a fixed point. In

infinite dimensional spaces compactness is used to define boundedness.

Definition 2.20. A subset S of a normed linear space X is compact if every infinite sequence of elements of S has a sub-sequence which converges to an element of S.

In finite dimensional spaces, closed and bounded sets are the same as compact sets. However, closed and boundedness are not sufficient to give compactness in the infinite dimensional spaces.

Linear Transformations

Definition 2.21. A linear transformation, T, from a linear space X to a linear space Y over F is a map $T : X \rightarrow Y$, such that

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for all $x, y \epsilon X$.

Definition 2.22. $T : X \rightarrow Y$, were X and Y are normed linear spaces, is said to be bounded if

$$\| Tx \| \leq K \| x \|_x$$

for some constant $K > 0$ and all $x \epsilon X$.

Definition 2.23. If T is a bounded linear transformation between two normed linear spaces X and Y, one can define $\| T \|$ by

$$\| T \| = \operatorname*{Sup}_{\substack{x \neq 0 \\ x \epsilon X}} \left\{ \frac{\| Tx \|_y}{\| x \|_x} \right\}$$

Definition 2.24. If X and Y are normed linear spaces, then one can define the space of all bounded linear transformations $T : X \rightarrow Y$ by $\mathcal{L}(X,Y)$ with the norm defined in

Definition 2.23.

Definition 2.25. If T is a linear operator (transformation) defined on its domain $D(T) \subset X$ with range $R(T) \subset Y$, then the graph $G(T)$ is the set

$$G(T) = \left\{ (x,\ Tx),\ x \epsilon D(T),\ Tx \epsilon R(T) \right\}$$

in the product space $X \times Y$. The linear operator T is closed if its graph $G(T)$ is a linear subspace of $X \times Y$.

Definition 2.26. A linear transformation over space X, which maps it into F, is called a linear functional

$$T : X \rightarrow F$$

Theorem 2.5 (Hahn-Banach Theorem). Every continuous linear functional $h : M \rightarrow F$ defined on a linear subspace M of a normed linear space X can be extended to a continuous linear functional H on all of X with preservation of norm.

A useful corollary of this theorem, which will be used in the controllability and observability analysis of the system, is given as follows:

Corollary 2.2. If E is an arbitrary subset of a normed linear space X, then $\overline{\text{span}}\ E = X$ if and only if the zero functional is the only functional which vanishes on all of E.

Definition 2.27. Dual (conjugate) space of a normed linear space X is the normed linear space of all bounded linear functionals on X. One can write this space as $\mathscr{L}(X,F)$ or X^*.

Definition 2.28: Inner Product. An inner product on a linear space H defined over the complex or real field F is a map $<\cdot,\cdot>: H \times H \rightarrow F$ such that

(a) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

(b) $\overline{\langle x, y \rangle} = \langle y, x \rangle$

(c) $\langle x, x \rangle \geq 0$ and $\langle x, y \rangle = 0$ if $x = 0$

for $x, y, z, \epsilon H$, and $\alpha, \beta \epsilon F$.

A linear space H with an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space.

Definition 2.29. A Hilbert space is an inner product space, which is complete as a normed linear space with the norm defined by inner product.

$$\|x\|_H = \langle x, x \rangle$$

Definition 2.30. An orthonormal space of a separable Hilbert space, H, is maximal if $H = \overline{\text{span}}\{\phi_i\}$. Then for any $x \epsilon H$, the Fourier expansion is

$$x = \sum_{i=1}^{\infty} \langle x, \phi_i \rangle \phi_i$$

One of the most important properties of a Hilbert space is the simple representation of its dual. The following theorems illustrates this representation.

Theorem 2.6: (Riesz Representation Theorem). Every bounded linear functional in a Hilbert space H can be written in the form $\langle x, y_o \rangle$ where $y_o \epsilon H$ is uniquely determined from T(x).

$$T(x) = \langle x, y_o \rangle, \quad \forall x \epsilon H$$

Theorem 2.7: (Lax-Milgram Theorem). Let H be a Hilbert space and let B(x,y) be a complex valued functional defined on the product Hilbert space $H \times H$, satisfying the following conditions:

i. Sesqui-linearity, i.e.,

$$B\left[(\alpha_1 x_1 + \alpha_2 x_2), y\right] = \alpha_1 B(x_1, y) + \alpha_2 B(x_2, y)$$

and

$$B\left[x, (\beta_1 y_1 + \beta_2 y_2)\right] = \overline{\beta}_1 B(x, y_1) + \overline{\beta}_2 B(x_1, y_2)$$

where $\overline{\beta}_1$, $\overline{\beta}_2$ are complex conjugates of β_1, β_2 respectively.

ii. Boundedness, i.e., there exists a positive constant γ such that

$$vb\, B(x,y)\, vb \leq \gamma \|x\| \cdot \|y\|$$

iii. Positivity, i.e., there exists a positive constant δ such that

$$B(x,x) \geq \delta \|x\|^2$$

Then there exists a uniquely determined bounded linear operator $S \epsilon \mathcal{L}(H,H)$ with a bounded linear inverse $S^{-1} \epsilon \mathcal{L}(H,H)$ such that

$$\langle x,y \rangle = B(x,Sy); \qquad \|S\| < 1/\delta$$

and

$$\langle x, S^{-1}y \rangle = B(x,y); \qquad \|S^{-1}\| < \gamma$$

This theorem can be used in relation with equivalent inner products and norms.

Theorem 2.8. Two inner products defined on a real linear vector space H are equivalent if and only if there exists a symmetric bounded positive definite linear operator $S \epsilon \mathcal{L}(H,H)$ such that

$$\langle x,y \rangle_2 = \langle x, Sy \rangle_1$$

where the indices identify the inner product in Hilbert spaces 1 and 2. Because $B(x,y) = \langle x, Sy \rangle_1$, it can be concluded that $\langle x,y \rangle_2$ is sesqui-linear, bounded and positive due to the properties of S, and that $\langle x,y \rangle_2$ satisfies all the properties of an inner product. Hence the norms defined by these inner products are equivalent. The

application of equivalent norms yields the derivation of the stability properties of the general operators (A) in differential equations of the form

$$\dot{x} = Ax ,$$

from the knowledge of the properties of the operator A. The concept of equivalent inner products can be extended to complex Hilbert spaces.

Definition 2.31. Let $A \in \mathcal{L}(H)$, then the adjoint operator A^* is defined by

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \text{ for all } x,\ y \in H .$$

This suggests that for bounded operator A, the adjoint is a transpose of its complex conjugate. For unbounded operators, where $x \in D(A)$ and $y \in D(A^*)$ with D(A) being dense in H, it can be shown that if A is closed, then $D(A^*)$ is dense in H and A^* is closed. Hence

$$A^{**} = A$$

For example, if

$$Au = \frac{du}{dt} , \quad \frac{du}{dt} \in H = L_2(0,T) , \qquad \text{with } u(0) = 0$$

$$\langle Au, v \rangle = \int_0^T \frac{du}{dt}\, v(t)\, dt$$

$$= u(T)\, v(T) - \int_0^T u(t)\, \frac{dv}{dt}\, dt$$

$$\text{then} \qquad A^*u = -\frac{du}{dt} ,$$

where u, and $\frac{du}{dt} \in H$ with $u(T) = 0$.

Distributions

Definition 2.32. The support of a function $f: R \rightarrow C$ is the set $\{x : f(x) \neq 0\}$, written as

supp(f). A function has bounded support if there exists numbers a,b such that supp(f) $\subset [a,b]$.

Definition 2.33. Letting Ω be an open set in R^n, then the function $f:\Omega \rightarrow C$ is n times continuously differentialable if its first n derivatives exist and are continuous. The function f is denoted by $f \epsilon C^n(\Omega)$. f is smooth or infinitely differentiable if $f \epsilon C^\infty$. The set $C_0^n(\Omega)$ consists of functions in $C^n(\Omega)$, which vanish outside a compact subset $\Omega' \subset \Omega$.

Definition 2.34. A test function is a smooth $R \rightarrow C$ function with bounded support. The set of all test functions is called $\mathcal{D}$.

Definition 2.35. A linear functional on $\mathcal{D}$ is a map $f:\mathcal{D} \rightarrow C$ such that $f(a\phi + b\psi) = a\, f(\phi) + bf(t)$ for all $a,b \epsilon C$ and $\phi, \psi \epsilon \mathcal{D}$.

Definition 2.36. A sequence of test functions ϕ_n converges to a limit point ϕ in $\mathcal{D}$, if (i) there is an interval [a,b] containing supp(ϕ) and supp(ϕ_n) for all n, and (ii) for any k, $\phi_n^{(k)}(x) \rightarrow \phi^{(k)}(x)$ as $n \rightarrow \infty$, uniformly for $x \epsilon [a,b]$, where $\phi^{(k)}$ is the k-th derivative of ϕ.

Definition 2.37. A functional f on $\mathcal{D}(\Omega)$ is continuous if it maps every convergent sequence in $\mathcal{D}(\Omega)$ into a convergent sequence in C; that is, $f(\phi_n) \rightarrow f(\phi)$ whenever $\phi_n \rightarrow \phi$ in $\mathcal{D}$. A continuous linear functional on $\mathcal{D}$ is called a distribution, namely

$$f = \int_\Omega f(x)\, \phi(x)\, dx = \langle f, \phi \rangle = f(\phi) \quad \forall \phi \epsilon \mathcal{D}(\Omega)$$

The set of all distribution in the compact support Ω is denoted by $\mathcal{D}'(\Omega)$.

Definition 2.38. Two distributions f and g are equal on (a,b) if $\langle f, \phi \rangle = \langle g, \phi \rangle$ for all $\phi \epsilon \mathcal{D}$ such that supp(ϕ) $\subset$ (a,b).

The delta distribution, δ, is defined as

$$<\delta, \phi> = \phi(o) \qquad \text{for all } \phi \text{ in } \mathcal{D} .$$

Definition 2.39. The derivative of a distribution f is a distribution f' defined by $<f', \phi> = -<f, \phi'>$ for all $\phi \epsilon \mathcal{D}$.

Similarly, higher order derivatives can be written as

$$<D^n f, \phi> = (-1)^n <f, D^n \phi> .$$

A formal partial differential operator defined in an open subset Ω of R^n can be shown in abbreviation by

$$\tau = \sum_{vb\, J\, vb \leq m} a_J (x)\, \partial^J$$

where the highest order of differentiation is m, symbol J denotes an index, i.e., a k-tuple, $J = (j_1, j_2, \ldots, j_k)$, and $vb\, J\, vb = \sum_{i=1}^{k} j_i = k$,

$$\partial^J = \frac{\partial^k}{\partial x_1^{j_1}\, \partial x_2^{j_2} \cdots \partial x_n^{j_k}}$$

The adjoint operator of τ is

$$\tau^* = \sum_{vb\, J\, vb \leq m} (-1)^J\, \partial^J \left[a_J(x)\, \partial^J \right]$$

Definition 2.40. Let τ be a formal partial differential operator defined in open set $\Omega \epsilon R^n$ with real coefficients $a_J(x) \epsilon C^\infty(\Omega)$. If $f \epsilon \mathcal{D}'(\Omega)$, then τf will denote a distribution defined by the following equation

$$(\tau f)\, (\phi) = f(\tau^* \phi) , \quad \phi \epsilon C_o^\infty(\Omega)$$

Sobolev Spaces

Definition 2.41. Letting Ω be an open subset of R^n and let K be a non-negative integer,

then

(i) the set of all distribution $f \in \mathcal{D}'$ such that $\partial^J f \in L_2(\Omega)$ for all $vb\,J\,vb \leq K$ will be denoted by $H^K(\Omega)$. This space is called Sobolev space, which is an inner product space mapping each pair f,g

$$\langle f,g\rangle_K = \sum_{vb\,J\,vb \leq K} \int_{\Omega} \partial^J f(x)\, \partial^J g(x)\, dx$$

The norm based on this inner product is

$$\|f\|_K = \left(\langle f,f\rangle_K\right)^{1/2}$$

(ii) the space $H^K(\Omega)$ is the closure of $C^K(\Omega)$ in the above norm, similar to the space $H^\circ(\Omega) = L_2(\Omega)$, which is the closure of $C(\Omega)$ in $\|\cdot\|_{L_2}$ norm.

(iii) the symbol $H_o^K(\Omega)$ will denote the closure of $C_o^K(\Omega)$ in $\|\cdot\|_{H^K}$ norm.

Lemma 2.2: Letting Ω be an open set in R^n, then the space $H^K(\Omega)$ of the preceding definition is a complete Hilbert space and the space $H_o^K(\Omega)$ is a closed subspace of $H^K(\Omega)$. In addition,

$$H_o^o(\Omega) \subseteq H^o(\Omega) = L^2(\Omega)$$

$$H^{K+1}(\Omega) \subseteq H^K(\Omega) \qquad K \geq 0$$

$$H_o^{K+1}(\Omega) \subseteq H_o^K(\Omega) \qquad K \geq 0$$

Theorem 2.9. Letting Ω be a bounded open set in R^n and $\partial\Omega$, the boundary of Ω, be a smooth surface with no interior point to $\bar{\Omega}(=\Omega \oplus \partial\Omega)$, then $H^m(\Omega) \subset C^K(\Omega)$, where m and K are integers with $m > K + n/2$.

Theorem 2.10. If Ω is as given in the preceding theorem, then the mapping $H^m(\Omega) \rightarrow H^{m-1}(\Omega)$ is compact.

CHAPTER 3 - DYNAMICAL SYSTEMS AND LYAPUNOV THEOREMS

Distributed Parameter System (DPS)

In this section DPS and lumped parameter systems are compared and general properties of these systems are described. The dynamics of distributed systems are often expressed by a set of partial differential, integral or differential delay equations, as opposed to lumped parameter systems which are described by a set of ordinary differential equations. For distributed systems, the state variables are functions of time and another set of parameters, whereas lumped parameter systems are only dependent on time. In many physical systems the states of DPS are functions of time and spatial coordinates. Fig. 3.1 illustrates the analogy between DPS and lumped parameter systems. In this figure, at each instance of time, the state of a lumped parameter system is $v \epsilon R^2$. However, the state of DPS is a vector function of one spatial coordinate, x. Therefore, each of the two states of the DPS, shown in Fig. 3.1, consists of infinite points and belongs to an infinite dimensional functional space. This example reveals the infinite dimensional nature of the DPS as opposed to finite dimensionality of lumped parameter systems.

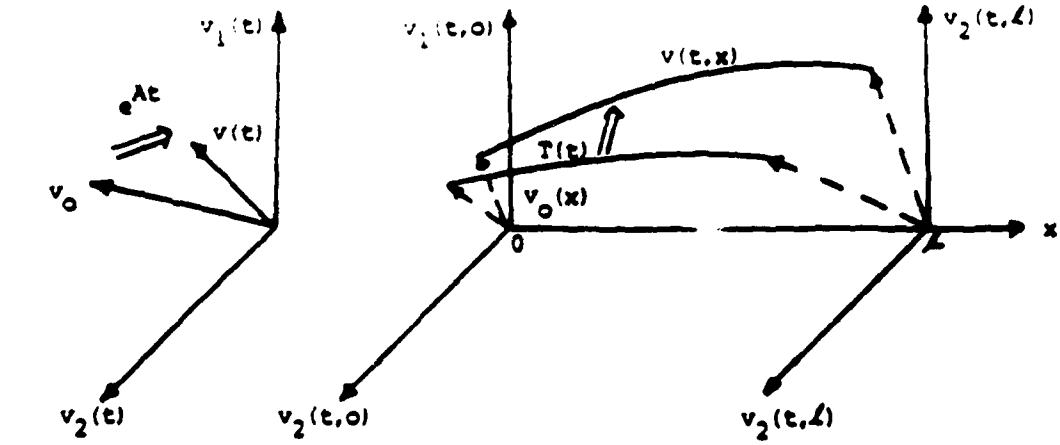
Semigroups and Abstract Evolution Equations

The simplest finite dimensional systems are linear autonomous systems which can be formulated by

$$\dot{v} = A v \qquad v \epsilon R^n \tag{3-1}$$

$$v(to) = v_o$$

where $v : [o,T] \rightarrow R^n$ is the solution of the system. The operator (matrix) A is bounded, $A \epsilon \mathcal{L}(R^n, R^n)$. Solution of the homogeneous system (3-1) is



$v \in R^2$

$e^{At}: v_0 \to v$

$$v(t) = e^{At} v_0 + \int_0^t e^{A(t-S)} Bu$$

$v \in V([0,\ell], E^2)$

$T(t) : v_0(x) \longrightarrow v(t,x)$

$$v(t,x) = T(t)\, v_0 + \int_0^t T(t-S) Bu$$

Figure 3.1 Comparison of finite and infinite dimensional systems.

$$v = e^{At}v_o$$

which satisfies

$$\frac{d}{dt}\left(e^{At}v_o\right) = A\,e^{At}v_o - Av$$

where e^{At} is the transition matrix defined by,

$$e^{At} = I + At + \frac{A^2t^2}{2!} + ... + \frac{A^nt^n}{n!} + \cdots$$

Usually, external effects such as body forces can be included in the above system representation by a vector valued function $g : t\epsilon[0,T] \longrightarrow \mathcal{R}^n$

$$\dot{v} = Av + g(t) \qquad v(0) = v_o \tag{3-2}$$

The solution is then

$$v(t) = e^{At}v_o + \int_o^t e^{A(t-s)}\, g(s)ds \tag{3-3}$$

The excitation term g(t) can be represented in terms of a vector valued (input) function $u(t) : [0,T] \longrightarrow \mathcal{R}^m$ with $g(t) = Bu(t)$ where $B : \mathcal{R}^m \longrightarrow \mathcal{R}^n$.

In the case of distributed parameter systems, the mathematical description is usually given by a set of partial differential equations. In order to generalize the results from the finite-dimensional systems, the set of partial differential equations can be transformed into an abstract form of equations (3-1) and (3-2).

$$\dot{v} = Av \qquad v\epsilon V\left([0,\ell]\,,\, E^n\right) \tag{3-4}$$

$$v(t_o) = v_o$$

This formulation of the system is called the evolution or the state space equation. The state space of the system V belongs to some Banach space or more commonly, to a Hilbert space. The evolution operator A in equation (3-4) is applied to the v and it may be a bounded or unbounded operator. An example of bounded operator is Fredholm

integral operator for which,

$$Av = \int_{\alpha}^{\beta} a(x,\xi)v(\xi,t)\,d\xi \tag{3-5}$$

where $a(x,\xi)$ is the kernel of the operation. The unbounded operator is often in the form of a derivative action which could map a bounded variable into an unbounded one. For example, the wave equation

$$\frac{\partial^2 v}{\partial t^2} = C^2 \frac{\partial^2 v}{\partial x^2} + f(x,t) \tag{3-6}$$

can be written as

$$\frac{\partial v_1}{\partial t} = v_2$$

$$\frac{\partial v_2}{\partial t} = C^2 \frac{\partial^2 v_1}{\partial x^2} + f(x,t)\,. \tag{3-7}$$

Then the evolution operator A would be

$$A = \begin{bmatrix} 0 & 1 \\ C^2 \dfrac{\partial^2(\cdot)}{\partial x^2} & 0 \end{bmatrix}$$

which is an unbounded operator because the $\dfrac{\partial^2}{\partial x^2}$ is unbounded.

The solution of the evolution equation (3-4) for infinite dimensional systems can be given as

$$v(t,x) = T(t)\, v_o \tag{3-8}$$

where the operator T(t) (strongly continuous semigroup or group) is an abstraction of operator e^{At} with similar properties [66], namely:

(a) $v(0;v_o) = v_o$ (i.e., at time 0, initial state is v_o)

(b) $v(t_1 + t_2; v_o) = v\left(t_1; v(t_2; v_o)\right) = v\left(t_2; v(t_1; v_o)\right)$

(c) $v(t, v_o)$ is continuous in t and v_o (i.e., if one of them changes slightly, then the solution should not be changed drastically).

Based on abstraction of the above conditions, the following definition can be given:

Definition 3.1. A C_o (strongly continuous) semigroup is an operator $T(t)$: $R^+ \rightarrow \mathcal{L}(V)$, where $\mathcal{L}(V)$ is the space of all bounded operators on V into V. The following properties characterize a semigroup:

(a) $T(0) = I$

(b) $T(t_1 + t_2) = T(t_1)T(t_2) \qquad t_1, t_2 > 0$

(c) $\lim_{t \rightarrow 0^+} T(t)v = v$, for all $v \epsilon V$ (i.e., T is strongly continuous at $t = 0$).

The operator T(t) will be defined as a C_o - group provided $t \epsilon (-\infty, \infty)$.

Fig. 3.1 depicts the similarity between e^{At} and T(t) in deriving the finite and infinite dimensional systems from an initial state v to a final state v(t). However, the operator T(t) is applied to the state v instead of mere multiplication. Two examples of semigroups are given here. These semigroups are generated from the evolution equations below.

For the diffusion process in heat or mass transfer, the governing dynamics is given by

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + f(x,t), \quad v(0,x) = v_o(x) \tag{3-9}$$

The evolution operator A in this case is

$$A = a^2 \frac{\partial^2(\cdot)}{\partial x^2}$$

By applying the Laplace transform to the homogeneous evolution equation, one can determine the semigroup T(t) generated by this system. If $\hat{v}(s,x)$ is a Laplace transform of $v(t,x)$, then

$$s\hat{v} - v_o = a^2 \frac{\partial^2 \hat{v}}{\partial x^2} \tag{3-10}$$

$$s\hat{v} - a^2 \frac{\partial^2 \hat{v}}{\partial x^2} = v_o(x)$$

if v_h is the green's function for the system

$$\hat{v} = \int_{-\infty}^{\infty} v_h(s,\xi)\, v_o(x-\xi)\, d\xi = \int_{-\infty}^{\infty} v_h \left[s,(x-\xi)\right] v_o(\xi)\, d\xi \tag{3-11}$$

$$sv_h - a^2 \frac{\partial^2 v_h}{\partial x^2} = \delta(x)$$

$$v_h\,(+\infty) = v_h(-\infty) = 0$$

$$v_h = C\left[e^{-\frac{\sqrt{s}\, vb \times vb}{a}}\right], \qquad \int_{-\epsilon}^{\epsilon} v_h\, dx = 0$$

$$\lim_{|\epsilon| \to 0} -a^2 \frac{\partial v_h}{\partial x}\, vb_{-\epsilon}^{+\epsilon} = +a^2\, C\left[\frac{\sqrt{s}}{a}\, e^{\frac{-\sqrt{s}\, vb \times vb}{a}}\, vb_{\epsilon} + \frac{\sqrt{s}}{a}\, e^{\frac{-\sqrt{s}\, vb \times vb}{a}}\, vb_{-\epsilon}\right]$$

$$= +\,2a^2\, C\left[\frac{\sqrt{s}}{a}\, e^{\frac{-\sqrt{s}\, 0}{a}}\right] = +\,2a^2\, C\frac{\sqrt{s}}{a} = 1$$

$$C = \frac{1}{2a\sqrt{s}}$$

$$v_h = \frac{1}{2a\sqrt{s}} \exp\left(-\frac{\sqrt{s}}{a}\, vb \times vb\right) \tag{3-12}$$

From equation (3-11), $\hat{v}$ can be calculated

$$\hat{v} = \frac{1}{2a\sqrt{s}} \int_{-\infty}^{\infty} \exp\left[-\frac{\sqrt{s}}{a}\, vb\, x - \xi\, vb\right] v_o(\xi) d\xi \tag{3-13}$$

$$v = \mathcal{L}^{-1}\left\{\frac{1}{2a\sqrt{s}} \int_{-\infty}^{\infty} \exp\left[-\frac{\sqrt{s}}{a}\, vb\, x - \xi\, vb\right] v_o(\xi)\, d\xi\right\}$$

$$v = \int_{-\infty}^{\infty} d\xi v_o(\xi) \mathcal{L}^{-1}\left\{\frac{1}{2a\sqrt{s}} \exp\left[-\frac{\sqrt{s}}{a}\, vb\, x - \xi\, vb\right]\right\}$$

The inverse Laplace transform is [67]:

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}} \exp\left[-\sqrt{s}\,\lambda\right]\right\} = \frac{1}{\sqrt{\pi t}} \exp\left(-\lambda^2/4t\right) \tag{3-14}$$

Therefore, for $\lambda = \dfrac{vb\, x - \xi\, vb}{a}$

$$v = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4a^2 t}\right] v_o(\xi)\, d\xi \tag{3-15}$$

Hence, T(t) will be

$$T(t)[\cdot] = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4a^2 t}\right] [\cdot]\, d\xi \tag{3-16}$$

The semigroup for the first order wave equation can be derived in a similar way:

$$\frac{\partial v}{\partial t} = -u \frac{\partial v}{\partial x} + f\ , \qquad v(o,x) = v_o(x) \tag{3-17}$$

$$A = -u \frac{\partial(\cdot)}{\partial x}$$

Similar calculations results in the following form for the state of the system v at time t:

$$v(t,x) = T(t)\, v_o(x) = v_o(x - ut) \tag{3-18}$$

Theorem 3.1. Letting T(t) be a semigroup on R^+ to $\mathcal{L}$ (V), where V is a Banach Space, then

(a) $\|T(t)\|$ is bounded on every compact interval of $[0,\infty]$ such that $\|T(t)\| \leq Me^{-t}$ for some M, $\omega \epsilon \mathcal{R}$.

(b) T(t) is strongly continuous, on $[0,\infty]$.

The proof of this theorem is given in [66].

Definition 3.2. The infinitesimal generator of a strongly continuous semigroup T(t) is defined by A when

$$Av = \lim_{t \to 0^+} \left\{ \frac{1}{t} (T(t)v - v) \right\}$$

$$v \epsilon D(A) \subseteq V$$

It should be noted that in general, A will be an unbounded operator. The operator A is closed, i.e., its range and domain converge to some element of their respective spaces. In addition, the closure of the D(A) covers the space V, namely it is dense in V.

Theorem 3.2. Let T(t) be a strongly continuous semigroup on a Banach space V with infinitesimal generator A. If $v_o \epsilon D(A)$, then

(a) $T(t)v_o \epsilon D(A)$ for $t \geq 0$

(b) $\frac{d}{dt}(T(t)v_o) = A(T(t)v_o) = T(t)Av_o, \; t > 0$

(c) $\frac{d^n}{dt^n}(T(t)v_o) = A^n(T(t)v_o) = T(t)A^n v_o$ for $v_o \epsilon D(A^n)$; $t > 0$

(d) $T(t)v_o - v_o = -\int_o^t T(s)Av_o \, ds; T > 0$

(e) $D(A^n)$ is dense in V for $n = 1, 2, \cdots$ and A is closed.

The proof is given in [62].

For finite dimensional systems, the Laplace transform of e^{At} is $L(e^{At}) = (SI-A)^{-1}$. This can be generalized to semigroups by the following proposition:

Proposition 3.1. If $\bar{T}(t)$ is a strongly continuous semigroup with infinitesimal generator A, then $Re(s) > \omega$ for $s \epsilon \rho(A)$. Where, $\rho(A)$ is defined as the resolvent set of A, namely

$$\rho(A) = \left\{ S:(SI-A)^{-1} \epsilon \mathcal{L}(V); \text{ bounded linear operator on } V \right\}.$$

Note that ω is given by

$$\omega = \text{Inf} \left\{ \omega: \| T(t) \| \leq M e^{\omega t}; M, \ \omega \epsilon \mathcal{R} \right\}.$$

where Inf is greatest lower bound of the set of ω's.

Theorem 3.3: (Hille-Yoshida Theorem). If

(a) A is a closed linear operator on V such that D(A) is dense in V,

(b) $(\lambda I-A)^{-1}$ exists for some $\omega \epsilon \mathcal{R}$ and for every $\lambda \epsilon \mathcal{R}$ such that $\lambda > \omega$,

(c) $\| (\lambda I-A)^{-1} \| \leq \frac{M}{\lambda-\omega}$; $\lambda > \omega$,

then A generates a strongly continuous semigroup T(t) with the norm $\| T(t) \| \leq M e^{\omega t}$. The proof of this theorem is given by [68].

Definition 3.3. Considering the strongly continuous semigroup T(t) with

$$\| T(t) \| \leq M e^{\omega t} \qquad M > 0, \ \omega < \infty$$

if $\omega = 0$, then $\| T(t) \| \leq M$ and T(t) is called an equi-bounded semi-group with $t \geq 0$.

Definition 3.4. For equi-bounded semigroup when $M = 1$,

$$\| T(t) \| \leq 1, \; t \geq 0 \, ,$$

therefore T(t) is called contraction semigroup.

Contraction semigroups are very important in the stability theory of semigroups. The contraction semigroups are closely related to the dissipative property of the infinitesimal generator A of the semigroup T. To study the dissipative property of an operator A, one needs to apply inner products. Hence, the Hilbert space would be a natural space for the study of the dissipative property of operators.

Definition 3.5. An operator A defined on $D(A) \subseteq H$ (Hilbert Space) is dissipative if $Re\langle Av, v\rangle \leq 0$ for every $v \epsilon D(A)$.

Theorem 3.4: (Phillips and Lumer Theorem). Letting A be a linear operator with domain D(A) and range of A, both of which are in the Hilbert Space H where $\overline{D(A)} = H$ (i.e., the domain of A denses in H), then A generates a contraction semigroup on H if and only if A is dissipative with respect to the inner product defined on H and there exists a $\lambda > 0$ such that $R(\lambda I - A) = H$.

The proof of this theorem is given in [68]. It should be mentioned that if the hypothesis of the above theorem is true, then one can derive

$$\| \frac{1}{(\lambda I - A)} \| \leq \lambda^{-1} \, ,$$

for all $\lambda > 0$. Hence, the Hille-Yoshida theorem could be used. By setting $M = 1$, $\omega = 0$ one can find

$$\| T(t) \| \leq M e^{\omega t} = 1$$

which indicates that the operator A generates a contraction semigroup.

Corollary 3.1. If A is a closed linear operator with dense domain in H, then A generates

a contraction semigroup if and only if A and A^* are dissipative. Note that A^* is the adjoint operator of A, given by:

$$\langle x, Ay \rangle = \langle A^* x, y \rangle$$

Thus far, the theory has covered the properties of an operator A with regard to semigroups. In general, the operator A of an evolution equation can represent a hyperbolic or parabolic problem. The specific problem which will be considered in Chapter 5 is a hyperbolic system of the class

$$\dot{v} = Lv \qquad v \epsilon (L_2(\Omega), E^n)$$

The states, v, form an Euchidean vector space. The vector valued function v, where each element of the vector forms a function which belongs to an infinite dimensional Hilbert space, is defined on one dimensional spatial region (domain) $\Omega = x \epsilon [0, \ell]$. The operator L is similar to the operator A previously discussed and is unbounded with the specific form:

$$Lv = A \frac{\partial v}{\partial x} + Bv$$

It is possible to establish the solution of this system of evolution equations in terms of semigroups. The solutions of this general system, depending on the conditions specified by dimension of E^n, can result in either groups or semigroups [69].

Dynamical Systems

The physical systems are modeled by the evolution equation of integro-differential structure. It is often difficult to investigate properties of the motion of a physical system based on its solution derived from a specific condition. To avoid this obstacle in the evaluation of selected properties of the system, such as boundedness, stability and asymptotic behavior, an abstraction of the physical systems is considered. This

abstraction of mathematical models for the evolution of physical systems was first proposed by Zubov [10] as dynamical systems. The significance of such systems is that Lyapunov stability theorems can be applied to them.

Definition 3.6. A dynamical system on a metric space V is a mapping u: $R^+ \times V \rightarrow V$ such that

(i) $u(\cdot,v) : R^+ \rightarrow V$ is continuous (right continuous at $t = 0$)

(ii) $u(t,\cdot) : V \rightarrow V$ is continuous,

(iii) $u(o,v) = v$,

(iv) $u(t+s,v) = u(t, u(s,v))$,

for all $t, s \epsilon R^+$, $v \epsilon V$.

For a dynamical system on a metric space V, the mapping $u(\cdot,v) : R^+ \rightarrow V$ is called the motion starting from $v \epsilon V$.

From the properties of a C_o-semigroup in definition 3.1, it is clear that every C_o-semigroup T(t) determines a dynamical system and vice versa.

Definition 3.7: (Linear dynamical systems). A dynamical system T(t) on a Banach space V is linear if $T(t) \epsilon \mathcal{L}(V,V)$ for every $t \epsilon R^+$, i.e., T(t) is a bounded linear operator on V. From the study of semigroups it can be shown that for a linear dynamical system there exists real numbers M, ω, $M \geq 1$ and $w \epsilon R$ such that

$$\| T(t) \| \leq M e^{\omega t} \qquad \forall\, t \epsilon R^+$$

A dynamical system may also be defined on a finite dimensional state space of a finite order system. In this case, the linear dynamical system is denoted by the semigroup exp(At). Therefore, the generation of finite and infinite dimensional

dynamical systems can be investigated by using the theorems regarding the generation of semigroups and from the equivalency between semigroups and dynamical systems.

Some definitions related to the motion of a dynamical system are provided below to enhance the analysis of the general motion of a system.

Definition 3.8. An orbit (positive orbit) $\gamma^+(v_o)$ in V is the union of all solutions of the dynamical system T(t) for all times $t \geq o$, i.e.,

$$\gamma^+(v) = \underset{t\geq 0}{U} [T(t)v_o]$$

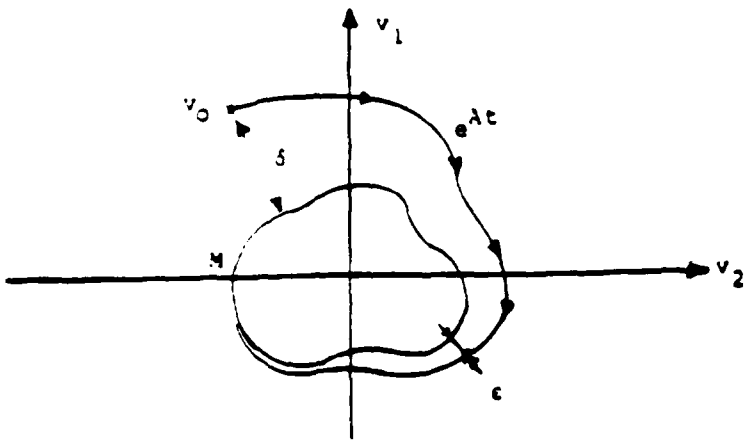
This definition represents a specific trajectory of the system for all times $t \geq o$.

Definition 3.9. If T(t) is a dynamical system on V, then the set $\Pi \subset V$ is called a positive invariant under T(t), where for every $v\epsilon\Pi$ there exists a $T(t)v\epsilon\Pi$ for all $t\epsilon R^+$. The set $\Pi_{(v)}$, which is positive invariant under T(t), is called an invariant set. Clearly, an invariant set Π exists if and only if the positive orbit $\gamma(v)$ is bounded. In that case, $\Pi_{(v)}$ covers $\gamma(v)$; therefore, $\Pi_{(v)}$ can be chosen as any bounded set that covers $\gamma(v)$. In a study of asymptotic behavior of the motion of a dynamical system as $t \rightarrow \infty$ it is often of interest to find the smallest subset of $\Pi(v)$ such that it has the same properties as Π. The task of finding this smallest subset could be achieved by considering a sequence of projections on Π. This leads to a limit set, namely, the smallest set of invariant sets. Therefore, this process would become a basis for the definition of a positive limit set below.

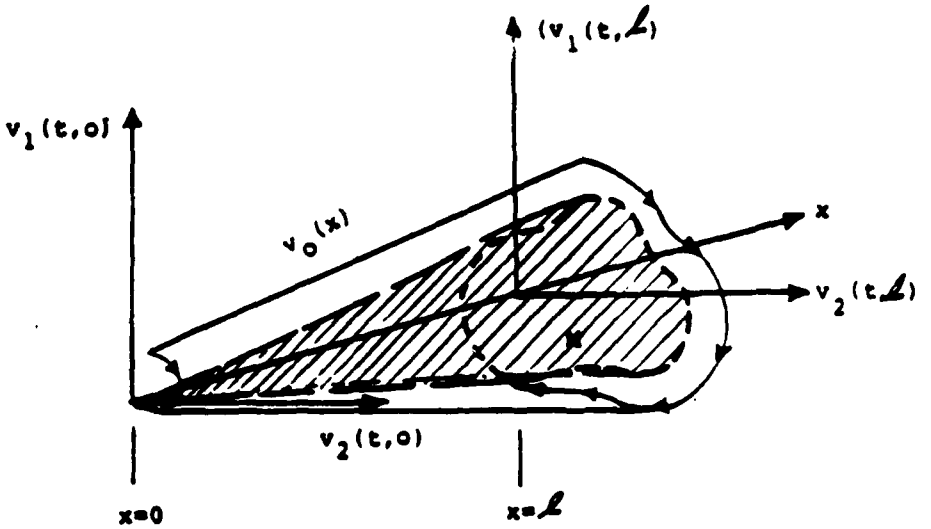
Definition 3.10. If for every $v\epsilon V$ there exists a set $\Omega(v)$ such that as $t_n \rightarrow \infty$, then $T(t_n)v \rightarrow \Omega(v)$, that set is called a positive limit set. Alternatively, it can be shown that

$$\Omega(v) = \bigcap_{r \in R^+} [C\ell \bigcup_{t \geq r} T(t)v] = \bigcap_{r \in R^+} [C\ell\gamma(T(t)v)]$$

where $C\ell$ mean the closure of the set. Fig. 3.1 shows the schematic of invariant sets and limit sets for finite dimensional systems in $V \equiv R^2$ and an infinite dimensional system in $V \equiv ([o,\ell],E^2)$.



a) Invariant set M in $V = R^2$



$d(v_o, M) = \delta$ $\quad$ $d(V, M) = \epsilon$

b) Invariant set M in $V = L^2([o,\ell], E^2)$

Figure 3.2 Invariant sets of finite and infinite dimensional systems.

From the definition of limit set it is clear that $\Omega(v) \subset C\ell\gamma(v)$. As mentioned earlier, if $\gamma(v)$ is bounded, then $\Pi(v)$ exists for finite and infinite dimensional systems. For finite dimensional dynamical systems, the boundedness of $\gamma(v)$ would be sufficient to guarantee the existence of the limit set $\Omega(v)$ as a limit set of the invariant set $\Pi(v)$. However, for infinite dimensional dynamical systems, mere boundedness of $\gamma(v)$ would not be sufficient for the existence of $\Omega(v)$. In this case, the limit set $\Omega(v)$ is nonempty if $\gamma(v)$ is compact (precompact if V is complete). More specifically, this can be stated in the form of the following theorem:

Theorem 3.5. If V is complete and $\gamma(v)$ is precompact, then $\Omega(v)$ is nonempty, compact, connected and invariant, and $T(t)v \to \Omega(v)$ as $t \to \infty$. In addition, if $T(t)v \to \Pi \subset V$ as $t \to \infty$, then $\Omega(v) \subset C\ell\,\Pi$ [26]. □

Although most of the aforementioned materials can immediately be used to cover the invariance principle, this discussion will be continued after the presentation of the Lyapunov stability results. Some of the preliminary concepts and definitions which are instrumental in Lyapunov's direct method are provided in the section below.

Definition 3.11. For a dynamical system $\{T(t)\}_{t\geq 0}$ on a metric space V, the state v_e is called the equilibrium state if $T(t)v_e = v_e$ for all $t \epsilon R^+$. In the framework of limit sets $\Omega(v_e) = \{v_e\}$.

Definition 3.12. Letting $\{T(t)\}_{t\geq 0}$ be a dynamical system on a metric space V, the continuous functional $\mathscr{L}: V \to R$ is called a Lyapunov functional for $\{T(t)\}_{t\geq 0}$ on V if $\dot{\mathscr{L}} \leq 0$ for every $v \epsilon V$. The function $\dot{\mathscr{L}}: V \to \mathscr{R}$ is defined by

$$\dot{\mathscr{L}} \equiv \lim_{t \to o^+} \inf \frac{1}{t} [\mathscr{L}(T(t)v) - \mathscr{L}(v)]$$

From this definition, it is clear that $\dot{\mathscr{L}}$ does not need to be continuous and in this case, is taken as the right derivative of $\mathscr{L}$.

Theorem 3.6. If $\{T(t)\}_{t \geq o}$ is a dynamical system on metric space V and $\mathscr{L}: V \to \mathscr{R}$ is Lyapunov functional for all $T(t)v \epsilon V$ for $t \epsilon [0, \infty)$, then

(i) $\mathscr{L}(T(t)v)$ is non-increasing with

$$\mathscr{L}(T(t)v) = \mathscr{L}(v) + \int_o^t \dot{\mathscr{L}}(T(\tau)v) d\tau \quad \forall t \epsilon [0, \infty) \tag{3-19}$$

(ii) In addition, if $\dot{\mathscr{L}}(v) \leq -\alpha \mathscr{L}(v)$ for some $\alpha > o$, then

$$\mathscr{L}(T(t)v) \leq e^{-\alpha t} \mathscr{L}(v) . \tag{3-20}$$

Proof:

Part (i) - Since $\mathscr{L}(T(t)v)$ is absolutely continuous from the extended fundamental theorem of calculus, then equation (3-19) will be derived with its integration defined in the sense of Lebesgue. Therefore,

$$\mathscr{L}(T(t)v) - \mathscr{L}(v) = \int_o^t \dot{\mathscr{L}}(T(\tau)v) \, d\tau \leq 0$$

$$\mathscr{L}(T(t)v) \leq \mathscr{L}(v) \tag{3-21}$$

Part (ii) -

$$\mathscr{L}(v) - \mathscr{L}(T(t)v) = \int_o^t -\dot{\mathscr{L}}(T(\tau)v) \, d\tau \geq \int_o^t \alpha \, \mathscr{L}(T(\tau)v) \, d\tau$$

From equation (3-21)

$$\mathcal{L}(T(t)v) \leq \mathcal{L}(T(\tau)v) \qquad \forall \tau \epsilon [o,t]$$

$$\mathcal{L}(v) - \mathcal{L}(T(t)v) \geq \alpha t\, \mathcal{L}(T(t)v)$$

$$\mathcal{L}(v) \geq (1 + \alpha t)\, \mathcal{L}(T(t)v)$$

Dividing t by n and applying the same inequality for duration of t/n results in

$$\mathcal{L}(v) \geq (1 + \alpha t/n)\, \mathcal{L}(T(t/n)v) \geq (1 + \alpha t/n)^2\, \mathcal{L}(T(2t/n)v)$$

or

$$\mathcal{L}(v) \geq (1 + \alpha t/n)^n\, \mathcal{L}(T(t)v)\,.$$

As the number of time divisions approaches infinity, then the

$$\lim_{n \to \infty} (1 + \alpha t/n)^n = e^{\alpha t}$$

Hence

$$\mathcal{L}(T(t)v) \leq e^{-\alpha t}\, \mathcal{L}(v)\,. \qquad \square$$

If the Lyapunov functional $\mathcal{L}$ is defined on some subset $G \subset V$ instead of V, the subset G must contain the positive orbit of the system $\gamma(v)$; namely, G must be an invariant set.

Lyapunov's Direct Method

The importance of the Lyapunov stability method when using Lyapunov functionals is to make statements about the stability of a dynamical system. In the dynamical system $\{T(t)\}_{t \geq o}$, one must know the characteristics of the mapping T(t)v or the trajectory of the system. The Lyapunov approach is based on making statements about the system characteristic by showing non-increasing property of a Lyapunov functional $\mathcal{L}(T(t)v)$. This process does not require explicit apriori knowledge of the T(t)v. In this section, preliminary concepts and related definitions and theorems of Lyapunov stability are given.

Definition 3.13. For a metric space V with the metric $d(v_1, v_2)$, where $v_1, v_2 \epsilon V$, the

distance of a point v_1 from a set $K \subset V$ is defined as:

$$d(v_1,K) \equiv \inf_{v \epsilon K} d(v_1,v) \qquad \forall v_1 \epsilon V \text{ and } v \epsilon K .$$

If one knows the "distance" between a point and a set in a metric space, then the stability in the sense of Lyapunov can be determined.

Definition 3.14: (Stability of a motion). A particular motion of a dynamical system $\{T(t)\}_{t \geq o}$ is stable if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $d(v_1,v_2) < \delta$ implies $d(T(t)v_1, T(t)v_2) < \epsilon$ for every $v_1, v_2 \epsilon V$ and $t \epsilon R^+$. The motion $T(\cdot)v_1 : R^+ \rightarrow V$ is called asymptotically stable if it is stable and there exists $\delta > 0$, such that $d(v_1,v_2) < \delta$ implies that $d(T(t)v_1, T(t)v_2) \rightarrow 0$ as $t \rightarrow \infty$ for every $v_1, v_2 \epsilon V$. The motion $T(\cdot)v_1 : R^+ \rightarrow V$ is called exponentially stable if it is stable and there exists $\delta > 0$, $\alpha(\delta) > 0$, and $M(\delta) < \infty$, such that $d(v_1,v_2) < \delta$ implies $d(T(t)v_1, T(t)v_2) < Me^{-\alpha t} d(v_1,v_2)$ for every $v_1, v_2 \epsilon V$ and $t \epsilon R^+$.

Definition 3.15: (Stability of a set). A set G in metric space V of a dynamical system $\{T(t)\}_{t \geq o}$ is stable if for every $\epsilon > o$ there exists $\delta(\epsilon) > 0$ such that $d(v,G) < \delta$ implies $d(T(t)v,G) < \epsilon$ for all $t \epsilon R^+$ and $v \epsilon V$. The set G is called asymptotically stable if it is stable and if there exists $\delta > o$ such that $d(v,G) < \delta$ implies that $d(T(t)v,G) \rightarrow 0$ as $t \rightarrow \infty$ for all $v \epsilon V$. The set G is exponentially stable if it is stable and if there exists $\delta > 0$ such that $d(T(t)v,G) \leq Me^{-\alpha t} d(v,G)$ for some $\alpha(\delta) > 0$ and $M(\delta) < \infty$ and for all $v \epsilon V$.

Clearly, if the equilibrium state Ve exists for a dynamical system, then the stability of equilibrium can be regarded as the stability of the set $\{(T(t)v_e)\} = \{v_e\}$. The following theorem gives the stability conditions for equilibrium based on Lyapunov functional approach:

Theorem 3.7. The equilibrium state of a dynamical system $\{T(t)\}_{t\geq o}$ is stable on the set of all states v with $d(v,v_e) < r$ and $r > 0$, if there exists a Lyapunov functional $\mathscr{L}$ on the set G

$$G = \{V : d(v,v_e) < r\},$$

such that

$$\mathscr{L}(v) \geq \mathscr{L}(v_e) + f(d(v,v_e))$$

The functional $f(\cdot)$ is a monotone function, $f(\cdot) : [0,r) \longrightarrow R^+$ with $f(0) = 0$.

In addition if $\dot{\mathscr{L}}(v) \leq -g[d(v,v_e)]$ for all $v\epsilon G$ and some monotone function $g(\cdot) : [0,r) \longrightarrow R^+$, $g(0) = 0$, then the equilibrium state is asymptotically stable. The proof of this theorem is presented in Appendix A. □

An extension of this theorem for dynamical systems can be given accordingly in the space of perturbed states from the equilibrium, namely, $\hat{v}\epsilon\hat{V}$.

Theorem 3.8. An equilibrium state of a dynamical system is stable with respect to a metric d if there exists a Lyapunov functional $\mathscr{L}$ on the set $G = \{\hat{v}\epsilon\hat{V} \mid d(\hat{v},0) \leq r\}$, where $\hat{v} = (v - v_e)\epsilon\hat{V}$ (space of perturbation) and $\mathscr{L}$ satisfies the following conditions:

(i) $\mathscr{L}$ is continuous and positive definite with respect to $d(\hat{v},0)$ on the $G \subset \hat{V}$.

(ii) $\frac{d\mathscr{L}}{dt}$ is negative semidefinite. Proof is by a restriction of theorem 3.7.

Theorem 3.9. An equilibrium state of a dynamical system is asymptotically stable with respect to a metric d if there exists a Lyapunov functional $\mathscr{L}$ with the conditions given in theorem (3.8), as well as $\mathscr{L}(d(\hat{v},0)) \longrightarrow 0$ as $t \longrightarrow +\infty$. Proof is provided by theorem 3.8 and definition 3.15.

Lyapunov's Method for Stability of Linear Systems

Consider the abstract linear evolution equation of the previous section:

$$\dot{v} = Av \qquad v(0) = v_o \,, \tag{3-4}$$

where the operation A generates a semigroup T(t) on Banach Space V. One can define a stronger criterion for the stability of this system by defining the notion of exponential stability.

Definition 3.16. The system with the above evolution equation (3-4) is exponentially stable if for a given ω there exists

$$\|T(t)\| \leq Me^{-\omega t},\ t \geq 0.$$

Note that exponential stability implies asymptotic stability but the converse is not always true.

Theorem 3.10. Letting operator A in the evolution equation (3-4) be generator of a semigroup T(t), then the null solution of (3-4) is asymptotically stable if there exists a Lyapunov functional L(v) such that $L(v) > 0$ and $\dot{L}(v) \leq -\gamma \|v\|^2$ for $v \epsilon D(A)$.

As addressed in the Lax-Milgram theorem (2.7), if the $\|\cdot\|$ is a Hilbert norm with $\overline{D(A)} = H$, then any Lyapunov functional with bilinear form L: $(H \times H) \rightarrow R$ can be made into an inner product of the form $L : (v \times w) \rightarrow \langle v, Sw \rangle$, where S is symmetric real bounded operator.

Hence, an equivalent form can be defined $\|v\|_2 = \langle v, v \rangle_2 = \langle v, Sv \rangle$ and $\alpha \langle v, v \rangle < \|v\|_2^2 < \beta \langle v, v \rangle$.

By this equivalence relationship, the existence of a Lyapunov functional would result in the existence of an operator S with the aforementioned properties. The

resulting Lyapunov functional, by hypothesis, satisfies

$$L(v) = L(v,v) = \langle v,\ Sv \rangle .$$

$$\dot{L}(v) = \lim_{t \to o} \left\{ L(T(t)v,\ T(t)v) - L(v,v) \right\} \cdot 1/t$$

$$= \lim_{t \to o} \left\{ L((T(t) + I)v,\ (T(t) - I)v \right\} \cdot 1/t = 2L(v,Av) = 2\langle v, SAv \rangle = 2\langle v, Av \rangle_2$$

$$\dot{L}(v) = 2\langle v, Av \rangle_2 \leq -\gamma \| v \|^2 \leq -\frac{\gamma}{\alpha} \| v \|_2^2$$

In addition,

$$\lambda \langle v, v \rangle_2 - \langle Av, v \rangle_2 = \langle (\lambda I - A)v, v \rangle_2 \leq \| (\lambda I - A)v \|_2 \ \| v \|_2$$

or

$$\langle v, v \rangle_2 \left[\lambda - \frac{\langle Av, v \rangle_2}{\langle v, v \rangle_2} \right] \leq \| (\lambda I - A) \|_2 \ \| v \|_2^2$$

where the hypothesis $\dfrac{\langle Av, v \rangle_2}{\langle v, v \rangle_2} \leq -\dfrac{\gamma}{2\alpha}$.

$$\| \lambda I - A \|_2^{-1} \leq \left[\lambda - \frac{\langle Av, v \rangle_2}{\langle v, v \rangle_2} \right]^{-1} \leq \left[\lambda + \frac{\gamma}{2\alpha} \right]^{-1}$$

$$\| (\lambda I - A)^{-1} \|_2 \leq \frac{1}{\lambda + \dfrac{\gamma}{2\alpha}}$$

Using the Hille-Yoshida theorem, this relation illustrates that operator A is the generator of a semigroup T(t) such that

$$\| T(t) \| \leq M e^{-\frac{\gamma}{2\alpha} t}$$

which is a condition for exponential stability.

The Invariance Principle and Asymptotic Behavior

The Invariance Principle provides information about the asymptotic behavior of

motions as $t \longrightarrow \infty$, whether or not an equilibrium exists.

As shown in theorem (3-5), if the positive orbit $\gamma(v)$ is precompact in a complete state space V, there exists a limit set $\Omega(v)$ such that the motion of the dynamical system at infinity will approach it. The method used to locate this limit set in the absence of explicit knowledge about the motion of the dynamical system $\{T(t)\}_{t\geq 0}$ is provided by LaSalle's Invariance Principle [21].

Theorem 3.11: (Invariance Principle). If $\{T(t)\}_{t\geq 0}$ is a dynamical system on a metric space V and there exists a continuous Lyapunov functional $\mathscr{L} : V \longrightarrow \mathscr{R}$ on a set $G \subset V$ such that $\mathscr{L} \leq -W(v)$ for all $v\epsilon G$, where $W : \overline{G} \longrightarrow R^+$ is a positive lower semi-continuous and $\gamma(v) \subset G$, then $\Omega(v) \subset M^+$. The set M^+ is the largest invariant subset of the set M_1,

$$M_1 \equiv \left\{ v\epsilon\overline{G} \mid \dot{\mathscr{L}}(v) = 0 \right\}.$$

In addition, if V is complete and $\gamma(v)$ is precompact (compact), then $T(t)v \longrightarrow M^+$ as $t \longrightarrow +\infty$. As shown later in this manuscript, this theorem will be used to derive stabilizability conditions. The proof of this theorem is given in Appendix B. □

Stability from Spectral Method

Another alternative theorem for stability analysis of dynamical systems originates from the spectral properties of the evolution operators for these systems. One of the major insufficiencies in the application of the spectral method, as opposed to Lyapunov method, is the restriction of the former to linear dynamical systems. For infinite dimensional systems, spectral analysis gives the necessary conditions, whereas Lyapunov-type analysis provides sufficient conditions for the stability of the system.

For finite dimensional linear dynamical systems, the spectrum of the evolution matrix A consists of a finite number of eigenvalues λ (point spectrum). The system is stable if and only if no eigenvalue of the system has a positive real part, i.e.,

(a) $\quad \text{Sup Re}(\lambda) \leq 0 \qquad \lambda \epsilon \sigma(A)$

The system is asymptotically stable if and only if

(b) $\quad \text{Sup Re}(\lambda) < 0 \qquad \lambda \epsilon \sigma(A)$

and it is exponentially stable if

(c) $\quad \text{Sup Re}(\lambda) < -\delta \qquad$ for some $\delta > 0$ and $\lambda \epsilon \sigma(A)$.

For infinite dimensional systems with the evolution equation (3-4), the spectral analysis would give the following three types of spectrum:

(i) point spectrum $\sigma_P(A)$, which is the set

$$\sigma_P(A) = \left\{ \lambda : (\lambda I - A)^{-1} \text{ does not exist} \right\}$$

(ii) residual spectrum $\sigma_R(A)$, which is the set

$$\sigma_R(A) = \left\{ \lambda : \text{Range } (\lambda I - A) \text{ is not dense in state space V} \right\}$$

(iii) continuous spectrum $\sigma_C(A)$, which is the set

$$\sigma_C(A) = \left\{ \lambda : \text{Range } (\lambda I - A) \text{ is not continuous} \right\} .$$

Requirements (a), (b), (c) provide necessary conditions for the types of stability under consideration. To relate these conditions to sufficiency conditions, one can define:

$$\omega \equiv \text{Sup} \left\{ \text{Re } \lambda : \lambda \epsilon \sigma(A),\ \sigma(A) = \sigma_P(A) \cap \sigma_R(A) \cap \sigma_C(A) \right\}$$

This number is called the lower index of stability. The upper index of stability is defined as [53]

$$\bar{\omega} \equiv \inf \left\{ \omega : \| T(t) \| \leq M e^{-\omega t} \right\}$$

where T(t) is the semigroup generated by the system of equation (3-4). In general, ω_s is less than $\bar{\omega}$. However, the condition where these two indices are equal is called "spectrum-determined growth assumption." Under this assumption, conditions (a), (b), (c) will be necessary and sufficient conditions for their respective forms of stability. The spectrum-determined growth assumption is true when the semigroup T(t) is compact [54]. It is clear that the compactness of the semigroup T(t) is directly equivalent to the compactness of positive orbit of dynamical systems. Therefore, the compactness condition is a common necessity in both spectral and Lyapunov stability analysis. However, spectral analysis is not directly applicable to nonlinear systems unless the system is linearized and the analysis is performed in the local sense.

CHAPTER 4 - APPLICATION OF LYAPUNOV'S STABILITY TO MAGNETO-PLASMA DYNAMIC SYSTEMS

Motivation to Study MPD System

Recent increases in space missions and the construction of the space station has attracted attention to new alternatives to chemical propulsion systems. One such system is categorized and known as the electrical propulsion engine. In general, electric rockets should be able to develop considerably higher specific impulses than chemical or nuclear ones. However, this gain in specific impulse requires massive energy conversion mechanisms. To avoid this, the electrical rockets generally provide lower thrust for navigation in low gravitational fields.

The propellant of an electrical rocket consists of either charged particles, accelerated by electrostatic forces, or an electrical conducting fluid (plasma) accelerated by electromagnetic and/or pressure forces.

In the present work, the modeling and analysis of steady magneto-gas-dynamic flow accelerators, among other categories of electromagnetic accelerators, are considered. The study of MPD stability has been divided into two parts. In the first, addressed in the present chapter, the stability of wave motion due to the combination of an external (applied) magnetic field and induced electro-magnetic field on the plasma, perturbed from rest condition, has been investigated. The second part, addressed in the following chapter, includes a study of the stability of wave motion when the flow is perturbed from a non-zero velocity equilibrium state.

MPD System Model

The plasma dynamic equations for a magneto-plasma dynamic system are listed in Appendix C in their general form. These equations consist of Maxwell's equation, Ohm's law, conservation of electric charge, equation of state and a set of mass, momentum and energy equations [70]. It has been assumed that the plasma is originally at rest with pressure P_o, temperature T_o, and density ρ_o with a uniform external magnetic field H_o present, but no applied electrical field.

If the plasma is perturbed by a small disturbance, then the state of the system is a combination of equilibrium and perturbed states, hence the instantaneous pressure, temperature, density, electric and magnetic fields and current density can be written as

$$P = P_o + P'(x,t)$$

$$T = T_o + T'(x,t)$$

$$\rho = \rho_o + \rho'(x,t)$$

$$\vec{E} = \vec{i}\, E_x(x,t) + \vec{j}\, E_y(x,t) + \vec{k}\, E_z(x,t)$$

$$\vec{H} = \vec{i}\left[H_x + h_x(x,t)\right] + \vec{j}\left[H_y + h_y(x,t)\right] + \vec{k}\, h_z(x,t)$$

$$\vec{J} = \vec{i}\, J_x(x,t) + \vec{j}\, J_y(x,t) + \vec{k}\, J_z(x,t)\,.$$

In the case of a neutral plasma, i.e., $\rho_e \simeq 0$, the number of ions and electrons per volume of plasma are nearly equal. The application of perturbations results in a set of linearized governing dynamic equations of perturbed states [70]. Following the modeling results presented in Appendix C, and considering the fact that $\frac{\partial h_x}{\partial x} = \frac{\partial h_x}{\partial t} = 0$, it is then possible to distinguish between two modes of wave propagation, the transverse mode (z-direction) and the longitudinal mode (x-y direction).

(i) Transverse Mode. The equations governing the transverse mode are:

$$\frac{\partial Z}{\partial t} = AZ, \qquad Z = [h_z, w]^T, \qquad 0 \le x \le \ell\ , \ t \ge 0 \tag{4-1}$$

$$A = \begin{bmatrix} \nu_H \dfrac{\partial^2}{\partial x^2} & H_x \dfrac{\partial}{\partial x} \\ \dfrac{V_x^2}{H_x} \dfrac{\partial}{\partial x} & \nu \dfrac{\partial^2}{\partial x^2} \end{bmatrix} \tag{4-2}$$

where $\nu_H = \dfrac{1}{\sigma \mu_e}$ and $V_x = \sqrt{\dfrac{\mu_e}{\rho_o}} H_x$. The parameter V_x is defined as the x-component of Alfven wave speed [70].

(ii) Longitudinal Mode. The state equations for this mode can be reduced to

$$\frac{\partial Z}{\partial t} = AZ, \qquad Z = \left[h_y, \ v, \ u, \ \rho'', \quad T''\right]^T \ 0 \le x \le \ell, \ge 0 \tag{4-3}$$

$$A = \begin{bmatrix} \nu_H \dfrac{\partial^2}{\partial x^2} & H_x \dfrac{\partial}{\partial x} & -H_y \dfrac{\partial}{\partial x} & 0 & 0 \\ \dfrac{Vx^2}{H_x} \dfrac{\partial}{\partial x} & \nu \dfrac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ -\dfrac{Vy^2}{Hy} \dfrac{\partial}{\partial x} & 0 & \dfrac{4}{3}\nu \dfrac{\partial^2}{\partial x^2} & -RT_0 \dfrac{\partial}{\partial x} & -RT_0 \dfrac{\partial}{\partial x} \\ 0 & 0 & -\dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -\dfrac{R}{C_v} \dfrac{\partial}{\partial x} & 0 & \dfrac{K}{\rho_0 C_v} \dfrac{\partial^2}{\partial x^2} \end{bmatrix} \tag{4-5}$$

where $p'' = \dfrac{\rho'}{\rho_o}$, $T'' = \dfrac{T'}{T_o}$, $V_y = \sqrt{\dfrac{\mu_e}{\rho_o}} H_y$. The parameter V_y is defined as the y-component of the Alfven wave speed.

Lyapunov's Functional and Stability Analysis

In this section, the Lyapunov functional approach is applied to each mode of the plasma dynamics. The stability results are derived and discussed.

(i) For transverse mode, equation (4-1), the boundary conditions are

$$Z(0,t) = 0, \quad Z(\ell,t) = 0, \quad Z(x,0) = Z_o \in L^2([0,\ell],E^2) \tag{4-6}$$

In order to apply the Lyapunov stability theorem to the equation of motion for the transverse mode, one should first investigate whether this system generates a semigroup; namely, whether the wave propagation in the transverse direction represents a dynamical system. One way to study this property is to use the Corollary of the Phillips-Lumer theorem, for which the adjoint operator must be derived. The domain of operator A in (4-2), D(A) was defined as

$$D(A) = \left\{(h_z,w) : (h_z,w) \in H^2 \cap H_o^1 \times H^2 \cap H_o^1\right\}.$$

The variables h_z and w each belong to a set of twice differentiable functions in $L_2(x)$, for which the first derivative in $L_2(x)$ exists on the boundary $\partial\Omega(x=0, x=\ell)$ and $\partial\Omega$ is a compact support of functions (h_z,w). The set of functions in this domain, D(A), are dense in the state space defined by the norms $H_0^0 \times H_0^0$. The assignment of this normed space is related to the choice of proper space for the generation of dynamical system and stability results. Therefore, the inner product defined on $V = H_o^o \times H_o^o$ is $<\cdot,\cdot>_V$

$$<u,Av>_V = \int_o^\ell [u_1,u_2] \begin{bmatrix} \nu_H \partial^2(\cdot) & H_x \partial(\cdot) \\ \frac{V_x^2}{H_x} \partial(\cdot) & \nu \partial^2(\cdot) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} dx \tag{4-7}$$

$$= \int_o^\ell [u_1,u_2] \begin{bmatrix} \nu_H \partial^2 v_1 + H_x \partial v_2 \\ \frac{V_x^2}{H_x} \partial v_1 + \nu \partial^2 v_2 \end{bmatrix} dx$$

$$= \nu_H u_1 \partial v_1 \Big|_o^\ell - \nu_H \partial u_1 v_1 \Big|_o^\ell + \int_o^\ell \nu_H v_1 \partial^2 u_1 dx + H_x u_1 v_2 \Big|_o^\ell - \int_o^\ell H_x v_2 \partial u_1 \, dx$$

$$+ \frac{v_x^2}{H_x} u_2 v_1 \Big|_o^\ell - \int_o^\ell \frac{v_x^2}{H_x} v_1 \partial u_2 dx + \nu u_2 \partial v_2 \Big|_o^\ell - \nu \partial u_2 v_2 \Big|_o^\ell + \int_o^\ell \nu v_2 \partial^2 u_2 dx \tag{4-8}$$

After the substitution of boundary conditions

$$u_1(o) = u_2(o) = v_1(o) = v_2(o) = 0$$

$$u_1(\ell) = u_2(\ell) = v_1(\ell) = v_2(\ell) = 0$$

The inner product becomes:

$$<u,Av>_V = \int_o^\ell [v_1, v_2]^T \begin{bmatrix} \nu_H \partial^2(\cdot) & -\frac{V_x^2}{H_x} \partial(\cdot) \\ -H_x \partial(\cdot) & \nu \partial^2(\cdot) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dx$$

$$= <v, A^* u>_V = <A^* u, v>_V .$$

Hence,

$$A^* = \begin{bmatrix} \nu_H \partial^2(\cdot) & -\frac{v_x^2}{H_x} \partial(\cdot) \\ -H_x \partial(\cdot) & \nu \partial^2(\cdot) \end{bmatrix} \tag{4-9}$$

If A is dissipative, then, Re $<v,Av> \leq 0$. If, instead of $V \equiv H_o^o \times H_o^o$, an equivalent inner product space V' is considered as

$$<.,.>_{V'} \equiv <.,P.>_V$$

where P is positive definite and self-adjoint.

If

$$P = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \quad , \quad \alpha_1, \ \alpha_2 > 0$$

then

$$<\mathcal{L}Z,Z>_{V'} = <Z,PZ>_V \tag{4-10}$$

$$<Z,AZ>_{V'} = <Z,PAZ>_V$$

$$<Z,AZ>_{V'} = \int_0^{\ell} [h_z, w]^T \begin{bmatrix} \alpha_1 \nu_H \partial^2(\cdot) & \alpha_1 H_x \partial(\cdot) \\ \alpha_2 \dfrac{V_x^2}{H_x} \partial(\cdot) & \alpha_2 \nu \partial^2(\cdot) \end{bmatrix} \begin{bmatrix} h_z \\ w \end{bmatrix} dx \tag{4-11}$$

$$= \int_0^{\ell} \left\{ h_z \left[\alpha_z \nu_H \partial^2 h_z + \alpha_1 H_x \partial w \right] + w \left[\alpha_2 \frac{v_x^2}{H_x} \partial h_z + \alpha_2 \nu \partial^2 w \right] \right\} dx$$

$$= \alpha_1 \nu_H h_z \partial h_z \Big|_0^{\ell} - \int_0^{\ell} \alpha_1 \nu_H (\partial h_z)^2 dx + \alpha_2 \nu \omega \partial w \Big|_0^{\ell} - \int_0^{\ell} \alpha_2 \nu (\partial w)^2 dx$$

$$+ \int_0^{\ell} \left[\alpha_1 H_x h_z \partial w + \alpha_2 \frac{V_x^2}{H_x} w \partial h_z \right] dx$$

If α_1 and α_2 are chosen such that $\alpha_1 H_x = \alpha_2 \dfrac{v_x^2}{H_x}$, and $\alpha_2 = 1$, then $\alpha_1 = \dfrac{v_x^2}{H_x^2}$.

Therefore,

$$<Z,AZ>_{V'} = -\int_0^{\ell}\left[\nu_H \frac{V_x^2}{H_x^2}(\partial h_z)^2 + \nu(\partial w)^2\right] dx$$

$$+\int_0^{\ell} \frac{v_x^2}{H_x}\,\partial(h_z w)\,dx$$

where $h_z(0) = w(0) = 0 = w(\ell) = h_z(\ell)$, resulting in

$$<Z,AZ>_{V'} = -\int_0^{\ell}\left[\nu_H\left(\frac{v_x}{H_x}\right)^2(\partial h_z)^2 + \nu(\partial w)^2\right] dx\,. \tag{4-12}$$

It is clear that operator A is dissipative in the equivalent state space V'. Similarly, for the adjoint operator A^* in the space V',

$$<Z,A^*Z>_{V'} = \int_0^{\ell}[h_z, w]^T \begin{bmatrix} \alpha_1\nu_H\partial^2(\cdot) & -\alpha_1\frac{v_x^2}{H_x}\partial(\cdot) \\ -\alpha_2 H_x\partial(\cdot) & \alpha_2\nu\partial^2(\cdot)\end{bmatrix}\begin{bmatrix} h_z \\ w\end{bmatrix} dx \tag{4-13}$$

$$= -\int_0^{\ell}\left[\nu_H\left(\frac{v_x}{H_x}\right)^2(\partial h_z)^2 + \nu(\partial w)^2\right] dx$$

$$-\int_0^{\ell} \frac{v_x^2}{H_x}\,\partial(h_z w)\,dx$$

or

$$<Z,A^*Z>_{V'} = -\int_0^{\ell}\left[\nu_H\left(\frac{v_x^2}{H_x}\right)(\partial h_z)^2 + \nu(\partial w)^2\right] dx \tag{4-14}$$

This indicates that A^* is a dissipative operator in the inner product space V'. However, the dissipativity of A and A^* are found with respect to the norm $H_0^1 \times H_0^1$ instead of $H_0^0 \times H_0^0$. To derive the dissipativity of A and A^* with respect to $H_0^0 \times H_0^0$, the relation between these norms will be sought. Let h_z and w be expanded in terms of their Hilbert

space coordinates. This expansion is possible because D(A), as defined, is compactly embedded in $H_0^0 \times H_0^0$ (Sobolev embedding theorem 2.9).

$$h_z = \sum_{n=1}^{n=\infty} a_n(t) \operatorname{Sin} \frac{n\pi x}{\ell} \tag{4-15}$$

$$w = \sum_{n=1}^{n=\infty} b_n(t) \operatorname{Sin} \frac{n\pi x}{\ell} \tag{4-16}$$

$$\int_0^\ell (\partial h_z)^2 \, dx = \int_0^\ell \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n\, m \frac{\pi^2}{\ell^2} a_n(t)\, a_m(t) \operatorname{Cos}\frac{n\pi x}{\ell} \operatorname{Cos}\frac{m\pi x}{\ell} \, dx$$

$$\sum_{n=1}^{\infty} n^2 \frac{\pi^2}{\ell^2} a_n^2(t) \int_0^\ell \operatorname{Cos}^2 \frac{n\pi x}{\ell} \, dx$$

$$\int_0^\ell \operatorname{Cos}^2 \frac{n\pi x}{\ell} \, dx = \int_0^\ell \operatorname{Sin}^2 \frac{n\pi x}{\ell} \, dx$$

$$\int_0^\ell (\partial h_z)^2 \, dx = \sum_{n=1}^{\infty} n^2 \frac{\pi^2}{\ell^2} a_n^2(t) \int_0^\ell \operatorname{Sin}^2 \frac{n\pi x}{\ell} \, dx$$

$$= \int_0^\ell \sum_{n=1}^{\infty} n^2 \frac{\pi^2}{\ell^2} a_n^2(t) \operatorname{Sin}^2 \left(\frac{n\pi x}{\ell} \right) dx$$

$$\geq \int_0^\ell \sum_{n=1}^{\infty} \frac{\pi^2}{\ell^2} a_n^2(t) \operatorname{Sin}^2 \left(\frac{n\pi x}{\ell} \right) dx$$

$$\geq \int_0^\ell \frac{\pi^2}{\ell^2} \left(\sum_{n=1}^{\infty} a_n(t) \operatorname{Sin} \left(\frac{n\pi x}{\ell} \right) dx \right) \left(\sum_{m=1}^{\infty} a_m(t) \operatorname{Sin} \left(\frac{m\pi x}{\ell} \right) \right) dx$$

$$\int_0^\ell (\partial h_z)^2 \, dx \geq \left(\frac{\pi}{\ell} \right)^2 \int_0^\ell h_z^2 \, dx \tag{4-17}$$

The same would be applied to w. Therefore,

$$\langle Z, AZ\rangle_V = -\int_0^{\ell} \left[\nu_H \left(\frac{v_x}{H_x} \right)^2 (\partial h_z)^2 + \nu (\partial w)^2 \right] dx$$

$$\langle Z, A^*Z\rangle_V = \langle Z, AZ\rangle_V \leq -\frac{\pi^2}{\ell^2} \int_0^{\ell} \left[\nu_H \left(\frac{v_x}{H_x} \right)^2 h_z^2 + \nu w^2 \right] dx \qquad (4\text{-}18)$$

Considering

$$\nu_{min} \equiv \min (\nu_H, \nu) > 0 \qquad (4\text{-}19)$$

then

$$\langle Z, A^*Z\rangle_V = \langle Z, AZ\rangle_V \leq \frac{-\pi^2}{\ell^2} \nu_{min} \langle z, z\rangle_V \qquad (4\text{-}20)$$

where

$$\langle Z, Z\rangle_V = \int_0^{\ell} \left[\left(\frac{v_x}{H_x} \right)^2 h_z^2 + w^2 \right] dx \qquad (4\text{-}21)$$

From the dissipativity of A and A^* in V' and the fact that D(A) is dense in V', Corollary 3.1 leads one to the conclusion that the operator A is an infinitesimal generator of a C_o-semigroup in V'. The normed spaces V and V' are equivalent, therefore, it is clear from Theorem 3.10 that A also generates a c_o-semigroup in V.

Clearly, the Lyapunov functional is the norm in equivalent normed space V'; namely,

$$\mathcal{L} = \langle Z, Z\rangle_V = \int_0^{\ell} \left[\left(\frac{v_x}{H_x} \right)^2 h_z^2 + w^2 \right] dx \qquad (4\text{-}22)$$

Therefore,

$$\dot{\mathcal{L}} = 2\langle \dot{Z}, Z \rangle_V = 2\langle AZ, Z \rangle_V \tag{4-23}$$

Equation (4-12) leads to the conclusion that $\dot{\mathcal{L}} \leq 0$, i.e., the perturbations in transverse mode are stable. However, stronger results are conclusive from inequality (4-20), which indicates the exponential stability of the system with

$$\| T(t) \|_{\mathcal{L}(V,V)} \leq e^{-\omega t} \tag{4-24}$$

where $\omega = \dfrac{-\pi^2}{\ell^2} \nu_{min}$ and ν_{min} is given by (4-19).

(ii) For longitudinal mode of wave propagation, the generic form of the evolution equation (4-3) is considered with $Z = \left[h_y,\ v,\ u,\ \rho'',\ T''\right]^T \in L^2(0,\ell),\ Z(0,t) = Z(\ell,t) = 0$. The evolution operator is given by (4-5). An approach is taken similar to the transverse mode to construct the domain of operator A and normed space V

$$D(A) = \left\{ Z :\ Z_i \in H^2 \cap H_0^1\ ,\ \ i=1,2,3,4,5 \right\} \tag{4-25}$$

$$V \equiv H_0^0 \times H_0^0 \times H_0^0 \times H_0^0 \times H_0^0 \equiv (H_0^0, E^5) \tag{4-26}$$

such that

$$\langle \cdot, \cdot \rangle_V = \langle \cdot, \cdot \rangle_{(H_0^0,\ E^5)}$$

The adjoint operator can be found similar to the case of the transverse mode but with a more laborious calculation.

$$A^* = \begin{bmatrix} \nu_H \frac{\partial^2}{\partial x^2} & -\frac{v_x^2}{H_x}\frac{\partial}{\partial x} & +\frac{v_y^2}{H_y}\frac{\partial}{\partial x} & 0 & 0 \\ -H_x\frac{\partial}{\partial x} & \nu\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ +H_y\frac{\partial}{\partial x} & 0 & \frac{4}{3}\nu\frac{\partial^2}{\partial x^2} & +\frac{\partial}{\partial x} & +\frac{R}{C_v}\frac{\partial}{\partial x} \\ 0 & 0 & +RT_o\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & RT_o\frac{\partial}{\partial x} & 0 & \frac{K}{\rho_o c_v}\frac{\partial^2}{\partial x^2} \end{bmatrix} \tag{4-27}$$

To show the dissipativity of A and A*, the equivalent normed space is defined as:

$$\langle \cdot, \cdot \rangle_{V'} = \langle \cdot, P\cdot \rangle_V$$

where $P = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_5)$ and $\alpha_i > 0$,

$$\alpha_1 \triangleq 1, \quad \alpha_2 = \frac{H_x^2}{V_x^2}, \quad \alpha_3 = \frac{H_y^2}{V_y^2}, \quad \alpha_4 = \frac{H_y^2}{V_y^2} RT, \quad \alpha_5 = \frac{H_y^2}{V_y^2} C_v T . \tag{4-28}$$

Based on these values for α_i's and similar algebraic techniques used in the transverse mode, the following can be derived:

$$\langle Z, AZ \rangle_{V'} = \langle Z, A^*Z \rangle_{V'} = -\left[\alpha_1 \nu_H \| \frac{\partial h_y}{\partial x} \|^2 + \alpha_2 \nu \| \frac{\partial v}{\partial x} \|^2 + \frac{4}{3}\alpha_3 \nu \| \frac{\partial u}{\partial x} \|^2 + \alpha_5 \frac{K}{\rho_o C_v} \| T'' \|^2\right] \tag{4-29}$$

$$\leq -\frac{\pi^2}{\ell^2}\left[\alpha_1 \nu_H \| h_y \|^2 + \alpha_2 \nu \| v \|^2 + \frac{4}{3}\alpha_3 \nu \| u \|^2 + \alpha_5 \frac{K}{\rho_o C_v} \| T'' \|^2\right] \tag{4-30}$$

Therefore, the operator A generates a semigroup and perturbations in longitudinal modes represent a dynamical system on V'. If one uses the Lyapunov stability method with

$$\mathcal{L} = \langle Z, Z \rangle_{V'}$$

then

$$\dot{\mathcal{L}} = 2\langle Z, AZ\rangle_{\mathcal{V}} \leq 0$$

which indicates the stability of system. However, due to the zero coefficient for $\|\rho''\|^2$ in (4-30), the results cannot be extended to exponential stability and the semigroup generated by A in this mode is a contraction semigroup, i.e.,

$$\|T(x)\|_{\mathcal{L}(\mathcal{V},\mathcal{V})} \leq 1$$

Application of Spectral Analysis

By means of the separation of variables, the semi-group property generated by A can be constructed. The solution for $Z = \begin{bmatrix} T_h X_h \\ T_w X_w \end{bmatrix}$ will exist if

$$\det \begin{bmatrix} \frac{\dot{T}_h}{T_h} - \nu_H \frac{X_h''}{X_h} & H_x \frac{X_w'}{X_w} \\ \frac{V_x^2}{H_x} \frac{X_h'}{X_h} & \frac{\dot{T}_w}{T_w} - \nu \frac{X_w''}{X_w} \end{bmatrix} = 0$$

Whenever T and X independent functions of x and t, respectively, the terms $\frac{\dot{T}}{T}$, $\frac{X'}{X}$ and $\frac{X''}{X}$ should be constant. This results in X being represented by a real periodic function. Hence, $\frac{x''}{X} = -\frac{\lambda^2}{\ell^2}$, $\frac{X'}{X} = \frac{i\lambda}{\ell}$ and $\frac{\dot{T}}{T} = S$. The application of boundary conditions yields $\lambda_n = n\pi$ for $n = \pm 1, \pm 2, \ldots\ldots$ Therefore, the characteristic equation for the transverse mode can be reduced to

$$S^2 + S(\nu_H + \nu)\frac{\lambda_n^2}{\ell^2} + V_x^2 \frac{\lambda_n^2}{\ell^2} + \nu\,\nu_H \frac{\lambda_n^4}{\ell^4} = 0\,,$$

with the following roots.

$$S_n\,|_{1,2} = \frac{1}{2}\left[-(\nu_H + \nu)\frac{\lambda_n^2}{\ell^2} \pm \sqrt{(\nu_H - \nu)^2\frac{\lambda_n^4}{\ell^4} - 4V_x^2\frac{\lambda_n^2}{\ell^2}}\right]$$

The above expression for S_n indicates that sup $[\mathrm{Re}\,\sigma(A)] < 0$, which is the necessary condition for the equilibrium solution of system equations to be exponentially stable, i.e., an equivalence to uniform asymptotic stability for the system of equation (4-1).

The application of the spectral analysis to the longitudinal mode results in the following equation:

$$\left\{K\left(\frac{1}{\rho_o} + \frac{4}{3}\frac{\nu}{p_o}S\right)\frac{\lambda_n^4}{\ell^4} + \left[S^2\frac{K}{p_o} + \frac{4}{3}\frac{\nu S^2}{T_o(\gamma-1)} + SC_p\right]\frac{\lambda_n^2}{\ell^2} + \frac{S^3}{T_o(\gamma-1)}\right\}$$

$$\left\{(\nu_H\frac{\lambda_n^2}{\ell^2} + S)(\nu\frac{\lambda_n^2}{\ell^2} + S) + V_x^2\frac{\lambda_n^2}{\ell^2}\right\} + \left\{\frac{\lambda_n^2}{\ell^2}V_y^2(S + \nu\frac{\lambda_n^2}{\ell^2})\right\}\left\{\frac{S^2}{T_o(\gamma-1)} + \frac{S\,K\,\lambda_n^2}{p_o\,\ell^2}\right\} = 0\,.$$

where p_o is the pressure related to ρ_o and T_o, and γ is the specific heat ratio C_p/C_v.

This characteristic equation does not have a closed form solution. Therefore, with the exception of very simplified and special cases in general, the spectrum approach is very complex and requires cumbersome, symbolic manipulations to provide stability results. Furthermore, λ's, in general, can be both positive and negative numbers, resulting in two sets of characteristic equations with positive and negative coefficients, whereas the outlined Lyapunov approach would provide stability analysis for the system without the knowledge of system solutions, eigenvalues or the form of wave numbers.

Wave Speed

From the characteristic equations derived for transverse and longitudinal modes of

motion it is possible to comment on the characteristics of the speed of wave propagation in both directions. In the case of transverse waves, the eigenvalues are given by the following equation.

$$S_n = \frac{1}{2}\left[-(\nu_h + \nu)\frac{\lambda_n^2}{\ell^2} \pm \sqrt{(\nu_H - \nu)^2 \frac{\lambda_n^4}{\ell^4} - 4V_x^2 \frac{\lambda_n^2}{\ell^2}}\right]$$

In general, S_n can be written as a combination of real and imaginary parts.

$$S_n = \mathrm{Re}(S_n) + i\ \mathrm{Im}(S_n)\ .$$

This results in a wave with frequency $\omega_n = \mathrm{Im}(S_n)$, where

$$\omega_n^2 = V_x^2 \frac{\lambda_n^2}{\ell^2} - \frac{(\nu_H - \nu)^2}{4} \frac{\lambda_n^4}{\ell^4}$$

Obviously, in order to have a wave (under damped conditions), ω_n^2 must be positive. If $\omega_n^2 \leq 0$, there is no wave propagation in the transverse mode. Hence, wave speed can be defined as

$$\hat{V}^2 = \frac{\ell^2 \omega_n^2}{\lambda_n^2} = V_x^2 - \frac{(\nu_H - \nu)^2}{4} \frac{\lambda_n^2}{\ell^2}$$

or

$$\frac{\hat{V}^2}{V_x^2} + \frac{(\nu_H - \nu)^2}{\frac{4\ell^2}{\lambda_n^2} V_x^2} = 1$$

which has an elliptical shape, as shown in Figure 4.1. When $\nu_H = \nu$, the maximum value of the transverse wave speed is limited by Alfven speed in the x-direction. Moreover, for the transverse wave to exist, the maximum difference in $(\nu_H - \nu)$ is caused by minimum λ_n (i.e., $\lambda_{n,min} = \pi$). Therefore, if

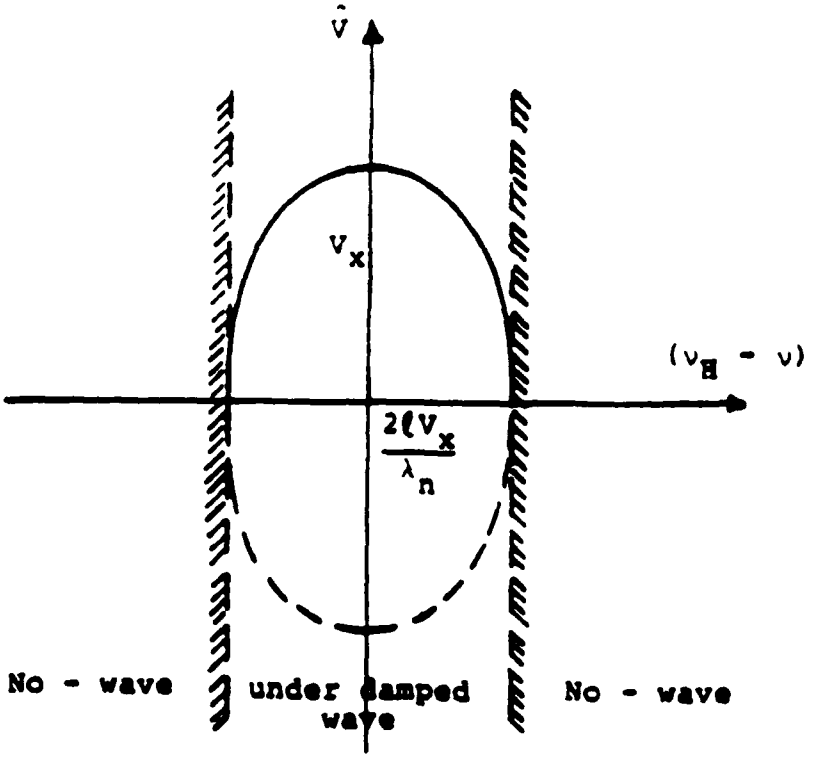


Figure 4.1 Wave speed for transverse mode

$$|\nu_H - \nu| \leq \frac{2\ell V_x}{\pi},$$

the generation of transverse wave is plausible.

Due to the complexity of the characteristic equation, only the special case where $\nu = K = 0$ and $\nu_H \neq 0$ are considered for longitudinal waves. The characteristic equation for this case would be:

$$S^4 + \frac{\lambda^2}{\ell^2} S^2 [\gamma RT_o + (V_x^2 + V_y^2)] + \gamma RT_o V_x^2 + \nu_H \left\{ S^3 \frac{\lambda^2}{\ell^2} + S \frac{\lambda^4}{\ell^4} \gamma RT_o \right\} = 0$$

In the complex $\frac{\ell^2 S^2}{\lambda^2}$ plane for $\nu_H = 0$, the following can be obtained:

$$S^4 + \frac{\lambda^2}{\ell^2} S^2 [\gamma RT_o + V^2] + \gamma RT_o V_x^2 = 0$$

$$\frac{\ell^2 S_{1,2}^2}{\lambda^2} = \frac{1}{2} \left[-(\gamma RT_o + V^2) \pm \sqrt{(\gamma RT_o + V^2)^2 - 4\gamma RT_o V_x^2} \right]$$

where $V^2 = V_x^2 + V_y^2$. For the case of ν_H approaching infinity S_3, the third root would be zero and $\frac{\ell^2 S_4^2}{\lambda^2}$ approaches $-\gamma RT_o$, as shown in Figure 4.2. Hence for $\nu_H \neq 0$, there are two wave speeds, $\hat{V}_{fast}$ and $\hat{V}_{slow}$, where

$$\hat{V}_{fast}^2 < \frac{1}{2} \left[(\gamma RT_o + V^2) + \sqrt{(\gamma RT_o + V^2)^2 - 4\gamma RT_o V_x^2} \right]$$

$$\hat{V}_{slow}^2 < \frac{1}{2} \left[(\gamma RT_o + V^2) - \sqrt{(\gamma RT_o + V^2)^2 - 4\gamma RT_o V_x^2} \right].$$

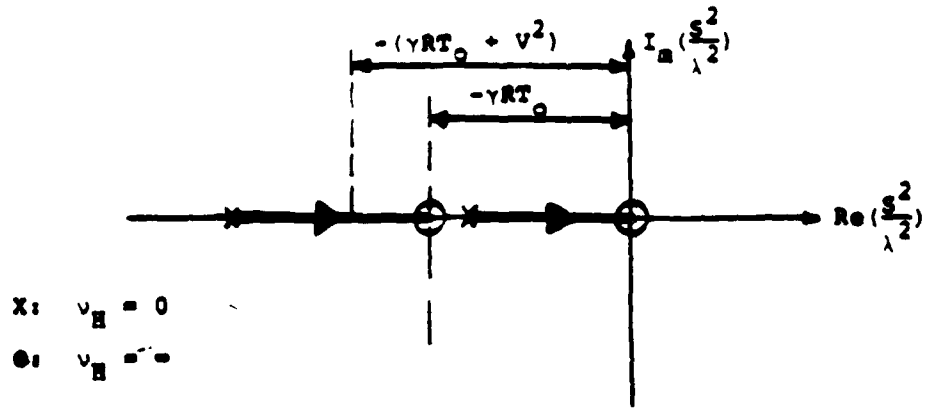


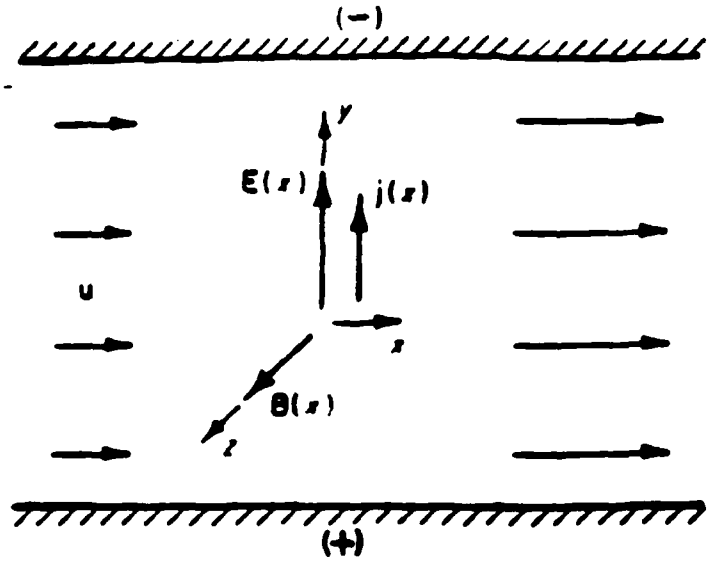
Figure 4.2 Eigenvalues for longitudinal mode

CHAPTER 5 - LYAPUNOV'S STABILITY AND CONTROL OF MPD THRUSTERS

Modeling of MPD Thrusters

In this section the model for MPD thrusters, used as accelerators, is derived. This is a simplified model of the MPD engine and other researchers are studying the derivation of a more complete model for the system. The task of complete MPD modeling is beyond the scope of this research. The simplified model, derived in this section, is used to study the feasibility of the proposed stability and control analysis on the MPD thruster. Figure 5.1 is a schematic of such a system. The flow of ionized gas enters the thruster and is subjected to an electric field E and a magnetic field B, which are perpendicular to each other and to the gas velocity. The electromagnetic acceleration process is an aggregate of effects from compressible gas dynamics, ionized gas physics, electromagnetic field theory and particle electrodynamics. The individual analytic complexity of each of these phenomenon adds to the level of difficulty in an adequate theoretical model for this composite system. Analytical progress normally stems from simplified models which preserve the essential physical aspects of a specific situation.

The description of the motion of plasma in terms of Maxwell-Boltzmann distribution function is too detailed to be useful for many practical problems in the electromagnetic acceleration process. In these cases, the ionized gas medium can be considered as a continuum fluid whose macroscopic physical properties may be described by the conservation laws and Maxwell's equations. These governing equations, for the motion of plasma, will be gas dynamic equations with the interaction terms due to



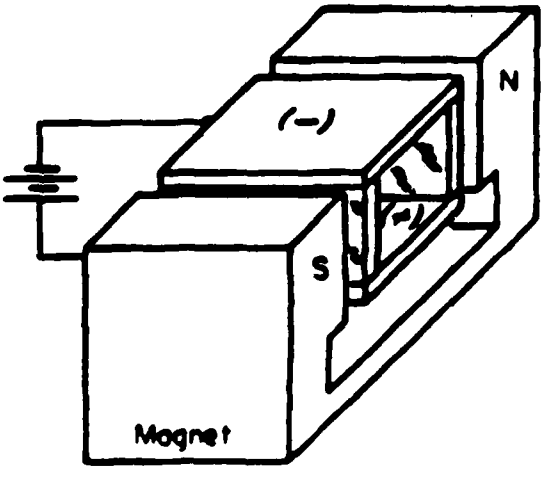


Figure 5.1. Schematic of configuration of flow and fields for one-dimensional electromagnetic steady flow accelerator.

electromagnetic forces. With this approach, a simplified model for the problem can be derived, as depicted in Figure 5.1. As shown, the plasma is flowing through the constant area channel along the x-axis. The channel is formed by two conducting walls connected to the cathode and anode poles, respectively. Between these walls an electric field E(x) is maintained in the y-direction. Normal to this electric field is a magnetic field B(x), applied in the z-direction. The model is assumed to be one dimensional, i.e. only variations in the x-direction are considered. Applying this assumption in the general set of governing equations in Appendix C, one finds

$$\text{Conservation of mass: } \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{5-1}$$

where ρ and u are the density and the velocity of the gas, respectively. The momentum equation results in:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} = -\frac{\partial p}{\partial x} + F_e + F_v + \frac{\partial \tau}{\partial x}$$

where p is the thermodynamic pressure, τ is the shear stress, F_e is the electromagnetic force (Lorentz force) per unit volume, i.e. $F_e = J \times B$, and F_v represents a combination of collisional forces and is assumed negligible compared to the other terms. It is also assumed that the shear stress τ is negligible. Expansion of the momentum equation and substitution from mass conservation equation results in the following:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + JB \ . \tag{5-2}$$

Due to the one-dimensional assumptions, the variations in the y and z directions are neglected, and the gas velocity components in y and z directions are omitted. The energy equation results in the following:

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial(\rho u h)}{\partial x} = JE + \frac{\partial(u\tau)}{\partial x} + \frac{\partial Q}{\partial x}$$

where

e = specific total energy (internal energy + potential energy + kinetic energy)

h = specific total enthalpy ($h = e + P/\rho$)

Q = heat flux due to convection and radiation, assumed negligible.

On the right hand side of the energy equation above the term JE represents the Joule heating due to the application of electric field. This term can be considered as the dominant form of dissipation of energy. The plasma can be assumed as a perfect gas, with the state equation

$$P = \rho RT,\ R = R_A/m$$

where temperature is denoted by T, and R_A and m are the gas constant of plasma and molecular weight of the plasma, respectively. As a result, the specific energy, e, and specific enthalpy, h, can be represented as

$$e = c_v T + u^2/2$$

$$h = c_p T + u^2/2$$

assuming the potential energy terms are negligible; c_p and c_v are the specific heat coefficients

$$c_v = R/(\gamma - 1)$$

$$c_p = \gamma R/(\gamma - 1)$$

and γ is the specific heat ratio of the plasma.

Hence, the energy equation can be simplified as follows:

$$\frac{\partial}{\partial t}\left(\rho c_v T + \rho\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(\rho u c_p T + \rho\frac{u^3}{2}\right) = JE \ . \tag{5-3}$$

From a practical point of view, it is reasonable to assume that the displaced current, $\frac{\partial \epsilon E}{\partial t}$, and excess electric charge, ρ_e, are negligible terms and that the energy in the electric field is much smaller than that of the magnetic field. This results in the following simplified form of Maxwell's equations in Appendix C:

$$\frac{1}{\mu_e} \cdot \nabla \times B = J$$

where $B = \mu_e H$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{5-4}$$

$$\nabla \cdot B = 0$$

$$\nabla \cdot E = 0$$

If one specifies that the magnetic field generated by the current flowing in the gas is negligible with respect to the applied field B(x), then the electromagnetic field equations (5-4) can be decoupled from the dynamics of the plasma [71].

By Ohm's law, one can relate current density J with the applied fields as:

$$J = \sigma(E - uB), \qquad \sigma = \sigma(\rho, T)$$

where σ is a transport coefficient and is called the electrical conductivity.

After substitution for $\frac{\partial \rho}{\partial t}$, $\frac{\partial u}{\partial t}$ in the energy equation (5-3), one can arrive at the following set of equations (5-5) to (5-7) as the set of dynamical governing equations for

the MPD thruster:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{5-5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{R}{\rho}\frac{\partial(\rho T)}{\partial x} + \frac{JB}{\rho} \tag{5-6}$$

$$\frac{\partial T}{\partial t} + \frac{RT}{c_v}\frac{\partial u}{\partial x} + u\frac{\partial T}{\partial x} + \frac{J(E - uB)}{\rho c_v} \tag{5-7}$$

where J is given by Ohm's law.

Control of MPD Thruster at Equilibrium State

In the set of non-steady equations derived in the previous section, i.e. equation (5-5) to (5-7), the parameters B and E can be regarded as input functions to the system. One of the crucial questions about the behavior of this system is how to characterize the relationship between system response and those inputs. The general response characterization of these systems, which change with both time t and spatial coordinate x, is a complex problem. However, one can break the problem into steps by trying to make assessments about the equilibrium states of the system of the partial differential equations. Hence, one can pose the question of what choice of inputs would lead the system to a set of equilibrium states, and under what conditions such controls would be inplausible. In order to answer these questions about the equilibrium states of the system, one must derive the equilibrium set of equations from the original partial differential equations. If the states of the system are represented by a vector valued function v, as following:

$$v = \begin{bmatrix} \rho(t,x) \\ u(t,x) \\ T(t,x) \end{bmatrix}$$

then the equilibrium vector would be denoted by v_e where

$$v_e = \begin{bmatrix} \rho_e(x) \\ u_e(x) \\ T_e(x) \end{bmatrix}$$

Similarly, the inputs (controls) can be defined by a vector valued function U as:

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} JB \\ JE \end{bmatrix}$$

At equilibrium, system states reach their steady state values and their time derivatives would be zero, i.e. $\frac{\partial v_e}{\partial t} = 0$. Equations (5-5) to (5-7) can then be simplified to the following equations:

$$\frac{d}{dx}(\rho_e u_e) = 0 \quad \text{or} \quad \frac{d\rho_e}{dx} = -\frac{\rho_e}{u_e} d\frac{u_e}{dx} \tag{5-8}$$

$$u_e \frac{du_e}{dx} + \frac{R}{\rho_e} \frac{d(\rho_e T_e)}{dx} = \frac{U_1}{\rho_e} \tag{5-9}$$

$$\frac{RT_e}{c_v} \frac{du_e}{dx} + u_e \frac{dT_e}{dx} = \frac{U_2 - u_e U_1}{\rho_e c_v} \tag{5-10}$$

Substitution of equation (5-8) into (5-9) results in the following equations:

$$\left(u_e - \frac{RT_e}{u_e}\right) \frac{du_e}{dx} + R \frac{dT_e}{dx} = \frac{U_1}{\rho_e}$$

$$\left(\frac{RT_e}{c_v}\right) \frac{du_e}{dx} + u_e \frac{dT_e}{dx} = \frac{U_2 - u_e U_1}{\rho_e c_v}$$

$$\frac{du_e}{dx} = \frac{\gamma u_e U_1 - (\gamma-1)U_2}{\rho_e(U_e^2 - \gamma RT_e)} \tag{5-11}$$

$$\frac{dT_e}{dx} = \frac{-u_e^3 U_1 + (u_e^2 - RT_e)U_2}{\rho_e u_e c_v(u_e^2 - \gamma RT_e)} \tag{5-12}$$

If the speed of sound a_e is defined by $a_e^2 = \gamma RT_e$, then (5-11) and (5-12) can be rewritten in vector differential equation form such that

$$\frac{dv_e}{dx} = \frac{1}{u_e^2 - a_e^2} \begin{bmatrix} \frac{\gamma u_e}{\rho_e} & -\frac{\gamma-1}{\rho_e} \\ -\frac{u_e^2}{\rho_e c_v} & \frac{u_e^2 - RT_e}{\rho_e u_e c_v} \end{bmatrix} U \tag{5-13}$$

$$U = \rho_e^2 c_v \begin{bmatrix} \frac{u_e^2 - RT_e}{\rho_e u_e c_v} & \frac{\gamma-1}{\rho_e} \\ \frac{u_e^2}{\rho_e c_v} & \frac{\gamma u_e}{\rho_e} \end{bmatrix} \frac{dv_e}{dx} \tag{5-14}$$

Therefore, for any equilibrium state it is possible to arrive at a control vector in the local sense with respect to the coordinate x. However, in transition from subsonic flow to supersonic flow, where $u_e = a_e$, i.e. unity Mach number, the control inputs U_1 and U_2 become dependent on each other. In this case, one should choose U_1 and U_2 according to a certain relationship such that the transition from subsonic to supersonic velocities and vise versa would be plausible, namely,

$$U_2 = \frac{\gamma u_e U_1}{(\gamma-1)}, \text{ at } u_e = \gamma RT_e$$

or

$$\frac{E}{B} = \frac{\gamma u_e}{(\gamma-1)} .$$

This effect is the same as the choking condition in gas dynamics, which is extended to magneto gas dynamics [72].

Based on the definition of Mach number as $M = \frac{u_e}{\sqrt{\gamma R T_e}}$, it can be shown that

$$M^2 = \frac{u_e^2}{\gamma R T_e} \text{ and } \frac{dM^2}{M^2} = \frac{du_e^2}{u_e^2} - \frac{dT_e}{T_e}$$

or

$$\frac{dM^2}{M^2} = 2\frac{du_e}{u_e} - \frac{dT_e}{T_e} \tag{5-15}$$

$$\frac{1}{M^2}\frac{d(M^2)}{dx} = \frac{2}{u_e}\frac{du_e}{dx} - \frac{1}{T_e}\frac{dT_e}{dx} \tag{5-16}$$

By substituting for $\frac{du_e}{dx}$ and $\frac{dT_e}{dx}$ from equations (5-11) and (5-12), one can derive the following results:

$$\frac{(1-M^2)}{M^2}\frac{dM^2}{dx} = \frac{\gamma M^2 + 1}{\rho_e u_e c_p T_e} U_2 - \frac{2\left[\frac{(\gamma-1)}{2}M^2 + 1\right]}{p_e} U_1 \tag{5-17}$$

For subsonic flow $M < 1$, an increase in M is possible provided the following inequality for control inputs is satisfied:

$$\frac{U_2}{U_1} > \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]}\frac{\rho_e u_e c_p T_e}{p_e} = \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]}\frac{\gamma u_e}{(\gamma-1)} \tag{5-18}$$

For supersonic flow $M > 1$, an increase in M along the thruster occurs if

$$\frac{U_2}{U_1} < \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]}\frac{\gamma u_e}{\gamma-1}. \tag{5-19}$$

At sonic condition the ratio of control inputs $\frac{U_2}{U_1}$ should comply with the

aforementioned quantity, $\left(\frac{\gamma u_e}{\gamma-1}\right)$. Clearly, for the decelerating flow (when M decreases along the flow) the direction of inequalities (5-18) and (5-19) would be reversed.

One approach to constructing acceptable control inputs U_1 and U_2, based on the exclusion of singularity at choking, is to consider

$$U_2 = \frac{[(\gamma-1)M^2 + 2]}{[\gamma M^2 + 1]} \frac{\gamma u_e}{(\gamma-1)} U_1 + \frac{U_4}{(1-M^2)} \tag{5-20}$$

where $U_4 > 0$. Substitution of (5-20) into (5-11) and (5-12) will result in the following equations:

$$\frac{du_e}{dx} = \frac{u_e U_1}{p_e(\gamma M^2+1)} + \frac{(\gamma-1)\, U_4}{\gamma p_e} \tag{5-21}$$

$$\frac{dT_e}{dx} = \frac{-2U_1}{\rho_e R(\gamma M^2+1)} - \frac{(M^2-1/\gamma)\, U_4}{\rho_e c_v u_e} \tag{5-22}$$

Since the Joule heating (U_2) is a positive function, for supersonic flow, from condition (5-18) one would have:

$$0 < \frac{U_4}{\left(\frac{\gamma u_e U_1}{\gamma-1}\right)} < \frac{[(\gamma-1)M^2 + 2]}{(M^2-1)(\gamma M^2+1)} \tag{5-23}$$

For subsonic flow, i.e., $M^2 \leq 1$, U_2 in equation (5-20) is positive for any $U_4 > 0$. Also, the current density, J, should be positive in the chosen direction to accelerate the flow; hence, one can find the following conditions from Ohm's law:

$$J = \sigma(E - uB) > 0$$

$$E > uB \tag{5-24}$$

or

$$U_2 > u_e U_1$$

From equation (5-20), U_2 can be substituted into the inequality (5-24). Hence,

$$\text{for } M < 1 \qquad \frac{U_4(1-M^2)}{\left(\frac{\gamma u_e U_1}{\gamma-1}\right)} + \frac{[(\gamma-1)M^2+2]}{(\gamma M^2-1)} - \frac{(\gamma-1)}{\gamma} > 0 \tag{5-25}$$

$$\text{for } M > 1 \qquad \frac{U_4}{\left(\frac{\gamma u_e U_1}{\gamma-1}\right)} < \frac{[(\gamma-1)M^2+2]}{(\gamma M^2+1)(M^2-1)} - \frac{(\gamma-1)}{\gamma(M^2-1)} \tag{5-26}$$

It is clear that in subsonic flow, the inequality (5-25) is always satisfied. Therefore, there is no constraint on U_4 in this regime. However, for supersonic flow, the inequality (5-26) represents a more restrictive constraint on U_4 than the inequality (5-23). Therefore, a proper choice of U_4 can be selected to satisfy the inequality (5-26). The steady state response can be found from equations (5-21) and (5-22) for some arbitrary choice of $U_1 > 0$ and by the selection of U_4 according to the aforementioned process. It would be an interesting proposition to apply the theory of optimal control to find the optimal control inputs U_1 and U_4 among an arbitrary class of functions. These optimal values for U_1 and U_4 can be obtained in terms of the minimization of a cost function. For example, the cost function can be selected from the group of "fuel optimal" problems, i.e.,

$$IP = \int_{x=0}^{x=\ell} \left[U_4^2 + (u_{e\,out} U_1)^2\right] dx$$

keeping in mind that the optimal control problem is involved with control constraints of the form of the inequality (5-26), while U_4, U_1 are positive quantities.

Perturbation of Nonlinear Unsteady Equations

Based on the equilibrium state represented by $v_e(x)$, it would be interesting to evaluate the system behavior in the neighborhood of its equilibrium states. If the perturbation of states with respect to equilibrium is denoted by $\hat{v}$, then

$$v = \hat{v} + v_e = \begin{bmatrix} \hat{\rho}(t,x) \\ \hat{u}(t,x) \\ \hat{T}(t,x) \end{bmatrix} + \begin{bmatrix} \rho_e(x) \\ u_e(x) \\ T_e(x) \end{bmatrix}$$

where

$$\hat{\rho} \ll \rho_e$$

$$\hat{u} \ll u_e$$

$$\hat{T} \ll T_e$$

By substitution of v into the dynamic equations (5-5) to (5-7), one would obtain the following:

$$\frac{\partial \hat{\rho}}{\partial t} + (\hat{\rho} u_e + \hat{u}\rho_e + \rho_e u_e + \hat{\rho}\hat{u}) = 0$$

$$\frac{\partial \hat{u}}{\partial t} + (u_e + \hat{u}) \frac{\partial (u_e + \hat{u})}{\partial x} + \frac{R}{\rho_e + \hat{\rho}} \frac{\partial (\rho_e T_e + \hat{\rho} T_e + \rho_e \hat{T} + \hat{\rho}\,\hat{T})}{\partial x} = \frac{U_1}{\rho_e + \hat{\rho}}$$

$$\frac{\partial \hat{T}}{\partial t} + \frac{R(T_e + \hat{T})}{c_v} \frac{\partial (u_e + \hat{u})}{\partial x} + (u_e + \hat{u}) \frac{\partial (T_e + \hat{T})}{\partial x} = \frac{U_2 - (u_e + \hat{u})U_1}{(\rho_e + \hat{\rho})c_v}$$

In order to simplify the notation, perturbation states are denoted without the "^" sign and should not be mistaken for their state functions. Moreover, the nonlinear terms are assumed negligible in a sufficiently small neighborhood of the equilibrium states. As an example, one observes

$$\hat{\rho}\,\frac{\partial \hat{u}}{\partial x} \ll \rho_e\,\frac{\partial \hat{u}}{\partial x}\,,\ \hat{\rho}u_e\,\frac{\partial \hat{u}}{\partial x} \ll \rho_e u_e\,\frac{\partial \hat{u}}{\partial x}\,.$$

Similar statements would be true about other nonlinear terms. Integration of these results with the steady state equations (5-8) and (5-10) yields

$$\frac{\partial \rho}{\partial t} + u_e\,\frac{\partial \rho}{\partial x} + \rho\left(\frac{du_e}{dx}\right) + \rho_e\,\frac{\partial u}{\partial x} - \left(\frac{\rho_e}{u_e}\,\frac{du_e}{dx}\right)u = 0 \tag{5-27}$$

$$\frac{\partial u}{\partial t} + \left(\frac{RT_e}{\rho_e}\right)\frac{\partial \rho}{\partial x} + \left(\frac{u_e}{\rho_e}\,\frac{du_e}{dx}\right)\rho + \left(\frac{R}{\rho e}\,\frac{dT_e}{dx}\right)\rho + u_e\,\frac{\partial u}{\partial x} + \left(\frac{du_e}{dx}\right)u + R\,\frac{\partial T}{\partial x} - \left(\frac{R}{u_e}\,\frac{du_e}{dx}\right)T = 0 \tag{5-28}$$

$$\frac{\partial T}{\partial t} + \left[(\gamma-1)\,\frac{T_e}{\rho_e}\,\frac{du_e}{dx}\right]\rho + \left(\frac{u_e}{\rho_e}\,\frac{dT_e}{dx}\right)\rho + \left(\frac{U_1}{\rho_e c_v}\right)u + \left(\frac{RT_e}{c_v}\right)\frac{\partial u}{\partial x} + \left(\frac{du_e}{dx}\right)T + u_e\frac{\partial T}{\partial x} = 0 \tag{5-29}$$

This set of perturbed equations are linear and their characteristics depend on both equilibrium states and their derivatives.

Lyapunov's Stability for DPS Applied to MPD Model

In this section, the Lyapunov stability theorem is modified and applied to the system of partial differential equations of (5-27) to (5-29). These equations represent the dynamics of the perturbation state v about an equilibrium state vector v_e. Hence, by Lyapunov method's one can determine whether any deviation from the equilibrium state is stable or grows unboundedly with time. To be consistent with the notation in Chapter 3, one could rearrange equations (5-27) through (5-29) to the canonical form of the evolution equation. Here the notation $(')$ is used to represent $\frac{d(\cdot)}{dx}$:

$$\frac{\partial \mathbf{v}}{\partial t} = -\begin{bmatrix} u_e & \rho_e & 0 \\ \dfrac{RT_e}{\rho_e} & u_e & R \\ 0 & \dfrac{RT_e}{c_v} & u_e \end{bmatrix} \frac{\partial \mathbf{v}}{\partial x} - \begin{bmatrix} u'_e & -\dfrac{\rho_e}{u_e} u'_e & 0 \\ \dfrac{u_e}{\rho_e} u'_e + \dfrac{R}{\rho_e} T'_e & u'_e & -\dfrac{R}{u_e} u'_e \\ (\gamma-1)\dfrac{T_e}{\rho_e} u'_e + \dfrac{u_e}{\rho_e} T'_e & \dfrac{U_1}{\rho_e c_v} + T'_e & u'_e \end{bmatrix} \mathbf{v} \tag{5-30}$$

It is clear that $\mathbf{v} = 0$, i.e., the null solution, represents the condition where there is no variation from the equilibrium state.

The characteristic directions (eigenvalues) of this system of equations are:

$$\lambda_1 = u_e \quad , \quad \lambda_2 = u_e + \sqrt{\gamma RT_e} \quad , \quad \lambda_3 = u_e - \sqrt{\gamma RT_e} \quad .$$

Therefore, (5-30) represents a system of hyperbolic partial differential equations. For a general hyperbolic system of the form

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{A \partial \mathbf{v}}{\partial x} + B\,\mathbf{v} \triangleq L\,\mathbf{v} \tag{5-31}$$

where

$$L\,\mathbf{v} = A\,\frac{\partial \mathbf{v}}{\partial x} + B\,\mathbf{v} \tag{5-32}$$

it is possible to show that the system of equation (5-30) can be made into the form of (5-31) with a symmetric A matrix.

For simplicity, $\frac{\partial(\cdot)}{\partial t}$ and $\frac{\partial(\cdot)}{\partial x}$ are denoted by subscripts $(\cdot)_t$ and $(\cdot)_x$. As shown below, by dividing equation (5-25) by (ρ_e), (5-28) by $(\sqrt{RT_e})$, and (5-29) by $(\sqrt{\gamma-1}\ T_e)$, the following symmetric A matrix can be obtained:

$$0 = \begin{bmatrix} \frac{\rho_t}{\rho_e} \\ \frac{u_t}{\sqrt{RT_e}} \\ \frac{T_t}{T_e\sqrt{\gamma-1}} \end{bmatrix} + \begin{bmatrix} u_e & \sqrt{RT_e} & 0 \\ \sqrt{RT_e} & u_e & \sqrt{(\gamma-1)RT_e} \\ 0 & \sqrt{(\gamma-1)RT_e} & u_e \end{bmatrix} \begin{bmatrix} \frac{\rho_x}{\rho_e} \\ \frac{u_x}{\sqrt{RT_e}} \\ \frac{T_x}{\sqrt{(\gamma-1)\ T_e}} \end{bmatrix} +$$

$$\begin{bmatrix} u'_e & -\sqrt{RT'_e}\ \frac{u'_e}{u_e} & 0 \\ \frac{(u_e u'_e + RT'_e)}{\sqrt{RT_e}} & u'_e & -\sqrt{(\gamma-1)RT_e}\ \frac{u'_e}{u_e} \\ \frac{(\gamma-1)T_e u'_e + u_e T'_e}{\sqrt{\gamma-1}\ T_e} & \frac{\sqrt{R}\left(\frac{U_1}{\rho c_v} + T'_e\right)}{\sqrt{(\gamma-1)T_e}} & u'_e \end{bmatrix} \begin{bmatrix} \rho/\rho_e \\ \frac{u}{\sqrt{RT_e}} \\ \frac{T}{\sqrt{\gamma-1}\ T_e} \end{bmatrix} \tag{5-33}$$

It can be shown that

$$\begin{bmatrix} \frac{\rho_x}{\rho_e} \\ \frac{u_x}{\sqrt{RT_e}} \\ \frac{T_x}{T_e\sqrt{\gamma-1}} \end{bmatrix} = \begin{bmatrix} (\rho/\rho_e)_x & + \frac{\rho\rho'_e}{\rho_e^2} \\ \left(\frac{u}{\sqrt{RT_e}}\right)_x & + \frac{1}{2}\ \frac{u\ T'_e}{T_e^{3/2}\sqrt{R}} \\ \left(\frac{T}{T_e\sqrt{\gamma-1}}\right)_x & + \frac{T}{\sqrt{\gamma-1}}\ \frac{T'_e}{T_e^2} \end{bmatrix} \tag{5-34}$$

The vector function v is then redefined by following new vector

$$v = \begin{bmatrix} \rho/\rho_e \\ \frac{u}{\sqrt{RT_e}} \\ \frac{T}{T_e\sqrt{\gamma-1}} \end{bmatrix} \tag{5-35}$$

Substitution of (5-34) into (5-33) results in

$$v_t + A\ v_x + B\ v = 0 \tag{5-36}$$

where

$$A = \begin{bmatrix} u_e & \sqrt{RT_e} & 0 \\ \sqrt{RT_e} & u_e & \sqrt{(\gamma-1)RT_e} \\ 0 & \sqrt{(\gamma-1)RT_e} & u_e \end{bmatrix} = A^T \tag{5-37}$$

and

$$B = \begin{bmatrix} u'_e + \frac{\rho'_e}{\rho_e} u_e & \sqrt{RT_e}\left(-\frac{u'_e}{u_e} + \frac{1}{2}\frac{T'_e}{T_e}\right) & 0 \\ \frac{(u_e u'_e + RT'_e)}{\sqrt{RT_e}} + \sqrt{RT_e}\frac{\rho'_e}{\rho_e} & u'_e + \frac{1}{2} u_e \frac{T'_e}{T_e} & \sqrt{(\gamma-1)RT_e}\left(\frac{-u'_e}{u_e} + \frac{T'_e}{T_e}\right) \\ \frac{(\gamma-1)T_e u'_e + u_e T'_e}{T_e\sqrt{\gamma-1}} & \sqrt{(\gamma-1)RT_e}\left[\frac{1}{2}\frac{T'_e}{T_e} + \frac{\left(\frac{U_1}{\rho_e c_v} + T'_e\right)}{(\gamma-1)T_e}\right] & u'_e + \frac{u_e T'_e}{T_e} \end{bmatrix}$$

(5-38)

To construct a Lyapunov functional for this system, it is sufficient to select the functional as an equivalent norm of a Hilbert space of states of equations (5-36), i.e. from the discussion in Chapter 3 this functional would be a bilinear form as follows

$$\mathcal{L} = \langle v, v\rangle_2 = \langle v, Sv\rangle = \int_0^{\ell} v^T S(x)\, v\, dx \tag{5-39}$$

where S is a symmetric positive definite and bounded linear operator. In order to satisfy the sufficiency condition of Zubov's theorem for the stability of system of (5-31), one has to show that there exists a S(x) for which $\frac{d\mathcal{L}}{dt} \leq 0$.

Equation (5-39) yields:

$$\frac{d\mathcal{L}}{dt} = \int_0^{\ell} \dot{v}^T S(x)\, v\, dx + \int_0^{\ell} v^T S(x)\, \dot{v}\, dx$$

Conjugate operators are defined based on the bilinear form operation as follows

$$\langle z, Sy \rangle = \langle S^* z, y \rangle$$

where S^* is defined as the conjugate of S. Since S is a real operator, $S^* = S^T$, and

$$\int_0^\ell v^T S(x) \dot{v}\, dx = \langle v, S\dot{v} \rangle = \langle S^T v, \dot{v} \rangle = \int_0^\ell (S^T v)^T \dot{v}\, dx$$

then

$$\frac{d\mathcal{L}}{dt} = \int_0^\ell \dot{v}^T S(x)\, v\, dx + \int_0^\ell (S^T(x)\, v)^T \dot{v}\, dx$$

$$= \langle v, Sv \rangle + \langle S^T v, \dot{v} \rangle$$

If the operator S(x) is chosen to be symmetric $S = S^T$, then

$$\frac{d\mathcal{L}}{dt} = 2 \langle Sv, \dot{v} \rangle = 2 \int_0^\ell v^T S(x) \dot{v}\, dx \tag{5-40}$$

Now, from equations (5-36), one can substitute for $\dot{v}$ (i.e., v_t) into (5-40). This results in

$$\frac{d\mathcal{L}}{dt} = 2 \langle Sv, (-Av_x) \rangle + 2 \langle Sv, (-Bv) \rangle$$

$$\frac{d\mathcal{L}}{dt} = -2 \int_0^\ell v^T S(x) [A\, v_x]\, dx - 2 \int_0^\ell v^T S(x) [B\, v]\, dx$$

where

$$\int_0^\ell v^T S(x) A \frac{\partial v}{\partial x} = \int_0^\ell \frac{\partial}{\partial x} [v^T S(x) A\, v]\, dx - \int_0^\ell v^T \left[\frac{\partial S(x)}{\partial x}\right] A\, v\, dx - \int_0^\ell v^T S(x) \left[\frac{\partial A}{\partial x}\right] v\, dx$$

$$- \int_0^\ell \left[\frac{\partial v^T}{\partial x}\right] S(x)\, A\, v\, dx \tag{5-41}$$

Using the same discussion about conjugate of the operator [SA], the following can be derived:

$$\int_0^{\ell} v^T S(x) A \frac{\partial v}{\partial x} dx = \int_0^{\ell} \left\{ [S(x) A]^T v \right\}^T \frac{\partial v}{\partial x} dx . \tag{5-42}$$

and

$$\int_0^{\ell} \left[\frac{\partial v^T}{\partial x} \right] S(x) A v dx = \int_0^{\ell} \left\{ [S(x) A] v \right\}^T \cdot \frac{\partial v}{\partial x} dx . \tag{5-43}$$

If $[S(x) A] = [S(x) A]^T$, equation (5-42) becomes identical to (5-43) and substitution into (5-41) results in

$$2 \int_0^{\ell} v^T \cdot [S(x) A] \frac{\partial v}{\partial x} dx = \int_0^{\ell} \frac{\partial}{\partial x} [v^T S(x) A v] dx - \int_0^{\ell} v^T \left[\frac{\partial S(x)}{\partial x} \right] A v dx - \int_0^{\ell} v^T S(x) \left[\frac{\partial A}{\partial x} \right] v dx$$

CHAPTER 6 - CONTROLLABILITY AND OBSERVABILITY OF DPS

In the design and analysis of a dynamic system, stability is often the most important objective, in terms of a desired performance. In cases where the system is not stable or the performance is not satisfactory one needs to know whether or not application of an external input, namely, a control action, will provide the desired result. In case of feedback control, implementation of such control action requires observation(s) of the system behavior. In specific terms, one must be able to determine a control action such that the system can be directed toward the desired performance. This property depends on the system characteristics and the way in which it interacts with the control inputs. On the other hand, the acquisition of data may not always lead to the proper anticipation of the system's behavior. The former property, which determines whether a system can be controlled to achieve a desired performance, is called "controllability". The latter characteristic, which determines whether the system behavior can be "estimated" from observation of the system output(s), is called "observability".

In this chapter these properties of finite and infinite dimensional systems and their relations to one another are reviewed. Once these properties of the system are investigated, then one can study whether an unstable system can be stabilized by addition of a proper control action, namely, whether the system is "stabilizable".

Controllability of Dynamical Systems

The abstract evolution system of Chapter 3 with the addition of control action is considered in this chapter. For linear time invariant finite dimensional systems

$$\dot{v} = Av + Bu \tag{6-1}$$

and

$$v(o) = v_o$$

where $v \epsilon R^n$, $u \epsilon R^m$, $A \epsilon \mathcal{L}(R^n, R^n)$, and $B \epsilon \mathcal{L}(R^m, R^n)$. The solution of this equation at time t is

$$v(t) = e^{At} v_o + \int_o^t e^{A(t-s)} Bu(s)\, ds \tag{6-2}$$

The system of equation (6-1) is called controllable if, for each pair of $v(t)$ and $v_o \epsilon R^n$ and finite time t, there exists a bounded control u which directs v_o toward $v(t)$. Without loss of generality v_o can be selected as zero. The set of states $v(t) \epsilon R^n$, which can be controlled from $v_o = 0$ by the control input in a finite time t, form controllability space $\mathscr{C}$.

Theorem 6.1. The controllability space $\mathscr{C}$ is a linear subspace of R^n.

Proof. If v_1 and $v_2 \epsilon \mathscr{C} \subset R^n$, where each state is attained from the application of controls u_1 and u_2 at times t_1 and t_2, respectively, then

$$v_1 = \int_o^{t_1} e^{A(t_1 - s)}\, Bu_1(s)\, ds$$

$$v_2 = \int_o^{t_2} e^{A(t_2 - s)}\, Bu_2(s)\, ds$$

Letting $0 \leq t_1 \leq t_2$, then $u_1(t)$ is nonzero for $0 \leq t \leq t_1$ and zero elsewhere. Consider an arbitrary control action $u(t)$ for $0 \leq t \leq t_2$ such that

$$u(t) = \alpha u_1(t) + \beta u_2(t) \qquad \alpha, \beta \epsilon R < \infty .$$

Hence, the new control u(t) directs the system to a new state $v(t_2) \epsilon \mathscr{C}$

$$v(t_2) = \int_0^{t_2} e^{A(t_2-s)} \left[\alpha u_1(s) + \beta u_2(s)\right] ds$$

$$= \int_0^{t_2} \alpha e^{A(t_2-s)} u,(s)\, ds + \beta v_2(t_2)$$

Letting $s' = -t_2 + t_1 + s$, then

$$\int_0^{t_2} e^{A(t_2-s)} u_1(s)\, ds = \int_{s'=t_1-t_2}^{s'=t_1} e^{A(t_1-s')} u_1(s' + t_2 - t_1)\, ds'$$

where $u_1(s' + t_2 - t_1)$ in the range of $t_1 - t_2 \leq s' \leq t_1$ is the same as $u_1(s')$ in the range of $0 \leq s' \leq t_1$. Therefore,

$$\int_0^{t_2} e^{A(t_2-s)} u_1(s)\, ds = \int_{s'=0}^{s'=t_1} e^{A(t_1-s')} u_1(s')\, ds' = v_1(t_1)$$

or

$$v(t_2) = \alpha v_1(t_1) + \beta v_2(t_2) .$$

This states that $\mathscr{C}$ is a linear closed subset (subspace) of R^n. □

A system with state space $V = R^n$ is controllable when every point in the space V is reachable. This implies that for every state in space V there exists a control action $u \epsilon U = R^m$. In turn, the controllability subspace must be all of V and the dimension of $\mathscr{C}$ must be equal to the dimension of V, i.e., n. This idea can be implemented in the following theorem to derive controllability criteria for finite dimensional state spaces.

Theorem 6.2. The system of equation (6-1) is controllable if and only if $B^* e^{A^*(-s)} v = 0$

implies

$v \equiv 0.$

Proof. Let an input u(t) control the system toward state v_1 at time t_1, where $v_1 \epsilon \mathscr{C}$. If state $v \epsilon V$ is considered such that v is perpendicular to $\mathscr{C}$, then

$$<V, V_1> = <v, \int_0^{t_1} e^{A(t_1-s)} Bu(s)\, ds> = 0$$

$$= \int_0^{t_1} <v,\ e^{A(t_1-s)} Bu(s)>\, ds = 0$$

$$= \int_0^{t_1} <B^*\, e^{A^*(t_1-s)} v,\ u(s)>\, ds = 0$$

Hence, for arbitrary choice of $u \epsilon U$, one should have

$$B^*\, e^{A^*(t_1-s)} v = 0\ , \tag{6-3}$$

so that the value of inner product remains zero. For the sufficiency portion of the proof, equation (6-3) implies $v = 0$. Therefore, the only vector perpendicular to $\mathscr{C}$ is zero, i.e., every vector in V is also contained in $\mathscr{C}$. The proof of necessary portion is obvious, because if $\mathscr{C} = V$, $v = 0$ and $B^*\, e^{A^*(t_1-s)}\, v = 0$. □

The well known controllability test can be derived from this theorem. If equation (6-3) is expanded then

$$B^*\, e^{A^*(t_1-s)}\, v = B^* \left[v + (t_1-s)A^* v + \frac{(t_1-s)^2}{2!} A^{*2} v + \cdots \right] = 0$$

Since (t_1-s) is an arbitrary time duration, then $\left[1, (t_1-s), \frac{(t_1-s)^2}{2!}, \ldots\right]^T$ is linearly independent vector. Therefore, the coefficient vector will be zero.

Or,

$$\begin{bmatrix} B^* \\ B^*A^* \\ B^*A^{*2} \\ \vdots \\ \vdots \end{bmatrix} v = 0 \qquad (6\text{-}4)$$

From the Cayley-Hamilton theorem, orders of A^* from n and higher are linear combinations of orders (n-1) and less. Therefore, the system of equation (6-1) is controllable if and only if equation (6-4) implies $v = 0$. This is equivalent to the controllability test of

$$C^* = \begin{bmatrix} B^* \\ B^*A^* \\ B^*A^{*2} \\ \vdots \\ B^*A^{*n-1} \end{bmatrix} \qquad \text{having rank n.}$$

From the above discussion it is clear that the concept of controllability is basically geometric and it is independent of coordinate system. To show this let $w = Pv$ in equation (6-1), where P is a non-singular projection operator, then

$$P^{-1}\dot{w} = AP^{-1}w + Bu$$
$$\dot{w} = PAP^{-1}w + PBu$$

$$\dot{w} = \hat{A}\,w + \hat{B}u$$

$$\hat{C} = \left[PB,\ PAP^{-1}PB,\ (PAP^{-1})^2PB,...,(PAP^{-1})^{n-1}PB\right]$$

$$\hat{C} = P\left[B,AB,A^2B,...,A^{n-1}B\right] = PC$$

rank $\hat{C}$ = rank PC = min(rank P, rank C) = min(n,K), where rank C = K and $K \leq n$, rank $\hat{C}$ = rank C. □

In the specific case that P is the similarity transformation, $PAP^{-1} = \Lambda$

$$\hat{C} = \left[I,\ \Lambda,\ \Lambda^2,\ \cdots,\ \Lambda^{n-1}\right] PB$$

Due to the diagonality of powers of Λ, matrix $\hat{C}$ has the following form:

$$\hat{C} = \left[\begin{array}{c|c|c|c|c} (PB)_1 & \lambda_1(PB)_1 & \lambda_1^2(PB)_1 & \cdots & \lambda_1^{n-1}(PB)_1 \\ \vdots & \vdots & \vdots & & \vdots \\ (PB)_n & \lambda_n(PB)_n & \lambda_n^2(PB)_n & & \lambda_1^{n-1}(PB)_n \end{array}\right]$$

Clearly, rank of $\hat{C}$ is n if and only if there is not any row of PB which consists of all zero elements

$$(PB)_i \neq [0]^T$$

In infinite dimensional systems the concept of controllability is more complicated than those of finite dimensional systems. This complexity is a natural outcome of differences between finite and infinite dimensional spaces. Consider the abstract system of (6-1)

$$\dot{v} = Av + Bu \tag{6-1}$$

$$v(o) = v_o$$

where $v \epsilon V = L_2([o,\ell],E^n)$ and $u \epsilon U = L_2([o,\ell],E^m)$ are product infinite dimensional Hilbert spaces. Most of the ideas presented here can be extended to Banach spaces. The operator A is assumed to be generator of a semigroup of bounded operators T(t) for $t \geq o$ on the Hilbert space V. The operator $B : U \rightarrow V$ is a bounded linear operator. The solution of this system as given in Chapter 3 is

$$v_1(t_1) = T(t)\, v_o + \int_o^{t_1} T(t-s)\, Bu(s)\, ds \tag{6-4}$$

The integral operation in (6-4) is a transformation C_u of $u \epsilon U$ into $v \epsilon \mathscr{C} \subset Z$. Equation (6-4) can be rewritten as

$$v_1(t_1) - T(t)\, v_o = \int_o^{t_1} T(t-s)\, Bu(s)\, ds = C_u(u) \tag{6-5}$$

The left hand side of (6-5) can be considered as a transformation of v_0 and v_1 into Z. This transformation is then $S{:}X = (V \times V) \rightarrow Z$, i.e., it maps the pair v_o and v(t), and its image is $v = \int_o^{t_1} T(t-s)\, Bu(s)\, ds$. Earlier in the discussion of finite dimensional systems, the case of $v_o = 0$ was considered, or in terms of present notation, the space X was $V \times \{0\}$ and consequently $X = V$. Clearly, there is no loss of generality for the former representation. However, the present notation provides a better insight in distinction between Z and V, even though they might very well be the same space. Based on relationships between abstract spaces U,X,Z, the abstract linear control system can be established. Figure 6-1 shows this abstract control system. Therefore, the abstract concept of controllability could be addressed with the following definition:

Definition 6.1. The abstract linear control system $\{X, U, Z, C_u, S\}$

(a) is exactly controllable if $\mathscr{C} = \text{Range}\,(C_u) \supseteq \text{Range}(S)$

(b) is approximately controllable if $\overline{\text{Range}}(C_u) \supseteq \text{Range}(S)$.

The definition 6.1 (a) means that for every v_o and $v_1(t_1)$ there is a control u such that $C_u(u) = S(v_1, v_0)$ or equation (6-4) holds. This indicates that the control u steers the

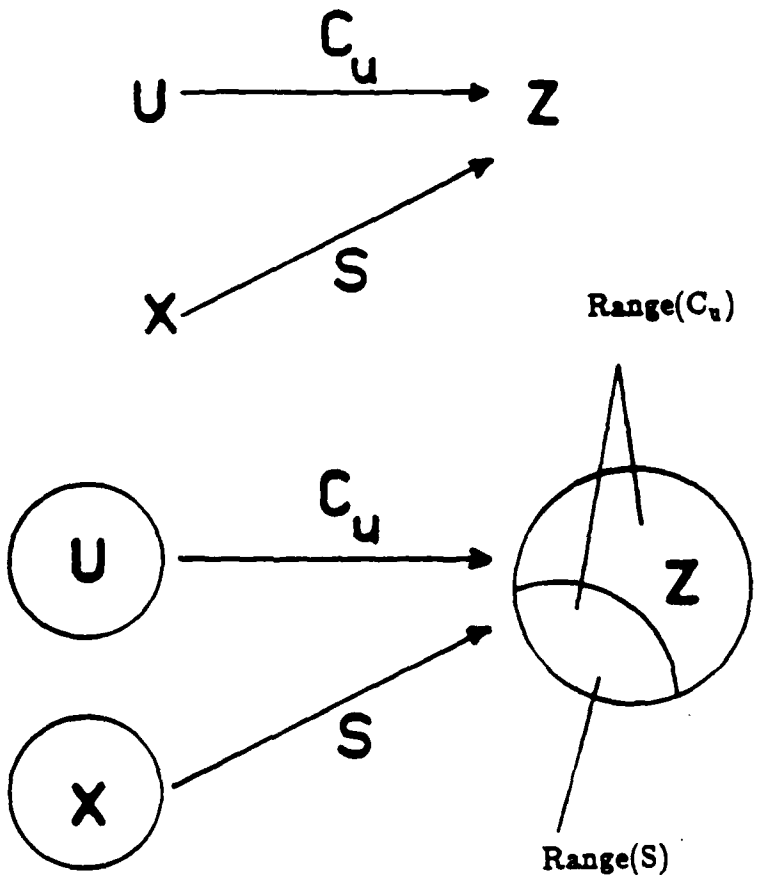


Figure 6.1 Abstract linear control system

system from v_0 to v_1 during $[0,t_1]$. Whenever Range(S) is contained in or is equal to the closure of Range(C_u), as denoted in definition 6.1(b), it is possible that the choice of v_0, v_1 would lead to an image point $S(v_0,v_1)$ in Range(S) which is in $\overline{\text{Range}}(C_u)$ but not in Range(C_u). Therefore, from relationship between Range(C_u) and its closure, for a point in $\overline{\text{Range}}(C_u)$, there is a point in Range(C_u) which is contained in the ball of radius ϵ for every $\epsilon > 0$ such that

$$\| C_u(u) - S(v_0,v_1) \|_Z \leq \epsilon . \tag{6-6}$$

This shows that Range(C_u) is dense with respect to Z. In finite dimensional system $Z = R^n$ both of definitions 6-1 (a) and (b) are equivalent, since every dense subspace of a finite dimensional vector space R^n is R^n. The controllability condition given by theorem 6.2, which indicates synonymous approximate and exact controllability in finite dimensional systems, relates to only one of the controllability concepts of definition 6.1 for infinite dimensional systems. For these systems, if there is an element z in Z such that it is perpendicular to $\mathscr{C} = \text{Range}(C_u)$, it can be shown the z is also perpendicular to $\overline{\text{Range}}(C_u)$, even if Range(C) is not closed. Consider an element of Range(C_u), v_n defined as

$$v_n = \int_0^{t_n} T(t_n - s)\, Bu(s)\, ds$$

$$v_n \in \text{Range}(C_u) \quad \text{and} \quad v_n \notin \overline{\text{Range}}(C_u)$$

For every $\epsilon > 0$ there is a v_n in Range(C_u) such that $\|v_n - v\| < \epsilon$ where v is the limit point of v_n's and $v \notin \text{Range}(C_u)$. Then

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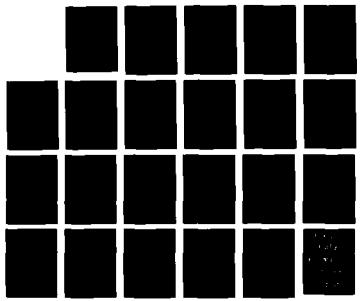

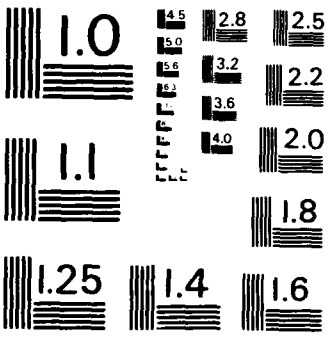
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4.5
5.0
5.6
6.3

$$<z, v_n> = 0$$

$$<z, v_n> - <z, v> + <z, v> = 0$$

$$<z, (v_n - v)> = -<z, v> \tag{6-7}$$

But $v_\epsilon = (v_n - v)$ is an element of Range(C_u). Hence, $<z, v> = -<z, (v_n - v)> = 0$.

In the case that $\epsilon \rightarrow 0$, $v_\epsilon \rightarrow 0$ such that

$$<z, v> = \lim_{\epsilon \rightarrow 0} <z, v_\epsilon> = 0$$

Therefore, z is perpendicular to $\overline{\text{Range}}(C_u)$. Now, as in theorem 6.2,

$$<z, \int_0^t T(t-s)\, Bu(s)\, ds> = 0$$

$$<B^*T^*(t-s)\, z,\ u(s)> = 0$$

where for arbitrary u, $B^*T^*(t-s)\, z = 0$ implies $z = 0$. In turn, this implies that the only element not in the closure of Range(C_u) is $z = 0$, namely $\overline{\text{Range}}(C_u) \supseteq \text{Range}(S)$. Therefore, the geometric concept used in controllability of finite dimensional systems is only good to derive the condition of approximate controllability of infinite dimensional systems.

Observability of Dynamical Systems

For the system of equation(s) 6-1, the question of observability is whether one can reconstruct the behavior of the system from the sensory measurements, or more specifically, whether it is possible to reconstruct the states of the system from the measured set of outputs. To introduce the output set for finite dimensional system of (6-1), let's consider the output set defined by

$$y = Cv \tag{6-8}$$

where $y \epsilon Y = R^p$ and C is a bounded operator from R^n to R^p, i.e., $C \epsilon \mathcal{L}(R^n, R^p)$, where $p \leq n$. The output y from (6-2) and (6-8) is

$$y(t) = Ce^{At}v_o + \int_o^t Ce^{A(t-s)}Bu(s)\, ds \tag{6-9}$$

since y(t) and u(t) are known, then $\bar{y}(t)$ defined by

$$\bar{y}(t) = y(t) - \int_o^t Ce^{A(t-s)}Bu(s)\, ds \tag{6-10}$$

is known and,

$$\bar{y}(t) = Ce^{A(t)}v_o\ . \tag{6-11}$$

If the state v_o of the system can be determined from $\bar{y}(t)$, then the states can be reconstructed from $\bar{y}(t)$, i.e., the system (6-1) is observable. The following definition gives a more specific notion of "observability":

Definition 6.2. The system (6-1) augmented with (6-8) is observable on $[o,t_1]$ if given u and y as an absolutely continuous functions, it is possible to determine v_o uniquely from (6-10). Clearly, for observable system of (6-1) and (6-8) for an arbitrary chosen u, the output y is zero if and only if $v_o = 0$. In other words, the system is observable if,

$$Ce^{At}v_o = 0 \quad \text{for } o \leq t \leq t_1 \quad \text{implies } v_o = 0\ . \tag{6-12}$$

Similar to the treatment of controllability, it is clear that condition (6-12) is the same as

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \qquad (6\text{-}13)$$

From the controllability condition in theorem 6.2 and the observability condition (6-12), a duality relation between controllable and observable systems can be concluded. This means that if a system is controllable, there exists a dual system for it which is observable. If system (6-1) is considered

$$\dot{v} = Av + Bu, \quad v(o) = v_o$$

The dual version of this system is its adjoint. The inner product $<w,v>$ between dual states represents an "energy type" scalar quantity. Therefore, the total energy of the system is $\int_o^{t_1} \frac{d}{dt}<w,v>dt = <w,v>\Big|_o^{t_1}$. From calculus of variations, minimizing the performance index $J = <w,v>\Big|_o^{t_1}$, the following Hamiltonian H would be derived:

$$H = \sum_{i=1}^{n} w_i \dot{v}_i = \sum_{i=1}^{n} w_i (Av + Bu)_i$$

$$\dot{v}_i = \frac{\partial H}{\partial w_i} = (Av + Bu)_i$$

$$\dot{w}_i = \frac{-\partial H}{\partial v_i} = -\sum_{i=1}^{n} (A^* w)_i \Rightarrow \dot{w} = -A^* w$$

From the transversality of the initial and terminal states and since v is not known at terminal time t_1, then $w(t_1)$ must be known, i.e., $w(t_1) = w_1$. The only source of variations in energy function is the control u in original system and the output y in the dual system. Therefore,

$$\frac{d}{dt}\langle w,v\rangle = \langle y,u\rangle$$

$$\langle \dot{w},v\rangle + \langle w,\dot{v}\rangle = \langle \dot{w},v\rangle + \langle A^*w,v\rangle + \langle B^*w,u\rangle = \langle y,u\rangle$$

Hence, $y = B^*w$ and the dual of system (6-1) is

$$\dot{w} = -A^*w\ ,\quad w(t_1) = w_1\ ,\quad w\epsilon R^n$$

$$y = B^*w \qquad\qquad y\epsilon R^m$$

From the duality between the observation and control, the abstract observed system as the dual to the abstract control system of previous section can be constructed. The operators S and C_u are defined as

$$S:\ X = V \times V \rightarrow Z$$

$$C_u:\ U \rightarrow Z$$

where S and C_u were bounded with dense domain. Therefore, the adjoint operator of those can be defined as $S^*:\ Z \rightarrow X$ and $C^*_u:\ D(C^*_u) \subseteq Z \rightarrow Y$, with

$$\langle Sx,z\rangle_Z = \langle x,S^*z\rangle_X \text{ for } x\epsilon X \text{ and } z\epsilon Z \tag{6-15}$$

$$\langle C_u y,z\rangle_Z = \langle y,C^*_u z\rangle_Y \text{ for } y\epsilon D(C_u) \text{ and } z\epsilon D(C^*_u) \tag{6-16}$$

Figure 6.2 shows this abstract linear observed system. This abstract system will be denoted by triple spaces $\{X,Y,Z,C^*_u,S^*\}$.

The observability for abstract linear system is more complex than the specific case of finite dimensional systems. The complexity arises for infinite dimensional systems due to the fact that dense subspaces of infinite dimensional space are not necessarily a whole space. Therefore, two concepts of observability would be considered.

Definition 6.3. The abstract linear observed system $\{X,Y,Z,C^*_u,S^*\}$ is

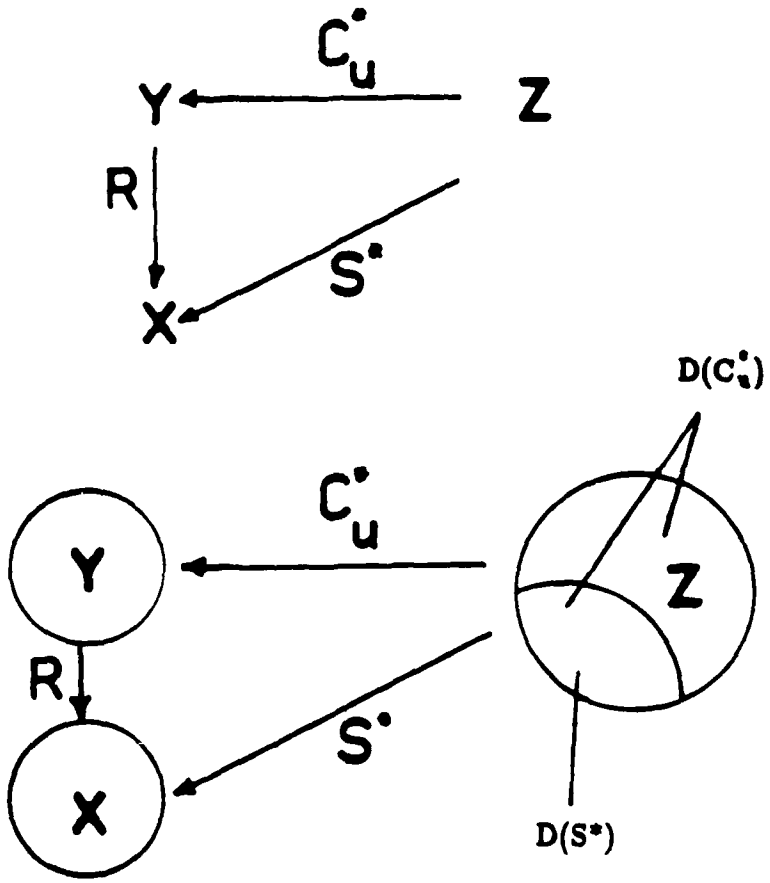


Figure 6.2 Abstract linear observed system

(a) is distinguishable if Ker $C^*_u \subseteq$ Ker S^*,

(b) is observable if, in addition to (a) there is a positive number K such that

$$\|C^*_u z\|_Y \geq K\|S^* z\|_X \qquad \text{for every } z\epsilon D(C^*_u) \subseteq Z \tag{6-17}$$

In definition 6.3(a), Ker C^* is the set of all z's in $D(C^*)$ such that $C^*z = 0$.

The linear observed dual system of an infinite dimensional control system (6-1) is

$$\begin{aligned} \dot{w} &= -A^*w \qquad & w\epsilon W &= L_2([o,\ell],E^n) \\ y &= B^*w \qquad & y\epsilon Y &= L_2([o,\ell],E^m) \\ w(t_1) &= w_1 \end{aligned} \tag{6-18}$$

The solution for y, as of equation (6-9), in the abstract sense becomes

$$y = B^*\, T^*(t_1 - t)w_1 \tag{6-19}$$

where $T^*(-t)$ is the semi-group generated by $-A^*$. The set of all w_1's to be reconstructed, i.e., the space of observed states, is given by X and the space of outputs y is Y and Z is the set of states of the system. It is clear in this case that $X = Z$ and S, S^* are identity on Z. The operator C^*_u from equation (6-19) is found by $C^*_u = B^*T^*(t_1 - t)$.

In definition 6.3(a), the term Ker $S^* \subseteq$ Ker C^*_u means that the set of w_1 for $y = C^*_u w_1 = 0$ is contained in or equal to the set of all states $w = 0$. This merely says that if $y = 0$, then $w_1 = 0$, leading to the uniqueness (distinguishability) of observations of states w. From these relations it is clear that the condition of distinguishability for abstract linear system is that $C^*_u\, w_1 = 0$ implies $w_1 = 0$.

Definition 6.3(b) is much stronger than 6.3(a) since it implies that for a bounded output there exists a bounded "reconstruction" of state, i.e., $w_1 \epsilon Z$, or,

$$\|C^*_u\, w_1\|_Y \geq K\|w_1\|_Z \tag{6-20}$$

This implies that there exists a bounded left inverse R for C^*_u such that

$$R\, C^*_u\, w_1 = w_1 \ .$$

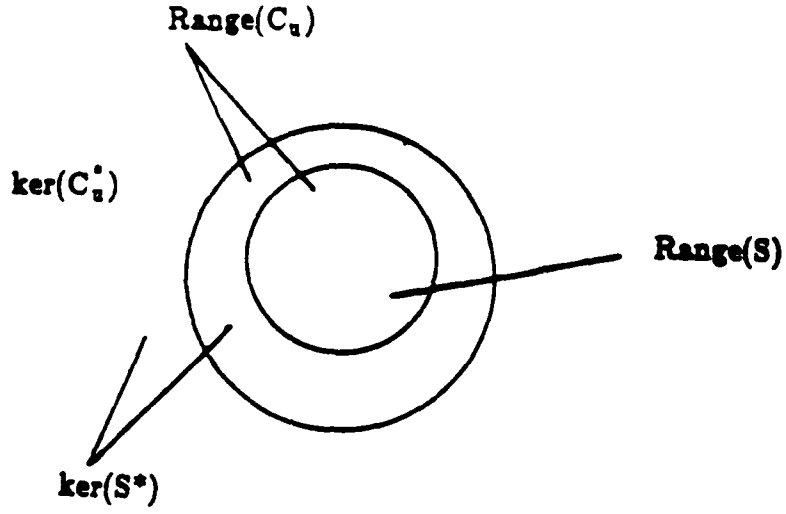
Therefore,

$$\|w_1\|_Z \leq \|R\|_{\mathcal{L}(Y,Y)}\ \|C^*_u\, w_1\|$$

which leads to (6-20) with $K = \frac{1}{\|R\|}$. The operator R is the reconstruction operator, as shown in Figure 6.2. Therefore, the state w_1 can be continuously reconstructed from the observation $y = C^*_u w_1$. Due to this duality relationship between controllable and observable abstract systems, the following theorems can be given:

Theorem 6.3. The abstract linear control system $\{X,U,Z,C_u,S\}$ is approximately controllable if and only if the abstract linear observed system $\{X,Y,Z,C^*_u,S^*\}$ is distinguishable. The proof can be seen from the definitions of approximate controllability and distinguishability. Namely, for every $u \in U$, it was shown that every $z \in$ perpendicular to $C_u u$ must be zero to give $\overline{\text{Range}}(C_u) \supseteq \text{Range}(S)$. Hence $\langle z, C_u u\rangle_Z = 0$ implies $z = 0$, but this inner product is same as $\langle C^*_u z, u\rangle_U = 0$, where for arbitrary $u, C^*_u z = 0$. Therefore $C^*_u z = 0$ implies $z = 0$. Distinguishability of the dual system of $\{X,U,Z,C_u,S\}$ where the observation operator is C^*_u is the same as $C^*_u z = 0$ implying $z = 0$, i.e., the controllability and observability is the same for dual systems. Figure 6.3 shows this concept in a diagram form.

Theorem 6.4. The abstract linear control system $\{X,U,Z,C_u,S\}$ is exactly controllable if and only if the abstract linear observed system $\{X,Y,Z,C^*_u,S^*\}$ is observable. The proof of this theorem is given in [40]. □



$$\text{Range}\,(S) \subseteq \text{Range}\,(C_u) \Leftrightarrow \ker\,(C_u^*) \subseteq \ker\,(S^*)$$

Figure 6.3 Schematic representation of observability

CHAPTER 7 - LYAPUNOV-BASED STABILIZATION OF DPS

One of the interesting topics in control theory is the relationship between controllability and stabilizability of dynamical systems. When some or all of the states of a homogeneous system is not asymptotically stable, it is desirable to control the system trajectory such that it becomes asymptotically stable, i.e., stabilize the system by application of a proper control action. In this chapter stabilizability of DPS is investigated. However, to simplify this analysis, the stabilizability of finite dimensional systems are first reformulated for ease of extension to distributed systems.

Stabilization of Finite Dimensional Systems

Considering the finite dimensional system of Chapter 6 when the system with null control is not asymptotically stable and a control action u will be applied to the system, then the closed loop system equations are:

$$\dot{v} = Av + Bu \qquad v\epsilon R^n,\ u\epsilon R^m \tag{7-1}$$

$$v(0) = v_o$$

where $A\epsilon\mathcal{L}(R^n,R^n)$ and $B = \mathcal{L}(R^m,R^n)$.

Assuming u is composed of the feedbacks of states, i.e., $u = Kv$, then this control system is stabilizable if the closed loop system

$$\dot{v} = (A + BK)v \tag{7-2}$$

$$v(0) = v_o\ ,$$

where $(A + BK)\epsilon\ \mathcal{L}(R^n,R^n)$, generates an asymptotically stable system.

If system (7-1) is controllable, then it can be shown that it is stabilizable. Namely,

controllability is a sufficient condition for stabilizability of the system (7-1). This can be shown by application of controllability properties. For a controllable system, it was shown that for an arbitrary u

$$B^* e^{A^*(t_1-t)} z = 0 \qquad \text{implies } z = 0$$

Moreover, the following inner product in space Z

$$\langle z, \int_0^{t_1} e^{A(t_1-t)} Bu \, dt \rangle_Z \,, \tag{7-3}$$

is equivalent to the inner product in space U

$$\int_0^{t_1} \langle B^* e^{A^*(t_1-t)} z, u \rangle_U \, dt \,. \tag{7-4}$$

The corresponding norm of (7-4) can be written as

$$\int_0^{t_1} \langle B^* e^{A^*(t_1-t)} z, \, B^* e^{A^*(t_1-t)} z \rangle_U \, dt = \int_0^{t_1} \| B^* e^{A^*(t_1-t)} z \|_U^2 \, dt \tag{7-5}$$

For finite dimensional systems, the condition of approximate and exact controllability are the same, i.e.,

$$\| B^* e^{A^*(t_1-t)} z \|_U \geq \delta \, \| z \|_Z \tag{7-6}$$

and from equation (7-5) it can be shown that

$$\int_0^{t_1} \langle e^{A(t_1-t)} B \, B^* \, e^{A^*(-t+t_1)} z, \, z \rangle_U \, dt \geq \beta \| z \|_Z$$

for some $\beta > 0$, or

$$< \left[\int_0^{t_1} (e^{-At} B\, B^*\, e^{-A^*t})\, dt\right] e^{A^*t_1} z,\ e^{A^*t_1} z >_U\, dt \geq \beta \|z\|_Z \tag{7-7}$$

An operator D can be defined as

$$D = \int_0^{t_1} e^{-At}\, B\, B^*\, e^{-A^*t}\, dt\,, \tag{7-8}$$

where D is a self-adjoint operator. When the system (7-1) is controllable, inequality (7-6) and (7-7) are satisfied and, consequently, D has a bounded inverse. From a Lyapunov approach one can show how the operator D could be used to assure asymptotic stability of the closed loop system. In fact, at this point, one can notice the concurrence of three concepts: controllability, stabilizability and Lyapunov stability. To show this, the candidate gain operator for feedback is considered as $K = -B^*\, D^{-1}$. Therefore,

$$u = -B^*\, D^{-1}\, v \tag{7-9}$$

The open loop system of (7-1) generates a solution of the form $v = e^{At}\, v_o$. The closed loop system, based on equation (7-2) and (7-9), becomes

$$\dot{v} = (A - B\, B^*\, D^{-1})v \tag{7-10}$$

$$v(0) = v_o\,.$$

Consider a system with the following evolution equation

$$\dot{y} = (A - B\, B^*\, D^{-1})^* y \tag{7-11}$$

$$y(0) = v_o$$

where y is in the space $V = R^n$, i.e., system (7-10) and (7-11) share a common space. The open loop system (7-1) generates the semigroup T(t) and the closed loop system (7-9) with evolution operator $(A - B\, B^*\, D^{-1})$ generates S(t). Therefore, the system of (7-

11) with evolution operator $(A - B\,B^*\,D^{-1})^*$ is a generator of semigroup $S^*(t)$. Let the Lyapunov function be selected as the following weighted inner product on $V \times V$

$$\mathcal{L}(y) = \langle y, Dy \rangle_V \tag{7-12}$$

$$= \langle y, (\int_0^{t_1} e^{-At}\, B\, B^*\, e^{-A^*t}\, dt)\, y \rangle$$

$$= \int_0^{t_1} \langle B^* e^{-A^*t} y\,,\; B^* e^{-A^*t} y \rangle_U\, dt \geq t_1\, \delta \|y\|_V^2 . \tag{7-13}$$

Since D is self-adjoint, then the time derivative of $\mathcal{L}$ becomes

$$\dot{\mathcal{L}}(y) = 2\langle \dot{y}, Dy \rangle_V$$

$$= 2\langle (A - B\,B^*\,D^{-1})^* y, Dy \rangle_V$$

$$= 2\langle A^* y, Dy \rangle_V - 2\langle D^{-1} B\, B^* y, Dy \rangle_V$$

$$\dot{\mathcal{L}}(y) = 2\langle D\, A^* y, y \rangle_V - 2\|B^* y\|_U^2 \tag{7-14}$$

where

$$D\, A^* y = (\int_0^{t_1} e^{-At}\, BB^*\, e^{-A^*t}\, A^*\, dt) y$$

$$= -[\int_0^{t_1} e^{-At}\, B\, B^* \frac{d}{dt}(e^{-A^*t})\, dt]$$

Operators B and B^* are time invariant, hence,

$$D\, A^* y = -\int_0^{t_1} e^{-At}\, B \frac{d}{dt}(B^*\, e^{-A^*t})\, dt\,,$$

and

$$\langle DA^*y, y\rangle_V = -\int_0^{t_1} \langle \frac{d}{dt}(B^*e^{-A^*t}y),\ B^*e^{-A^*t}y\rangle_U\, dt$$

$$= -\frac{1}{2}\int_0^{t_1} \frac{d}{dt}\ \langle B^*e^{-A^*t}y,\ B^*e^{-A^*t}y\rangle_U\, dt$$

$$= -\frac{1}{2}\ \|B^*e^{-A^*t_1}y\|_U + \frac{1}{2}\ \|B^*y\|_U \qquad (7\text{-}15)$$

Therefore, by substitution of (7-15) into (7-14), it can be seen that

$$\dot{\mathscr{L}}(y) = -\|B^*e^{-A^*t_1}y\|_U - \|B^*y\|_U \leq 0 \qquad (7\text{-}16)$$

From (7-6) it is clear that

$$\|B^*e^{-A^*t_1}y\|_U \geq \delta_1 \|y\|_V \qquad (7\text{-}17)$$

$$\|B^*y\|_N \geq \delta_2 \|y\|_V \qquad (7\text{-}18)$$

Therefore $\dot{\mathscr{L}}(y)$ is not just negative semidefinite, but for finite dimensional systems due to (7-17) and (7-18), it satisfies the more restrictive condition of being negative definite.

$$\dot{\mathscr{L}}(y) \leq -(\delta_1 + \delta_2)\ \|y\|_V \qquad (7\text{-}19)$$

From this Lyapunov functional and the results of the Lyapunov's direct method applied to linear systems, as shown in Chapter 3, system (7-11) can be found exponentially stable with the

$$\|e^{(A^*-D^{-1}BB^*)t}\| \leq e^{-\alpha t} \quad \text{for some } \alpha > 0 \qquad (7\text{-}20)$$

Therefore system (7-10) which is the closed loop system with the control law of equation (7-9) is exponentially stable, i.e.,

$$\|e^{(A-BB^*D^{-1})t}\| = \|e^{(A^*-D^{-1}BB^*)t}\|$$

or from (7-20)

$$\|e^{(A-BB^*D^{-1})t}\| \leq e^{-\alpha t} \qquad \text{for some } \alpha > 0 . \tag{7-21}$$

Therefore, controllability of a finite dimensional system leads to stabilizability of that system. However, this relationship does not exist in the case of infinite dimensional systems, as will be discussed later.

When the condition for existence of feedback gain operator K is given, it is often required that the system characteristics behave in a desired manner, namely, the eigenvalues of the closed loop system being specified apriori. The theorem below proves the existence of an operator K such that the closed loop system has this placement property.

Theorem 7.1. Letting the system (7-1) be controllable and operator B=b be an n×1 matrix (single input system), then the closed loop system

$$\dot{v} = (A+bK)v \tag{7-22}$$

$$v = v_o$$

has the eigenvalue placement property. This property can be stated as: given any n-tuple of eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$, there is one and only one row vector $K = [K_1, ..., K_n]$ such that the matrix A+bK has the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

Clearly, this theorem can be extended to the case of a general n×m B-matrix. The proof of theorem 7.1 is given in [73].

In general, stabilizability of system (7-1) does not guarantee controllability of this system. The counter example would be a homogeneous stable system, which is clearly uncontrollable. However, the pole placement property, i.e., being able to select the eigenvalues of the closed loop matrix A+BK, implies controllability.

The pole placement property of controllable systems can be considered as an algebraic property of matrix operators. From this property it was shown that for a set of selected eigenvalues of system (7-2) there exists at least a matrix K, provided the system is controllable. This means that the characteristic equation

$$\det [\lambda I - (A+BK)] = 0 , \tag{7-23}$$

for the given λ_i's, $i = 1,2,...,n$, would result in a set of n characteristic equations which lead to determination of K, provided system (7-2) is controllable. Therefore, the controllability of the pair (A,B) is the necessary and sufficient condition for matrix (A+BK) to have eigenvalues as specified and this property is independent of (A+BK) being an evolution matrix. In the literature, the pole placement of general matrix (A+BK) is not usually looked upon in the framework of controllability of the pair (A,B). Despite the fact that placement property of eigenvalues of matrix (A+BK) and evolution system (7-2) is the same, here this distinction is made because of the usage of this property applied to the matrix (A+BK) in derivation of stabilization of a class of DPS. These views of stabilization in finite dimensional systems set a basis for their comparison with infinite dimensional systems.

Stabilization of Infinite Dimensional Systems

If the abstract evolution equation of an infinite dimensional system given in the form of equation (3-4) with evolution operator A generates a semigroup T(t) which is not asymptotically stable, then a control u will be applied to the system and the nonhomogeneous system can be represented by

$$\dot{v} = Av + Bu \tag{7-24}$$

where

$$v \epsilon V = L_2([0,\ell],E^n)$$

$$u \epsilon U = L_2([0,\ell],E^m)$$

$$v(0) = v_o$$

The state v and control u are vector functions of spatial domain $x \epsilon [0,\ell]$. Hence, they belong to n and m-tuple function spaces V and U, respectively.

The control system is then stabilizable if there exists an operator $K : V \longrightarrow U$ such that the control $u = Kv$ generates the closed loop system

$$\dot{v} = (A + BK)v \tag{7-25}$$

$$v(0) = v_o$$

$$v \epsilon V$$

and the closed loop evolution operator $(A + BK)$ is the generator of an asymptotically stable semigroup S(t).

Unlike the finite dimensional systems, approximate controllability of system (7-24) does not guarantee stabilizability. However, if the system is exactly controllable as discussed in Chapter 6, then

$$\|C^*_u \, z\|_U \geq \delta \|z\|_Z \, . \tag{7-26}$$

For the linear system (7-24), this condition can be represented as

$$\|B^* \, T^*(t_1 - t)z\|_U \geq \delta \, \|z\|_Z \, , \tag{7-27}$$

for every $t_1 \epsilon R^+$ and $0 \leq t \leq t_1$. From this condition the norm

$$\int_0^{t_1} \|B^*T^*(t_1-t)\, z\|_U^2 \, dt \tag{7-28}$$

will satisfy the following

$$\int_0^{t_1} \|B^*T^*(t_1-t)\, z\|_U^2 \, dt \geq \beta \, \|z\|_Z^2 \tag{7-29}$$

for some $\beta > 0$.

The following can be written from the properties of semigroups T and T*.

$$T(t_1-t) = T(t_1)\, T(-t) = T(-t)T(t_1)$$

$$T^*(t_1-t) = T^*(t_1)\, T^*(-t) = T^*(-t)\, T^*(t_1)$$

Therefore,

$$\int_0^{t_1} \|B^*T^*(t_1-t)z\|_U^2 dt = \int_0^{t_1} \langle B^*T^*(-t)\, T^*(t_1)z,\, B^*T^*(-t)\, T^*(t_1)z\rangle_U dt$$

Similar to the treatment of finite dimensional systems, it has been determined that

$$\int_0^{t_1} \|B^*T^*(t_1-t)z\|_U^2 dt = \langle (\int_0^{t_1} T(-t)BB^*T^*(-t)dt)T_1^* z,\, T_1^* z\rangle_U \, . \tag{7-30}$$

If an operator D is considered as

$$D(\cdot) = (\int_0^{t_1} T(-t)BB^*T^*(-t)dt)\,(\cdot)$$

then $D(\cdot)$ is a self-adjoint operator and from (7-29) and (7-30)

$$\langle D\, T^*(t_1)z,\, T^*(t_1)z\rangle_U \geq \beta \|z\|_Z^2 \, . \tag{7-31}$$

From (7-31) it follows that $D(\cdot)$ will have a bounded inverse.

If the control action is considered as a feedback control with the gain operator K

$$K(\cdot) = -B^*D^{-1}(\cdot) \tag{7-32}$$

then

$$u = Kv = -B^*D^{-1}(v) \tag{7-33}$$

Since both operators $D(\cdot)$ and $D^{-1}(\cdot)$ operate on a distributed state function and generate a distributed function, then u is a distributed feedback control. The closed loop operator of system (7-25) generates a semigroup S(t) and, similar to finite dimensional case, a Lyapunov functional $\mathscr{L}$ can be defined as

$$\mathscr{L}(y) = \langle y, Dy \rangle \tag{7-34}$$

where y is the state of the following system

$$\dot{y} = (A - BB^*D^{-1})^* y \tag{7-35}$$

$$y(0) = v_o$$

$$y \epsilon V = L_2([0,\ell],E^n)$$

The closed loop operators of systems (7-25) and (7-35) are adjoint of each other and (7-35) generates the adjoint semigroup $S^*(t)$. The time derivative of $\mathscr{L}$ becomes,

$$\dot{\mathscr{L}}(y) = 2\langle DA^*y, y\rangle_V - 2\langle B^*y, B^*y\rangle_U$$

where

$$DA^*y = \int_o^{t_1} T(-t)BB^*T^*(-t)A^*y$$

Since the open loop evolution operator of system (7-35) is A^*, which generates the semigroup $T^*(t)$, where

$$\frac{d}{dt}T^*(-t) = -T^*(-t)A^* ,$$

then

$$DA^*y = -\int_o^{t_1} T(-t)B\frac{d}{dt}(B^*T^*(-t))\,dt$$

Finally, similar to the derivations of finite dimensional systems, the following can be obtained

$$\dot{\mathscr{L}}(y) = -\|B^*T^*(-t_1)y\|_U^2 - \|B^*y\|_U^2 \leq 0 \qquad (7\text{-}36)$$

Due to the exact controllability and (7-27), the right hand side of (7-36) is bounded from below in the state space V, i.e., $\dot{\mathscr{L}}$ is negative definite with respect to $\|y\|_V^2$. Hence, the system (7-35) is asymptotically stable. It can be seen that

$$\|S^*(t)\| \leq Me^{-\alpha t} \qquad \text{for some } M \geq 1 \text{ and } \alpha > 0$$

Since $\|S(t)\| = \|S^*(t)\|$, then the same statement applies to the system (7-25).

Therefore, the exact controllability is a sufficient condition for stabilizability of a general distributed parameter system. However, the exact controllability condition, as used in (7-27), is often very difficult to satisfy for general distributed systems.

Stabilization of Symmetric Hyperbolic DPS

The general class of systems studied in Chapter 5 are represented by

$$\frac{\partial v}{\partial t} + A_1\frac{\partial v}{\partial x} + B_1\, v = 0 \qquad (7\text{-}37)$$

$$v(x,0) = v_o , \qquad v(o,t) = 0$$

where $v\epsilon V([0,\ell],E^n)$ and matrices A_1 and B_1 are denoted by A and B in Chapter 5. This

class of systems, with A_1 being symmetric, shares a common form with the linearized model of the MPD system studied in this chapter. Let $A_1(x)$ possess eigenvalues of the general form: $\lambda_1(x) \leq \lambda_2(x) \leq \ \ldots \ \leq \lambda_p(x) < \ 0 < \lambda_{p+1}(x) \leq \ \ldots \ \leq \lambda_n(x)$. The eigenvalues show the directions of the characteristic lines of partial differential equations. In addition, there exists a continuously differentiable matrix $O(x)$, based on the system eigenvectors such that $O^{-1}(x)\, A(x)\, O(x) = \Lambda(x)$. If one considers a new set of states w, such that

$$v = O(x)w \ ,$$

then the substitution of v into (7-37) results in

$$O(x)\frac{\partial w}{\partial t} + A_1(x)O(x)\frac{\partial w}{\partial x} + \left[A_1(x)\frac{\partial O(x)}{\partial x} + B_1 O(x)\right]w = 0$$

$$\frac{\partial w}{\partial t} + O^{-1}(x)A_1(x)O(x)\frac{\partial w}{\partial x} + O^{-1}\left[A_1(x)\frac{\partial O(x)}{\partial x} + B_1 O(x)\right]w = 0$$

Therefore, one can conclude

$$\frac{\partial w}{\partial t} + \Lambda(x)\,\frac{\partial w}{\partial x} + \beta(x)w = 0 \qquad (7\text{-}38)$$

Since $\Lambda(x)$ is a diagonal matrix of the system eigenvalues, and since λ_1 to λ_p are negative and λ_{p+1} to λ_n are positive, then Λ can be decomposed as

$$\Lambda = \begin{pmatrix} \Lambda^- & O \\ 0 & \Lambda^+ \end{pmatrix}, \quad \begin{aligned} \Lambda^- &= \mathrm{diag}(\lambda_1, \ldots, \lambda_p) \\ \Lambda^+ &= \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_n) \end{aligned} \qquad (7\text{-}39)$$

The corresponding decomposition can be applied to the states

$$w = \begin{bmatrix} w^- \\ w^+ \end{bmatrix} \qquad (7\text{-}40)$$

Therefore, (7-38) can be reduced to a set of ordinary differential equations of the following form:

$$\frac{dw_d}{dt} = \frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = \frac{\partial w_k}{\partial t} + \left(\frac{dx}{dt}\right)_k \frac{\partial w_k}{\partial x} = -\beta_k(x)w \qquad (7\text{-}41)$$

$$k = 1,2,...,n$$

where w_k is the kth element of the vector valued function w. Moreover, λ_k represents the direction of the kth characteristic line, hence $\frac{d(\cdot)}{dt}$ is the directional derivative along the corresponding characteristic line. On the right hand side of (7-41), β_k is the kth row of the matrix valued function $\beta(x)$. The set of n ordinary differential equations (7-41) are coupled by the term $\beta_k(x)w$. Initial values for $t = 0$ can be written as

$$w(x,0) = w_o \in L_2([0,\ell];E^n)$$

Considering the boundary conditions at a point on the boundary $x = 0$ and $t = t_o$, as shown in Figure 7.1, one finds the characteristics with negative and positive eigenvalues arrive at that point with negative and positive slopes, respectively [47]. The "incoming information" consists of values of w_k associated with characteristic line $C_k(0,t_o)$ with negative slope for $k = 1,...,p$. The "outgoing information" consists of values of w_k associated with the positive slope characteristic $C_k(o, t_o)$, $k = p+1,...,n$. Hence, along the boundary $x = 0$, the values of w^+ should be known. It is clear that along the boundary $x = \ell$, the orientation of characteristics will be reversed and hence the values of w^- should be known. When system (7-37) is unstable it can become subjected to stabilization with addition of a control action u, such that the resulting nonhomogeneous system will have the generic form of the abstract system (7-24)

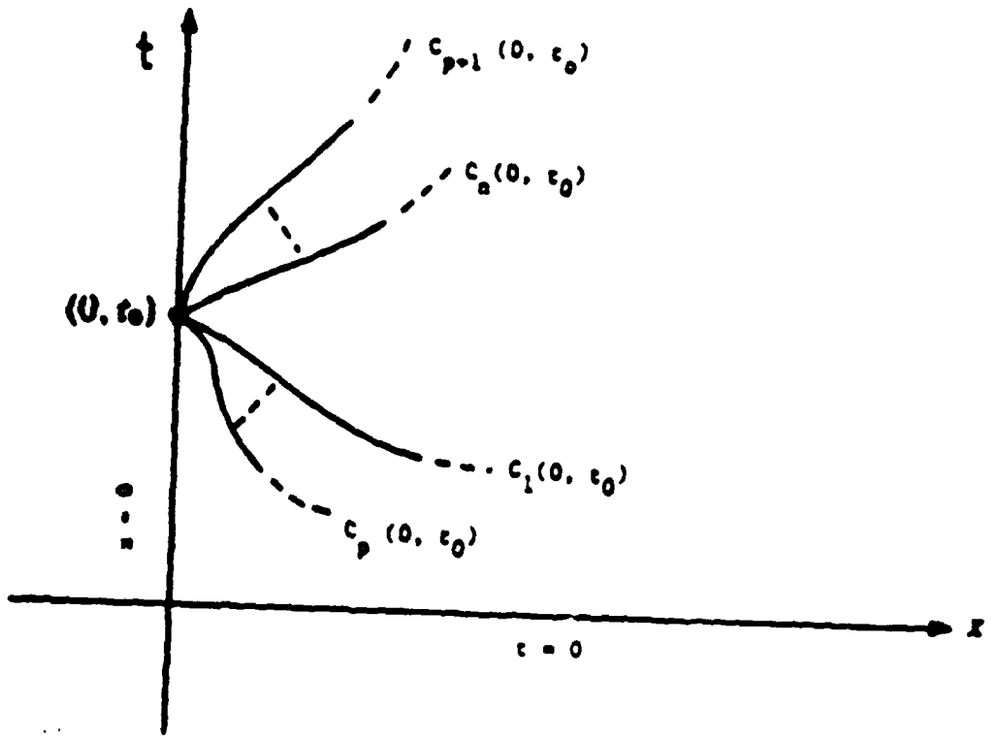


Figure 7.1 Characteristics configuration at $x = 0$ and $t = t_o$.

$$\frac{\partial v}{\partial t} = Av + Bu$$

where $Av = -A_1 \frac{\partial v}{\partial x} - B_1 v$

$$v(0) = v_o$$

The input operator B depends on the way that the control is interacting with the system.

In general, the system of equation (7-37) is assumed to be stabilizable by some form of feedback control as $u = Kv$, where $K : v \longrightarrow u$. Hence, the operator A+BK would be an infinitesimal generator of an asymptotically stable semigroup. If one defines a Lyapunov function as $\mathcal{L} = \langle v, v \rangle$, similar to (5-39) with $S = I$, then the following would result:

$$\dot{\mathcal{L}} = -v^T A_1 v \Big|_o^\ell + \int_o^\ell v^T \left\{ \frac{\partial A_1}{\partial x} - 2B_1 + 2BK \right\} v dx \tag{7-42}$$

From transformation $v = O(x)\, w$ we have

$$v^T A_1 v \Big|_o^\ell = w^T O^T A_1 O w \Big|_o^\ell = w^T \Lambda w \Big|_o^\ell \tag{7-43}$$

where

$$w^T \Lambda w = \begin{bmatrix} w^- \\ w^+ \end{bmatrix}^T \begin{bmatrix} \Lambda^- & 0 \\ 0 & \Lambda^+ \end{bmatrix} \begin{bmatrix} w^- \\ w^+ \end{bmatrix}$$

$$w^T \Lambda w \Big|_o^\ell = w^{-T} \Lambda^- w^- \Big|_o^\ell + w^{+T} \Lambda^+ w^+ \Big|_o^\ell \tag{7-44}$$

At $x = 0$, and at $x = \ell$, w^+ and w^-, respectively, must be known or given from the boundary conditions, i.e.,

$$w^+(o,t) = D_o w^-(o,t) \tag{7-45}$$

$$w^-(\ell,t) = D_\ell w^+(\ell,t) \tag{7-46}$$

Hence

$$w^T \Lambda w \Big|_o^\ell = w^{+^T}(\ell,t)\Lambda^+(\ell)w^+(\ell,t) + w^{+^T}(\ell,t)\, D_\ell^T \Lambda^-(\ell)\, D_\ell\, w^+(\ell,t)$$

$$-[w^{-^T}(o,t)\Lambda^-(o)w^-(o,t) + w^{-^T}(o,t)D_o^T \Lambda^+(o)D_o w^-(o,t)]$$

To guarantee stability in the sense of Lyapunov, the following must be satisfied:

$$-w^{+^T}(\ell,t)\, [\Lambda^+(\ell) + D_\ell^T \Lambda^-(\ell)D_\ell]\, w^+(\ell,t) \leq 0 \tag{7-47}$$

$$w^{-^T}(o,t)\, [\Lambda^-(o) + D_o^T \Lambda^+(o)D_o]\, w^-(o,t) \leq 0 \tag{7-48}$$

In the case that $w^+(o,t) = 0$ and $w^-(\ell,t) = 0$, (7-43) reduces to

$$w^T \Lambda w \Big|_o^\ell = w^+(\ell,t)^T \Lambda^+(\ell)w^+(\ell,t)$$

$$-[w^-(o,t)^T \Lambda^-(o)w^-(o,t)] \geq 0 \tag{7-49}$$

Hence, $-v^T A_1 v \Big|_o^\ell \leq 0$.

Another condition to be satisfied is that the following matrix should be negative definite

$$M_c = \left\{ \frac{\partial A_1}{\partial x} - 2B_1 + 2BK \right\} : \text{ negative definite} \tag{7-50}$$

Therefore, by using a proper feedback gain K, such that conditions (7-49) and (7-50) are satisfied, the Lyapunov functional leads to

$$\dot{\mathscr{L}} \leq \langle v, M_c v \rangle \leq -\gamma \, \|v\|_V \tag{7-51}$$

which results in the asymptotic stability of system (7-37). If M_c is negative definite, then

the supremum of its eigenvalues must be less than zero. Since the operators $\frac{\partial A_1}{\partial x}$ and B_1 are only matrix functions of x, stabilization of the infinite dimensional system (7-37) reduces to the pole placement property of a finite dimensional matrix. As mentioned earlier in the discussion of finite dimensional systems, if the pair (A,B) is controllable, then there exists a K such that the eigenvalues of $A + BK$ can be placed in desired locations. Therefore, there will exist at least a matrix K such that M_c is negative definite if and only if the controllability matrix C, below,

$$C = \left[2B \, , \, 2\left[\frac{\partial A_1}{\partial x} - 2B_1\right]B \, , \, 2\left[\frac{\partial A_1}{\partial x} - 2B_1\right]^2 B \, , \, \ldots . \, , \, 2\left[\frac{\partial A_1}{\partial x} - 2B_1\right]^{n-1} B \right] \tag{7-52}$$

has a full rank. In the case of the MPD thruster, where A_1 is given by (5-32) and B_1 is given by (5-38), its control is a distributed (body force) type input. Addition of the distributed control inputs can be achieved by imposing a perturbation to the equilibrium controls in equations (5-27) to (5-29). Hence, the control vector in (7-24) will be defined as

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7-53}$$

Therefore, the input operator B, which couples u to the system of (7-24), will be calculated as

$$B = \begin{bmatrix} 0 & 0 \\ \dfrac{1}{\rho_e\sqrt{RT_e}} & 0 \\ \dfrac{-u_e}{\rho_e T_e\sqrt{\gamma-1}\,C_v} & \dfrac{1}{\rho_e T_e\sqrt{\gamma-1}\,C_v} \end{bmatrix} \tag{7-54}$$

From the values of A_1 and B_1 given by equations (5-37) and (5-38), respectively, and B given by (7-54), one can construct the controllability matrix for the MPD model. A detailed computation leads to the condition that $\dfrac{T_e'}{T_e}$ and $\dfrac{u_e'}{u_e}$ should not be zero simultaneously so that the system of the MPD model would be stabilizable. This indicates that although there is not a formal input term influencing the density evolution equation (5-27), i.e., the continuity equation, it is possible to control and stabilize the density, velocity and temperature of the plasma with the input controls appearing only in the momentum and energy equations.

Stabilization of DPS Represented by Contraction Semigroups

An important class of distributed parameter systems which covers a wide range of hypo-eliptic and structural systems are discussed in this section. Consider a nonhomogeneous infinite dimensional system represented by the evolution equation (7-24), where the open loop evolution operator A generates a contraction semigroup T(t), defined in Chapter 3 as $\|T(t)\| \leq 1$. From the Hille-Yoshida theorem it has been determined that the generator A must be dissipative with respect to an appropriate norm so that this contraction exists. For those dissipative generators with $\|T(t)\| < 1$ the evolution process is already asymptotically stable and stabilization has no meaning. However, for the case where $\langle v, Av\rangle_V = 0$ and $\|T(t)\| = 1$, the dynamical system is not asymptotically stable and stabilization can be addressed. This category of problems

contains a majority of structural systems with negligible damping which are often subject to undamped oscillations. Consider the system (7-24)

$$\dot{v} = Av + Bu$$

$$v(0) = v_o$$

$$v \epsilon V$$

which is subjected to the following condition: $\langle Av, v \rangle_V = 0$.

Assuming a Lyapunov functional $\mathscr{L}$ on a set G, where the positive orbit of the system is contained in G, then

$$\mathscr{L} = \langle v, v \rangle_v$$

$$\dot{\mathscr{L}} = 2\langle \dot{v}, v \rangle_V = 2\langle Av, v \rangle_V + 2\langle Bu, v \rangle_V$$

$$\dot{\mathscr{L}} = 2\langle Bu, v \rangle_V = 2\langle u, B^* v \rangle_U \qquad (7\text{-}56)$$

This leads to construction of a feedback control u, where

$$u = -B^* v \qquad (7\text{-}57)$$

Hence,

$$\dot{\mathscr{L}} = -2\|B^* v\|_U^2 \leq 0 \qquad (7\text{-}58)$$

Since $\dot{\mathscr{L}}$ is negative with respect to the norm in the control space U, no further implications can be made about the negative definiteness of $\dot{\mathscr{L}}$ in terms of the norms in state space V. This means that the Lyapunov direct method is not applicable to such systems. However, from the invariance principle it can be concluded that if the positive orbit of the system is compact, then the closed loop system $S(t)v \longrightarrow M^+$ as $t \longrightarrow \infty$, where M^+ is the largest invariant set on which $\dot{\mathscr{L}} = 0$. The motion of the dynamical

system is asymptotically stable and $S(t)v \longrightarrow 0$ as $t \longrightarrow 0$ if $M^+ = \{0\}$. Therefore, it is sufficient to add approximate controllability in order to arrive at stabilizability of this system, as shown by [38] in the following theorem:

Theorem 7.2. Let A in system (7-24) be the infinitesimal generator of a contraction C_o-semigroup T(t) on V for $t \geq 0$. If (i) for every $y \epsilon V$, S(t)y remains in a compact set of V where S(t) is the semi-group generated by closed loop operator as described above, and (ii) (7-24) is approximately controllable, then $S(t)y \longrightarrow 0$ as $t \longrightarrow \infty$, i.e., system (7-24) is weakly stabilizable.

Proof. If $C = A - BB^*$ is the closed loop operator with the control law (7-57) and S(t) is the semigroup generated by C, then $C^* = A^* - BB^*$ is the generator of $S^*(t)$ for the following system

$$\dot{y} = C^* y \tag{7-59}$$

$$y(0) = v_o \qquad y \epsilon V$$

Consider the Lyapunov functional

$$\mathscr{L} = \langle y, y \rangle_V ,$$

then $\dot{\mathscr{L}} = -2\|B^* y\|_U = -2\|B^* S^*(t) v_o\|_U$, similar to (7-58). From condition (i) and $\dot{\mathscr{L}} \leq 0$, it can be concluded that $S^*(t)v_o \longrightarrow M^+$ as $t \longrightarrow \infty$, where M^+ is the largest invariant set on which $\dot{\mathscr{L}} = 0$. From condition (ii) it can be concluded that $M^+ = \{0\}$. Let $m \epsilon M^+$ and define z(t) as

$$z(t) = \int_o^t S^*(s) m \, ds \tag{7-60}$$

Since C^* is closed, $z(t) \epsilon D(C^*) = D(A^*) \subset V$, and $z(0) = 0$, then

$$\dot{z}(t) = (A^* - BB^*)z(t) + m \tag{7-61}$$

from the definition of M^+, it follows that

$$\dot{\mathscr{L}} = -2\|B^*S^*(t)\, m\|_U^2 = 0 \tag{7-62}$$

or

$$B^*S^*(t)\, m = 0, \quad \text{for all } t \geq 0 \tag{7-63}$$

This implies that $B^*[z(t) - z(0)] = 0$, or

$$B^*z(t) = 0 \tag{7-64}$$

Hence,

$$\dot{z} = A^*z(t) + m \tag{7-65}$$

and

$$z(t) = \int_0^t T^*(s)\, m\, ds \tag{7-66}$$

From (7-64), it can be concluded that

$$B^*z(t) = 0 = \int_0^t B^*T^*(s)\, m\, ds \tag{7-67}$$

Therefore,

$$B^*T^*(s)\, m = 0 \tag{7-68}$$

However, if system (7-24), by assumption (ii), is approximately controllable, then

$$B^*T^*(s)\, m = 0 \quad \text{implies} \quad m = 0$$

This indicates that every element of the invariant set M^+ is zero. Therefore, from the

invariance principle, $S^*(t)v_o \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, $S(t)v_o \longrightarrow 0$ weakly in V as $t \longrightarrow \infty$.

□

The analysis in this chapter indicates that, unlike the case of finite dimensional systems where controllability implies stabilizability, for infinite dimensional systems such an implication is not appropriate. In general for infinite dimensional systems, exact controllability provides sufficiency for existence of a stabilizing control. The special case of an infinite dimensional dynamical system which has the characteristics of a contraction semigroup is considered in the last section of this chapter. In this case, approximate controllability along with compactness of the closed loop motion provided sufficient conditions leading to the existence of a stabilizing control. It can be shown that under the condition of contraction, the compactness of closed loop motion would imply the close range of the control operator C_u. This in turn implies that the approximate controllability would lead to an exact controllability.

CHAPTER 8 - SUMMARY AND CONCLUSIONS

A summary of the research presented in the previous chapters and the results are reviewed in this section. The major conclusions drawn from the study are presented, and some recommendations for future directions are described.

Summary

The semigroup properties of states of dynamical systems were presented. The extension of the Lyapunov direct stability method for distributed parameter systems was presented. It was shown that whenever the Lyapunov functional subject to the direct method or the equilibrium point does not exist, then asymptotic behavior of the distributed parameter systems can be predicted by extension of the invariance principle.

A simplified model of the MPD thruster using conservation laws and Maxwell's equations was derived. This model was used for stability and controllability analysis of the MPD systems. The theorems and concepts of the stability of DPS was applied to a parabolic model of a magneto-plasma dynamic system, subject to perturbations about its null equilibrium velocity.

Asymptotic stability of the transverse and longitudinal modes of motion were derived. Similar treatment was applied to the case of the MPD accelerator with nonzero equilibrium flow velocity. The stability of the linearized model was derived based on an equivalent norm. In addition, the stability of the original nonlinear DPS model was investigated and derived.

The concepts of controllability and observability for linear time invariant DPS were compared with the corresponding concepts for finite dimensional system.

Fundamental differences between these two types of systems from controllability and observability points of view were described.

The stabilizability of DPS in the absence of asymptotic stability was analyzed and the resulting theorems were applied to a class of linear symmetric hyperbolic systems. It was shown that if the system is exactly controllable, then existence of a stabilizing distributed control can be achieved. For a special class of DPS where states of the system form a contraction semigroup, stabilization was proven based on the invariance principle and approximate controllability.

Conclusions

Following items are concluded from the research presented in this manuscript.

In general, Lyapunov stability theorem provides sufficient condition(s) for the stability or asymptotic stability of systems. In the case of systems represented by a large number of partial differential equations, Lyapunov's method provides an applicable process for stability analysis as opposed to the spectrum analysis of high order characteristic equations with the presence of wave number in the characteristic equation. Moreover, in infinite dimensional systems, negative definiteness of supremum of the spectrum provides only the necessary conditions for asymptotic stability. However, the extra condition from the spectrum determined growth assumption should exist to guarantee the existence of asymptotic stability.

In a case that asymptotic stability does not exist, then exact controllability provides a sufficient condition for the stabilizability of the system. In general, exact controllability is very difficult to obtain and has been proven only for a few special cases. This study has shown that the stabilization of linear symmetric hyperbolic systems with

distributed control is plausible without the need for exact controllability.

The results of this study can be applied to more elaborate models of MPD thrusters, and more definite answers on the subject of system observability and stabilization can be derived.

Recommendations

The following issues are related to the materials discussed during the course of this research and are recommended as topics for future research in continuation of this study on distributed parameter systems:

1. determination of different types of strategic points for general classes of DPS and their applications to sensor/actuator optimal locations,
2. closed-loop robust control of DPS with respect to noise and random excitations,
3. effects of delay in DPS and robust control design for delayed DPS,
4. stabilization of nonlinear structural systems based on DPS models,
5. experimental investigation of the proposed stabilizability technique.
6. implementation of the results on an actual MPD thruster at one of the Air Force Laboratories

REFERENCES

1. Butkovskii, A.G. and Lerner, A.Y. Soviet Physics English Translated, Vol. 5, p. 936, 1961.
2. Egorov, Y.V. *Some Problems in the Theory of Optimal Control,*oSoviet Math., Vol. 3, pp. 1080-1084, 1962.
3. Wang, P.K.C. *Control of Distributed Parameter Systems,* Advances in control systems, New York, Academic Press, Vol. 1, pp. 75-120, 1964.
4. Courant, R. and Hilbert, D. *Methods of Mathematical Physics, Vol. II Partial Differential Equations,* New York, Interscience, 1962.
5. Courant, R. and Fredrichs, K.O. *Supersonic Flow and Shock Waves,* New York, Interscience, 1948.
6. Hormander, L. *Linear Partial Differential Operators,* New York, Academic Press, 1964.
7. Lax, P.D. *Hyperbolic Systems of Conservation Laws, II,* Communs. Pure and Appl. Math., Vol. 10, pp. 537-566, 1957.
8. Sobolev, S.L. *Partial Differential Equations of Mathematical Physics,* Reading, Massachusetts, Addison-Wesley, 1964.
9. Bers, L., John, F. and Schecter, M. *Partial Differential Equations,* New York, Interscience, 1964.
10. Zubov, V.I. *Methods of A.M. Lyapunov and their Application,* The Netherlands, P. Noordhoff Ltd., 1964.
11. Lyapunov, A.M. *Stability of Motion,* New York, Academic Press, 1966.
12. Massera, J.L. *Contributions to Stability Theory,* Ann. of Math., Vol. 64, pp. 182-206, 1956.
13. Hsu, C. *Stability Analysis of Reactor Systems via Lyapunov's Second Method,* Paper No. 66-WA/AUT-7, ASME-Winter Annual Meeting 1966.
14. Parks, P.C. and Pritchard, A.J. *On the Construction and use of Lyapunov Functionals,* Proc. IVth IFAC Congress, Warsaw, 1969.
15. Movchan, A.A. *The Direct Method of Lyapunov in Stability Problems of Elastic Systems,* Prikladnaya Matematika i Mekhanika, Vol. 26, No. 2, pp. 356-58, 1962.
16. Wang, P.K.C. *Stability Analysis of Aeroelastic Systems via Lyapunov's Direct Method,* Journal of Franklin Institute, Vol. 281, No. 1, pp. 51-72, Jan. 1966.
17. Buis, G.R., Vogt, W.G. and Eisen, M.M. *Lyapunov Stability for Partial Differential Equations,* NASA CR-1100, 1967.
18. Pau, C.V. *Stability Theory of Nonlinear Operational Differential Equations in Hilbert Spaces,* Ph.D. Thesis, University of Pittsburg, 1968.
19. Shoureshi, R. and Pourki, F. *Stability Analysis of Electro-Magneto Plasma Dynamics,* Accepted for publication in the AIAA Journal of Guidance and Control.

20. LaSalle, J.P. *Stability Theory for Ordinary Differential Equations,* J. of Differential Equations, Vol. 4, pp. 57-65, 1968.

21. Hale, J.K. *Dynamical Systems and Stability,* Journal of Math. Anal. Application, Vol. 26, pp. 39-59, 1969.

22. Dafermos, C.M. *Application of Invariance Principle for Compact Processes II: Asymptotic Behavior of Solutions of a Hyperbolic Conservation Law,* J. of Differential Equations, Vol. 11, pp. 416-424, 1972.

23. Dafermos, C.M. and Slemrod, M. *Asymptotic Behavior of Nonlinear Contraction Semigroups,* J. of Functional Analysis, Vol. 13, pp. 97-106, 1973.

24. Walker, J.A. *On the Application of Lyapunov's Direct Method to Linear Dynamical Systems,* J. of Math. Anal. and Appl., Vol. 53, pp. 187-220, 1976.

25. Walker, J.A. *Some Results on Lyapunov Functions and Generated Dynamical Systems,* J. of Differential Equations, Vol. 30, pp. 424-440, 1978.

26. Walker, J.A. *Dynamical Systems and Evolution Equations Theory and Applications,* New York, Plenum Press, 1980.

27. Kalman, R.E. *Contributions to the Theory of Optimal Control,* Bol. Soc. Math. Mexicana, Vol. 5, pp. 102-119, 1960.

28. Kalman, R.E. and Bucy, R.S. *New Results in Linear Prediction and Filtering Theory,* Trans. ASME Ser. D, J. Basic Engineering, Vol. 83, pp. 95-100, 1961.

29. Fattorini, H.O. *Control in Finite Time of Differential Equations in Banach Space,* Comm. Pure and Appl. Math., Vol. 19, pp. 17-34, 1966.

30. Fattorini, H.O. *Some Remarks on Complete Controllability,* SIAM J. Control, Vol. 4, pp. 684-694, 1966.

31. Fattorini, H.O. *On Complete Controllability of Linear Systems, J. Differential Equations, Vol. 3, pp. 391-402, 1967.*

32. *Fattorini, H.O. Boundary Control Systems,* SIAM J. Control, Vol. 6, pp. 349-388, 1968.

33. Fattorini, H.O. and Russell, D.L. *Exact Controllability Theorems for Linear Parabolic Equations in One Space Dimension,* Arch. Rational Mech. Anal., Vol. 43, pp. 272-292, 1971.

34. Fattorini, H.O. *The Time-Optimal Problem for Boundary Control of the Heat Equation,* Calculus of Variations and Control Theory, Academic Press, 1976.

35. Triggiani, R. *Extension of Rank Conditions for Controllability and Observability to Banach Spaces and Unbounded Operators,* SIAM J. Control, Vol. 14, pp. 313-338, 1976.

36. Russell, D.L. *Cn Boundary-Value Controllability of Linear Symmetric Hyperbolic Systems,* Mathematical Theory of Control, New York, Academic Press, pp. 312-321, 1967.

37. Russell, D.L. *Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems,* J. Math. Anal. Appl., Vol. 18, pp. 542-559, 1967.

38. Russell, D.L. *Boundary Value Control Theory of the Higher Dimensional Wave Equation,* SIAM J. Control, Vol. 9, pp. 29-42, 1971.

39. Russell, D.L. *Control Theory of Hyperbolic Equations Related to Certain Questions in Harmonic Analysis and Spectral Theory,* J. Math. Anal. Appl., Vol. 40, pp. 336-368, 1972.

40. Dolecki, S. and Russell, D. *A General Theory of Observation and Control,* SIAM J. Control, Vol. 15, pp. 185-220, 1977.

41. Seidman, T.I. *Observation and Prediction for the Heat Equation, IV,* SIAM J. Control, Vol. 15, pp. 412-427, 1977.

42. Seidman, T.I. *Boundary Observation and Control for the Heat Equation,* Calculus of Variations and Control Theory, Academic Press, 1976.

43. Sakawa, Y. *Observability and Related Problems for Partial Differential Equations of Parabolic Type,* SIAM J. Control, Vol. 13, pp. 14-27, 1975.

44. Butkovskii, A.G. *Theory of Optimal Control of Distributed Parameter Systems,* New York, American Elsevier, 1969.

45. Lions, J.L. *Optimal Control of Systems Governed by Partial Differential Equations,* New York-Heidelberg-Berlin, Springer-Verlag, translated by Miter, S.K., 1971.

46. Russell, D.L. *Quadratic Performance Criteria in Boundary Control of Linear Symmetric Hyperbolic Systems,* SIAM J. Control, Vol. 11, pp. 475-509, 1973.

47. Russell, D.L. *Controllability and Stabilizability Theory for Partial Differential Equations: Recent Progress and Open Questions,* SIAM Review, Vol. 20, pp. 639-739, 1978.

48. Bensoussan, A. *Optimization of Sensors' Location in a Distributed Filtering Problem,* Stability of Stochastic Dynamical Systems, New York, Springer-Verlag, 1972.

49. Yu, T.K. and Seinfeld, J.H. *Observability and Optimal Measurement Location in Linear Distributed Parameter Systems,* Int. J. Control, Vol. 18, 1973.

50. Amouroux, M. and Babary, J.P. *Optimal Pointwise Control for a Class of Distributed Parameter Systems,* Proc. IFAC 6th World Congress, 1975.

51. Slemrod, M. *The Linear Stabilization Problem in Hilbert Space,* Journal of Functional Anal., Vol. 11, pp. 334-345, 1972.

52. Slemrod, M. *A Note on Complete Controllability and Stabilizability for Linear Control Systems in Hilbert Spaces,* SIAM J. Control, Vol. 12, pp. 500-508, 1974.

53. Triggiani, R. *On the Stabilizability Problem in Banach Space,* Journal of Math. Anal. and Applications, Vol. 52, pp. 383-403, 1975.

54. Pritchard, A.J. and Zabczyk, J. *Stability and Stabilizability of Infinite Dimensional Systems,* SIAM Review, Vol. 23, pp. 25-52, 1981.

55. Lassiecka, I. and Triggiani, R. *Stabilization and Structural Assignment of Dirichlet Boundary Feedback Parabolic Equations,* SIAM J. Control, Vol. 21, pp. 766-803, 1983.

56. Lasiecka, I. and Triggiani, R. *Stabilization of Newmann Boundary Feedback of Parabolic Equations: The Case of Trace in the Feedback Loop,* Appl. Math. Optim., Vol. 10, pp. 307-350, 1983.

57. Balas, M.J. *Feedback Control of Flexible Systems,* IEEE Transaction on Automatic Control, Vol. AC-23, No. 4, August 1978.

58. Slemrod, M. *Boundary Feedback Stabilization for a Quasi-linear Wave Equation,* Lecture Notes in Control and Information Sciences, Vol. 54, pp. 221-237, 1982.

59. Slemrod, M. *Feedback Stabilization of a Linear Control System in a Hilbert Space with an Apriori Bounded Control,* To appear in Math. Control, Signals, Systems.

60. Royden, H.L. *Real Analysis,* New York: The Macmillan Company, 1963.

61. Naylor, A.W. and Sell, G.R. *Linear Operator Theory in Engineering and Science,* Holt, Rinehart and Winston, Inc., 1971.

62. Curtain, R.F. and Pritchard, A.J. *Functional Analysis in Modern Applied Mathematics,* London, Academic Press, 1977.

63. Balakrishnan, A.V. *Applied Functional Analysis,* 2nd edition, New York, Springer-Verlag, 1981.

64. Schechter, M. *Principles of Functional Analysis,* New York, Academic Press, 1971.

65. Showalter, R.E. *Hilbert Space Methods for Partial Differential Equations,* London, Pitman, 1977.

66. Goldstein, J.A. *Semigroups of Linear Operators and Applications,* New York: Oxford Mathematical, Monographs, 1985.

67. Carrier, G.F. and Pearson, C.E. *Partial Differential Equations Theory and Technique,* New York, Academic Press, 1976.

68. Pazy, A, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* New York, Springer-Verlag, 1983.

69. Tanabe, H. *Equations of Evolution,* London, Pitman, 1979.

70. Pai, S.I. *Magneto-gas Dynamics and Plasma Dynamics,* Springer-Verlag, 1962.

71. Jahn, R.C. *Physics of Electric Propulsion,* McGraw-Hill, 1968.

72. King, D.Q., Clark, K.E. and Jahn, R.G. *Effect of Chocked Flow on Terminal Characteristics of MPD Thrusters,* AIAA-81-0686, 15th International Electric Propulsion Conf., Las Vegas, NV, April 1981.

73. Wonham, W.M. *Linear Multivariable Control: A Geometric Approach,* Springer-Verlag, 1974.

74. Lions, J.L. *Exact Controllability, Stabilization and Perturbation for Distributed Systems,* SIAM Review, Vol. 30, No. 1, pp. 1-68, 1988.

Appendix A: Proof of Theorem 3.7

Given any $\epsilon > 0$, and without any loss of generality, let $\epsilon < r$. Defining $\alpha(\epsilon) \equiv \inf_{\|v-v_e\|=\epsilon} \mathcal{L}$, and noting that $\alpha \geq \mathcal{L}(v_e) + f(\epsilon)$, it is clear that the ball of radius ϵ at v_e i.e., $B_\epsilon(v_e)$ contains a disjoint component G_α of the set $\{v \epsilon V \mid \mathcal{L}(v) < \alpha\}$ such that $v_e \epsilon G_\alpha$. Since $\mathcal{L}: V \rightarrow R$ is continuous, G_α is an open and positive invariant set and there exists $\delta(\epsilon) > 0$ such that $B_\delta(v_e) \subset G_\alpha$. Therefore, given any $v_o \epsilon B_\delta(v_e) \subset G_\alpha$, the positive orbit is contained in G_α, i.e., $\gamma(v_o) \subset G_\alpha \subset B_\epsilon(v_e)$. Therefore, the point v_e is stable.

With α and δ defined as above, and $0 < \epsilon < r$, consider any fixed point $v_o \epsilon B_\delta(v_e) \subset G_\alpha$. Theorem 3.6 implies that $\mathcal{L}(T(t)v_o)$ is non-increasing on R^+. Therefore, as $t \rightarrow \infty$, $\mathcal{L}(T(t)v_o) \rightarrow \inf_{t \epsilon R^+} \mathcal{L}(T(t)v_o) \equiv \beta(v_o)$. Clearly, $\alpha \geq \beta \geq \mathcal{L}(V_e)$ because $\mathcal{L}(v) \geq \mathcal{L}(v_e)$ for all $v \epsilon G_\alpha \subset B_r(v_e)$. Either β must be $\mathcal{L}(v_e)$ or $\gamma(v_o) \cap \{v \epsilon G_\alpha \mid \mathcal{L}(v) < \beta\}$ for $\beta > \mathcal{L}(v_e)$ is empty. In the latter case, continuity of $\mathcal{L}$ implies that there exists a $\nu > 0$ such that $\gamma(v_o) \cap B_\nu(v_e)$ is empty, assuming that $\beta \neq \mathcal{L}(v_e)$ and $\dot{\mathcal{L}}(v) \leq -g(d(v, v_e))$ for all $v \epsilon G_\alpha \subset B_r(v)$. From the hypothesis, $g(\cdot)$ is a monotone function, hence $g(\nu) > 0$ and

$$\mathcal{L}(v_e) < \mathcal{L}(T(t)v_o) = \mathcal{L}(v_o) - \int_o^t g(\nu)$$

$$\leq \mathcal{L}(v_o) - t\, g(\nu) \rightarrow -\infty \text{ as } t \rightarrow \infty$$

which is impossible. Hence, $\beta = \mathcal{L}(v_e)$ and $\mathcal{L}(T(t)v_o) \rightarrow \mathcal{L}(v_e)$ as $t \rightarrow \infty$. This implies that $f(d(t(t)v_o, v_e)) \rightarrow 0$ as $t \rightarrow \infty$. Since f is a monotone function, $T(t)v_o \rightarrow v_e$ as $t \rightarrow \infty$ and v_e is asymptotically stable [26].

Appendix B: Proof of Theorem 3.11

If V is complete and $\gamma(v)$ is precompact, then $\Omega(v)$ is nonempty and invariant; moreover, $d(T(t)v,\Omega(v)) \rightarrow 0$ as $t \rightarrow \infty$. Assuming that $\gamma(v) \subset \overline{G}$, one has $\Omega(v) \subset \overline{\gamma}(v) \subset \overline{G}$. If $\gamma(v)$ is not precompact, $\Omega(v)$ may be empty. In this case, the theorem is obviously true but meaningless. Hence, the case that $\Omega(v)$ is nonempty is considered. Since $\mathscr{L}(v) < \infty$ and $\mathscr{L}(T(t)v)$ is nonincreasing, then $\mathscr{L}(T(t)v)$ has a finite value for $t \epsilon R^+$. This implies that $\mathscr{L} \rightarrow \beta < \infty$ as $t \rightarrow \infty$, where $\beta = \inf_{t \epsilon R^+} \mathscr{L}(T(t)v)$. Since $\Omega(v)$ is assumed to be nonempty, $\Omega(v) \subset \overline{\gamma}(v) \subset \overline{G}$, and since $\mathscr{L}(v)$ is continuous from the definition of $\Omega(v)$, it follows that $\mathscr{L}(z) = \beta$ for every $z \epsilon \Omega(v)$. Furthermore, since $\Omega(v)$ is positive invariant, $\dot{\mathscr{L}}(z) = 0$ for every $z \epsilon \Omega(v)$ and proof is complete [26].

Appendix C: Derivation of Model for MPD Systems

The plasma dynamic equations for this system consist of Maxwell's equations, Ohm's law, conservation of electric charge, equation of state (ideal gas law) and a set of mass, momentum and energy equations [70]. In derivation of these equations it is assumed that the bulk properties of the gas (plasma) are shared by all species contained in the plasma.

Maxwell equations:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \epsilon \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \mu_e \vec{H}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon} \rho_e$$

Ohm's law:

$$J_i = \sigma \left[E_i + \mu_e (\vec{u^*} \times \vec{H})_i \right] + \rho_e u^*_i \qquad i \triangleq \text{direction index } x,y,z \text{ or } x_1, x_2, x_3$$

Conservation of electric charge:

$$\frac{\partial \rho_e}{\partial t} + \sum_{j=1}^{3} \frac{\partial J_j}{\partial x_j} = 0 \qquad j \triangleq \text{direction index}$$

Equation of state (ideal gas law):

$$P = R \rho T$$

$$P = \text{thermodynamic pressure}$$

Conservation of Mass:

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\rho u^*_j) = 0$$

Conservation of Momentum:

$$\rho \frac{Du^*_i}{Dt} = - \frac{\partial P_t}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} + F_{e_i} + F_{g_i} + F_v$$

where,

P = sum of thermodynic pressure and radiation pressure, the latter is negligible.

$F_{e_i} = \rho_e E_i + \mu_e (\vec{J} \times \vec{H})_i \qquad i \triangleq x,y,z \text{ or } x_1,x_2,x_3$

F_{g_i} = gravity force per volume, negligible.

$\tau_{ij} = \mu \frac{\partial u^*_i}{\partial x_j}$

F_v = collisional forces, negligible.

Energy Equation:

$$\frac{\partial \rho \bar{e}_m}{\partial t} + \sum_{j=1}^{3} \frac{\partial \rho \bar{e}_m \, u^*_j}{\partial x_j} = - \frac{\partial u^*_j P_t}{\partial x_j} + \frac{\partial u^*_j \tau_{ij}}{\partial x_j} + E_j J_j + \frac{\partial Q_j}{\partial x_j}$$

where $\bar{e}_m$ = total energy per unit mass $\simeq c_v T$.

Q_j = heat flux in direction j.

In this model, it is assumed that the plasma is originally at rest with the pressure P_o, temperature T_o, and density ρ_o. An external uniform magnetic field $\vec{H}_o$ is applied to the system, where

$$\vec{H}_o = \vec{i} H_x + \vec{j} H_y + \vec{k} 0 .$$

There is no electric field applied to the system. Plasma is perturbed by a small disturbance and, as a result, the state of the system is a combination of the stationary (equilibrium) part and the perturbed portion. The velocity vector for the basic flow is zero. Therefore,

$$u^* = i\,u(x,t) + jv(x,t) + k\,w(x,t)$$

However, an assumption is made that the variations of variables are only functions of one spatial dimension, x, and time. Therefore, instantaneous pressure, temperature and density can be written as

$$P = P_o + P'(x,t)$$
$$T = T_o + T'(x,t)$$
$$\rho = \rho_o + \rho'(x,t)$$

Electric and magnetic fields can be represented as

$$\vec{E} = i\,E_x(x,t) + j\,E_y(x,t) + k\,E_z(x,t)$$
$$\vec{H} = \vec{H}_o + \vec{h}(x,t)$$
$$= i\left[H_x + h_x(x,t)\right] + j\left[H_y + h_y(x,t)\right] + k\,h_z(x,t)$$

Current density J and net electric charge ρ_e are

$$\vec{J} = \vec{J}(x,t),\ \rho_e = \rho_e(x,t)\,.$$

The one-dimensional assumption results in: $\frac{\partial(\cdot)}{\partial y} = 0,\ \frac{\partial(\cdot)}{\partial z} = 0.$

Inserting the above simplification into the set of general dynamic equations, the following describing equations can be derived:

Maxwell's equations:

$$J_x + \epsilon \frac{\partial E_x}{\partial t} = 0 \tag{C-1}$$

$$J_y + \epsilon \frac{\partial E_y}{\partial t} = - \frac{\partial h_z}{\partial x} \tag{C-2}$$

$$J_z + \epsilon \frac{\partial E_z}{\partial t} = \frac{\partial h_y}{\partial x} \tag{C-3}$$

$$\mu_e \frac{\partial h_x}{\partial t} = 0 \tag{C-4}$$

$$\mu_e \frac{\partial h_y}{\partial t} = \frac{\partial E_z}{\partial x} \tag{C-5}$$

$$\mu_e \frac{\partial h_z}{\partial t} = - \frac{\partial E_y}{\partial x} \tag{C-6}$$

Generalized Ohm's law:

$$J_x = \sigma(E_x - \mu_e \, wH_y) + \rho_e u \tag{C-7}$$

$$J_y = \sigma(E_y + \mu_e \, wH_x) + \rho_e v \tag{C-8}$$

$$J_z = \sigma(E_z + \mu_e u H_y - \mu_e v H_x) + \rho_e w \tag{C-9}$$

Conservation of electric charge:

$$\frac{\partial \rho_e}{\partial t} + \frac{\partial J_x}{\partial x} = 0 \tag{C-10}$$

The equation of state for perturbed variables is

$$\frac{P'}{P_o} = \frac{\rho'}{\rho_o} + \frac{T'}{T_o}, \qquad \text{where } P_o = \rho_o R T_o \tag{C-11}$$

The linearized continuity equation becomes:

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u}{\partial x} = 0 \tag{C-12}$$

The linearized equations of momentum are:

$$\rho_o \frac{\partial u}{\partial t} = - \frac{\partial P'}{\partial x} + \frac{4}{3} \mu \frac{\partial^2 u}{\partial x^2} - \mu_e J_z H_y + \rho_e E_x \tag{C-13}$$

$$\rho_o \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} + \mu_e J_z H_x + \rho_e E_y \tag{C-14}$$

$$\rho_o \frac{\partial w}{\partial t} = \mu \frac{\partial^2 w}{\partial x^2} + \mu_e (J_x H_y - H_x J_y) + \rho_e E_z \tag{C-15}$$

It is assumed that the nonlinear perturbation terms are negligible in comparison with the linear terms. Therefore, the energy equation becomes

$$\rho_o c_v \frac{\partial T'}{\partial t} = - \rho_o R T_o \frac{\partial u}{\partial x} + K \frac{\partial^2 T'}{\partial x^2} \tag{C-16}$$

Decoupled Modes of Motion

In the case of a neutral plasma, i.e., $\rho_e \simeq 0$, the number of ions and electrons per volume of plasma are nearly equal. For this case, if one considers the fact that $\frac{\partial h_x}{\partial t} = 0$, $\frac{\partial h_x}{\partial x} = 0$, then it is possible to distinguish between two modes of wave propagation: transverse mode (z-direction) and longitudinal mode. In the transverse mode, the states are found to be h_z and w, and the state equations can be formed from the reduction of equations (C-1), (C-2), (C-6), (C-7), (C-8), (C-10), and (C-15). The rest of the equations can be reduced to the form state equations for the longitudinal mode.

(i) Transverse Mode

The magneto-gas-dynamic assumption results in an insignificant magnetic induction effect in Maxwell's equations from the terms carrying variations of electric field with time. This is due to the fact that nondimensional parameters $R_t = \frac{t_o u^*}{\ell}$ and $R_E = \frac{E_o}{\mu_e u^* H_o}$ are of the order of one or smaller, and

$R_c = \frac{u^{*2}}{c^2} = u^{*2}\mu_e\epsilon \ll 1$ [70]. The resulting equations are

$$\frac{\partial h_z}{\partial t} = \mu_H \frac{\partial^2 h_z}{\partial x^2} + H_x \frac{\partial w}{\partial x} \tag{C-17}$$

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial x^2} + \frac{V_x^2}{H_x} \frac{\partial h_z}{\partial x} \tag{C-18}$$

where $\nu_H = \frac{1}{\sigma\mu_e}$ and $V_x = \sqrt{\frac{\mu_e}{\rho_o}}\, H_x$. The parameter V_x is defined as the x-component of the speed of the Alfven wave.

(ii) Longitudinal Mode

The state equation for this mode can be reduced to

$$\frac{\partial h_y}{\partial t} = \nu_H \frac{\partial^2 h_y}{\partial x^2} - H_y \frac{\partial u}{\partial x} + H_x \frac{\partial v}{\partial x} \tag{C-19}$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} + \frac{V_x^2}{H_x} \frac{\partial h_y}{\partial x} \tag{C-20}$$

$$\frac{\partial u}{\partial t} = -RT_o \left[\frac{\partial \rho''}{\partial x} + \frac{\partial T''}{\partial x}\right] + \frac{4}{3}\nu \frac{\partial^2 u}{\partial x^2} - \frac{V^2 y}{H_y} \frac{\partial h_y}{\partial x} \tag{C-21}$$

$$\frac{\partial p''}{\partial t} = -\frac{\partial u}{\partial x} \tag{C-22}$$

$$\frac{\partial T''}{\partial t} = \frac{K}{\rho_o c_\nu} \frac{\partial^2 T''}{\partial x^2} - \frac{R}{c_\nu} \frac{\partial u}{\partial x} \tag{C-23}$$

where $\rho'' = \frac{\rho'}{\rho_o}$, $\frac{T''}{T_o}$, $V_y = \sqrt{\frac{\mu_e}{\rho_o}}\, H_y$. The parameter V_y is defined as the y-component of the speed of the Alfven wave.